New York Journal of Mathematics

New York J. Math. **15** (2009) 97–123.

Isoperimetric problems on the sphere and on surfaces with density

Max Engelstein, Anthony Marcuccio, Quinn Maurmann and Taryn Pritchard

Abstract. We discuss partitions of the sphere and other ellipsoids into equal areas and isoperimetric problems on surfaces with density. We prove that the least-perimeter partition of any ellipsoid into two equal areas is by division along the shortest equator. We extend the work of C. Quinn, 2007, and give a new sufficient condition for a perimeter-minimizing partition of $\mathbf{S}^2$ into four regions of equal area to be the tetrahedral arrangement of geodesic triangles. We solve the isoperimetric problem on the plane with density $|y|^\alpha$ for $\alpha > 0$ and solve the double bubble problem when α is a positive integer. We also identify isoperimetric regions on cylinders with densities e^z and $|\theta|^\alpha$. Next, we investigate stable curves on surfaces of revolution with radially symmetric densities. Finally, we give an asymptotic estimate for the minimal perimeter of a partition of any smooth, compact surface with density into n regions of equal area, generalizing the previous work of Maurmann *et al.* (to appear).

Contents

1. Introduction 98
2. Partitions of the ellipsoid into two equal areas 99
3. Geodesics in partitions of $\mathbf{S}^2$ 101
4. The plane with density $|y|^\alpha$ 104
5. Densities on the cylinder 111
6. Stability of circles in surfaces of revolution with density 113
7. Asymptotic estimates 118
References 121

Received July 1, 2008.

Mathematics Subject Classification. 53C42.

Key words and phrases. Minimal partitions, isoperimetric problem, surfaces with density, stability.

ISSN 1076-9803/09

1. Introduction

The spherical partition problem asks for the least-perimeter way to partition a sphere into n regions of equal area. The solution is known only in three cases. When $n = 2$, any great circle partitions the sphere with least perimeter (for example, see Section 2: Proposition 2.3 states that the shortest curve partitioning the ellipsoid into two regions of equal area is the shortest equator). For $n = 3$, three meridinal arcs meeting at 120 degrees at the poles are best (Masters [Ma96]). Finally for $n = 12$, a dodecahedral arrangement is minimizing (Hales [H02]). All three solutions are geodesic and are pictured in Figure 2. Section 3 extends the work of C. Quinn [Q07] on the $n = 4$ case, where the tetrahedral partition is conjectured to be minimizing. Proposition 3.6 proves this conjecture under the assumption that some component of the minimizing partition is geodesic.

The isoperimetric problem in a Riemannian surface asks for the shortest curve enclosing a given area; we investigate this problem in surfaces with density. A density is a smooth positive function which weights perimeter and area equally. Note that giving a surface a density is not the same as scaling its metric, which scales area and perimeter by different factors. By standard geometric measure theory, on a smooth surface with density and finite weighted area, isoperimetric regions exist and are bounded by smooth curves of constant generalized curvature (see Morgan [M08] and [M03], Section 3.10). The generalized curvature κ_ψ at a point on a curve in a surface with density e^ψ is defined as $\kappa_\psi = \kappa - \hat{n} \cdot \nabla\psi$, where $\hat{n}$ is a unit normal.

Proposition 4.6 and Corollary 4.9 use Steiner symmetrization to show that isoperimetric regions in the plane with density $|y|^\alpha$ for $\alpha > 0$ are semicircles perpendicular to the x-axis. An essential step is Lemma 4.7, which gives a sufficient condition for Steiner symmetrization (in a fairly general setting) to give isoperimetric results with uniqueness. Proposition 4.3 finds that half of the standard double bubble is the least-perimeter way to enclose two areas in the plane with density $|y|^n$, where $n \in \mathbf{N}$. Proposition 5.1 uses a novel projection argument to prove that isoperimetric regions on the cylinder $\mathbf{S}^n \times \mathbf{R}$ with density e^z are half-cylinders bounded above by horizontal spherical slices. Finally, Proposition 5.2 shows that for small area on the cylinder with density $|\theta|^\alpha$, semicircles on the line $\theta = 0$ solve the isoperimetric problem.

Section 6 examines stable curves in surfaces of revolution. Theorem 6.3 gives necessary and sufficient conditions for circles of revolution to be stable: for a surface of revolution with metric $ds^2 = dr^2 + f(r)^2 d\theta^2$ and radial density $e^{\psi(r)}$, a circle of revolution is stable if and only if

$$f'^2 - ff'' - f^2\psi'' \leq 1$$

on that circle. An intriguing consequence, Corollary 6.4, finds that in a disk of revolution with radial density e^ψ, decreasing Gauss curvature and ψ''

nonnegative imply circles are stable, while increasing Gauss curvature and stable circles imply ψ'' nonnegative. In the constant-curvature spaces $\mathbf{R}^2$, $\mathbf{S}^2$, and $\mathbf{H}^2$, circles of revolution are stable if and only if ψ'' is nonnegative. (In $\mathbf{R}^2$, this result was known by Rosales *et al.* ([RCB08], Theorem 3.10).) However, we dismiss the possibility that the equivalence holds in general surfaces of revolution. Corollary 6.6 finds it to be impossible for spaces with strictly monotone curvature, and Corollary 6.7 finds there is no condition on the density which is sufficient for circles of revolution to be stable in every surface of revolution. Given any radial density function, Proposition 6.8 constructs an annulus in which all circles of revolution are stable, though this surface turns out to be fairly tame: it is just the cylinder $\mathbf{S}^1 \times \mathbf{R}$ with area density (weights area but not length).

Section 7 generalizes previous work on the asymptotic estimates of perimeter-minimizing partitions of surfaces into n regions of equal area. Maurmann *et al.* [MEM08] found that the least perimeter $P(n)$ of partitions of a compact surface M with area $|M|$ into n regions of area $|M|/n$ is asymptotic to half the perimeter of n regular planar hexagons of area $|M|/n$ (this is related to the fact that regular hexagons partition the plane most efficiently [H01]). That is, $P(n)$ is asymptotic to $12^{1/4}\sqrt{|M|n}$. In the present paper, Theorem 7.5 and Corollary 7.6 generalize the result to surfaces with density and again find that $P(n)$ is asymptotic to $C\sqrt{n}$ for some constant C given explicitly in terms of the surface and its density.

This paper represents continuing work of the 2007 SMALL undergraduate research Geometry Group. We would like to thank Williams College and the National Science Foundation Research Experiences for Undergraduates for funding SMALL. We would also like to thank the Canadian Undergraduate Mathematics Conference and MAA MathFest for supporting our travels and giving us the opportunity to present our research. Finally we thank Professor Frank Morgan, whose guidance and patience have been invaluable in writing this paper.

2. Partitions of the ellipsoid into two equal areas

Proposition 2.3 shows that the short equator provides the least-perimeter partition of an ellipsoid into two regions of equal area. For ellipsoids of revolution, a stronger result is known: that any isoperimetric region is a disk centered at a pole of revolution ([R01], Theorem 3.5). We begin with two lemmas governing the regularity of minimizing partitions.

Lemma 2.1. *An isoperimetric curve which partitions an ellipsoid into two regions must be connected.*

Proof. Suppose not. Then there exists some minimizing curve C that partitions the ellipsoid into two regions R_1 and R_2 and is disconnected. Examine two components of the curve, C_1 and C_2 with lengths L_1 and L_2 respectively. Deform C_1 toward R_1 with unit rate $1/L_1$ and deform C_2 toward R_2 with unit rate $1/L_2$. This deformation initially preserves area and is nontrivial if $C_1 \neq C_2$. Let u be the normal component of this deformation and examine the second variation: $-\int \kappa^2 u^2 - \int G u^2 < 0$ because the Gauss curvature G is positive. Negative second variation shows that C is unstable and therefore not minimizing. □

Lemma 2.2. *Any isoperimetric curve which divides an ellipsoid into two regions of equal area must contain a pair of antipodal points.*

Proof. Consider an isoperimetric curve γ which bounds two regions of equal area (γ is connected by Lemma 2.1). Define a map ϕ which maps each point p to its antipodal point $-p$, and let $-\gamma$ be the image of γ under ϕ. Then $-\gamma$ also bounds two regions of equal area. If the intersection of γ and $-\gamma$ is empty, then one of the regions bounded by γ must be entirely contained in one of the regions bounded by $-\gamma$, a contradiction since all of the bounded regions are of equal area. Thus the intersection of γ and $-\gamma$ contains at least one point, so γ contains a pair of antipodal points. □

We now prove the main result.

Proposition 2.3. *Consider the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $a \geq b \geq c$. Then the short equator, the ellipse $\{x = 0\}$, gives a least-perimeter partition of the ellipsoid into two regions of equal area, and does so uniquely if $a > b$.

Proof. First consider the oblate spheroid $a = b > c$, on which the boundary of any partition into two equal areas must contain a pair of antipodal points by Lemma 2.2. Geodesics on the oblate spheroid are either (i) intersections of the spheroid with vertical and horizontal planes through the origin, the shortest of which are the vertical meridinal ellipses or (ii) undulating curves which oscillate between two parallels equidistant from the equator, as cited by A. Cayley ([C1894], page 15). On these undulating geodesics, $\Delta\theta < \pi/2$ between a minimum and a maximum vertex, so such geodesics could only connect a pair of antipodal points after one and a half cycles of undulation. Such curves are unstable on the spheroid. Thus on the oblate spheroid, the shortest path between any two antipodal points is a meridinal half-ellipse through a pole, and hence any pair of antipodal points is the same distance apart, as illustrated in Figure 1.

