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On a symmetry of the category of integrable modules

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Abstract. Haisheng Li showed that given a module $(W, Y_W(\cdot, x))$ for a vertex algebra $(V, Y(\cdot, x))$, one can obtain a new V-module

$$W^{\Delta} = (W, Y_W(\Delta(x)\cdot, x))$$

if $\Delta(x)$ satisfies certain natural conditions. Li presented a collection of such Δ-operators for $V = L(k, 0)$ (a vertex operator algebra associated with an affine Lie algebra, k a positive integer). In this paper, for each irreducible $L(k, 0)$-module W, we find a highest weight vector of W^{Δ} when Δ is associated with a minuscule coweight. From this we completely determine the action of these Δ-operators on the set of isomorphism equivalence classes of $L(k, 0)$-modules.

Contents

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1. Introduction

Haisheng Li introduced his Δ-operators in a very general setting in [Li1]. These operators allow one to obtain new vertex algebra modules from old ones by modifying the vertex algebra's action on the module while leaving the underlying vector space unchanged. Thus, given a vertex algebra V and a collection of Δ-operators, we obtain (usually quite interesting) symmetries of the category of V-modules.

Consider the $\hat{\mathfrak{g}}$-module $L(k,0) = L(k\Lambda_0)$ where k is a positive integer and $\hat{\mathfrak{g}}$ is an untwisted affine Lie algebra. It is well-known that $L(k,0)$ has the structure of a vertex operator algebra (VOA), and that $\hat{\mathfrak{g}}$-modules $L(k,\lambda)$ for certain λ are modules for this VOA. Define

$$\Delta(H,x) = x^{H(0)} \exp\left(\sum_{n\geq 1} \frac{H(n)}{-n}(-x)^{-n}\right)$$

where H is a coweight of $\mathfrak{g}$ (the underlying finite-dimensional Lie algebra). If $(W, Y(\cdot,x))$ is an $L(k,0)$-module, then $(W, Y(\Delta(H,x)\cdot,x))$ (call this module $W^{(H)}$) is also an $L(k,0)$-module. In fact, Li proved [Li2] that these two modules are equivalent if H is a coroot. However, when H is not a coroot, we may get a new (inequivalent) module. So, using Li's Δ-operators we can induce an action on the equivalence classes of $L(k,0)$-modules. In his thesis [C] (see also [CLM]), the first author was able to use these operators to obtain recurrence relations for characters of integrable highest weight $\hat{\mathfrak{g}}$-modules which in turn led to interesting combinatorial identities. These recurrence relations were obtained by studying the effect of $\Delta(H,x)$ (for H a coroot) on characters. If we consider a coweight instead of a coroot, we obtain a relation between the characters of two *different* integrable $\hat{\mathfrak{g}}$-modules.

In this paper, we completely determine the action of $\Delta(H,x)$ on equivalence classes of $L(k,0)$-modules for all coweights H.

First, recall that $L(k,0)$ is a *regular* VOA. This implies (among other things) that its modules are completely reducible. This means that we just need to determine $\Delta(H,x)$'s action on irreducible $L(k,0)$-modules (these are precisely the integrable $\hat{\mathfrak{g}}$-modules of level k).

Next, we know that the minuscule coweights provide a complete set of coset representatives for the coweight lattice modulo the coroot lattice. So, since the coroots give back equivalent modules, it is enough to study the action of $\Delta(H,x)$ on an irreducible $L(k,0)$-module $L(k,\lambda)$ where H is one of the minuscule coweights. To determine which module $L(k,\lambda^{(H)})$ is obtained from $L(k,\lambda)$ via the $\Delta(H,x)$-action, it is enough to identify a highest weight vector and "measure" its weight.

In this paper we explicitly determine a highest weight vector for $L(k,\lambda)^{(H)}$ — that is, the module $(L(k,\lambda), Y(\Delta(H,x)\cdot,x))$ — if $L(k,\lambda)$ is an $L(k,0)$-module and H is a minuscule coweight (of $\mathfrak{g}$).

Recall that the $\hat{\mathfrak{g}}$-modules $L(k,\lambda)$ are induced up from $\mathfrak{g}$-modules $L(\lambda)$ and in fact $L(\lambda)$ (the "finite-dimensional part") makes up the lowest conformal weight space of $L(k,\lambda)$. Now let us change the action of $L(k,\lambda)$ from $Y(\cdot,x)$ to $Y(\Delta(H,x)\cdot,x)$ where H is a minuscule coweight. We get a new $L(k,0)$-module action while leaving the underlying vector space fixed. It is interesting to note that in each case, the highest weight vector stays inside the lowest conformal weight space of $L(k,\lambda)$. Li in [Li2] considered the special case, $L(k,0)$, and found that the old highest weight vector (which is the vaccuum vector) is also the new highest weight vector. This happens because the lowest conformal weight space is 1-dimensional — that is, the vector has nowhere to go. On the other hand, when we consider $L(k,\lambda)$, $\lambda \neq 0$, the lowest conformal weight space is bigger and so the highest weight vector moves and thus is harder to find.

One rather interesting feature of the action of $\Delta(H,x)$ is that the restriction on λ so that $L(k,\lambda)$ is an $L(k,0)$-module shows up explicitly in this action. We know that if $L(k,\lambda)$ is an $L(k,0)$-module, then $\langle\lambda,\theta\rangle \leq k$ where θ is the highest long root of $\mathfrak{g}$. In each case, when acting on $L(k,\lambda)$ with $\Delta(H,x)$ where H is the j^{th} minuscule coweight, we see that the coefficent of λ_j (the j^{th} fundamental weight) is replaced by $k - \langle\lambda,\theta\rangle$.

Without making any changes, the calculations and proofs involved in determining the new highest weight vectors of the $L(k,0)$-modules $L(k,\lambda)^{(H)}$ (H the j^{th} minuscule coweight) apply to any $V(k,0)$-module as well ($V(k,0)$ is a generalized Verma module which also has VOA structure). It is well-known that $L(k,\lambda)$, where λ is a dominant integral weight of $\mathfrak{g}$, is a $V(k,0)$-module. However, if $\langle\lambda,\theta\rangle > k$ (so that $L(k,\lambda)$ is *not* an $L(k,0)$-module), then the $\Delta(H,x)$-action will produce an (irreducible) *weak* $V(k,0)$-module $L(k,\lambda^{(H)})$. But $L(k,\lambda^{(H)})$ is not a $V(k,0)$-module (as a VOA) since the coffecient of λ_j in $\lambda^{(H)}$ is negative (and thus not a dominant integral weight). So for $V(k,0)$-modules, the Δ-action can send (strong) modules to weak modules.

It is interesting to note that Li's Δ-operators also allow one to create new intertwining operators from old ones ([Li2], Proposition 2.12). In fact, since the Δ-operators are invertible, they give isomorphisms between spaces of intertwining operators. Therefore, using the results above, one can obtain symmetries of fusion rules. This is the topic of a future project of the authors with Sjuvon Chung and Yi-Zhi Huang.

The contents of the paper are organized as follows:

In the second section, we begin by reviewing the definition of a vertex operator algebra and its modules. Then, we set up all of the necessary notations related to finite-dimensional simple Lie algebras and untwisted affine Lie algebras and conclude by introducing Li's Δ-operators and performing some preliminary calculations.

The third section illustrates our results in the most basic case — that of $\hat{\mathfrak{g}} = \widehat{sl_2}$. The fourth section tackles the general case where we must consider the effects of the Δ-operators one type at a time. In the course of deriving these results, we need to perform some tedious Weyl group calculations. The calculations for types B_ℓ, C_ℓ, and D_ℓ are given in an appendix (the fifth section). The appendix also contains a summary of results for types E_6 and E_7. The calculations for types E_6 and E_7 were performed in Maple using a modified version of a worksheet developed by the first author for another project [CMS].

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2. Definitions and background

We will begin by reviewing the definition of a vertex operator algebra and its modules. Further details can be found in [LL].

2.1. Vertex algebras. A *vertex algebra* is a complex vector space V equipped with a linear map,

$$\begin{aligned} Y(\cdot, x) : V &\to (\operatorname{End} V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \end{aligned}$$

and a distinguished vector $\mathbf{1} \in V$ (the *vacuum vector*) such that for $u, v \in V$, $u_n v = 0$ for n sufficiently large. The operator u_n is called the n^{th}*-mode* of u.

It is assumed that the vacuum vector behaves like an identity in that $Y(\mathbf{1}, x) = \mathrm{Id}_V$ and for $v \in V$

$$Y(v,x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v,x)\mathbf{1} = v \qquad \text{(the creation axiom)}.$$

Finally, we also must require that the *Jacobi identity* holds: for all $u, v \in V$

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u,x_1)Y(v,x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v,x_2)Y(u,x_1) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u,x_0)v, x_2).$$

Please note that $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$ is the *formal delta function* and we adopt the binomial expansion convention, namely, $(x+y)^n$ should be expanded in nonnegative powers of y.

The vertex algebras that we will consider have additional structure making them vertex operator algebras. A *vertex operator algebra* is a vertex algebra V with the following additional data:

V is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ (over $\mathbb{C}$) such that $\dim V_{(n)} < \infty$ for all $n \in \mathbb{Z}$ and $V_{(n)} = 0$ for n sufficiently negative.

Elements of $V_{(n)}$ are said to have *conformal weight* n. The vacuum must have conformal weight 0 (e.g., $\mathbf{1} \in V_{(0)}$).

V has a second distinguished vector $\omega \in V_{(2)}$ (the *conformal vector*), where

$$Y(\omega, x) = \sum_{n\in\mathbb{Z}} \omega_n x^{-n-1} = \sum_{n\in\mathbb{Z}} L(n) x^{-n-2}.$$

The modes of the conformal vector satisfy the *Virasoro* relations:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}\, c_V \quad \text{for } m, n \in \mathbb{Z}. \tag{2.1}$$

The scalar $c_V \in \mathbb{C}$ is called the *central charge* (or *rank*) of V. Finally, we must require that

$$\begin{aligned} Y(L(-1)v, x) &= \frac{d}{dx} Y(v, x) \quad \text{for } v \in V, \\ V_{(n)} &= \{v \in V \mid L(0)v = nv\} \quad \text{for } n \in \mathbb{Z}. \end{aligned} \tag{2.2}$$

Let V be a vertex algebra. A V-module is a complex vector space W equipped with a linear map

$$\begin{aligned} Y_W(\cdot, x) : V &\to (\operatorname{End} W)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \end{aligned}$$

such that for $v \in V$ and $w \in W$, $v_n w = 0$ for n sufficiently large. Also, $Y_W(\mathbf{1}, x) = \mathrm{Id}_W$ and the Jacobi identity holds.

If V is a vertex operator algebra, we say W is a (strong) V-module if W is a module for V thought of as a vertex algebra and in addition W is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{Z}} W_{(n)}$ where $W_{(n)} = \{w \in W \mid L(0)w = nw\}$ such that for all $n \in \mathbb{C}$, $\dim W_{(n)} < \infty$ and $W_{(n+r)} = 0$ for r sufficiently negative.

If W is a vertex algebra module for a vertex operator algebra V, we say that W is a *weak* V-module.

