New York Journal of Mathematics
New York J. Math. **15** (2009) 73–95.

Convergence of the dual greedy algorithm in Banach spaces

M. Ganichev and N. J. Kalton

Abstract. We show convergence of the weak dual greedy algorithm in wide class of Banach spaces, extending our previous result where it was shown to converge in subspaces of quotients of L_p (for $1 < p < \infty$). In particular, we show it converges in the Schatten ideals S_p when $1 < p < \infty$ and in any Banach lattice which is p-convex and q-concave with constants one, where $1 < p < q < \infty$. We also discuss convergence of the algorithm for general convex functions.

Contents

1. Introduction

Suppose X is a real Banach space. A *dictionary* is a subset D of X such that:

(i) $d \in D \implies \|d\| = 1$.
(ii) $d \in D \implies -d \in D$.
(iii) $x^* \in X^*,\ \langle d, x^*\rangle = 0\ \forall d \in D \implies x^* = 0$.

Received June 13, 2008.
Mathematics Subject Classification. 41A65, 41A46, 46B20, 52A41.
Key words and phrases. Dual greedy algorithm, convex functions, uniformly convex, uniformly smooth Banach spaces.
The authors were supported by NSF grant DMS-0244515 and DMS-0555670.

ISSN 1076-9803/09

Here (iii) is equivalent to the statement that the closed linear span of D is X. For complex Banach spaces X we define D to be a dictionary if it is dictionary for the underlying real Banach space $X_{\mathbb{R}}$. This means that (iii) is replaced by

(iv) $x^* \in X^*,\ \mathrm{Re}\,\langle d, x^*\rangle = 0\ \forall d \in D \implies x^* = 0.$

If the dictionary D satisfies

(v) $d \in D \implies e^{i\theta} d \in D,\ 0 \le \theta < 2\pi,$

then (iv) is equivalent to (iii). Thus we treat complex Banach spaces throughout as well as real Banach spaces, by simply forgetting their complex structure.

If $f : X \to \mathbb{R}$ is a continuous convex function we denote by $\nabla f(x)$ the *subdifferential* of f at x, i.e., the set of $x^* \in X^*$ such that

$$f(x) + x^*(y - x) \le f(y), \qquad y \in X.$$

If f is Gâteaux differentiable then ∇f is single-valued and we consider

$$\nabla f : X \to X^*$$

as a mapping.

Now suppose $f : X \to \mathbb{R}$ is a continuous convex function which is Gâteaux differentiable. Assume further that f is *proper*, i.e., that

$$\lim_{\|x\| \to \infty} f(x) = \infty.$$

The *weak dual greedy algorithm* with dictionary D and weakness $0 < c < 1$ is designed to locate the minimum of f. We select an initial point $x_0 \in X$. Then for $n \in \mathbb{N}$ so that x_{n-1} has been selected we choose $d_n \in D$ to nearly optimize the rate of descent. Precisely we choose d_n so that

$$\langle d_n, \nabla f(x_{n-1}\rangle \ge c \sup_{d \in D} \langle d, \nabla f(x_n)\rangle.$$

We then choose $t_n > 0$ so that

$$f(x_{n-1} - t_n d_n) = \min_{t \ge 0} f(x_{n-1} - t d_n).$$

We say the algorithm converges if, for any initial point x_0 and weakness c, the sequence $(x_n)_{n=0}^{\infty}$ always converges in norm to a point $a \in X$ at which f assumes its minimum.

This algorithm has been studied in the literature (see [4], [16] and [17]) in the special case when $f(x) = \|x\|$ on a space X with a Gâteaux differentiable norm. Strictly speaking this does not quite fit our hypotheses since the norm is never Gâteaux differentiable at the origin (where it attains its minimum); however it would be equivalent to consider the algorithm for $f(x) = \|x\|^2$ which then is Gâteaux differentiable everywhere. The aim in this case is to give an expansion of the initial point $x_0 = \sum_{n=1}^{\infty} t_n d_n$ in terms of the dictionary.