Now consider the original ellipsoid $a \geq b \geq c$. Let γ be an isoperimetric curve partitioning the ellipsoid into two regions of equal area. Then γ is connected (by regularity, it is homeomorphic to a circle) and contains a pair of antipodal points, so can be decomposed into two curves C_1 and C_2 running between those antipodal points. Let L be half the length of the ellipsoid's short equator; we claim that neither C_1 nor C_2 can have length less than L. To see this, scale the ellipsoid down in the x-direction by a factor of $b/a \leq 1$. This deformation leaves fixed the short equator $\{x = 0\}$ without increasing the length of C_1. Moreover, this deformation turns the given ellipsoid into an oblate spheroid, where no curve between antipodal points has length less than L. That is, if L_1 is the length of C_1, we have just shown that $L \leq L_1 b/a \leq L_1$. Similarly, the length of C_2 is no less than L, so γ has length at least $2L$. But there certainly exists a curve of length $2L$ which partitions the ellipsoid into two regions of equal area: the short equator $\{x = 0\}$. For uniqueness when $a > b$, we simply note that scaling by $b/a < 1$ in the x-direction strictly decreases the lengths of C_1 and C_2 unless they are arcs of the short equator. □

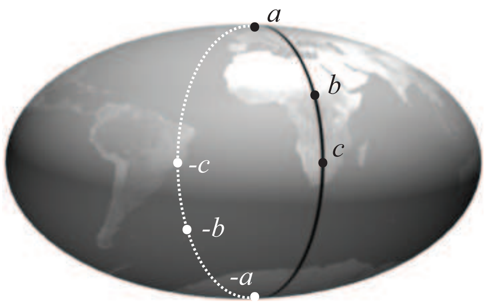

FIGURE 1. On an oblate spheroid (like the earth) all pairs of antipodal points are the same distance apart

3. Geodesics in partitions of $\mathbf{S}^2$

We make some progress on proving the conjecture that a perimeter-minimizing partition of the sphere $\mathbf{S}^2$ into four equal areas is the tetrahedral partition, seen in the center of the top row in Figure 2. Proposition 3.6 finds that if a minimizing partition contains any geodesic component, then it is tetrahedral. The following theorem due to C. Quinn [Q07] identifies several other conditions which are sufficient to show that a minimizing partition is tetrahedral. We follow Quinn's notation, labeling the region of highest pressure R_1, the region of second-highest pressure R_2, and so on.

Theorem 3.1 ([Q07], Theorem 5.2)**.** *A perimeter-minimizing partition of the sphere into four equal areas is tetrahedral if any of these five conditions is met:*

(i) *The high-pressure region R_1 is connected.*
(ii) *The low-pressure region R_4 contains a triangle.*
(iii) *The partition contains a geodesic m-gon with m odd.*
(iv) *The high-pressure region R_1 has the same pressure as some other region.*
(v) *The partition is geodesic.*

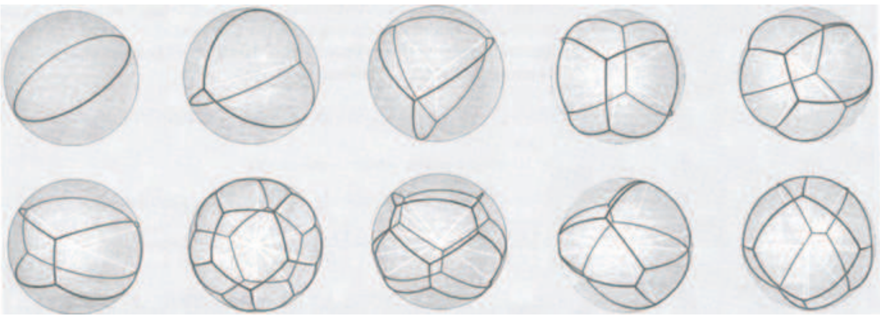

FIGURE 2. The ten partitions of the sphere by geodesics meeting in threes at 120 degrees (picture originally from Almgren and Taylor [AT76], ©1976 Scientific American).

The proof of Proposition 3.6 will require several lemmas restricting the components of R_1 and the components of the lower pressure regions.

Lemma 3.2. *No perimeter-minimizing partition into four equal areas can contain a geodesic m-gon with $m \geq 6$.*

Proof. Suppose a geodesic m-gon exists for $m \geq 6$ and apply Gauss–Bonnet. Each edge has curvature $\kappa = 0$, and $\mathbf{S}^2$ has Gauss curvature $G = 1$, so

$$A + \sum_{i=1}^{m} (\pi - \alpha_i) = 2\pi,$$

where A is the area enclosed by the m-gon and $\alpha_1, \ldots, \alpha_m$ are its angles. By regularity, each $\alpha_i = 2\pi/3$, so $A = 2\pi - m\pi/3 \leq 0$, a contradiction. □

Lemma 3.3. *If a perimeter-minimizing partition into four equal areas contains a geodesic polygon, then R_1 has at most two components.*

Proof. Every component of R_1 is convex, as every edge has nonnegative curvature. Quinn ([Q07], Corollary 2.22) proves that the partition can have at most three convex components total, or else the partition is geodesic. If R_1 contains any geodesic edge, it must have the same pressure as another region, and by Theorem 3.1 the partition must be tetrahedral (in which case R_1 is connected). Otherwise, the geodesic polygon is a convex component of another region, which together with R_1 can have at most three convex components, so R_1 can have no more than two components. □

We cite two more results of C. Quinn, and then prove the main result of this section.

Lemma 3.4 (C. Quinn [Q07], Lemma 5.11)**.** *In a perimeter-minimizing partition of $\mathbf{S}^2$ into four equal areas, the lowest pressure region R_4 satisfies:*

(i) *If R_4 contains a 3-gon, then R_4 is a geodesic 3-gon.*
(ii) *R_4 cannot contain more than one 4-gon.*

(iii) R_4 *cannot contain more than three* 5*-gons, and if it contains three* 5*-gons, then these are its only components and they are geodesic.*
(iv) *If* R_4 *contains a* 4*-gon and a* 5*-gon, then these are its only components and they are geodesic.*

The second result of Quinn's is a summary of several propositions, in which all the possible configurations for R_1 are listed, then a few are shown to be impossible. Those remaining which are relevant to the proof of Proposition 3.6 share one important characteristic:

Lemma 3.5 (C. Quinn [Q07], Proposition 5.9, Lemma 5.14, Lemma 5.15)**.** *In a perimeter-minimizing partition of* $\mathbf{S}^2$ *into four equal areas, if the highest pressure region* R_1 *has exactly two connected components, then* R_1 *has a total of either* 7 *or* 8 *edges.*

This leads to the main result of the section:

Proposition 3.6. *In a least-perimeter partition of* $\mathbf{S}^2$ *into four equal areas, if any region contains a geodesic polygon, then the partition is tetrahedral.*

Proof. By Lemma 3.2, any geodesic m-gon has at most 5 sides. If $m = 3$ or 5, then the partition is tetrahedral by Theorem 3.1. If $m = 2$, then the polygon can be slid (preserving area and perimeter) until it touches another component, contradicting regularity, so the only case to consider is when $m = 4$. If R_1 contains or is adjacent to the geodesic polygon, then R_1 has the same pressure as another region, and the partition is tetrahedral by 3.1. Then we can safely assume that the geodesic quadrilateral is in a lower-pressure region and is surrounded only by components of lower-pressure regions. When this is the case, all three lower pressure regions must have the same pressure; by relabeling, we assume R_2 contains the geodesic quadrilateral.

Let l be the total length of the perimeter of R_1, and let κ be the curvature of the edges of R_1 (the curvature is the same for all edges since R_2 through R_4 have equal pressure). By Lemma 3.3, R_1 has at most two components. If R_1 has just one component, then Theorem 3.1 proves the partition to be tetrahedral, so we treat the case where R_1 has two components. Then by Lemma 3.5, R_1 has either 7 or 8 edges. By Gauss–Bonnet, $\pi+\kappa l+8\pi/3 \geq 4\pi$ and $\pi + \kappa l + 7\pi/3 \leq 4\pi$, so that $2\pi/3 \geq \kappa l \geq \pi/3$.

We turn our attention to R_3 and R_4. Let C be their total number of components and n their total number of edges (count an edge twice if it borders both regions). Let l' be the perimeter of their edges incident with R_1. Then $2\pi - \kappa l' + n\pi/3 = 2C\pi$, which implies $2\pi(1-C)+n\pi/3 = \kappa l'$. Because the geodesic quadrilateral has area less than π, there must be at least one other component of R_2. Any such component must touch R_1, otherwise it would be convex and R_2 and R_1 would have more than three convex components between them, contradicting Quinn [Q07], Corollary 2.22. Since

R_2 borders R_1, we have $l' < l$, hence $2\pi(1 - C) + n\pi/3 < \kappa l \leq 2\pi/3$. Then $6\pi(1 - C) + n\pi < 2\pi$, so $n - 6C < -4$. We analyze three final cases.

Case 1. R_3 and R_4 each contain a 4-gon. By Lemma 3.4, every other component of R_3 and R_4 must have at least six edges (or else the partition contains a geodesic 5-gon and the partition must be tetrahedral). When this is the case, $n - 6C \geq -4$, a contradiction.