2.2. Affine Lie algebras. Following [H] and [K], we now establish some notation and review some basic definitions and facts concerning (untwisted) affine Lie algebras.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank ℓ (over $\mathbb{C}$). Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and let $\langle \cdot, \cdot \rangle$ be the standard form such that

$\|\alpha\|^2 = \langle \alpha, \alpha \rangle = 2$ for any long root α. Let Δ be the set of roots of $\mathfrak{g}$. Fix a set of simple roots $\{\alpha_1, \dots, \alpha_\ell\}$, simple coroots $\{H_1, \dots, H_\ell\}$, and Chevalley generators $\{E_i, F_i, H_i \,|\, i = 1, \dots, \ell\}$. Recall that $\alpha_j(H_i) = a_{ij}$ where $C = (a_{ij})_{i,j=1}^{\ell}$ is the Cartan matrix of $\mathfrak{g}$ Let $\Delta^+ \subset \Delta$ be the system of positive roots corresponding to the α's. We have the triangular decomposition:

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_- \qquad \text{where} \qquad \mathfrak{g}_\pm = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{\pm\alpha}.$$

Let $\{\lambda_1, \dots, \lambda_\ell\} \subset \mathfrak{h}^*$ be the fundamental weights and let $\{H^{(1)}, \dots, H^{(\ell)}\}$ be the fundmental coweights. Of course, the fundamental weights are dual to the simple coroots (i.e., $\lambda_i(H_j) = \delta_{ij}$) and the fundamental coweights are dual to the simple roots (i.e., $\alpha_i(H^{(j)}) = \delta_{ij}$). We should also note that $H_i = \sum_{i=1}^{\ell} a_{ji} H^{(j)}$.

Let Q and P denote the root lattice and weight lattice of $\mathfrak{g}$, respectively:

$$Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_\ell \quad \text{and} \quad P = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_\ell$$

and let $Q^\vee$ and $P^\vee$ denote the coroot lattice and coweight lattice of $\mathfrak{g}$, respectively:

$$Q^\vee = \mathbb{Z}H_1 + \cdots + \mathbb{Z}H_\ell \quad \text{and} \quad P^\vee = \mathbb{Z}H^{(1)} + \cdots + \mathbb{Z}H^{(\ell)}.$$

Define the set of *dominant integral weights* by:

$$P_+ = \{\lambda \in \mathfrak{h}^* | \lambda(H_i) \in \mathbb{Z}_{\geq 0},\ 1 \leq i \leq \ell\}.$$

For $\lambda \in \mathfrak{h}^*$, define the *Verma module* of highest weight λ by:

$$V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{g}_+)} \mathbb{C}_\lambda, \tag{2.3}$$

where $\mathbb{C}_\lambda$ is a 1-dimensional $\mathfrak{h} \oplus \mathfrak{g}_+$-module given by:

$$\begin{aligned} h \cdot 1 &= \lambda(h) && \text{for all } h \in \mathfrak{h} \\ g \cdot 1 &= 0 && \text{for all } g \in \mathfrak{g}_+. \end{aligned}$$

Let $J(\lambda)$ be the maximal proper submodule of $V(\lambda)$. Then

$$L(\lambda) = V(\lambda)/J(\lambda)$$

is an irreducible (highest weight) $\mathfrak{g}$-module. Of course, $L(\lambda)$ is finite-dimensional if and only if $\lambda \in P_+$.

Let α be a root, then the map $\sigma_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ defined by

$$\sigma_\alpha(\lambda) = \lambda - \frac{2\langle \alpha, \lambda \rangle}{\|\alpha\|^2}\alpha = \lambda - \frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

is called the *reflection* associated with α (notice that $\sigma_\alpha(\alpha) = -\alpha$ and $\sigma_\alpha^2 = \mathrm{Id}_{\mathfrak{h}^*}$). The group W generated by these reflections is called the *Weyl group* of $\mathfrak{g}$. Let $\sigma_i = \sigma_{\alpha_i}$ for each simple root α_i. The σ_i's are called *simple reflections*. It is well-known that W is generated by simple reflections. The elements of W are isometries (with respect to the standard form) of $\mathfrak{h}^*$. Note: we can (and do) transport the action of W on $\mathfrak{h}^*$ to an action on $\mathfrak{h}$ using the standard form.

The (untwisted) affine Lie algebra associated with $\mathfrak{g}$ is given by

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c, \tag{2.4}$$

where for $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$,

$$\begin{aligned}[a \otimes t^m, b \otimes t^n] &= [a, b] \otimes t^{m+n} + m\langle a, b\rangle \delta_{m+n,0}c,\\ [\hat{\mathfrak{g}}, c] &= 0.\end{aligned}$$

For $a \in \mathfrak{g}, n \in \mathbb{Z}$, let $a(n)$ denote the action of $a \otimes t^n \in \hat{\mathfrak{g}}$ on a $\hat{\mathfrak{g}}$-module. Let $\theta = \sum_{i=1}^{\ell} a_i\alpha_i$ be the highest long root of $\mathfrak{g}$, and choose (nonzero) vectors $E_\theta \in \mathfrak{g}_\theta$ and $F_\theta \in \mathfrak{g}_{-\theta}$ such that $\langle E_\theta, F_\theta\rangle = 1$. Let $H_\theta = [E_\theta, F_\theta]$. Define

$$e_0 = F_\theta \otimes t, \qquad f_0 = E_\theta \otimes t^{-1}, \qquad \text{and} \qquad h_0 = [e_0, f_0] \tag{2.5}$$

and for $i = 1, \ldots, \ell$ define:

$$e_i = E_i \otimes 1, \qquad f_i = F_i \otimes 1, \qquad \text{and} \qquad h_i = H_i \otimes 1. \tag{2.6}$$

Then, by [K], $\{e_i, f_i, h_i \,|\, 0 \le i \le \ell\}$ is a set of Chevalley generators for $\hat{\mathfrak{g}}$.

Let $\hat{\mathfrak{g}}_{>0} = \mathfrak{g} \otimes t\mathbb{C}[t]$ and $\hat{\mathfrak{g}}_0 = \mathfrak{g} \otimes 1 \oplus \mathbb{C}c$. Also, let

$$\hat{\mathfrak{g}}_{\ge 0} = \hat{\mathfrak{g}}_{>0} \oplus \hat{\mathfrak{g}}_0 = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c$$

and fix a scalar $k \in \mathbb{C}$. We can make $L(\lambda)$ into a $\hat{\mathfrak{g}}_{\ge 0}$-module by extending the action of $\mathfrak{g} = \mathfrak{g} \otimes 1$ as follows:

$$\begin{aligned} c \cdot v &= kv \qquad \text{for all } v \in L(\lambda),\\ x \cdot v &= 0 \qquad \text{for all } x \in \mathfrak{g}_{>0}.\end{aligned}$$

We now induce up to a $\hat{\mathfrak{g}}$-module:

$$V(k, \lambda) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\ge 0})} L(\lambda).$$

$V(k, \lambda)$ is a *generalized Verma module.*

Unless k is the negative of the dual Coxeter number of $\mathfrak{g}$, $V(k, 0)$ has the structure of a vertex operator algebra. For such k, $V(k, \lambda)$ is a weak $V(k, 0)$-module for all $\lambda \in \mathfrak{h}^*$. Moreover, $V(k, \lambda)$ is a (strong) $V(k, 0)$-module if (and only if) $\lambda \in P_+$.

Let $J(k, \lambda)$ be the maximal proper submodule of $V(k, \lambda)$, and define

$$L(k, \lambda) = V(k, \lambda)/J(k, \lambda).$$

If k is not the negative of the dual Coxeter number of $\mathfrak{g}$, then $L(k, 0)$ has the structure of a simple vertex operator algebra, and each $L(k, \lambda)$ is an irreducible weak (unless $\lambda \in P_+$) $V(k, 0)$-module.

If k is a positive integer, then:

Theorem 2.1 ([DL])**.** *$L(k, \lambda)$ is an $L(k, 0)$-module if and only if $\langle \lambda, \theta\rangle \le k$, where θ is the highest long root of $\mathfrak{g}$.*

2.3. Li's $\mathbf{\Delta}$-operators. Now, let Let $H \in P^\vee$. Set

$$\Delta(H,x) = x^{H(0)} \exp\left(\sum_{n\geq 1} \frac{H(n)}{-n}(-x)^{-n}\right) \tag{2.7}$$

(recall the $H(n)$ is the action of $H \otimes t^n$ on a $\hat{\mathfrak{g}}$-module).

Note that $\Delta(H,x)$ enjoys the following properties:

$$\begin{aligned} \Delta(H_1+H_2,x) &= \Delta(H_1,x)\Delta(H_2,x), \\ \Delta(0,x) &= \mathrm{Id}. \end{aligned}$$

We fix the notation

$$(L(k,\lambda)^{(H)}, Y^{(H)}_{L(k,\lambda)}(\cdot,x)) = (L(k,\lambda), Y_{L(k,\lambda)}(\Delta(H,x)\cdot,x)).$$

For $v \in L(k,0)$, set

$$Y^{(H)}_{L(k,\lambda)}(v,x) = Y_{L(k,\lambda)}(\Delta(H,x)v,x) = \sum_{n\in\mathbb{Z}} v^{(H)}(n)x^{-n-1}. \tag{2.8}$$

Let us take care of some preliminary calculations involving $\Delta(H,x)$. For $g \in \mathfrak{g}_\beta$, $\beta \in \Delta \cup \{0\}$, so that $[h,g] = \beta(h)g$ for all $h \in \mathfrak{h}$ (Here, $\mathfrak{g}_0 = \mathfrak{h}$), we have:

$$\begin{aligned} [h(m), g(-1)] &= [h,g](m-1)\mathbf{1} + m\langle h,g\rangle(m-1)\mathbf{1} + \delta_{m-1,0}k\mathbf{1} \\ &= \beta(h)g(m-1)\mathbf{1} + \langle h,g\rangle\delta_{m,1}k\mathbf{1} \\ &= \begin{cases} \beta(h)g(-1)\mathbf{1} & m=0 \\ \langle h,g\rangle k\delta_{m,1}\mathbf{1} & m>0. \end{cases} \end{aligned}$$

Note that $g(m-1)\mathbf{1} = 0$ for $m > 0$ by the creation axiom.

We also have (for any $g \in \mathfrak{g}$):

$$\left(\sum_{m\geq 1} \frac{H(m)}{-m}(-x)^{-m}\right) g = \langle H,g\rangle k\mathbf{1}x^{-1}.$$

and for $n \geq 2$

$$\begin{aligned} \left(\sum_{m\geq 1} \frac{H(m)}{-m}(-x)^{-m}\right)^n g &= k\langle H,g\rangle x^{-1}\left(\sum_{m\geq 1} \frac{H(m)}{-m}(-x)^{-m}\right)^{n-1}\mathbf{1} \\ &= 0 \end{aligned} \tag{2.9}$$

(again using the creation axiom, $H(m)\mathbf{1} = 0$ for $m \geq 1$).

Therefore,

$$\begin{aligned} \Delta(H,x)g &= x^{H(0)}g + x^{H(0)}\langle H,g\rangle k\mathbf{1}x^{-1} \\ &= x^{\beta(H)}g + \langle H,g\rangle k\mathbf{1}x^{-1} \end{aligned} \tag{2.10}$$

if $g \in \mathfrak{g}_\beta$.

Applying this to our vertex operator map, we have:

$$
\begin{aligned}
Y^{(H)}(g,x) &= Y(\Delta(H,x)g,v) \\
&= x^{\beta(H)}Y(g,x) + \langle H,g\rangle k(\mathrm{Id})x^{-1}.
\end{aligned}
\tag{2.11}
$$

Let k be a positive integer and $\lambda \in P_+$ (a dominant integral weight). In addition, assume that $\langle \lambda, \theta\rangle \le k$ so that $L(k,\lambda)$ is an (irreducible) $L(k,0)$-module. It was proved in ([Li2], Proposition 2.9) that $(L(k,\lambda)^{(H)}, Y^{(H)}_{L(k,\lambda)})$ carries the structure of an irreducible (weak) $L(k,0)$-module. However, since $L(k,0)$ is a regular vertex operator algebra, its weak modules are in fact (strong) modules. So we have that:

Proposition 2.2. *There exists a (unique) $\lambda^{(H)} \in P_+$ such that $\langle \lambda^{(H)}, \theta\rangle \le k$ and $(L(k,\lambda))^{(H)} \cong L(k,\lambda^{(H)})$ as $L(k,0)$-modules.*

Also, Li established that:

Theorem 2.3 ([Li2], Proposition 2.25). *For $H \in Q^\vee$, $(L(k,\lambda))^{(H)}$ and $L(k,\lambda)$ are isomorphic as $L(k,0)$ modules.*

That is, in our notation, $\lambda^{(H)} = \lambda$ when $H \in Q^\vee$.