Historically this algorithm was first considered and shown to converge for $f(x) = \|x\|^2$ when X is a Hilbert space (see [9], [10] and [14]). In 2003, the current authors showed that the algorithm converges provided X has a Fréchet differentiable norm and property (Γ) ([7] Theorem 4). To define property (Γ), assume X has a Gâteaux differentiable norm and let $J : X \setminus \{0\} \to X^*$ be the duality map, i.e., $J = \nabla N$ where $N(x) = \|x\|$. X has property (Γ) provided there is a constant C such that:

$$\|x\| = 1,\ y \in X,\ \langle y, Jx\rangle = 0 \implies \langle y, J(x+y)\rangle \le C(\|x+y\| - 1). \tag{1.1}$$

In fact the assumption of a Fréchet differentiable norm in Theorem 4 of [7] is redundant because this is implied by property (Γ), as will be seen in this paper. It turns out that the classical spaces $L_p(0,1)$ enjoy property (Γ) as long as $1 < p < \infty$. Furthermore the property passes to subspaces and quotients, so that the algorithm converges for all subspaces of quotients of L_p (Theorem 4 of [7]). This result was the main conclusion of [7], and it appeared at the time that property (Γ) was a rather specialized property that could only be established for a restricted class of Banach spaces. (This class does, however, include the complex L_p-spaces $(1 < p < \infty)$ because these are isometric to subspaces of the corresponding real spaces.) Later Temlyakov [17] studied modifications of the (WDGA) which converge in spaces which are assumed only to be uniformly smooth with a certain degree of smoothness. See also the recent preprint [5] for a discussion of problems of weak convergence.

In this paper we will develop further the study of spaces with property (Γ). We first introduce the notion of a *tame* convex function. A convex function $f : X \to \mathbb{R}$ is tame if there is a constant γ such that we have

$$f(x+2y) + f(x-2y) - 2f(x) \le \gamma(f(x+y) + f(x-y) - 2f(x)), \qquad x, y \in X. \tag{1.2}$$

We show that if f is a continuous tame convex function then f is continuously Fréchet differentiable. Furthermore the (WDGA) converges to the necessarily unique minimizer of f for any proper tame continuous convex function (Theorem 3.6 below).

The connection with property (Γ) is that, if $r > 1$, X has property (Γ) if and only if $\|x\|^r$ is tame (Theorem 4.3). It turns out that this provides a much better way to deal with property (Γ). The advantage of dealing with tame functions is that (1.2) is much easier to handle than (1.1). Using this approach it is quite easy to see that a space with property (Γ) is both uniformly convex and uniformly smooth (and hence superreflexive), and that X^* must also have property (Γ) (Theorem 4.4).

We can then expand the list of spaces with property (Γ) quite substantially. We show that a Banach lattice which is p-convex and q-concave with constants one where $1 < p \le q < \infty$ always has a property (Γ) (see Theorem 5.2). We also show that an Orlicz space $L_F(0, \infty)$ (with either the

Luxemburg or the Orlicz norm) has property (Γ) if and only if the function $t \to F(|t|)$ is tame on $\mathbb{R}$; this is equivalent to the statement that the second derivative of F is a doubling measure (see Proposition 2.10 and Theorem 5.1). We study stability of property (Γ) under interpolation and use these results to deduce that the Schatten ideals S_p for $1 < p < \infty$ have property (Γ).

2. Tame convex functions

We shall say that a function $\varphi : [0, \infty) \to [0, \infty)$ is an *Orlicz function* if φ is continuous, convex function and satisfies $\varphi(0) = 0$. We allow the degenerate case when φ is identically zero. φ satisfies a Δ_2-*condition* with constant $\beta \geq 2$ if

$$\varphi(2t) \leq \beta\varphi(t) \qquad t > 0. \tag{2.1}$$

It then follows that $t^{-b}\varphi(t)$ is a decreasing function of $t > 0$ where $b = \beta - 1$ and hence that (at points of differentiability)

$$t\varphi'(t) \leq b\varphi(t) \qquad t > 0. \tag{2.2}$$

If φ obeys (2.2) then it obeys (2.1) with $\beta = 2^b$.