Case 2. Exactly one of R_3 or R_4 contains a 4-gon, say R_3. By Lemma 3.4, every other component in R_3 has at least six edges, and R_4 contains one or two 5-gons, the rest of its components also having at least six edges each. Then $n - 6C \geq -4$, a contradiction.

Case 3. Neither R_3 nor R_4 contains a 4-gon. Then each region can have at most two 5-gons, and the rest of their components must have at least six edges each. Again, $n - 6C \geq -4$, a contradiction.

Having shown that every possible case either implies the partition is tetrahedral or leads to a contradiction, it follows that the partition is tetrahedral. □

4. The plane with density $|y|^\alpha$

Section 4 examines isoperimetric regions in the plane with density $|y|^\alpha$ with $\alpha > 0$. We start with the simplifying Lemma 4.1 which allows us to work in just the upper half-plane. Proposition 4.6 and Corollary 4.9 then prove that isoperimetric regions are semicircular half-disks centered on the x-axis. In proving the solution is unique, Lemma 4.7 gives a rather general result about Steiner symmetrization.

Along the way, Proposition 4.2 solves the problem in the much easier case $\alpha \in \mathbf{N}$, where the problem has an interpretation in terms of hypersurfaces of revolution in $\mathbf{R}^{\alpha+2}$. Proposition 4.3 uses this same interpretation and recent work by Ben Reichardt [Re08] to prove that the least-perimeter way to enclose and separate two areas is half of the standard double bubble perpendicular to the x-axis as in Figure 3.

We define a *cluster* in a Riemannian surface as a collection of disjoint open sets (*regions*) of prescribed areas and its *perimeter* as the length of the union of its regions' boundaries.

Lemma 4.1. *Let ψ be a nonnegative continuous function on $[0, \infty)$ which vanishes precisely at 0. Suppose that for any prescribed areas $a_1, \ldots, a_m \geq 0$ there is a bounded minimizing cluster in the half-plane $\{y > 0\}$ with density $\psi(y)$. Then every minimizer in the half-plane is minimizing in the whole plane with density $\psi(|y|)$, and uniqueness in the half-plane implies uniqueness in the whole plane (up to horizontal translation and reflection across the x-axis of components of clusters).*

Proof. Let C be any cluster enclosing areas $a_1, \ldots, a_m$ in the whole plane with density $\psi(|y|)$. Let C^+ be the cluster of m regions in the upper half-plane whose regions are the intersections of the regions of C with the upper

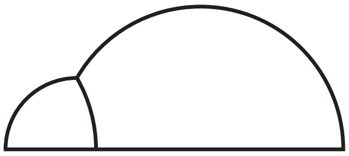

Figure 3. In the upper half-plane with density y^n, half of the standard double bubble encloses and separates two areas with least perimeter. Illustration modified from Reichardt ([Re08], Figure 1).

half-plane, and let C^- be the analogous cluster in the lower half-plane. By hypothesis, C^+ and C^- may be replaced by bounded minimizing clusters Γ^+ and Γ^- enclosing the same areas. Reflect Γ^- across the x-axis and translate it horizontally until the regions of Γ^+ and Γ^- are disjoint (this is possible, since each is bounded). Then $\Gamma^+ \cup \Gamma^-$ is a cluster in the upper half-plane enclosing areas $a_1, \ldots, a_m$ with perimeter no greater than C. Thus every minimizer in the half-plane is minimizing in the whole plane.

Now assume uniqueness in the half-plane. Supposing there is another minimizer in the whole plane, the construction above yields a minimizer in the half-plane of the form $\Gamma^+ \cup \Gamma^-$ with Γ^+ and Γ^- nonempty. Now translating Γ^+ horizontally while leaving Γ^- fixed contradicts uniqueness in the half-plane. $\square$

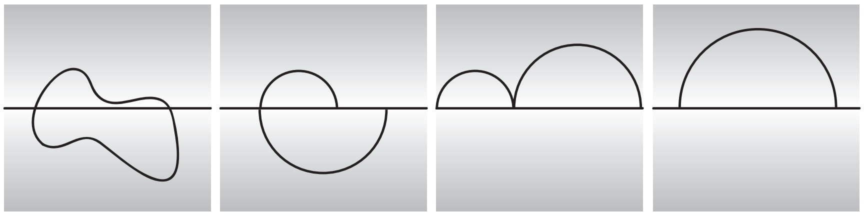

Figure 4. Lemma 4.1 shows that if semicircles on the x-axis are minimizing in the upper half-plane with density y^α for $\alpha > 0$, then they are minimizing in the plane with density $|y|^\alpha$.

Proposition 4.2. *Given a natural number n, on the plane with density $|y|^n$, semicircles on the x-axis uniquely solve the isoperimetric problem up to horizontal translation and reflection across the x-axis.*

Proof. By Lemma 4.1, it suffices to prove the result in the half-plane with density y^n (see Figure 4). Given a closed curve C in the half-plane bounding a region R, the surface area of and the volume enclosed by its $(n+1)$-dimensional surface of revolution around the x-axis are proportional to $\int_C y^n\,ds$ and $\int_R y^n\,dA$ respectively. These integrals give the weighted perimeter and weighted area of the curve in the half-plane with density y^n. So minimizing weighted perimeter given a weighted area is equivalent to finding the least-area $(n+1)$-dimensional surface of revolution enclosing a given volume. Since the round $(n+1)$-dimensional sphere $\mathbf{S}^{n+1}$ uniquely minimizes surface area for such surfaces of revolution, a curve will be isoperimetric if and only if the curve is a semicircle on the x-axis. $\square$

Before treating the general case where α is any positive real number, we can use the techniques in Proposition 4.2 to solve the double bubble problem in the plane with density $|y|^n$.

Proposition 4.3. *In the plane with density $|y|^n$, the least-perimeter way to enclose two regions of prescribed area is half the standard double bubble; namely three circular arcs perpendicular to the x-axis and meeting one another at* 120 *degrees as in Figure* 3 *above. This shape is uniquely minimizing up to horizontal translation and reflection across the x-axis.*

Proof. By Lemma 4.1, it suffices to prove the result in the upper half-plane with density y^n. Let C be a graph which encloses two regions, R_1 and R_2. The area of the $(n+1)$-dimensional surface of revolution generated by C is proportional to the weighted length of C, and the volumes enclosed by this surface of revolution are proportional to the weighted areas of R_1 and R_2, so the proposition follows from Reichardt's result ([Re08], Theorem 1.1) that the standard double bubble in $\mathbf{R}^{n+2}$ uniquely minimizes surface area among all surfaces enclosing two prescribed volumes. $\square$

We now move to the general case where α is any positive real number. The most important technique used in the proofs that follow is Steiner symmetrization. Classical Steiner symmetrization says a region R in $\mathbf{R}^n$ can be symmetrized over an $(n-1)$-dimensional hyperplane H by replacing each 1-dimensional slice of R perpendicular to H by a line segment of the same length centered on H. This process preserves the volume of R without increasing its surface area. For product manifolds with density, [Ro05], Proposition 8 guarantees that slices of a region in the product may be replaced by minimizers in one of the factors, provided these minimizers grow by uniform enlargement. This new region will have no greater perimeter, but a characterization of equality is not given; hence the need for Lemma 4.7.

Lemma 4.4. *On the half-plane $\{(x,y) : y > 0\}$ with density y^α, $\alpha > 0$, for every region R with finite area, there exists an $f : \mathbf{R} \to [0, +\infty]$ satisfying:*

(i) *The region $R' = \{(x,y) : 0 < y < f(x)\}$ has the same weighted area as R.*

(ii) *R' is symmetric with respect to the y-axis.*
(iii) *$f(x)$ is monotonic in $|x|$.*
(iv) $\lim_{|x|\to\infty} f(x) = 0$.
(v) *The perimeter of R' is no greater than the perimeter of R.*

Proof. We use a Steiner symmetrization argument. Slice the half-plane with vertical half-lines, and replace each vertical slice of R with an initial interval $(0, a)$ of the same weighted length, preserving the weighted area of R. Since initial intervals solve the isoperimetric problem in the half-line with density y^α and grow by uniform enlargement, this symmetrization does not increase perimeter. Similarly replace horizontal slices of R with intervals centered on the y-axis, still preserving area. By the classical symmetrization, this does not increase *unweighted* perimeter; since density is constant along horizontal slices, the symmetrization does not increase weighted perimeter either. Call the new region R', and conditions (i), (ii), (iii) and (v) follow directly from the symmetrization (see Figure 5), while condition (iv) follows from the requirement that R has finite area. □

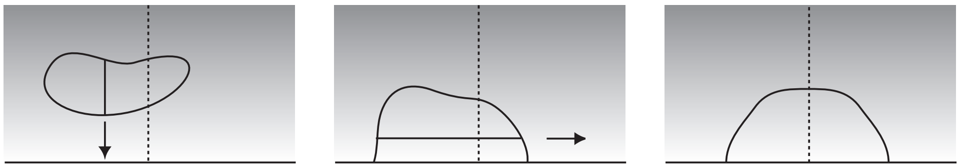

FIGURE 5. By replacing vertical slices with initial intervals and horizontal slices with centered intervals, perimeter is reduced while area is maintained.