Our objective is to see what happens when we allow H to be any element of $P^\vee$ (not just $Q^\vee$).

Because $\Delta(H'+H'',x) = \Delta(H',x)\Delta(H'',x)$, we have the following:

$$
\begin{aligned}
L(k,\lambda^{(H'+H'')}) &\cong L(k,\lambda)^{(H'+H'')} \\
&= (L(k,\lambda)^{(H')})^{(H'')} \\
&\cong L(k,\lambda^{(H')})^{(H'')} \\
&\cong L(k,(\lambda^{(H')})^{(H'')}).
\end{aligned}
$$

Therefore, $\lambda^{(H'+H'')} = (\lambda^{(H')})^{(H'')}$.

The coweights $\{H^{(1)},\dots,H^{(\ell)}\}$ form a basis for $P^\vee$ and so, by the observation above, all we need to find is action of $\Delta(H^{(i)},x)$. Let us fix the notation $(L(k,\lambda))^{(i)} = (L(k,\lambda))^{(H^{(i)})}$ and $\lambda^{(i)} = \lambda^{(H^{(i)})}$ for $i = 1,\dots,\ell$ that is $L(k,\lambda^{(i)}) \cong (L(k,\lambda))^{(i)} = (L(k,\lambda))^{(H^{(i)})}$.

Actually, since the operators associated with coroots act trivially, we only need to consider one representative for each coset of $P^\vee/Q^\vee$. This pares down the list of coweights considerably.

We know that the irreducible $L(k,0)$-modules are precisely the irreducible integrable highest weight modules (i.e., standard modules) for $\hat{\mathfrak{g}}$ ([DL]). Now, an irreducible highest weight module is completely determined by its highest weight. Therefore, if we can locate a highest weight vector in $(L(k,\lambda))^{(i)}$ and measure its weight, we have determined $\lambda^{(i)}$ and thus the action of $\Delta(H^{(i)},x)$ on $L(k,\lambda)$.

Now, for our Chevalley generators e_i, f_i, h_i $(0 \le i \le \ell)$, we define ${e_i}^{(j)}$, ${f_i}^{(j)}$, and ${h_i}^{(j)}$ to be the actions of e_i, f_i, and h_i on $L(k,\lambda)^{(j)}$. Let us calculate these actions:

$$\begin{aligned} Y^{(j)}(H_i,x) &= Y(\Delta(H^{(j)},x)H_i,x) \\ &= Y(H_i,x) + \frac{2k}{\|\alpha_i\|^2}\delta_{i,j}x^{-1} \end{aligned}$$

and so we have that

$$(H_i)^{(j)}(0) = H_i(0) + \frac{2k}{\|\alpha_i\|^2}\delta_{i,j}. \tag{2.12}$$

For $e_i,\ 1 \le i \le \ell$, we have

$$\begin{aligned} Y^{(j)}(E_i,x) &= Y(\Delta(H^{(j)},x)E_i,x) \\ &= x^{\alpha_i(H^{(j)})}Y(E_i,x) \\ &= x^{\delta_{i,j}}Y(E_i,x) \end{aligned}$$

and so

$${e_i}^{(j)} = {E_i}^{(H^{(j)})}(0) = E_i(\delta_{i,j}). \tag{2.13}$$

Finally, for e_0, we have:

$$\begin{aligned} Y^{(j)}(F_\theta,x) &= Y(\Delta(H^{(j)},x)F_\theta,x) \\ &= x^{-\theta(H^{(j)})}Y(F_\theta,x) \\ &= x^{-a_j}Y(F_\theta,x), \end{aligned}$$

(recall that $\theta = \sum_{i=1}^{\ell} a_i\alpha_i$) and so

$${e_0}^{(j)} = {F_\theta}^{(j)}(1) = F_\theta(1-a_j). \tag{2.14}$$

3. The $sl_2(\mathbb{C})$ case

Before answering our question for all finite-dimensional simple Lie algebras let us consider the simplest case — that of $\mathfrak{g} = sl_2(\mathbb{C})$.

Let $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We know that E, F, H are Chevalley generators of $sl_2(\mathbb{C})$. Let α be the fundamental root of $sl_2(\mathbb{C})$ (that is, $\alpha(H) = 2$), and $\lambda = \frac{1}{2}\alpha$ be the fundamental weight of $sl_2(\mathbb{C})$. Since α is the only positive root, it is the highest long root — that is $\alpha = \theta$.

Then, H is the coroot corresponding to α and $H^{(1)} = \frac{1}{2}H$ is the coweight corresponding to λ.

In the $sl_2(\mathbb{C})$ case, we denote $L(k, n\lambda)$ by $L(k,n)$. Let k be a positive integer and $n \in \mathbb{Z}$ such that $0 \le n \le k$.

Theorem 3.1. $(L(k,n))^{(1)} \cong L(k,k-n)$. *Moreover, if v is a highest weight vector for $L(k,n)$, then $F(0)^n v$ is a highest weight vector for $(L(k,n))^{(1)}$ (with weight $\Lambda = (k,(k-n)\lambda)$).*

Proof. Since $-\theta = -\alpha$, we have $F = F_\theta$ and so, recalling (2.5), we have $e_0 = F_\theta \otimes t = F \otimes t$. By (2.14), we have that ${e_0}^{(1)} \cdot w = F(0) \cdot w$. Therefore,

$$ {e_0}^{(1)} \cdot F(0)^n v = F(0) \cdot F(0)^n v = F(0)^{n+1} \cdot v = 0 $$

by the representation theory of $sl_2(\mathbb{C})$.

Next, we have

$$ {e_1}^{(1)} \cdot F(0)^n v = E(1)F(0)^n v. $$

Since $L(k,n)$ is a vertex operator algebra module, we can consider conformal weights. Notice that as operators wt $F(0) =$ wt $(F) - 0 - 1 = 0$ and wt $E(1) =$ wt $(E) - 1 - 1 = -1$. This implies that ${e_1}^{(1)} \cdot F(0)^n v$ has a lower (conformal) weight than v. However, the highest weight vector occupies the lowest (conformal) weight space of $L(k,n)$. Thus ${e_1}^{(1)} F(0)^n v = 0$. Therefore, $F(0)^n v$ is a highest weight vector for $(L(k,n))^{(1)}$.

Now let us determine the weight of $F(0)^n v$. We will use the fact that $[H, F^n] = -2nF^n$ (in $U(sl_2)$).

$$ \begin{aligned} (H)^{(1)} \cdot F(0)^n v &= H(0)F(0)^n v + kF(0)^n v \\ &= F(0)^n H(0) v + [H(0), F(0)^n] v + kF(0)^n v \\ &= nF(0)^n v - 2nF(0)^n v + kF(0)^n v \\ &= (k-n)F(0)^n v. \end{aligned} $$

Hence, $(L(k,n))^{(1)} \cong L(k,k-n)$. □

Remark 3.2. The lowest conformal weight space of $L(k,n)$ is a copy of $L(n\lambda)$ (the finite-dimensional sl_2-module with highest weight $n\lambda$). The highest weight vector of $L(k,n)$ is located in this (lowest) conformal weight space.

Observe that the new highest weight vector for $(L(k,n))^{(1)}$ is also located in this copy of $L(n\lambda)$. In fact, we can obtain the new highest weight vector from the old one by applying the (only) reflection in the Weyl group of sl_2.

4. The general case

Recall that, given an irreducible $L(k,0)$-module W and coroot H, $W^{(H)}$ and W are isomorphic as $L(k,0)$-modules. This, along with the fact that $W^{(H'+H'')}$ is isomorphic to $(W^{(H')})^{(H'')}$, implies that we only need to consider one representative from each distinct coset of $P^\vee/Q^\vee$. It is well-known that $H = 0$ (which acts as the identity — $W^{(0)} = W$), along with the minuscule coweights, give us a complete set of coset representatives. We give a list of such coweights in Table 4.1.

Table 4.1.

Type	A_ℓ	B_ℓ	C_ℓ	D_ℓ	E_6	E_7
Coweights	$H^{(1)},\ldots,H^{(\ell)}$	$H^{(1)}$	$H^{(\ell)}$	$H^{(1)}$, $H^{(\ell-1)}$, $H^{(\ell)}$	$H^{(1)}$, $H^{(6)}$	$H^{(7)}$

Types E_8, F_4, and G_2 have no minuscule coweights, so the action of $\Delta(H,x)$ is always trivial.

Given any coweight H from this list, we wish to determine $\lambda^{(H)}$ such that $(L(k,\lambda))^{(H)}$ is isomorphic to $L(k,\lambda^{(H)})$. To do this, we need to identify a highest weight vector for $(L(k,\lambda))^{(H)}$ and then measure its weight. We will see that in each case our "new" highest weight vector (for $(L(k,\lambda))^{(H)}$) is located in the lowest (conformal) weight space. This lowest (conformal) weight space is a copy of $L(\lambda)$ — the finite-dimensional $\mathfrak{g}$-module from which $L(k,\lambda)$ is built. In fact, our "new" highest weight vector can be obtained from our "old" highest weight vector by applying the Weyl group elements given in Table 4.2. (For each type X_ℓ and coweight $H^{(j)}$, define an element $\sigma_X^{(j)}$ as in the table.)

Table 4.2.

Type	Weyl Group Element
A_ℓ	$1\le j\le \ell,\quad \sigma_A^{(j)}=(\sigma_1\sigma_2\cdots\sigma_\ell)^j$
B_ℓ	$\sigma_B^{(1)}=\sigma_1\sigma_2\cdots\sigma_{\ell-1}\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2\sigma_1$
C_ℓ	$\sigma_C^{(\ell)}=(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2)\cdots(\sigma_\ell\sigma_{\ell-1})(\sigma_\ell)$
D_ℓ	$\sigma_D^{(1)}=\sigma_1\sigma_2\cdots\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2\sigma_1$
ℓ even	$\sigma_D^{(\ell-1)}=(\sigma_{\ell-1}\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2\sigma_1)(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_3\sigma_2)$ $(\sigma_{\ell-1}\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_3)\cdots(\sigma_{\ell-1})$
	$\sigma_D^{(\ell)}=(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2\sigma_1)(\sigma_{\ell-1}\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_3\sigma_2)$ $(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_3)\cdots(\sigma_\ell)$
ℓ odd	$\sigma_D^{(\ell-1)}=(\sigma_{\ell-1}\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2\sigma_1)(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_3\sigma_2)$ $(\sigma_{\ell-1}\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_4\sigma_3)\cdots(\sigma_\ell)$
	$\sigma_D^{(\ell)}=(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2\sigma_1)(\sigma_{\ell-1}\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_3\sigma_2)$ $(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_4\sigma_3)\cdots(\sigma_{\ell-1})$
E_6	$\sigma_E^{(1)}=\sigma_1\sigma_3\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_1\sigma_6\sigma_5\sigma_4\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6$
E_6	$\sigma_E^{(6)}=\sigma_6\sigma_5\sigma_4\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_1\sigma_3\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_1$
E_7	$\sigma_E^{(7)}=\sigma_7\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_4\sigma_5\sigma_6\sigma_7\sigma_1\sigma_3\sigma_4\sigma_5\sigma_6\sigma_2\sigma_4\sigma_5\sigma_3\sigma_4\sigma_1\sigma_3\sigma_2\sigma_4\sigma_5\sigma_6\sigma_7$

Let w be a Weyl group element and $\mathbf{v}\in L(\lambda)_\mu$ (a vector of weight μ). Then $w(\mathbf{v})\in L(\lambda)_{w(\mu)}$. We begin by calculating the effects of the Weyl group elements defined above on the fundamental weights and simple roots of $\mathfrak{g}$. To simplify computations, we will use the following conventions:

$$\lambda_0 = 0 \qquad \lambda_j = \lambda_{j(\mathrm{mod}(l+1))}$$
$$\alpha_0 = -\theta \qquad \alpha_j = \alpha_{j(\mathrm{mod}(l+1))}.$$

We will repeatedly use the fact that $\sigma_i(\lambda_j) = \lambda_j - \delta_{i,j}\alpha_j$.