Lemma 4.5 (F. Morgan)**.** *On the half-plane $\{(x, y) : y > 0\}$ with density y^α, $\alpha > 0$, for any given area, there exists an isoperimetric region.*

Proof. Consider a sequence of regions of area A with perimeter approaching the infimum. By Lemma 4.4 we may assume that each region is bounded by the x-axis and the graph of a symmetric function f monotone in $|x|$, approaching zero as $|x| \to \infty$. The perimeter P and area A satisfy

$$P \geq \int_{-\infty}^{\infty} f^\alpha \, dx, \tag{1}$$

$$A = \int_{-\infty}^{\infty} \int_0^{f(x)} y^\alpha \, dy \, dx = \frac{1}{1+\alpha} \int_{-\infty}^{\infty} f^{\alpha+1} \, dx \leq f(0) \frac{P}{1+\alpha}. \tag{2}$$

By standard compactness arguments of geometric measure theory (see Morgan [M08]) we may assume that these regions converge weakly without perimeter cancelation to a perimeter-minimizing region of area $A_0 \leq A$. By

(2) and P bounded we have $f(0)$ bounded below, which gives $A_0 > 0$. By scaling, minimizers exist for all areas. □

With these first two lemmas, we prove that semicircles are minimizing, though we are not yet ready to prove uniqueness.

Proposition 4.6. *On the half-plane with density y^α, $\alpha > 0$, the vertical Steiner symmetrization of any isoperimetric curve is a semicircle perpendicular to the x-axis. In particular, the semicircle is a solution to the isoperimetric problem.*

Proof. Suppose that C is the boundary of a Steiner-symmetrized isoperimetric region in the half-plane (isoperimetric regions exist for all areas by Lemma 4.5). Let $C(s) = (x(s), y(s))$ be a parameterization by unweighted arc length so that C has curvature $\kappa = x''y' - x'y''$ and outward normal $\hat{n} = (y', -x')$. Then C must have constant generalized curvature (see Introduction)

$$\kappa_\varphi = \kappa - \hat{n} \cdot \nabla(\log y^\alpha) = x''y' - x'y'' + \frac{x'\alpha}{y}.$$

We set up a model used in a paper by Hsiang [H82] on surfaces of Delaunay. Since C is isoperimetric, it is smooth, so we can define θ at every s as the angle clockwise from vertically upward to the unit tangent. Taking derivatives with respect to s then gives $y' = \cos\theta$ and $x' = \sin\theta$, hence

$$\kappa_\varphi = \theta'(\cos^2\theta + \sin^2\theta) + \frac{\alpha \sin\theta}{y} = \theta' + \frac{\alpha\sin\theta}{y}.$$

Recall that κ_φ must be constant. Put

$$F(s) = y^\alpha \sin(\theta) - \kappa_\varphi \frac{y^{\alpha+1}}{\alpha+1},$$

so that

$$\begin{aligned} F'(s) &= \alpha y^{\alpha-1} y' \sin\theta + y^\alpha \theta' \cos\theta - \kappa_\varphi y^\alpha y' \\ &= \alpha y^{\alpha-1}\cos\theta\sin\theta + y^\alpha\theta'\cos\theta - \left(\theta' + \frac{\alpha\sin\theta}{y}\right) y^\alpha \cos\theta \\ &= 0. \end{aligned}$$

Then in fact

$$F = y^\alpha\left(\sin\theta - y\frac{\kappa_\varphi}{\alpha+1}\right)$$

is constant. Since C has finite weighted length, there exists a sequence of points $C(s_n)$ on C such that the y-coordinate $y(s_n) \to 0$ as $n \to \infty$. Then $F(s_n) \to 0$, and because F is constant, $F = 0$. Putting $c = \kappa_\varphi/(\alpha+1)$, the identity $F = 0$ gives the last equality in the string below:

$$\sqrt{1 - (y')^2} = x' = \sin\theta = cy,$$

from which it follows that $y' = \pm\sqrt{1 - c^2y^2}$. Then every component of C is a semicircle on the x-axis of radius $1/c$. Since two semicircles may be replaced by a single larger semicircle with less perimeter, C must consist of a single semicircle. □

We need a final tool to conclude that the above solution is unique.

Lemma 4.7. *Let ψ be a smooth density on $I = (0, \infty)$, and suppose that initial intervals uniquely minimize perimeter for every prescribed volume. Let M be a smooth n-dimensional Riemannian manifold with density φ, and consider the product $M \times I$ with density $\varphi \times \psi$. Let R be an isoperimetric region in $M \times I$, and let R' denote its Steiner symmetrization. Suppose that at every point R' is smooth with a nonvertical tangent plane. Then $R = R'$.*

Proof. The crux of this argument follows Rosales *et al.* ([RCB08], proof of Theorem 5.2). By the near-smoothness of minimizers (the set of singularities is closed, with dimension no greater than $n - 7$ by [M03], Section 3.10) and by Sard's theorem, the set of all $x \in M$ such that R has any nonsmooth points or any vertical tangent planes above x has Lebesgue measure 0. (This does not imply that R has nonvertical tangent planes a.e., but only that the irregular points project to a null set in M.) Then over almost all points in M, R can be written as the region between the graphs of some number (depending on the point) of smooth positive functions $h_1, \dots, h_m : M \to I$, and the surface area of R is at least

$$\int_M \sum_i \varphi\psi(h_i)\sqrt{1 + |\nabla h_i|^2}\, dA.$$

On the other hand, its symmetrization R' is the region under the graph of a smooth function f, and by the hypothesis of nonvertical tangent planes, all of its surface area is captured by the integral

$$\int_M \varphi\psi(f)\sqrt{1 + |\nabla f|^2}\, dA.$$

Since R and R' must have the same surface area (they are both isoperimetric), the desired result will follow by proving that the inequality for integrands

$$\psi(f)\sqrt{1 + |\nabla f|^2} \le \sum_i \psi(h_i)\sqrt{1 + |\nabla h_i|^2} \tag{3}$$

holds pointwise (a.e., where quantities are defined), with equality only when the sum on the right consists of a single term with $h_1 = f$, as this will show $R = R'$.

The symmetrization gives relations between f and the h_i. First, since vertical slices of R' have the same weighted length as vertical slices of R, we have for a.e. point in M that

$$\int_0^f \psi(z)\,dz = \int_{h_{m-1}}^{h_m} \psi(z)\,dz + \int_{h_{m-3}}^{h_{m-2}} \psi(z)\,dz + \cdots,$$

formally taking $h_0 = 0$ if m is odd. Differentiating this equality gives

$$\psi(f)\nabla f = \sum_i \pm\psi(h_i)\nabla h_i,$$

so by the triangle inequality,

$$|\nabla f| \le \sum_i \frac{\psi(h_i)}{\psi(f)}|\nabla h_i|. \tag{4}$$

By the assumption that intervals $(0, f)$ are uniquely minimizing in I, we have

$$\psi(f) \le \sum_i \psi(h_i) \tag{5}$$

with equality only if the sum on the right consists of $h_1 = f$ alone. Put $\lambda = \sum_i \psi(h_i)/\psi(f) \ge 1$, and define $F(t) = \sqrt{1+t^2}$, so that F is strictly convex and is monotone in $|t|$. By this monotonicity and then by convexity, we calculate

$$\begin{aligned} F(|\nabla f|) &\le F\left(\sum_i \frac{\psi(h_i)}{\psi(f)}|\nabla h_i|\right) = F\left(\sum_i \frac{\psi(h_i)}{\lambda\psi(f)}\lambda|\nabla h_i|\right) \\ &\le \sum_i \frac{\psi(h_i)}{\lambda\psi(f)} F(\lambda|\nabla h_i|) = \frac{1}{\psi(f)}\sum_i \psi(h_i)\sqrt{\lambda^{-2}+|\nabla h_i|^2} \\ &\le \frac{1}{\psi(f)}\sum_i \psi(h_i)\sqrt{1+|\nabla h_i|^2}, \end{aligned}$$

which proves (3) with the desired characterization of equality in the last line. □

Remark 4.8. We note that trivial modifications of the proof show that Lemma 4.7 can also be applied where I is the real line with density such that either:

(i) symmetric intervals are uniquely minimizing for all volumes, or
(ii) initial intervals $(-\infty, a)$ are uniquely minimizing for all volumes.

Such a lemma simplifies the proof of Rosales *et al.* ([RCB08], Proposition 5.2) that balls about the origin are uniquely isoperimetric in $\mathbf{R}^n$ with density $\exp(r^2)$.

Corollary 4.9. *On the half-plane with density y^α, $\alpha > 0$, a semicircle perpendicular to the x-axis uniquely solves the isoperimetric problem.*

Proof. The vertical Steiner symmetrization of any isoperimetric region is exactly such a semicircle (Proposition 4.6), whose tangent is everywhere nonvertical, satisfying the conditions of Lemma 4.7. □

5. Densities on the cylinder

This Section 5 first considers the isoperimetric problem on the $(n+1)$-dimensional cylinder $\mathbf{S}^n \times \mathbf{R}$ with density e^z. We also examine isoperimetric curves on the 2-dimensional cylinder $\mathbf{S}^1 \times \mathbf{R}$ with density $|\theta|^\alpha$.

Proposition 5.1. *On the $(n+1)$-dimensional cylinder, $\mathbf{S}^n \times \mathbf{R}$, with density e^z a horizontal n-sphere, $\mathbf{S}^n \times \{z\}$, uniquely solves the isoperimetric problem among smooth hypersurfaces.*

Proof. We will prove both that a horizontal n-sphere has weighted surface area equal to the weighted volume it bounds below and that any other surface has weighted surface area strictly greater than the weighted volume it bounds.

Given any smooth closed hypersurface $S \subset \mathbf{S}^n \times \mathbf{R}$, the weighted surface area of S is given by the equation $P = \int_S e^z \, dA$. If $\alpha(p) \in [0, \pi/2]$ is the angle between S and the horizontal then

$$P = \int_S e^z \, dA \geq \int_S e^z \cos(\alpha(p)) \, dA = \int_{\mathbf{S}^n} \left(\sum_{\{z:(\Theta,z)\in S\}} e^z \right) dA \tag{6}$$

by the co-area formula (Morgan [M08]). Let

$$z_0(\Theta) = \sup\{z : (\Theta, z) \in S\}, \tag{7}$$

adopting the convention $\sup \emptyset = -\infty$. Then

$$P \geq \int_{\mathbf{S}^n} e^{z_0} \, dA = \int_{\mathbf{S}^n} \int_{-\infty}^{z_0} e^z \, dz \, dA \geq V. \tag{8}$$

Since equality holds throughout for a horizontal sphere, horizontal spheres minimize perimeter for given volume. Conversely, suppose equality holds throughout. By (6), $\alpha(p) = 0$ and S consists of horizontal spheres. By (8), S is a single sphere. This process is illustrated for $\mathbf{S}^1 \times \mathbf{R}$ in Figure 6 below. □

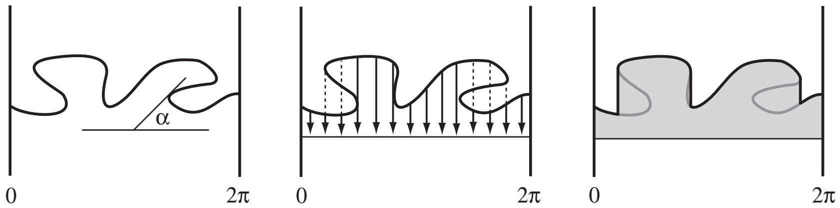

Figure 6. Any curve which is not a horizontal circle has perimeter strictly greater than area enclosed.

Finally, we examine the cylinder with density $|\theta|^\alpha$ and find that semicircles on the line $\theta = 0$ are minimizing for small areas.

Proposition 5.2. *Let $\alpha > 0$ be given, and consider the cylinder $\mathbf{S}^1 \times \mathbf{R}$ with density $|\theta|^\alpha$ for $\theta \in (-\pi, \pi]$. For small areas, semicircles on the line $\theta = 0$ solve the isoperimetric problem, and do so uniquely up to translation along and reflection across the line $\theta = 0$.*

Proof. Consider a curve γ enclosing area no greater than the area enclosed by a semicircle on the line $\theta = 0$ and tangent to the line $\theta = \pi$. First suppose no component of γ crosses the line $\theta = \pi$. By the assumption on the area enclosed by γ, semicircles on $\theta = 0$ enclosing no greater area fit inside the strips $[0, \pi] \times \mathbf{R}$ and $[-\pi, 0] \times \mathbf{R}$, so we can apply Proposition 4.6, to replace the parts of γ both above and below the line $\theta = 0$ with such semicircles, reducing perimeter. Reflect one of these semicircles over the line $\theta = 0$, and then Proposition 4.6 can be applied again to recombine the two semicircles into one larger semicircle with less perimeter, again fitting inside one of the half cylinders. Then we see that a semicircle encloses area more efficiently than any curve not crossing $\theta = \pi$.

It now suffices to show for small areas that semicircles on $\theta = 0$ are more efficient than curves crossing $\theta = \pi$. First let σ be the semicircle of radius R, $\sigma(t) = (R \sin t, R \cos t)$ for $t \in [0, \pi]$, which will be considered a curve on the strip $[0, \pi] \times \mathbf{R}$ for $0 < R \leq \pi$. Calculate the length of σ as

$$L(\sigma) = \int_\sigma \theta^\alpha \, ds = \int_0^\pi (R \sin t)^\alpha R \, dt = R^{\alpha+1} \int_0^\pi (\sin t)^\alpha \, dt$$

and the area enclosed by σ as

$$A(\sigma) = \int \theta^\alpha \, dA = \int_0^R \int_0^\pi (r \sin t)^\alpha r \, dt \, dr = \frac{R^{\alpha+2}}{\alpha+2} \int_0^\pi (\sin t)^\alpha \, dt.$$

In particular, $L(\sigma)$ is proportional to $A(\sigma)^{(\alpha+1)/(\alpha+2)}$. Now suppose ζ is a component of an area-enclosing curve which intersects the line $\theta = \pi$. For small area, we will see that ζ must stay close to $\theta = \pi$, say, ζ must stay in the strips where $\theta > \pi/2$ or $\theta < -\pi/2$, or else a semicircle on $\theta = 0$ would enclose that area more efficiently. If ζ did not stay close to $\theta = \pi$, its length would satisfy

$$L(\zeta) = \int_\zeta |\theta|^\alpha \, ds \geq 2 \int_{\pi/2}^\pi \theta^\alpha \, d\theta,$$

which is constant and positive. Yet we just calculated that a semicircle on $\theta = 0$ has perimeter proportional to a positive power of the area enclosed, so that the perimeter approaches to zero as area goes to zero. Thus we may suppose that ζ stays in the region with density between $(\pi/2)^\alpha$ and π^α. Let L_0 and A_0 denote unweighted perimeter and area respectively, and then by the planar isoperimetric inequality, $L_0(\zeta) \geq \sqrt{4\pi A_0(\zeta)}$. This gives for weighted perimeter and area

$$L(\zeta) > \left(\frac{\pi}{2}\right)^\alpha \sqrt{4\pi A_0(\zeta)} \geq \left(\frac{\pi}{2}\right)^\alpha \sqrt{4\pi^{1-\alpha} A(\zeta)}.$$

That is, $L(\zeta)$ is bounded below by a constant multiple of $A(\zeta)^{1/2}$. Then semicircles on $\theta = 0$ are more efficient for small areas, because they have length proportional to a higher power of area enclosed, so their lengths shrink to 0 more rapidly as area goes to 0. □

6. Stability of circles in surfaces of revolution with density

We turn to the problem of determining when a circle in a surface of revolution with radial density is stable, that is, when its second variation of weighted perimeter for fixed weighted area is nonnegative. We will say that a curve is *stable* if it locally minimizes perimeter for given area. We begin by stating Wirtinger's inequality in Lemma 6.1 and use it to generalize a proposition of Rosales *et al.* ([RCB08], Theorem 3.10), that spheres about the origin in Euclidean space with radial density are stable if and only if the density is log convex. That is, Theorem 6.3 finds necessary and sufficient conditions in disks, spheres, and annuli of revolution with radial density for the stability of its circles of revolution. An immediate corollary (6.4) demonstrates the symmetry of the condition in disks with increasing or decreasing Gauss curvature. In the borderline cases $\mathbf{S}^2$ and $\mathbf{H}^2$ where Gauss curvature is constant, Rosales' observation for Euclidean space still holds: circles about the origin are stable if and only if the radial density is log convex (Corollary 6.5). The constructions of 6.6 and 6.7 show that the equivalence does not hold in all surfaces of revolution. Proposition 6.8 then finds that for any positive density, there exists a Riemannian annulus of revolution where circles of revolution are trivially stable. We end with two conjectures as to when isoperimetric regions are bounded by circles of revolution.

Lemma 6.1 (Wirtinger's inequality [RW08]). *If $f : \mathbf{R} \to \mathbf{R}$ is C^1 and periodic with period 2π, and if $\int_0^{2\pi} f(t)\,dt = 0$, then $\int_0^{2\pi} f'(t)^2\,dt \geq \int_0^{2\pi} f(t)^2\,dt$. Equality holds if and only if $f(t) = a\sin t + b\cos t$ for some constants a and b.*

Lemma 6.2. *Let S be a smooth Riemannian disk, sphere, or annulus of revolution with metric $ds^2 = dr^2 + f(r)^2 d\theta^2$, where r is the Riemannian distance from the pole of revolution, or in the case of the annulus, r is the signed distance from some chosen circle of revolution. Then a circle of revolution (r constant) has classical geodesic curvature $\kappa = f'(r)/f(r)$, and the Gauss curvature of S along the circle is given by $G = -f''(r)/f(r)$.*

Proof. Set $P(r) = 2\pi f(r)$, the unweighted perimeter of a centered circle with radius r, and set $A(r) = \int_0^r P(t)\,dt$, the [signed] unweighted area of a disk [annulus] bounded by that circle. Then

$$\kappa = \frac{dP}{dA} = \frac{P'(r)}{A'(r)} = \frac{P'(r)}{P(r)} = \frac{f'(r)}{f(r)}.$$

To calculate G, we observe that

$$P'(r) = \frac{dP}{dA}A'(r) = \kappa P(r) = 2\pi - \int_0^r GP(t)\,dt$$

by Gauss–Bonnet. Thus $P''(r) = -GP(r)$, and $G = -f''(r)/f(r)$. □

Theorem 6.3. *Let S be a smooth Riemannian disk, sphere, or annulus of revolution with metric $ds^2 = dr^2 + f(r)^2 d\theta^2$ and density $e^{\psi(r)}$. Set*

$$Q(r) = f'(r)^2 - f(r)f''(r) - f(r)^2\psi''(r).$$

Then the circle of revolution at distance r is stable if and only if $Q(r) \leq 1$. Additionally, that circle has strictly positive second variation of weighted perimeter for fixed weighted area $(P - \kappa_\psi A)''(0)$ for all such (nontrivial) variation vector fields if and only if $Q(r) < 1$.