4.1. The Weyl group elements' actions. Let us begin by calculating the action of $\sigma_A^{(1)}$ on the fundamental weights and then use that to determine the action of our type A_ℓ Weyl group elements.

For λ_1, we have: $\sigma_1\sigma_2\cdots\sigma_\ell(\lambda_1) = \sigma_1(\lambda_1) = -\lambda_1 + \lambda_2$.

For λ_j, with $1 < j < \ell$:

$$\begin{aligned}
\sigma_1\sigma_2\cdots\sigma_\ell(\lambda_j) &= \sigma_1\sigma_2\cdots\sigma_j(\lambda_j)\\
&= \sigma_1\sigma_2\cdots\sigma_{j-1}(\lambda_{j-1} - \lambda_j + \lambda_{j+1})\\
&= \sigma_1\sigma_2\cdots\sigma_{j-2}(\lambda_{j-2} - \lambda_{j-1} + \lambda_{j+1})\\
&= \cdots\\
&= \sigma_1(\lambda_1 - \lambda_2 + \lambda_{j+1})\\
&= -\lambda_1 + \lambda_{j+1}.
\end{aligned}$$

Finally, for λ_ℓ:

$$\begin{aligned}
\sigma_1\sigma_2\cdots\sigma_\ell(\lambda_\ell) &= \sigma_1\sigma_2\cdots\sigma_{\ell-1}(\lambda_{\ell-1} - \lambda_\ell)\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-2}(\lambda_{\ell-2} - \lambda_{\ell-1})\\
&= \cdots\\
&= \sigma_1(\lambda_1 - \lambda_2)\\
&= -\lambda_1.
\end{aligned}$$

Adhering to our above convention, we conclude that for $1 \le j \le l$:

$$\sigma_1\sigma_2\cdots\sigma_\ell(\lambda_j) = -\lambda_1 + \lambda_{j+1}. \tag{4.1}$$

Therefore, for $1 \le j < \ell$ (recall that $\lambda_0 = 0$),

$$\begin{aligned}
\sigma_1\sigma_2\cdots\sigma_\ell(\alpha_j) &= \sigma_1\sigma_2\cdots\sigma_\ell(-\lambda_{j-1} + 2\lambda_j - \lambda_{j+1})\\
&= -(-\lambda_1 + \lambda_{j-1+1}) + 2(-\lambda_1 + \lambda_{j+1}) - (-\lambda_1 + \lambda_{j+1+1})\\
&= -\lambda_j + 2\lambda_{j+1} - \lambda_{j+2} = \alpha_{j+1}.
\end{aligned}$$

For α_ℓ, we have that

$$\begin{aligned}
\sigma_1\sigma_2\cdots\sigma_\ell(\alpha_\ell) &= \sigma_1\sigma_2\cdots\sigma_\ell(-\lambda_{\ell-1} + 2\lambda_\ell)\\
&= -(-\lambda_1 + \lambda\ell - 1 + 1) + 2(-\lambda_1 + \lambda_{\ell+1})\\
&= -\lambda_1 - \lambda_\ell = -\theta \ \ (= \alpha_0).
\end{aligned}$$

For $\alpha_0 = -\theta$, we have:

$$\begin{aligned}\sigma_1\sigma_2\cdots\sigma_\ell(-\theta) &= \sigma_1\sigma_2\cdots\sigma_\ell(-\lambda_1-\lambda_\ell)\\ &= -(-\lambda_1+\lambda_2)-(-\lambda_1+\lambda_{\ell+1})\\ &= 2\lambda_1-\lambda_2 = \alpha_1.\end{aligned}$$

So we have found that for all $0 \le i \le \ell$, $\sigma_A^{(1)}(\alpha_i) = \alpha_{i+1}$ and therefore

$$\sigma_A^{(j)}(\alpha_i) = (\sigma_1\sigma_2\cdots\sigma_\ell)^j(\alpha_i) = \alpha_{i+j}. \tag{4.2}$$

The calculations for types B_ℓ, C_ℓ, and D_ℓ are similar and can be found in the Appendix. Calculations for types E_6 and E_7 were done with the help of a Maple worksheet written and updated by the first author which was orginally written for [CMS]. This worksheet is available at

http://dimax.rutgers.edu/∼sadowski/LieAlgebraCalculations/index.html.

The Weyl group elements' actions are summed up in Table 4.3 (recall that $\alpha_0 = -\theta$).

4.2. The main theorems. Now we can apply our Weyl group calculations to determine the highest weight vector of $(L(k,\lambda))^{(i)}$ (which is simultaneously an $L(k,0)$ and $\hat{\mathfrak{g}}$ module). Suppose our underlying finite-dimensional simple Lie algebra $\mathfrak{g}$ is of type X_ℓ. Recall that $\theta = \sum_j a_j\alpha_j$ is the highest long root of $\mathfrak{g}$, k is a positive integer, $\lambda \in P_+$ with $\langle\lambda,\theta\rangle \le k$, and $H^{(i)}$ is one of the coweights appearing in Table 4.1. Also, note that for each valid choice of index i, we have $a_i = 1$.

The irreducible $\hat{\mathfrak{g}}$-module $L(k,\lambda)$ was built up from the irreducible $\mathfrak{g}$-module $L(\lambda)$. In fact, the lowest (conformal) weight space of $L(k,\lambda)$ is merely a copy of $L(\lambda)$. In what follows, let us identify $L(\lambda)$ with this subspace of $L(k,\lambda)$.

Let $\mathbf{v}$ be a highest weight vector for $L(k,\lambda)$. Such a vector is also homogeneous vector of lowest conformal weight in $L(k,\lambda)$. So we have that $\mathbf{v} \in L(\lambda) \subset L(k,\lambda)$.

Theorem 4.1. *$\sigma_X^{(i)}(\mathbf{v})$ is a highest weight vector for $(L(k,\lambda))^{(i)}$.*

Proof. Let $\mathbf{w} = \sigma_X^{(i)}(\mathbf{v})$ and $\mu = \sigma_X^{(i)}(\lambda)$.

Since $\sigma_X^{(i)}$ is an invertible map, $\mathbf{w} \neq \mathbf{0}$. The weight of $\mathbf{v}$ (as an element of $L(\lambda)$) is λ, so the weight of $\mathbf{w}$ is μ. To establish that $\mathbf{w}$ is a highest weight vector for $(L(k,\lambda))^{(i)}$, we need to show that $(e_j)^{(i)}\cdot\mathbf{w} = \mathbf{0}$ for $j = 0,\ldots,\ell$.

By (2.13), we have $(e_j)^{(i)}\cdot\mathbf{w} = E_j(\delta_{ij})(\mathbf{w})$ for $1 \le j \le \ell$.

When $j = i$, $(e_i)^{(i)}\cdot\mathbf{w} = E_i(1)(\mathbf{w})$. Now recall that the operator $E_i(1)$ lowers (conformal) weights by -1. Since $\mathbf{w}$ is already a lowest (conformal) weight vector, $(e_i)^{(i)}\cdot\mathbf{w} = \mathbf{0}$.

Next, when $j \neq i$, $(e_j)^{(i)}\cdot\mathbf{w} = E_j(0)(\mathbf{w})$ which is a vector of weight $\mu+\alpha_j$ (in the $\mathfrak{g}$-module $L(\lambda)$).

TABLE 4.3.

$A_\ell:$ $\sigma_A^{(j)}(\alpha_j) = \alpha_{j+i \,(\mathrm{mod}\ \ell+1)}$ $(0 \le j \le \ell$ and $1 \le i \le \ell)$.

$B_\ell:$ $\sigma_B^{(1)}(\alpha_0) = \alpha_1,\ \sigma_B^{(1)}(\alpha_1) = \alpha_0,$ and $\sigma_B^{(1)}(\alpha_j) = \alpha_j\ (1 < j \le \ell)$.

$C_\ell:$ $\sigma_C^{(\ell)}(\alpha_0) = \alpha_\ell,\ \sigma_C^{(\ell)}(\alpha_j) = \alpha_{\ell-j}\ (1 \le j < \ell),$ and $\sigma_C^{(\ell)}(\alpha_\ell) = \alpha_0$.

$D_\ell:$ $\sigma_D^{(1)}(\alpha_0) = \alpha_1,\ \sigma_D^{(1)}(\alpha_1) = \alpha_0,\ \sigma_D^{(1)}(\alpha_j) = \alpha_j\ (1 < j < \ell-1),$
$\sigma_D^{(1)}(\alpha_{\ell-1}) = \alpha_\ell,$ and $\sigma_D^{(1)}(\alpha_\ell) = \alpha_{\ell-1}$.

ℓ odd $\sigma_D^{(\ell-1)}(\alpha_0) = \alpha_{\ell-1},\ \sigma_D^{(\ell-1)}(\alpha_1) = \alpha_\ell,$ and
$\sigma_D^{(\ell-1)}(\alpha_j) = \alpha_{\ell-j}\ (1 < j \le \ell)$.
$\sigma_D^{(\ell)}(\alpha_0) = \alpha_\ell,\ \sigma_D^{(\ell)}(\alpha_j) = \alpha_{\ell-j}\ (1 \le j < \ell-1),$
$\sigma_D^{(\ell)}(\alpha_{\ell-1}) = \alpha_0,$ and $\sigma_D^{(\ell)}(\alpha_\ell) = \alpha_1$.

ℓ even $\sigma_D^{(\ell-1)}(\alpha_0) = \alpha_{\ell-1},\ \sigma_D^{(\ell-1)}(\alpha_1) = \alpha_\ell,$
$\sigma_D^{(\ell-1)}(\alpha_j) = \alpha_{\ell-j}\ (1 < j < \ell-1),$
$\sigma_D^{(\ell-1)}(\alpha_{\ell-1}) = \alpha_0,$ and $\sigma_D^{(\ell-1)}(\alpha_\ell) = \alpha_1$.

$\sigma_D^{(\ell)}(\alpha_j) = \alpha_{\ell-j}\ (0 \le j \le \ell)$.

$E_6:$ $\sigma_E^{(1)}(\alpha_0) = \alpha_1,\ \sigma_E^{(1)}(\alpha_1) = \alpha_6,\ \sigma_E^{(1)}(\alpha_2) = \alpha_3,\ \sigma_E^{(1)}(\alpha_3) = \alpha_5,$
$\sigma_E^{(1)}(\alpha_4) = \alpha_4,\ \sigma_E^{(1)}(\alpha_5) = \alpha_2,$ and $\sigma_E^{(1)}(\alpha_6) = \alpha_0$.

$\sigma_E^{(6)}(\alpha_0) = \alpha_6,\ \sigma_E^{(6)}(\alpha_1) = \alpha_0,\ \sigma_E^{(6)}(\alpha_2) = \alpha_5,\ \sigma_E^{(6)}(\alpha_3) = \alpha_2,$
$\sigma_E^{(6)}(\alpha_4) = \alpha_4,\ \sigma_E^{(6)}(\alpha_5) = \alpha_3,$ and $\sigma_E^{(6)}(\alpha_6) = \alpha_1$.