In the borderline case that $Q(r) = 1$, the second variation for the circle is positive for all area-preserving u except $u = a\sin\theta + b\cos\theta$, in which case it vanishes.

Proof. As given in Rosales *et al.* ([RCB08], Proposition 3.6), the second variation of such a circle of revolution γ centered at the pole is

$$(P - \kappa_\psi A)''(0) = \int_\gamma \left(\left(\frac{du}{ds}\right)^2 - u^2\left(\kappa^2 + G - \psi''\right)\right) e^\psi\, ds,$$

where κ is the classical geodesic curvature of γ, G is the classical Gauss curvature of S along γ, and u is the normal component of a smooth variation vector field with $\int_\gamma u\, ds = 0$. For such circles, u can be reparameterized in terms of θ with $\frac{ds}{d\theta} = f(r)$, so $\frac{du}{ds} = \frac{1}{f(r)}\frac{du}{d\theta}$. Now by Lemma 6.2, our second variation becomes

$$\frac{e^\psi}{f}\int_0^{2\pi} \left(\left(\frac{du}{d\theta}\right)^2 - u^2(f'^2 - ff'' - f^2\psi'')\right) d\theta$$
$$= \frac{e^\psi}{f}\int_0^{2\pi} \left(\left(\frac{du}{d\theta}\right)^2 - Qu^2\right) d\theta.$$

Now if $Q < 1$, we have

$$(P - \kappa_\psi A)''(0) > \frac{e^\psi}{f}\int_0^{2\pi} \left(\left(\frac{du}{d\theta}\right)^2 - u^2\right) d\theta,$$

which is nonnegative by Wirtinger's inequality (Lemma 6.1), since

$$\int_0^{2\pi} u\, d\theta = \frac{1}{f}\int_\gamma u\, ds = 0.$$

In the case that $Q = 1$, Wirtinger's inequality tells us the second variation is nonnegative for all appropriate u, vanishing only for $u = a\sin\theta + b\cos\theta$.

When $Q > 1$, we take $u = \sin\theta$ and now find that the second variation is negative.

Conversely, if $(P - \kappa_\psi A)''(0) \geq 0$ for all smooth area-preserving variation vector fields u, then $\int_0^{2\pi} \left(\left(\frac{du}{d\theta} \right)^2 - Qu^2 \right) d\theta \geq 0$ so that $Q \leq 1$, else the choice $u = \sin\theta$ gives a negative second variation, a contradiction. Similarly, if $(P - \kappa_\psi A)''(0) > 0$ for all such u, then $Q < 1$. $\square$

Corollary 6.4. *Let S be a smooth Riemannian disk of revolution with density $e^{\psi(r)}$ and monotone Gauss curvature $G(r)$. If G is nonincreasing and $\psi''(r) \geq 0$, then the circle of revolution with radius r is stable. If either G is strictly increasing or $\psi''(r) > 0$, then that circle has strictly positive second variation. On the other hand, if G is nondecreasing and the circle at radius r is stable, then $\psi''(r) \geq 0$. Similarly, if either G is strictly increasing or that circle has strictly positive second variation, then $\psi''(r) > 0$.*

Proof. As in Ritoré ([R01], Section 1), write the metric of S as $ds^2 = dr^2 + f(r)^2 d\theta^2$, and put $H = f'^2 - ff''$. For a smooth disk of revolution, $f(0) = 0$ and $f'(0) = 1$, so that $H(0) = 1$. By Lemma 6.2, we have $G = -f''/f$, so that $G' = (f'f'' - ff''')/f^2$. Observe that $H' = f'f'' - ff'''$, so that G and H increase or decrease together, and they have the same critical points. Thus when G is nonincreasing and $\psi''(r) \geq 0$, we have $H(r) \leq H(0) = 1$, so that $H(r) - f(r)^2\psi''(r) \leq 1$ and the circle at r is stable by Theorem 6.3. When G is nondecreasing and the circle at r is stable, we have $H(r) \geq 1$ and $H(r) - f(r)^2\psi''(r) \leq 1$, so that $\psi''(r) \geq 0$. The corresponding statements about strict inequalities follow by the same brand of reasoning. $\square$

Corollary 6.5. *In $\mathbf{R}^2$, $\mathbf{S}^2$, or $\mathbf{H}^2$ with radially symmetric density $e^{\psi(r)}$, a circle about the origin (about the pole for $\mathbf{S}^2$) is stable [respectively has positive second variation] if and only if ψ'' is nonnegative [positive] on that circle.*

Proof. All three surfaces have constant Gauss curvature, and any circle of revolution lies in an open disk, so the result follows from Corollary 6.4. Alternatively, the result can seen by the identity $f_S'^2 - f_S f_S'' - f_S^2\psi'' = 1 - f_S^2\psi''$ for each surface, where $f_{\mathbf{R}^2}(r) = r$, $f_{\mathbf{S}^2}(r) = \sin r$, and $f_{\mathbf{H}^2}(r) = \sinh r$. $\square$

For general surfaces of revolution, it is not true that the stability of circles of revolution is equivalent to the log convexity of a radial density. In the next corollary, we split hairs in our inequalities to show that the converses to both halves of Corollary 6.4 are false when Gauss curvature is strictly monotone. The following corollaries give more intuitive constructions: we see that for any radial density function, there exists a complete surface of revolution with that density where some circle of revolution is unstable, and there exists an annulus where every circle of revolution is trivially stable.

Corollary 6.6. *Let S be a smooth Riemannian disk of revolution with metric $ds^2 = dr^2 + f(r)^2 d\theta^2$, and strictly increasing or strictly decreasing Gauss curvature $G(r)$. Set $H = f'^2 - ff''$, and define the function ψ up to a first-degree polynomial in r by taking $\psi'' = (H-1)/2f^2$. Endow S with density e^ψ, which is smooth except possibly at the pole of revolution. If G is decreasing, then $\psi'' < 0$, and all circles of revolution have positive second variation. If G is increasing, then $\psi'' > 0$, and all circles of revolution are unstable.*

Proof. Recall that G and H have the same derivative up to multiplication by the positive function $1/f^2$, so they increase and decrease simultaneously. Recall also that for a smooth disk, we have $H(0) = 1$. If G is decreasing, then H decreases from 1 so that $\psi'' = (H-1)/2f^2 < 0$ for all $r > 0$, yet $H - f^2\psi'' = (H+1)/2 < 1$, so circles of revolution have positive second variation. Symmetrically, if G is increasing, then H increases from 1, and $\psi'' = (H-1)/2f^2 > 0$ although $H - f^2\psi'' = (H+1)/2 > 1$, so that circles of revolution are unstable. □

Corollary 6.7. *For any smooth real-valued function ψ, defined on $\mathbf{R}$, $[0,\infty)$, or some subinterval $[0,a]$, there exists a complete surface of revolution embedded in $\mathbf{R}^3$ with density $e^{\psi(r)}$ such that some circle of revolution is unstable.*

Proof. As in Figure 7, the idea is to take a surface with a highly curved lip, along which a circle will not be stable. In the case that ψ is defined on $\mathbf{R}$ or $[0,\infty)$, fix some $r_0 > 1$. Pick a generatrix in the plane to be rotated about the y-axis, $\gamma(r) = (x(r), y(r))$, parameterized by arc length, such that $x(r_0) = 1$, $x'(r_0) = 0$, $x''(r_0) < -\psi''(r_0) - 1$, and $x > 0$ except when ψ is defined on $[0,\infty]$, when we ask $x(0) = 0$. The surface has metric $ds^2 = dr^2 + x(r)^2 d\theta^2$, and we see that $x'(r_0)^2 - x(r_0)x''(r_0) - x(r_0)^2\psi''(r_0) > 1$, so the circle at r_0 is not stable.

In the case that ψ is only defined on $[0,a]$, we assume without loss of generality that $a > 2$ so that the generatrix can be constructed as claimed with $1 < r_0 < a-1$. Then the same construction goes forward, except we require that $x(0) = x(a) = 0$, and the surface is a sphere of revolution with an unstable circle. □

Proposition 6.8. *For any smooth real-valued function ψ, defined on $\mathbf{R}$ or some subinterval, the annulus of revolution S with metric $dr^2 + e^{-2\psi}d\theta^2$ and density e^ψ has the property that all of its circles of revolution have length 2π and are (uniquely) the shortest noncontractible curves (in particular, they have positive second variation). In this sense, S is essentially a cylinder with weighted area but unweighted perimeter.*

Proof. Note the surface is an annulus since $e^{-\psi} > 0$. We have weighted length $ds_\psi^2 = e^{2\psi}(dr^2 + e^{-2\psi}d\theta^2) = e^{2\psi}dr^2 + d\theta^2$. Any noncontractible curve

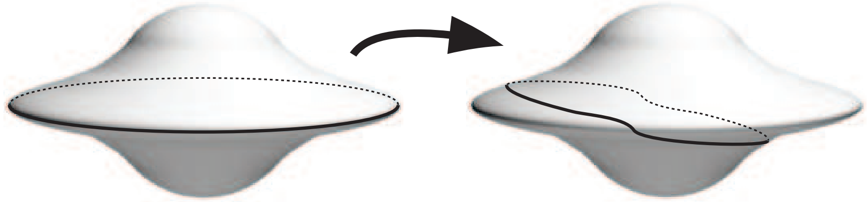

FIGURE 7. Even with favorable density, on a surface with a highly curved lip, a circle along the lip is unstable.

must sweep out at least angle 2π, hence have weighted length at least 2π, with equality holding if and only if the curve has no dr component in its arc length and hence is a circle of revolution. Also for $f = e^{-\psi}$, we have $f'^2 - ff'' - f^2\psi'' = 0$, so circles of revolution have positive second variation as expected.

The relationship between S and the cylinder $\mathbf{S}^1 \times \mathbf{R}$ (or a truncation of the cylinder) is made clear by reparameterization. Setting $dt = e^{\psi}dr$, we have length $ds_\psi^2 = dt^2 + d\theta^2$ and area $dA_\psi = e^{\psi}dr\, e^{-\psi}d\theta = e^{-\psi}dt\, d\theta$. □

With the notation of Theorem 6.3, a direct calculation shows that $Q(r)$ is identically 0 in the above example, giving another proof that the circles of revolution have positive second variation. We note that actually the minimality of circles in Proposition 6.8 is an example of a more general principle ([MM08], Proposition 2.1), by comparison with constant-density cylinders with no curvature. (When $f \leq 1$ or more generally is bounded, we may apply [MM08], Proposition 2.1 directly; if f is unbounded, we make the comparison on every compact truncation of the cylinder.)

We end this section with two conjectures as to when circles of revolution may be isoperimetric.

Conjecture 6.9. *On a complete, smooth Riemannian plane or sphere of revolution with nonincreasing Gauss curvature and increasing, log convex radial density, isoperimetric regions exist and are bounded by circles of revolution.*

We have reason to believe such a conjecture: in the classical case where the surface has constant density, decreasing Gauss curvature implies that isoperimetric regions exist and are bounded by such circles ([MHH00], [R01]), though the regions needn't be disks. By Corollary 6.4, we know at least that such circles are stable and hence local minima. Appealing to the common wisdom that isoperimetric curves prefer low-density regions, it seems there is no other place for an isoperimetric curve to be. To make a comparison, it was conjectured by Rosales ([RCB08], Conjecture 3.12) that isoperimetric regions in $\mathbf{R}^n$ with log convex radial density are balls centered at the origin. We provide an alternate, logically independent conjecture from Frank Morgan, motivated by the observation ([R01], Section 1) that Gauss curvature and $f'^2 - ff''$ (the constant-density version of $Q(r)$) increase and decrease together.

Conjecture 6.10 (F. Morgan)**.** *Consider a smooth Riemannian plane or sphere of revolution with radial density e^ψ and metric $ds^2 = dr^2 + f(r)^2 d\theta^2$. If $Q(r) = f'^2 - ff'' - f^2\psi''$ not only is less than or equal to 1 but also is nonincreasing, then isoperimetric regions exist and are bounded by circles of revolution.*

To see the logical independence, we start with any example surface satisfying the hypotheses of Conjecture 6.9 with ψ'' strictly positive, then introduce a small, sharp decrease in ψ'' so that the hypothesis of $Q(r)$ decreasing in Conjecture 6.10 is not met. Conversely, we can start with an example satisfying the hypotheses of Conjecture 6.10 with constant density 1 and $f'^2 - ff''$ strictly decreasing, then introduce a nearly constant density with ψ'' small enough to maintain the hypotheses of Conjecture 6.10 but somewhere negative.

The question arises whether either conjecture might also apply in some capacity to cylinders $\mathbf{R} \times \mathbf{S}^1$ with density. We first acknowledge that even the constant-density versions of the above conjectures (and even after some necessary restatement) would amount to new results. For work on this constant-density analogue, see [R01], Section 5, which treats the case where the end of maximal curvature has finite area but shows that circles can be unstable without this hypothesis (in contrast with planes and spheres, decreasing Gauss curvature does not imply stability of circles of revolution on cylinders). Of course, no such instability can occur in any generalization of Conjecture 6.10, but an additional hypothesis guaranteeing stability would be necessary to apply Conjecture 6.9 to cylinders.

For those cylinders whose ends both have infinite weighted area, we should often expect small areas to be more efficiently enclosed as nearly round disks than as annuli of revolution, so a more sensible restatement of the two conjectures would ask for curves of minimal weighted length among those enclosing a given (net) weighted area with some fixed circle of revolution. Moreover, such a restatement can apply just as well to truncated cylinders with no cause for concerns about the boundary. In any case, an appropriate generalization of Conjecture 6.10 would correctly predict the minimality of circles where the density is $1/f$ (as in Proposition 6.8), by the observation that $Q(r)$ vanishes in this scenario.

7. Asymptotic estimates

In this section we generalize to surfaces with density the main result of Maurmann *et al.* [MEM08] that the minimum perimeter $P(n)$ to partition a compact surface M with surface area $|M|$ into n regions of equal area is asymptotic to half the perimeter of n planar hexagons of area $|M|/n$. That is, $P(n)$ is asymptotic to $12^{1/4}\sqrt{|M|n}$, which we abbreviate by defining a constant, $H = 12^{1/4}\sqrt{|M|}$. The proof used area-preserving near-isometries between small regions in M and in the plane to exploit the fact that regular hexagons partition the plane most efficiently into unit areas (Hales [H01]). This argument gave the asymptotic lower bound; the upper bound followed by constructing a partition of M into n regions of equal area by mapping clusters of planar hexagons into M and patching up the interstices with asymptotically negligible cost. We begin by stating a fundamental inequality from Hales and the relevant results of Maurmann *et al.* Let M be a smooth, compact Riemannian surface throughout, possibly with boundary, and recall from Section 4 that a *cluster* is a collection of disjoint open sets of prescribed areas.

Theorem 7.1 (Hales, Theorem 2 [H01], Proposition 15.6 [M08])**.** *Any cluster of planar regions with areas $a_1, \ldots, a_n$ such that $0 < a_i \leq 1$ has perimeter greater than $12^{1/4}\sum_{i=1}^{n} a_i$.*

Lemma 7.2 allows for the partitioning of M into a finite number of regions which can be made as nearly planar as we wish, almost reducing the problem to the known planar case.

Lemma 7.2 ([MEM08], Lemma 7)**.** *For any $\varepsilon > 0$, there exists a partition of M into a finite number of disjoint regions $E_1, \ldots, E_n$ with piecewise smooth boundaries such that on each E_i there exists an area-preserving diffeomorphism Φ_i mapping E_i to a region in $\mathbf{R}^2$ while distorting length by no more than a factor of $1+\varepsilon$. Moreover, the E_i may be taken so that each consists of finitely many disks.*

In constructing an upper bound for $P(n)$, small regular hexagons can be mapped by Lemma 7.2 into the regions E_i. The purpose of Lemma 7.3 is then to partition the remaining area into k regions at cost $O(\sqrt{k})$, which will be asysmptotically insignificant compared to $\sqrt{n}$ because k will be $O(\sqrt{n})$, as in [MEM08], Lemma 4.

Lemma 7.3 ([MEM08], Lemma 8)**.** *Let M be a smooth, compact Riemannian surface. Then there exist constants c_1 and c_2 such that for any integer k and any measurable subset $A \subset M$, there exists a partition of M into k regions $R_1, \ldots, R_k$ with total perimeter less than $c_1\sqrt{k} + c_2$ such that the area of each $A \cap R_i$ equals $|A|/k$.*

These lemmas imply the main theorem:

Theorem 7.4 ([MEM08], Theorem 9)**.** *The least perimeter $P(n)$ to partition M into n regions of equal area is asymptotic to $n/2$ times the perimeter of a planar regular hexagon of area $|M|/n$:*

$$\lim_{n\to\infty} \frac{P(n)}{\sqrt{n}} = 12^{1/4}\sqrt{|M|}.$$

To generalize the result, we first consider surfaces with *length density*, in which length is weighted, but not area. A conformal change of metric will yield the analogous result for surfaces with density.

Theorem 7.5. *Let M be a smooth, compact Riemannian surface with length density φ. Now let $P(n)$ denote the minimum (weighted) perimeter of a partition of M into n regions of equal area. Then $P(n)$ is asymptotic to*

$$12^{1/4}\sqrt{\frac{n}{|M|}}\int_M \varphi\, dA.$$

That is, as $n \to \infty$, $P(n)/\sqrt{n}$ approaches $H = 12^{1/4}\sqrt{|M|}$ times the expected value of φ on M.

Proof. For the upper bound, we retrace our steps through the construction in Theorem 7.4. Let $\varepsilon > 0$ be given. Use Lemma 7.2 to partition M into regions $E_1, \ldots, E_m$ on which area-preserving diffeomorphisms $\Phi_1, \ldots, \Phi_m$ distort length by no more than a factor of $1+\varepsilon$. By the uniform continuity of φ on M, it is clear that we can also require the regions E_i to be sufficiently small that φ varies by less than ε on each E_i. That is, $J_i - j_i < \varepsilon$, where $J_i = \sup\{\varphi(x) : x \in E_i\}$ and $j_i = \inf\{\varphi(x) : x \in E_i\}$. At this point, exactly the same construction given in Theorem 7.4 gives the desired upper bound: each Φ_i^{-1} maps clusters of regular planar hexagons of size $|M|/n$ into E_i, and the partition is finished by an application of Lemma 7.3. As before, all lower error terms in the estimate will simply drop out in the limit; they are increased at most by a constant factor of $\sup\{\varphi(x) : x \in M\}$. That is, we need only explicitly treat the term corresponding to half the perimeter of the hexagons mapped in, and the rest will be carried as an $o(\sqrt{n})$ error term.