$E_7:$ $\sigma_E^{(7)}(\alpha_0) = \alpha_7,\ \sigma_E^{(7)}(\alpha_1) = \alpha_6,\ \sigma_E^{(7)}(\alpha_2) = \alpha_2,\ \sigma_E^{(7)}(\alpha_3) = \alpha_5,$
$\sigma_E^{(7)}(\alpha_4) = \alpha_4,\ \sigma_E^{(7)}(\alpha_5) = \alpha_3,\ \sigma_E^{(7)}(\alpha_6) = \alpha_1,$ and $\sigma_E^{(7)}(\alpha_7) = \alpha_0$.

By (2.14), we have $(e_0)^{(i)} \cdot \mathbf{w} = F_\theta(1-a_i)(\mathbf{w}) = F_\theta(0)(\mathbf{w})$ which has weight $\mu - \theta = \mu + \alpha_0$.

Therefore, we have reduced our problem to establishing that $\mu + \alpha_j$ for $j \ne i$, $0 \le j \le \ell$ are not weights of the $\mathfrak{g}$-module $L(\lambda)$.

Since Weyl group elements permute the weights of $L(\lambda)$, if $\mu + \alpha_j$ is a weight, then $(\sigma_X^{(i)})^{-1}(\mu + \alpha_j) = \lambda + (\sigma_X^{(i)})^{-1}(\alpha_j)$ must be a weight as well.

However, we can see by inspecting Table 4.3 that $\sigma_X^{(i)}$ permutes the set of simple roots along with $\alpha_0 = -\theta$. Notice that each case $\sigma_X^{(i)}(\alpha_0) = \alpha_i$. Therefore, $\left(\sigma_X^{(i)}\right)^{-1}$ maps each α_j where $j \neq i$, $0 \leq j \leq \ell$ to some α_k where $1 \leq k \leq \ell$. But λ is a highest weight vector for $L(\lambda)$, therefore $\lambda + \alpha_k$ $(1 \leq k \leq \ell)$ is not a weight. This implies that $\mu + \alpha_j$ (for $j \neq i$, $0 \leq j \leq \ell$) is not a weight. Therefore, $\mathbf{w}$ is annihilated by the action of each $(e_j)^{(i)}$ (for $0 \leq j \leq \ell$) and thus is a highest weight vector for $(L(k,\lambda))^{(i)}$. □

Remark 4.2. Elements of the Weyl group are linear transformations, so they always send the linear functional $\lambda = 0$ to itself. This implies that a highest weight vector for $L(k,0)$ (i.e., a nonzero scalar multiple of the vaccuum vector) is still a highest weight vector for $(L(k,0))^{(i)}$. This special case was discussed in [Li2].

Theorem 4.3. *Recall* $(L(k,\lambda))^{(i)} \cong L(k,\lambda^{(i)})$. *Let* $\lambda = \sum_{j=1}^{\ell} m_j\lambda_j$. *Then we have the following for the displayed types:*

A_ℓ
$$\lambda^{(i)} = \sum_{j=1}^{\ell} m_j\lambda_{j+i} + (k - \langle\lambda,\theta\rangle)\lambda_i$$

B_ℓ
$$\lambda^{(1)} = (k - \langle\lambda,\theta\rangle)\lambda_1 + \sum_{j=2}^{\ell} m_j\lambda_j$$

C_ℓ
$$\lambda^{(\ell)} = \sum_{j=1}^{\ell} m_j\lambda_{\ell-j} - (k - \langle\lambda,\theta\rangle)\lambda_\ell$$

D_ℓ
$$\lambda^{(1)} = (k - \langle\lambda,\theta\rangle)\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_j + m_{\ell-1}\lambda_\ell + m_\ell\lambda_{\ell-1}$$

(ℓ odd)
$$\lambda^{(\ell-1)} = m_{\ell-1}\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} + (k - \langle\lambda,\theta\rangle)\lambda_{\ell-1} + m_1\lambda_\ell$$

(ℓ even)
$$\lambda^{(\ell-1)} = m_\ell\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} + (k - \langle\lambda,\theta\rangle)\lambda_{\ell-1} + m_1\lambda_\ell$$

(ℓ odd)
$$\lambda^{(\ell)} = m_\ell\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} + m_1\lambda_{\ell-1} + (k - \langle\lambda,\theta\rangle)\lambda_\ell$$

(ℓ even)
$$\lambda^{(\ell)} = m_{\ell-1}\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} + m_1\lambda_{\ell-1} + (k - \langle\lambda,\theta\rangle)\lambda_\ell$$

E_6
$$\lambda^{(1)} = (k - \langle\lambda,\theta\rangle)\lambda_1 + m_5\lambda_2 + m_2\lambda_3 + m_4\lambda_4 + m_3\lambda_5 + m_1\lambda_6$$
$$\lambda^{(6)} = m_6\lambda_1 + m_3\lambda_2 + m_5\lambda_3 + m_4\lambda_4 + m_2\lambda_5 + (k - \langle\lambda,\theta\rangle)\lambda_6$$

E_7 $$\lambda^{(7)} = m_1\lambda_6 + m_2\lambda_2 + m_3\lambda_5 + m_4\lambda_4 + m_5\lambda_3 + m_6\lambda_1 + (k - \langle\lambda,\theta\rangle)\lambda_7.$$

Proof. Let $\mathfrak{g}$ be of type X_ℓ and let $H^{(i)}$ be a coweight from Table 4.1. Consider a vector $\mathbf{u} \in L(\lambda)_\beta \subset L(\lambda) \subset L(k,\lambda)$ (i.e., the $\mathfrak{h}$-weight of $\mathbf{u}$ is β). Consider $\mathbf{u}$ as a vector in $(L(k,\lambda))^{(i)}$ (which is the same vector space as $L(k,\lambda)$). We have a new associated $\hat{\mathfrak{g}}$-action and thus a new $\mathfrak{h}$-action. In particular, by (2.12), $(H_j)^{(i)}(0) = H_j(0) + \frac{2k}{\|\alpha_i\|^2}\delta_{i,j}$. If we restrict i to the indices appearing in Table 4.1, we see that in each case $\|\alpha_i\|^2 = 2$. Thus $(H_j)^{(i)}(0) = H_j(0) + k\delta_{i,j}$. Therefore, when $\mathbf{u}$ is thought of as a weight vector under the new $\mathfrak{h}$-action, the (new) $\mathfrak{h}$-weight of $\mathbf{u}$ is $\beta + k\lambda_i$.

Let $\mathbf{v}$ be a highest weight vector for $L(k,\lambda)$. Theorem 4.1 states that $\mathbf{w} = \sigma_X^{(i)}(\mathbf{v})$ is a highest weight vector for $(L(k,\lambda))^{(i)}$. As an element of the $\mathfrak{g}$-module $L(\lambda)$, $\mathbf{w}$ has $\mathfrak{h}$-weight $\mu = \sigma_X^{(i)}(\lambda)$. Therefore, the $\mathfrak{h}$-weight of $\mathbf{w}$ in terms of the new $\mathfrak{h}$-action is $\lambda^{(i)} = \mu + k\lambda_i$.

For $X_\ell = A_\ell$, using (4.1), we have

$$\begin{aligned}\mu &= \sum_{j=1}^{\ell} m_j(\lambda_{j+i} - \lambda_i) = \sum_{j=1}^{\ell} m_j\lambda_{j+i} - \left(\sum_{j=1}^{\ell} m_j\right)\lambda_i \\ &= \sum_{j=1}^{\ell} m_j\lambda_{j+i} - \langle\lambda,\theta\rangle\lambda_i.\end{aligned}$$

Therefore, $\lambda^{(i)} = \sum_{j=1}^{\ell} m_j\lambda_{j+i} - \langle\lambda,\theta\rangle\lambda_i + k\lambda_i$.

For type B_ℓ, by (A.1) and (A.2), we have

$$\begin{aligned}\mu &= \sum_{j=1}^{\ell-1} m_j(-2\lambda_1 + \lambda_j) + m_\ell(-\lambda_1 + \lambda_\ell) \\ &= \sum_{j=1}^{\ell} m_j\lambda_j - 2\left(\sum_{j=1}^{\ell-1} m_j\right)\lambda_1 + m_\ell\lambda_1 \\ &= \sum_{j=2}^{\ell} m_j\lambda_j - \langle\lambda,\theta\rangle\lambda_1.\end{aligned}$$

Therefore, $\lambda^{(1)} = \sum_{j=2}^{\ell} m_j\lambda_j - \langle\lambda,\theta\rangle\lambda_1 + k\lambda_1$.

For type C_ℓ, by (A.3), we have

$$\begin{aligned}\mu &= \sum_{j=1}^{\ell} m_j(\lambda_{\ell-j} - \lambda_\ell) = \sum_{j=1}^{\ell} m_j\lambda_{\ell-j} - \left(\sum_{j=1}^{\ell} m_j\right)\lambda_\ell \\ &= \sum_{j=1}^{\ell} m_j\lambda_{\ell-j} - \langle\lambda,\theta\rangle\lambda_\ell.\end{aligned}$$

Therefore, $\lambda^{(\ell)} = \sum_{j=1}^{\ell} m_j\lambda_{\ell-j} - \langle\lambda,\theta\rangle\lambda_\ell + k\lambda_\ell$.

For type D_ℓ and $i = 1$, by (A.4), (A.5), and (A.6), we have

$$\begin{aligned}
\mu &= \sum_{j=1}^{\ell-2} m_j(-2\lambda_1 + \lambda_j) + m_{\ell-1}(-\lambda_1 + \lambda_\ell) + m_\ell(-\lambda_1 + \lambda_{\ell-1}) \\
&= \sum_{j=1}^{\ell-2} m_j(-2\lambda_1 + \lambda_j) + m_{\ell-1}(-\lambda_1 + \lambda_\ell) + m_\ell(-\lambda_1 + \lambda_{\ell-1}) \\
&= -\langle\lambda,\theta\rangle\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_j + m_{\ell-1}\lambda_\ell + m_\ell\lambda_{\ell-1}.
\end{aligned}$$

Therefore, $\lambda^{(1)} = -\langle\lambda,\theta\rangle\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_j + m_{\ell-1}\lambda_\ell + m_\ell\lambda_{\ell-1} + k\lambda_1$.

Now, consider type D_ℓ when ℓ is odd and $i = \ell - 1$. By (A.9), (A.10), (A.11), and (A.12), we have

$$\begin{aligned}
\mu &= m_1(-\lambda_{\ell-1} + \lambda_\ell) + \sum_{j=2}^{\ell-2} m_j(\lambda_{\ell-j} - 2\lambda_{\ell-1} + m_{\ell-1}(\lambda_1 - \lambda_{\ell-1}) \\
&\quad + m_\ell(-\lambda_{\ell-1}) \\
&= m_{\ell-1}\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} - \left(m_1 + 2\sum_{j=2}^{\ell-2} m_j + m_{\ell-1} + m_\ell\right)\lambda_{\ell-1} + m_1\lambda_\ell \\
&= m_{\ell-1}\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} - \langle\lambda,\theta\rangle\lambda_{\ell-1} + m_1\lambda_\ell.
\end{aligned}$$

Therefore, $\lambda^{(\ell-1)} = m_{\ell-1}\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} - \langle\lambda,\theta\rangle\lambda_{\ell-1} + m_1\lambda_\ell + k\lambda_{\ell-1}$. Considering a Dynkin diagram symmetry (interchanging the roles of nodes $\ell-1$ and ℓ), we also have $\lambda^{(\ell)} = m_\ell\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} + m_1\lambda_{\ell-1} - \langle\lambda,\theta\rangle\lambda_\ell + k\lambda_\ell$.