We focus our attention on each E_i individually. Since the hexagons have size $|M|/n$, it is clear that no more than $|E_i|n/|M|$ hexagons are mapped into E_i. Half the perimeter of so many planar hexagons is no more than $12^{1/4}|E_i|\sqrt{n/|M|}$. Accounting for stretching and weighting, these hexagons are mapped into E_i to contribute less than $(1+\varepsilon)12^{1/4}J_i|E_i|\sqrt{n/|M|}$ to the weighted perimeter of our partition. Summing over i, we find

$$\begin{aligned}\frac{P(n)}{\sqrt{n}} &< \frac{(1+\varepsilon)12^{1/4}}{\sqrt{|M|}}\sum_{i=1}^{m} J_i|E_i| + \frac{o(\sqrt{n})}{\sqrt{n}}\\ &< \frac{(1+\varepsilon)12^{1/4}}{\sqrt{|M|}}\left(\int_M \varphi\, dA + \varepsilon|M|\right) + \frac{o(\sqrt{n})}{\sqrt{n}}.\end{aligned}$$

Taking the lim sup as $n \to \infty$ and then the limit as $\varepsilon \to 0$ gives the upper bound.

For the lower bound, we let $\varepsilon > 0$ and take E_i, Φ_i, J_i, and j_i as above, though we may assume each Φ_i maps each E_i into a different region of the plane. Let X_n be a graph which partitions M into n regions of equal area and minimizes weighted perimeter (minimizing graphs exist and satisfy the conditions of smoothness and finiteness, just as they do in the classical case when $\varphi = 1$). Let Q denote the total weighted perimeter of the ∂E_i. We first find a lower bound on the unweighted perimeter of X_n in each E_i. Let $Y_i = \partial\Phi_i(E_i) \cup \Phi_i(X_n \cap E_i)$. Then Y_i defines a cluster of planar regions with area no greater than $|M|/n$ by considering the connected open sets it encloses as its regions. Dilate by $\sqrt{n/|M|}$ so that the regions have area at most 1, and a quick application of Hales' Theorem 1 guarantees the perimeter of Y_i is greater than $12^{1/4}|E_i|\sqrt{n/|M|}$. Accounting for stretching by Φ and weighting by φ, the weighted length of $\partial E_i \cup (X_n \cap E_i)$ must then be greater than

$$\frac{12^{1/4}}{1+\varepsilon} j_i |E_i| \sqrt{\frac{n}{|M|}}.$$

To get a lower bound on $P(n)$, the weighted length of X_n, we take a union of the graphs $\partial E_i \cup (X_n \cap E_i)$ and remove all the twice-counted curves ∂E_i. Let Q denote the total weighted length of the ∂E_i, and we have

$$\begin{aligned} \frac{P(n)}{\sqrt{n}} &> \frac{12^{1/4}}{(1+\varepsilon)\sqrt{|M|}} \sum_{i=1}^{m} j_i |E_i| - \frac{2Q}{\sqrt{n}} \\ &> \frac{12^{1/4}}{(1+\varepsilon)\sqrt{|M|}} \left(\int_M \varphi \, dA - \varepsilon |M| \right) - \frac{2Q}{\sqrt{n}}, \end{aligned}$$

from which the result follows easily. □

The theorem can be easily modified to treat surfaces with density:

Corollary 7.6. *If M is a smooth, compact Riemannian surface with density φ, then the minimum weighted length of a partition of M into n regions of equal weighted area is asymptotic to*

$$12^{1/4} \sqrt{\frac{n}{|M|_\varphi}} \int_M \varphi^{3/2} \, dA,$$

where $|M|_\varphi = \int_M \varphi \, dA$, the weighted area of M.

Proof. With a conformal change of metric ds to $\varphi^{1/2} ds$, M can be thought of as a surface with length density $\varphi^{1/2}$. □

References

[AT76] Almgren, F.J. Jr.; Taylor, J.E. Geometry of soap films. *Sci. Amer.* **235** (1976) 82–93.

[BN06] Bezdek, Károly; Naszódi, Márton. Rigidity of ball-polyhedra in euclidean 3-space. *European J. Combin.* **27** (2006) 255–268. MR2199778 (2006k:51028), Zbl 1089.52010.

[CJQ07] Carroll, Colin; Jacob, Adam; Quinn, Conor; Walters, Robin. The isoperimetric problem on planes with density. *Bull. Austral. Math. Soc.* **78** (2008) 177–197. MR2466858.

[C1894] Cayley, Arthur. The collected mathematical papers of Arthur Cayley. Vol. VII. *Cambridge Univ. Press, Cambridge*, 1894.

[CHH06] Corwin, Ivan; Hoffman, Neil; Hurder, Stephanie; Šešum, Vojislav; Xu, Ya. Differential geometry of manifolds with density. *Rose-Hulman Und. Math. J.* **7** (2006), no. 1.

[H01] Hales, Thomas. The honeycomb conjecture. *Disc. Comput. Geom.* **25** (2001) 1–22. MR1797293 (2002a:52020), Zbl 1007.52008.

[H02] Hales, Thomas. The honeycomb problem on the sphere. arXiv:math/0211234v1, 2002.

[H82] Hsiang, Wu-Yi. On generalization of theorems of A.D. Alexandrov and C. Delaunay on hypersurfaces of constant mean curvature. *Duke Math. J.* **49** (1982) 485–496. MR0672494 (84k:53007), Zbl 0496.53006.

[Ma96] Masters, Joseph D. The perimeter-minimizing enclosure of two areas in $\mathbf{S}^2$. *Real Anal. Exchange* **22** (1996) 1–10. MR1460978 (99a:52010), Zbl 0946.52004.

[MEM08] Maurmann, Quinn; Engelstein, Max; Marcuccio, Anthony; Pritchard, Taryn. Asymptotics of perimeter-minimizing partitions. *Can. Math. Bull.*, to appear.

[MM08] Maurmann, Quinn; Morgan, Frank. Isoperimetrimetric comparison theorems for manifolds with density. *Calc. Var. Partial Differential Equations*, to appear.

[M08] Morgan, Frank. Geometric measure theory. A beginner's guide. Fourth edition. *Elsevier/Academic Press, Amsterdam*, 2009. viii+249 pp. ISBN: 978-0-12-374444-9. MR2455580, Zbl 0974.49025.

[M05] Morgan, Frank. Manifolds with density. *Notices Amer. Math. Soc.* **52** (2005) 853–858. MR2161354 (2006g:53044), Zbl 1118.53022.

[M03] Morgan, Frank. Regularity of isoperimetric hypersurfaces in Riemannian manifolds. *Trans. Amer. Math. Soc.* **355** (2003) 5041–5052. MR1997594 (2004j:49066), Zbl 1063.49031.

[M98] Morgan, Frank. Riemannian geometry. Second edition. *A.K. Peters, Ltd., Wellesley, MA*, 1998. x+156 pp. ISBN: 1-56881-073-3. MR1600519 (98i:53001), Zbl 0911.53001.

[M92] Morgan, Frank. Soap bubbles in $\mathbf{R}^2$ and in surfaces. *Pacific J. Math.* **165** (1994) 347–361. MR1300837 (96a:58064), Zbl 0820.53002.

[MHH00] Morgan, Frank; Hutchings, Michael; Howards, Hugh. The isoperimetric problem on surfaces of revolution of decreasing gauss curvature. *Trans. Amer. Math. Soc.* **352** (2000) 4889–4909. MR1661278 (2001b:58024), Zbl 0976.53082.

[Q07] Quinn, Conor. Least-perimeter partitions of the sphere. *Rose-Hulman Und. Math. J.* **8** (2007), no. 2.

[Re08] Reichardt, Ben. Proof of the double bubble conjecture in $\mathbf{R}^n$. *J. Geom. Anal.* **18** (2008) 172–191. MR2365672 (2008m:53018), Zbl 1149.53009.

[R01] Ritoré, Manuel. Constant geodesic curvature curves and isoperimetric domains in rotationally symmetric surfaces. *Comm. Anal. Geom.* **9** (2001) 1093–1138. MR1883725 (2003a:53018), Zbl 1018.53003.

[Ro05] Ros, Antonio. The isoperimetric problem. *Global theory of minimal surfaces*, Proceedings of the Clay Mathematics Institute 2001 summer school, Berkeley, CA, USA, June 25–July 27, 2001. *Amer. Math. Soc., Providence, RI*, 2005. Clay Mathematics Institute, Cambridge, MA. Clay Mathematics Proceedings **2** (2005) 175–209. MR2167260 (2006e:53023), Zbl 1125.49034.

[RCB08] Rosales, César; Cañete, Antonio; Bayle, Vincent; Morgan, Frank. On the isoperimetric problem in Euclidean space with density. *Calc. Var. Partial Differential Equations* **31** (2008) 27–46. MR2342613 (2008m:49212), Zbl 1126.49038.

[RW08] ROWLAND, TODD; WEISSTEIN, ERIC W. Wirtinger's Inequality. *MathWorld — A Wolfram Web Resource*.
http://mathworld.wolfram.com/WirtingersInequality.html. accessed 2008.

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CT 06520
max.engelstein@yale.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267
08anm@williams.edu

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912
quinn_maurmann@brown.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267
08tbp@williams.edu

This paper is available via http://nyjm.albany.edu/j/2009/15-5.html.