When ℓ is even and $i = \ell - 1$, by (A.14), we have

$$\begin{aligned}
\mu &= m_1(-\lambda_{\ell-1} + \lambda_\ell) + \sum_{j=2}^{\ell-2} m_j(\lambda_{\ell-j} - 2\lambda_{\ell-1}) + m_{\ell-1}(-\lambda_{\ell-1}) \\
&\quad + m_\ell(\lambda_1 - \lambda_{\ell-1}) \\
&= m_\ell\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} - \left(m_1 + 2\sum_{j=2}^{\ell-2} m_j + m_{\ell-1} + m_\ell\right)\lambda_{\ell-1} + m_1\lambda_\ell \\
&= m_\ell\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} - \langle\lambda,\theta\rangle\lambda_{\ell-1} + m_1\lambda_\ell.
\end{aligned}$$

Therefore, $\lambda^{(\ell-1)} = m_\ell\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} - \langle\lambda,\theta\rangle\lambda_{\ell-1} + m_1\lambda_\ell + k\lambda_{\ell-1}$. Again, considering a Dynkin diagram symmetry (interchanging the roles of nodes

$\ell-1$ and ℓ), we also have

$$\lambda^{(\ell)} = m_{\ell-1}\lambda_1 + \sum_{j=2}^{\ell-2} m_j\lambda_{\ell-j} + m_1\lambda_{\ell-1} - \langle\lambda,\theta\rangle\lambda_\ell + k\lambda_\ell.$$

Finally, let us consider types E_6 and E_7. For type E_6 and $i = 1$, by (A.16), we have

$$\begin{aligned}\mu &= m_1(-\lambda_1+\lambda_6) + m_2(-2\lambda_1+\lambda_3) + m_3(-2\lambda_1+\lambda_5) + m_4(-3\lambda_1+\lambda_4)\\ &\quad + m_5(-2\lambda_1+\lambda_2) + m_6(-\lambda_1)\\ &= -(m_1+2m_2+2m_3+3m_4+2m_5+m_6)\lambda_1 + m_5\lambda_2 + m_2\lambda_3 + m_4\lambda_4\\ &\quad + m_3\lambda_5 + m_1\lambda_6\\ &= -\langle\lambda,\theta\rangle\lambda_1 + m_5\lambda_2 + m_2\lambda_3 + m_4\lambda_4 + m_3\lambda_5 + m_1\lambda_6.\end{aligned}$$

Therefore, $\lambda^{(1)} = -\langle\lambda,\theta\rangle\lambda_1 + m_5\lambda_2 + m_2\lambda_3 + m_4\lambda_4 + m_3\lambda_5 + m_1\lambda_6 + k\lambda_1$. Now we can use the symmetry of the Dynkin diagram of E_6 (interchanging nodes 1 and 6 and also interchanging nodes 3 and 5). Therefore, for type E_6 with $i = 6$, we have $\lambda^{(6)} = m_6\lambda_1 + m_3\lambda_2 + m_5\lambda_3 + m_4\lambda_4 + m_2\lambda_5 - \langle\lambda,\theta\rangle\lambda_6 + k\lambda_6$.

For type E_7 and $i = 7$, by (A.18), we have

$$\begin{aligned}\mu &= m_1(-2\lambda_7+\lambda_6) + m_2(-2\lambda_7+\lambda_2) + m_3(-3\lambda_7+\lambda_5) + m_4(-4\lambda_7+\lambda_4)\\ &\quad + m_5(-3\lambda_7+\lambda_3) + m_6(-2\lambda_7+\lambda_1) + m_7(-\lambda_7)\\ &= m_1\lambda_6 + m_2\lambda_2 + m_3\lambda_5 + m_4\lambda_4 + m_5\lambda_3 + m_6\lambda_1\\ &\quad - (2m_1+2m_2+3m_3+4m_4+3m_5+2m_6+m_7)\lambda_7\\ &= m_1\lambda_6 + m_2\lambda_2 + m_3\lambda_5 + m_4\lambda_4 + m_5\lambda_3 + m_6\lambda_1 - \langle\lambda,\theta\rangle\lambda_7.\end{aligned}$$

Therefore,

$$\lambda^{(7)} = m_1\lambda_6 + m_2\lambda_2 + m_3\lambda_5 + m_4\lambda_4 + m_5\lambda_3 + m_6\lambda_1 - \langle\lambda,\theta\rangle\lambda_7 + k\lambda_7. \qquad \square$$

Appendix A. Weyl group calculations

All of these calculations repeatedly use the fact that $\sigma_i(\lambda_j) = \lambda_j - \delta_{i,j}\alpha_j$, so in particular, $\sigma_{j-1}\sigma_{j-2}\cdots\sigma_2\sigma_1(\lambda_j) = \lambda_j$. In addition, if $C = (a_{ij})$ is the Cartan matrix of our simple Lie algebra $\mathfrak{g}$, then $\alpha_i = \sum_j a_{ij}\lambda_j$. Also, recall our conventions that $\lambda_0 = 0$, $\alpha_0 = -\theta$ (the negative of the highest long root of $\mathfrak{g}$), and $\lambda_j = \lambda_{j \pmod{\ell+1}}$ for all $j \in \mathbb{Z}$.

A.1. Type B_ℓ. Looking at the Cartan matrix of type B_ℓ, we see that $\alpha_i = -\lambda_{i-1} + 2\lambda_i - \lambda_{i+1}$ for $i \neq \ell-1$ and $\alpha_{\ell-1} = -\lambda_{\ell-2} + 2\lambda_{\ell-1} - 2\lambda_\ell$. Also, recall that $\sigma_B^{(1)} = \sigma_1\sigma_2\cdots\sigma_{\ell-1}\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2\sigma_1$.

For λ_j, $1 \le j \le \ell-1$, we have:

$$\begin{aligned}\text{(A.1)}\qquad &\sigma_B^{(1)}(\lambda_j)\\ &= \sigma_1\sigma_2\cdots\sigma_{\ell-1}\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2\sigma_1(\lambda_j)\\ &= \sigma_1\sigma_2\cdots\sigma_{\ell-1}\sigma_\ell\sigma_{\ell-1}\cdots\sigma_{j+1}\sigma_j(\lambda_j)\end{aligned}$$

$$
\begin{aligned}
&= \sigma_1\sigma_2\cdots\sigma_{\ell-1}\sigma_\ell\sigma_{\ell-1}\cdots\sigma_{j+2}\sigma_{j+1}(\lambda_{j-1}-\lambda_j+\lambda_{j+1})\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-1}\sigma_\ell\sigma_{\ell-1}\cdots\sigma_{j+3}\sigma_{j+2}(\lambda_{j-1}-\lambda_{j+1}+\lambda_{j+2}) \;=\; \cdots\\
&= \sigma_1\sigma_2\cdots\sigma_\ell\sigma_{\ell-1}(\lambda_{j-1}-\lambda_{\ell-2}+\lambda_{\ell-1})\\
&= \sigma_1\sigma_2\cdots\sigma_\ell\sigma_\ell(\lambda_{j-1}-\lambda_{\ell-1}+2\lambda_\ell)\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-1}(\lambda_{j-1}+\lambda_{\ell-1}-2\lambda_\ell)\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-2}(\lambda_{j-1}+\lambda_{\ell-2}-\lambda_{\ell-1})\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-3}(\lambda_{j-1}+\lambda_{\ell-3}-\lambda_{\ell-2}) \;=\; \cdots\\
&= \sigma_1\sigma_2\cdots\sigma_j(\lambda_{j-1}+\lambda_j-\lambda_{j+1})\\
&= \sigma_1\sigma_2\cdots\sigma_{j-1}(2\lambda_{j-1}-\lambda_j)\\
&= \sigma_1\sigma_2\cdots\sigma_{j-2}(2\lambda_{j-2}-2\lambda_{j-1}+\lambda_j) = \cdots\\
&= \sigma_1(2\lambda_1-2\lambda_2+\lambda_j) = -2\lambda_1+\lambda_j.
\end{aligned}
$$

For λ_ℓ, we have:

$$
\begin{aligned}
\text{(A.2)}\qquad & \sigma_B^{(1)}(\lambda_\ell)\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-1}\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2\sigma_1(\lambda_\ell) = \sigma_1\sigma_2\cdots\sigma_{\ell-1}\sigma_\ell(\lambda_\ell)\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-1}(\lambda_{\ell-1}-\lambda_\ell) = \sigma_1\sigma_2\cdots\sigma_{\ell-2}(\lambda_{\ell-2}-\lambda_{\ell-1}+\lambda_\ell)\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-3}(\lambda_{\ell-3}-\lambda_{\ell-2}+\lambda_\ell) \;=\; \cdots\\
&= \sigma_1(\lambda_1-\lambda_2+\lambda_\ell) = -\lambda_1+\lambda_\ell.
\end{aligned}
$$

Applying these results to the fundamental roots and highest long root, we have:

$$
\begin{aligned}
&\sigma_B^{(1)}(\alpha_0) = \sigma_B^{(1)}(-\theta) = \sigma_B^{(1)}(-\lambda_2) = 2\lambda_1-\lambda_2 = \alpha_1\\
&\sigma_B^{(1)}(\alpha_1) = \sigma_B^{(1)}(2\lambda_j-\lambda_{j+1}) = -2\lambda_1+2\lambda_1-\lambda_2 = -\theta = \alpha_0\\
&\sigma_B^{(1)}(\alpha_j) = \alpha_j \qquad \text{for } 1<j\le\ell.
\end{aligned}
$$

A.2. Type C_ℓ. Looking at the Cartan matrix of type C_ℓ, we see that

$$
\begin{aligned}
\alpha_i &= -\lambda_{i-1}+2\lambda_i-\lambda_{i+1} \quad (1\le i<\ell),\\
\alpha_\ell &= -2\lambda_{\ell-1}+2\lambda_\ell.
\end{aligned}
$$

Also, recall that $\sigma_C^{(\ell)} = (\sigma_\ell\cdots\sigma_2\sigma_1)(\sigma_\ell\cdots\sigma_2)\cdots(\sigma_\ell\sigma_{\ell-1})(\sigma_\ell)$.

$$
\begin{aligned}
\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1(\lambda_j) &= \sigma_\ell\sigma_{\ell-1}\cdots\sigma_j(\lambda_j) = \sigma_\ell\sigma_{\ell-1}\cdots\sigma_{j+1}(\lambda_{j-1}-\lambda_j+\lambda_{j+1})\\
&= \sigma_\ell\sigma_{\ell-1}\cdots\sigma_{j+2}(\lambda_{j-1}-\lambda_{j+1}+\lambda_{j+2}) = \cdots\\
&= \sigma_\ell(\lambda_{j-1}-\lambda_{\ell-1}+\lambda_\ell) = \lambda_{j-1}+\lambda_{\ell-1}-\lambda_\ell
\end{aligned}
$$

where $1\le j<\ell$. Also, $\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1(\lambda_\ell) = \sigma(\lambda_\ell) = 2\lambda_{\ell-1}-\lambda_\ell$, so the above formula works for all j. Applying this result multiple times, we have (for $1\le j<\ell$):

$$
\begin{aligned}
&\sigma_C^{(\ell)}(\lambda_j)\\
&= (\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2)\cdots(\sigma_\ell\sigma_{\ell-1})(\sigma_\ell)(\lambda_j) = \cdots
\end{aligned}
$$

$$\begin{aligned}
&= (\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2)\cdots(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_j)(\lambda_j)\\
&\quad + (\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2)\cdots(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_j-1)(\lambda_{j-1}+\lambda_{\ell-1}-\lambda_\ell)\\
&= (\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2)\cdots(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_j-2)(\lambda_{j-2}+\lambda_{\ell-2}-\lambda_\ell)\\
&= \cdots = (\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1)(\lambda_1+\lambda_{\ell-(j-1)}-\lambda_\ell) = \lambda_{\ell-j}-\lambda_\ell.
\end{aligned}$$

For λ_ℓ, we have:

$$\begin{aligned}
\sigma_C^{(\ell)}(\lambda_\ell) &= (\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2)\cdots(\sigma_\ell\sigma_{\ell-1})(\sigma_\ell)(\lambda_\ell)\\
&= (\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2)\cdots(\sigma_\ell\sigma_{\ell-1})(2\lambda_{\ell-1}-\lambda_\ell)\\
&= (\sigma_\ell\sigma_{\ell-1}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2)\cdots(\sigma_\ell\sigma_{\ell-1}\sigma_{\ell-2})(2\lambda_{\ell-2}-\lambda_\ell) = \cdots\\
&= \sigma_\ell\sigma_{\ell-1}\cdots\sigma_1(2\lambda_1-\lambda_\ell) = 2\lambda_{\ell-1}-2\lambda_\ell-2\lambda_{\ell-1}+\lambda_\ell = -\lambda_\ell.
\end{aligned}$$

Adhering to our convention (i.e., $\lambda_0 = 0$), we have that

$$\sigma_C^{(\ell)}(\lambda_j) = \lambda_{\ell-j} - \lambda_\ell \tag{A.3}$$

for all $j = 1, \ldots, \ell$.

A quick calculation now shows that

$$\begin{aligned}
\sigma_C^{(\ell)}(\alpha_0) = \sigma_C^{(\ell)}(-\theta) \;&=\; \alpha_\ell\\
\sigma_C^{(\ell)}(\alpha_j) = \alpha_{\ell-j} \quad &(\text{for } 1 \le j < \ell)\\
\sigma_C^{(\ell)}(\alpha_\ell) = -\theta \;&=\; \alpha_0.
\end{aligned}$$

A.3. Type D_ℓ (any rank). Looking at the Cartan matrix of type D_ℓ, we see that

$$\begin{aligned}
\alpha_i &= -\lambda_{i-1}+2\lambda_i-\lambda_{i+1} \quad (1 \le i < \ell-2),\\
\alpha_{\ell-2} &= -\lambda_{\ell-3}+2\lambda_{\ell-2}-\lambda_{\ell-1}-\lambda_\ell,\\
\alpha_{\ell-1} &= -\lambda_{\ell-2}+2\lambda_{\ell-1},\\
\alpha_\ell &= -\lambda_{\ell-2}+2\lambda_\ell.
\end{aligned}$$

First, recall that $\sigma_D^{(1)} = \sigma_1\sigma_2\cdots\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2\sigma_1$.

As a first step, we determine that action of $\sigma_1\sigma_2\cdots\sigma_\ell$ on λ_j. For $1 \le j < \ell-2$ we have:

$$\begin{aligned}
\sigma_1\sigma_2\cdots\sigma_\ell(\lambda_j) &= \sigma_1\sigma_2\cdots\sigma_j(\lambda_j)\\
&= \sigma_1\sigma_2\cdots\sigma_{j-1}(\lambda_{j-1}-\lambda_j+\lambda_{j+1}) = \cdots\\
&= \sigma_1(\lambda_1-\lambda_2+\lambda_{j+1}) = -\lambda_1+\lambda_{j+1}.
\end{aligned}$$

$$\begin{aligned}
\sigma_1\sigma_2\cdots\sigma_\ell(\lambda_{\ell-2}) &= \sigma_1\sigma_2\cdots\sigma_{\ell-2}(\lambda_{\ell-2})\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-3}(\lambda_{\ell-3}-\lambda_{\ell-2}+\lambda_{\ell-1}+\lambda_\ell)\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-4}(\lambda_{\ell-4}-\lambda_{\ell-3}+\lambda_{\ell-1}+\lambda_\ell) = \cdots\\
&= \sigma_1(\lambda_1-\lambda_2+\lambda_{\ell-1}+\lambda_\ell) = -\lambda_1+\lambda_{\ell-1}+\lambda_\ell.
\end{aligned}$$

$$\sigma_1\sigma_2\cdots\sigma_\ell(\lambda_{\ell-1}) = \sigma_1\sigma_2\cdots\sigma_{\ell-1}(\lambda_{\ell-1}) = \sigma_1\sigma_2\cdots\sigma_{\ell-2}(\lambda_{\ell-2}-\lambda_{\ell-1})$$

$$
\begin{aligned}
&= \sigma_1\sigma_2\cdots\sigma_{\ell-3}(\lambda_{\ell-3}-\lambda_{\ell-2}+\lambda_\ell)\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-4}(\lambda_{\ell-4}-\lambda_{\ell-3}+\lambda_\ell)\\
&= \cdots = \sigma_1(\lambda_1-\lambda_2+\lambda_\ell) = -\lambda_1+\lambda_\ell.
\end{aligned}
$$

$$
\begin{aligned}
\sigma_1\sigma_2\cdots\sigma_\ell(\lambda_\ell) &= \sigma_1\sigma_2\cdots\sigma_{\ell-1}(\lambda_{\ell-2}-\lambda_\ell) = \sigma_1\sigma_2\cdots\sigma_{\ell-2}(\lambda_{\ell-2}-\lambda_\ell)\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-3}(\lambda_{\ell-3}-\lambda_{\ell-2}+\lambda_{\ell-1})\\
&= \sigma_1\sigma_2\cdots\sigma_{\ell-4}(\lambda_{\ell-4}-\lambda_{\ell-3}+\lambda_{\ell-1})\\
&= \cdots = \sigma_1(\lambda_1-\lambda_2+\lambda_{\ell-1}) = -\lambda_1+\lambda_{\ell-1}.
\end{aligned}
$$

For $1 \leq j \leq \ell-2$ we have:

$$
\begin{aligned}
&\sigma_D^{(1)}(\lambda_j) \qquad\qquad \text{(A.4)}\\
&= \sigma_1\sigma_2\cdots\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2\sigma_1(\lambda_j) \;=\; \sigma_1\sigma_2\cdots\sigma_\ell\sigma_{\ell-2}\cdots\sigma_j(\lambda_j)\\
&= \sigma_1\sigma_2\cdots\sigma_\ell\sigma_{\ell-2}\cdots\sigma_{j+1}(\lambda_{j-1}-\lambda_j+\lambda_{j+1})\\
&= \sigma_1\sigma_2\cdots\sigma_\ell\sigma_{\ell-2}\cdots\sigma_{j+2}(\lambda_{j-1}-\lambda_{j+1}+\lambda_{j+2}) = \cdots\\
&= \sigma_1\sigma_2\cdots\sigma_\ell\sigma_{\ell-2}(\lambda_{j-1}-\lambda_{\ell-3}+\lambda_{\ell-2})\\
&= \sigma_1\sigma_2\cdots\sigma_\ell(\lambda_{j-1}-\lambda_{\ell-2}+\lambda_{\ell-1}+\lambda\ell)\\
&= (-\lambda_1+\lambda_j)-(-\lambda_1+\lambda_{\ell-1}+\lambda_\ell)+(-\lambda_1+\lambda_\ell)+(-\lambda_1+\lambda_{\ell-1})\\
&= -2\lambda_1+\lambda_j.
\end{aligned}
$$

$$
\begin{aligned}
&\sigma_D^{(1)}(\lambda_{\ell-1}) \qquad\qquad \text{(A.5)}\\
&= \sigma_1\sigma_2\cdots\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2\sigma_1(\lambda_{\ell-1})\\
&= \sigma_1\sigma_2\cdots\sigma_\ell(\lambda_{\ell-1}) = -\lambda_1+\lambda_\ell.
\end{aligned}
$$

$$
\begin{aligned}
&\sigma_D^{(1)}(\lambda_\ell) \qquad\qquad \text{(A.6)}\\
&= \sigma_1\sigma_2\cdots\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2\sigma_1(\lambda_\ell)\\
&= \sigma_1\sigma_2\cdots\sigma_\ell(\lambda_\ell) = -\lambda_1+\lambda_{\ell-1}.
\end{aligned}
$$

A quick calculation shows:

$$
\begin{aligned}
\sigma_D^{(1)}(\alpha_0) &= \sigma_D^{(1)}(-\theta) \;=\; \alpha_1\\
\sigma_D^{(1)}(\alpha_1) &= -\theta \;=\; \alpha_0\\
\sigma_D^{(1)}(\alpha_j) &= \alpha_j \quad (\text{for } 2 \leq j \leq \ell-2)\\
\sigma_D^{(1)}(\alpha_{\ell-1}) &= \alpha_\ell\\
\sigma_D^{(1)}(\alpha_\ell) &= \alpha_{\ell-1}.
\end{aligned}
$$

To help determine the actions of $\sigma_D^{(\ell-1)}$ and $\sigma_D^{(\ell)}$ on each λ_j, we will first consider $\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1$:

$$
\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1(\lambda_j) = \sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_j(\lambda_j)
$$

$$
\begin{aligned}
&= \sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_{j+1}(\lambda_{j-1}-\lambda_j+\lambda_{j+1})\\
&= \sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_{j+2}(\lambda_{j-1}-\lambda_{j+1}+\lambda_{j+2}) = \cdots\\
&= \sigma_{\ell-2}(\lambda_{j-1}-\lambda_{\ell-3}+\lambda_{\ell-2})\\
&= \lambda_{j-1}-\lambda_{\ell-2}+\lambda_{\ell-1}+\lambda_\ell, \quad (1\le j\le \ell-2),\\
\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1(\lambda_{\ell-1}) &= \lambda_{\ell-1},\\
\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1(\lambda_\ell) &= \lambda_\ell.
\end{aligned}
$$

Next, we consider $\sigma_{\ell-1}\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1$:

$$
\begin{aligned}
\text{(A.7)}\qquad \sigma_{\ell-1}\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1(\lambda_j) &= \sigma_{\ell-1}(\lambda_{j-1}-\lambda_{\ell-2}+\lambda_{\ell-1}+\lambda_\ell)\\
&= \lambda_{j-1}-\lambda_{\ell-1}+\lambda_\ell, \quad (1\le j\le \ell-2),\\
\sigma_{\ell-1}\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1(\lambda_{\ell-1}) &= \lambda_{\ell-2}-\lambda_{\ell-1},\\
\sigma_{\ell-1}\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1(\lambda_\ell) &= \lambda_\ell.
\end{aligned}
$$

Finally, consider $\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1$:

$$
\begin{aligned}
\text{(A.8)}\qquad \sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1(\lambda_j) &= \sigma_\ell(\lambda_{j-1}-\lambda_{\ell-2}+\lambda_{\ell-1}+\lambda_\ell)\\
&= \lambda_{j-1}+\lambda_{\ell-1}-\lambda_\ell, \quad (1\le j\le \ell-2),\\
\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1(\lambda_{\ell-1}) &= \lambda_{\ell-1},\\
\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_1(\lambda_\ell) &= \lambda_{\ell-2}-\lambda_\ell.
\end{aligned}
$$

A.4. Type D_ℓ (odd rank). Now consider the case when ℓ is odd and recall that

$$\sigma_D^{(\ell-1)} = (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2)(\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_3)\cdots(\sigma_\ell).$$

First, consider the case of λ_1.

$$
\begin{aligned}
\text{(A.9)}\quad & \sigma_D^{(\ell-1)}(\lambda_1)\\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2)(\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_3)\cdots(\sigma_\ell)(\lambda_1)\\
&= \cdots = (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)(\lambda_1) = -\lambda_{\ell-1}+\lambda_\ell.
\end{aligned}
$$

(We obtain the last step by using (A.7) above.)

Next, consider the case of λ_j, $1<j\le \ell-2$ and j odd. By applying (A.7) and (A.8) successively, we obtain the following:

$$
\begin{aligned}
\sigma_D^{(\ell-1)}(\lambda_j) &= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2)(\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_3)\cdots(\sigma_\ell)(\lambda_j)\\
&= \cdots = (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)\cdots(\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_j)(\lambda_j)\\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)\cdots(\sigma_\ell\sigma_{\ell-2}\cdots\sigma_{j-1})(\lambda_{j-1}-\lambda_{\ell-1}+\lambda_\ell)\\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)\cdots(\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_{j-2})(\lambda_{j-2}+\lambda_{\ell-2}-2\lambda_\ell)\\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)\cdots(\sigma_\ell\sigma_{\ell-2}\cdots\sigma_{j-3})(\lambda_{j-3}+\lambda_{\ell-3}-2\lambda_{\ell-1})\\
&= \cdots = (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)(\lambda_1+\lambda_{\ell-(j-1)}-2\lambda_\ell)\\
&= -\lambda_{\ell-1}+\lambda_\ell+\lambda_{\ell-j}-\lambda_{\ell-1}+\lambda_\ell-2\lambda_\ell\\
&= \lambda_{\ell-j}-2\lambda_{\ell-1}.
\end{aligned}
$$

A similar calculation shows that the same holds for j even and $2 \leq j \leq \ell - 3$. Therefore, we have that

$$\sigma_D^{(\ell-1)}(\lambda_j) = \lambda_{\ell-j} - 2\lambda_{\ell-1} \quad (\text{for } 1 < j \leq \ell - 2). \tag{A.10}$$

This leaves the cases $j = \ell - 1$ and $j = \ell$.

$$\begin{aligned}
&\sigma_D^{(\ell-1)}(\lambda_{\ell-1}) \\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2)(\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_3) \\
&\quad \cdots(\sigma_\ell)(\lambda_{\ell-1}) \\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)\cdots(\sigma_{\ell-1}\sigma_{\ell-2})(\lambda_{\ell-1}) \\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)\cdots(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3})(\lambda_{\ell-2} - \lambda_{\ell-1}) \\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)\cdots(\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_{\ell-4})(\lambda_{\ell-3} - \lambda_\ell) = \cdots \\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)(\lambda_2 - \lambda_\ell) \\
&= \lambda_1 - \lambda_{\ell-1} + \lambda_\ell - \lambda_\ell \;=\; \lambda_1 - \lambda_{\ell-1}.
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
&\sigma_D^{(\ell-1)}(\lambda_\ell) \\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3}\cdots\sigma_2)(\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_3)\cdots(\sigma_\ell)(\lambda_\ell) \\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)\cdots(\sigma_{\ell-1}\sigma_{\ell-2})(\lambda_{\ell-2} - \lambda_\ell) \\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)\cdots(\sigma_\ell\sigma_{\ell-2}\sigma_{\ell-3})(\lambda_{\ell-3} - \lambda_{\ell-1}) \\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)\cdots(\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_{\ell-4})(\lambda_{\ell-4} - \lambda_\ell) = \cdots \\
&= (\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_1)(\lambda_1 - \lambda_\ell) \\
&= -\lambda_{\ell-1} + \lambda_\ell - \lambda_\ell \;=\; -\lambda_{\ell-1}.
\end{aligned} \tag{A.12}$$

A quick calculation shows:

$$\begin{aligned}
\sigma_D^{(\ell-1)}(\alpha_0) &= \sigma_D^{(\ell-1)}(-\theta) \;=\; \alpha_{\ell-1} \\
\sigma_D^{(\ell-1)}(\alpha_1) &= \alpha_\ell \\
\sigma_D^{(\ell-1)}(\alpha_j) &= \alpha_{\ell-j} \quad (\text{for } 2 \leq j \leq \ell - 1) \\
\sigma_D^{(\ell-1)}(\alpha_\ell) &= -\theta \;=\; \alpha_0.
\end{aligned}$$

Using a Dynkin diagram symmetry, we see that all of these results should still hold if we interchange the labels $\ell - 1$ and ℓ. So we also have that

$$\begin{aligned}
\sigma_D^{(\ell)}(\lambda_1) &= \lambda_{\ell-1} - \lambda_\ell \\
\sigma_D^{(\ell)}(\lambda_j) &= \lambda_{\ell-j} - 2\lambda_\ell \quad (\text{for } 2 \leq j \leq \ell - 2) \\
\sigma_D^{(\ell)}(\lambda_{\ell-1}) &= -\lambda_\ell \\
\sigma_D^{(\ell)}(\lambda_\ell) &= \lambda_1 - \lambda_\ell,
\end{aligned} \tag{A.13}$$

and also

$$\sigma_D^{(\ell)}(\alpha_0) = \sigma_D^{(\ell)}(-\theta) \;=\; \alpha_\ell$$

$$\begin{aligned}
\sigma_D^{(\ell)}(\alpha_j) &= \alpha_{\ell-j} \quad (\text{for } 1 \le j \le \ell-2)\\
\sigma_D^{(\ell)}(\alpha_{\ell-1}) &= -\theta \;=\; \alpha_0\\
\sigma_D^{(\ell)}(\alpha_\ell) &= \alpha_1.
\end{aligned}$$

A.5. Type D_ℓ (even rank). Almost identical calculations reveal that for even ℓ we have the following:

$$\begin{aligned}
\sigma_D^{(\ell-1)}(\lambda_1) &= -\lambda_{\ell-1} + \lambda_\ell\\
\sigma_D^{(\ell-1)}(\lambda_j) &= \lambda_{\ell-j} - 2\lambda_{\ell-1} \quad (\text{for } 2 \le j \le \ell-2)\\
\sigma_D^{(\ell-1)}(\lambda_{\ell-1}) &= -\lambda_{\ell-1}\\
\sigma_D^{(\ell-1)}(\lambda_\ell) &= \lambda_1 - \lambda_{\ell-1}
\end{aligned} \tag{A.14}$$

and it follows that:

$$\begin{aligned}
\sigma_D^{(\ell-1)}(\alpha_0) &= \sigma_D^{(\ell-1)}(-\theta) \;=\; \alpha_{\ell-1}\\
\sigma_D^{(\ell-1)}(\alpha_1) &= \alpha_\ell\\
\sigma_D^{(\ell-1)}(\alpha_j) &= \alpha_{\ell-j} \quad (\text{for } 2 \le j \le \ell-2)\\
\sigma_D^{(\ell-1)}(\alpha_{\ell-1}) &= -\theta \;=\; \alpha_0\\
\sigma_D^{(\ell-1)}(\alpha_\ell) &= \alpha_1.
\end{aligned}$$

Again using a Dynkin diagram symmetry, we see that all of these results should still hold if we interchange the labels $\ell-1$ and ℓ. So we also have that:

$$\begin{aligned}
\sigma_D^{(\ell)}(\lambda_1) &= \lambda_{\ell-1} - \lambda_\ell\\
\sigma_D^{(\ell)}(\lambda_j) &= \lambda_{\ell-j} - 2\lambda_\ell \quad (\text{for } 2 \le j \le \ell-2)\\
\sigma_D^{(\ell)}(\lambda_{\ell-1}) &= \lambda_1 - \lambda_\ell\\
\sigma_D^{(\ell)}(\lambda_\ell) &= -\lambda_\ell
\end{aligned} \tag{A.15}$$

and it follows that (recall $-\theta = \alpha_0$):

$$\sigma_D^{(\ell)}(\alpha_j) = \alpha_{\ell-j} \quad (\text{for } 1 \le j \le \ell).$$

A.6. Exceptional types. Recall that if $\mathfrak{g}$ is of type E_8, F_4 or G_2, then $\mathfrak{g}$ has no minuscule weights and so $P^\vee = Q^\vee$. Thus the action of $\Delta(H,x)$ is always trivial. So we only need to consider types E_6 and E_7.

The following calculations for types E_6 and E_7 were done with the help of a Maple worksheet which is available at

http://dimax.rutgers.edu/~sadowski/LieAlgebraCalculations/index.html.

For type E_6 using $H^{(1)}$, we have:

$$\sigma_E^{(1)}(\lambda_1) = -\lambda_1 + \lambda_6 \tag{A.16}$$

$$\sigma_E^{(1)}(\lambda_2) = -2\lambda_1 + \lambda_3$$
$$\sigma_E^{(1)}(\lambda_3) = -2\lambda_1 + \lambda_5$$
$$\sigma_E^{(1)}(\lambda_4) = -3\lambda_1 + \lambda_4$$
$$\sigma_E^{(1)}(\lambda_5) = -2\lambda_1 + \lambda_2$$
$$\sigma_E^{(1)}(\lambda_6) = -\lambda_1$$

and so

$$\sigma_E^{(1)}(\alpha_0) = \sigma_E^{(1)}(-\theta) = \alpha_1$$
$$\sigma_E^{(1)}(\alpha_1) = \alpha_6$$
$$\sigma_E^{(1)}(\alpha_2) = \alpha_3$$
$$\sigma_E^{(1)}(\alpha_3) = \alpha_5$$
$$\sigma_E^{(1)}(\alpha_4) = \alpha_4$$
$$\sigma_E^{(1)}(\alpha_5) = \alpha_2$$
$$\sigma_E^{(1)}(\alpha_6) = -\theta \ = \ \alpha_0.$$

For type E_6 using $H^{(6)}$, we have:

(A.17)
$$\sigma_E^{(6)}(\lambda_1) = -\lambda_6$$
$$\sigma_E^{(6)}(\lambda_2) = -2\lambda_6 + \lambda_5$$
$$\sigma_E^{(6)}(\lambda_3) = -2\lambda_6 + \lambda_2$$
$$\sigma_E^{(6)}(\lambda_4) = -3\lambda_6 + \lambda_4$$
$$\sigma_E^{(6)}(\lambda_5) = -2\lambda_6 + \lambda_3$$
$$\sigma_E^{(6)}(\lambda_6) = -\lambda_6 + \lambda_1$$

and so

$$\sigma_E^{(6)}(\alpha_0) = \sigma_E^{(6)}(-\theta) \ = \ \alpha_6$$
$$\sigma_E^{(6)}(\alpha_1) = \alpha_5$$
$$\sigma_E^{(6)}(\alpha_2) = \alpha_4$$
$$\sigma_E^{(6)}(\alpha_3) = \alpha_3$$
$$\sigma_E^{(6)}(\alpha_4) = \alpha_2$$
$$\sigma_E^{(6)}(\alpha_5) = \alpha_1$$
$$\sigma_E^{(6)}(\alpha_6) = -\theta \ = \ \alpha_0.$$

For type E_7 using $H^{(7)}$, we have:

$$\begin{aligned}\sigma_E^{(7)}(\lambda_1) &= -2\lambda_7 + \lambda_6\\ \sigma_E^{(7)}(\lambda_2) &= -2\lambda_7 + \lambda_2\\ \sigma_E^{(7)}(\lambda_3) &= -3\lambda_7 + \lambda_5\\ \sigma_E^{(7)}(\lambda_4) &= -4\lambda_7 + \lambda_4\\ \sigma_E^{(7)}(\lambda_5) &= -3\lambda_7 + \lambda_3\\ \sigma_E^{(7)}(\lambda_6) &= -2\lambda_7 + \lambda_1\\ \sigma_E^{(7)}(\lambda_7) &= -\lambda_7\end{aligned} \tag{A.18}$$

and so

$$\begin{aligned}\sigma_E^{(7)}(\alpha_0) &= \sigma_E^{(7)}(-\theta) = \alpha_7\\ \sigma_E^{(7)}(\alpha_1) &= \alpha_6\\ \sigma_E^{(7)}(\alpha_2) &= \alpha_2\\ \sigma_E^{(7)}(\alpha_3) &= \alpha_5\\ \sigma_E^{(7)}(\alpha_4) &= \alpha_4\\ \sigma_E^{(7)}(\alpha_5) &= \alpha_3\\ \sigma_E^{(7)}(\alpha_6) &= \alpha_1\\ \sigma_E^{(7)}(\alpha_7) &= -\theta \;=\; \alpha_0.\end{aligned}$$

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