Conversely φ satisfies a Δ_2^*-condition with constant $\alpha > 2$ if

$$\varphi(2t) \geq \alpha\varphi(t) \qquad t > 0. \tag{2.3}$$

It then follows that $t^{-a}\varphi(t)$ is an increasing function of $t > 0$ where $a = 2 - 2\alpha^{-1} > 1$.

Let V be a real vector space. We will say that a convex function $f : V \to \mathbb{R}$ is tame if the collection $\mathcal{F} = \{\varphi_{x,y} : \ x, y \in V\}$ of all functions

$$\varphi_{x,y}(t) = f(x + ty) + f(x - ty) - 2f(x) \qquad t \geq 0$$

obeys a uniform Δ_2-condition, i.e., for some $\gamma \geq 2$ we have:

$$f(x+2y) + f(x-2y) - 2f(x) \leq \gamma(f(x+y) + f(x-y) - 2f(x)) \qquad x, y \in V.$$

We then say f has is *tame with constant* γ. A collection of convex functions $\mathcal{F}$ is *uniformly tame* if there is a uniform constant γ such that each $f \in \mathcal{F}$ has is tame with constant γ.

Lemma 2.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a nonnegative convex function with $\varphi(0) = 0$. Assume φ is tame with constant γ. Then we have*

$$\alpha\varphi(t) \leq \varphi(2t) \leq \varphi(2t) + \varphi(-2t) \leq \beta\varphi(t) \qquad -\infty < t < \infty$$

where

$$\alpha = 2 + \gamma^{-1} > 2$$

and

$$\beta = \gamma^3.$$

In particular φ is differentiable at 0 and $\varphi'(0) = 0$.

Proof. We start by observing that for any t we have

$$\varphi(3t) + \varphi(-t) - 2\varphi(t) \leq \gamma(\varphi(2t) - 2\varphi(t)).$$

Hence

$$\varphi(-t) + \varphi(t) \leq \gamma(\varphi(2t) - 2\varphi(t)). \tag{2.4}$$

Thus

$$\gamma^2\varphi(2t) \geq \gamma(\varphi(-t) + \varphi(t)) \geq \varphi(2t) + \varphi(-2t).$$

Now we deduce

$$\begin{aligned}\varphi(2t) &\leq \varphi(2t) + \varphi(-2t)\\ &\leq \gamma(\varphi(t) + \varphi(-t))\\ &\leq \gamma^3\varphi(t).\end{aligned}$$

On the other hand by (2.4) we have

$$\varphi(2t) \geq \alpha\varphi(t).$$

Since $\alpha > 2$, it trivially follows that both the left- and right-derivatives of φ at 0 are 0. □

Proposition 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a tame convex function. Then f is continuously differentiable.*

Proof. If $s \in \mathbb{R}$ let λ be the right-derivative of f at s. Let

$$\varphi(t) = f(s+t) - \lambda t - f(s).$$

Then φ satisfies Lemma 2.1 for some constant γ. In particular φ is differentiable at 0 which implies that f is differentiable at s. Since f is convex, f must be continuously differentiable. □

Theorem 2.3. *Let $\mathcal{F}$ be a collection of continuously differentiable convex functions $f : \mathbb{R} \to \mathbb{R}$. The following conditions on $\mathcal{F}$ are equivalent:*

(i) *$\mathcal{F}$ is uniformly tame.*
(ii) *There is a constant λ such that*

$$(f'(t)-f'(s))(t-s) \leq \lambda(f(t)-f(s)-f'(s)(t-s)) \qquad f \in \mathcal{F},\ s,t \in \mathbb{R}. \tag{2.5}$$

Proof. (i) $\Longrightarrow$ (ii). Let γ be a uniform tameness constant for $\mathcal{F}$. For $s,t \in \mathbb{R}$ we define

$$\varphi_{s,t}(u) = f(s + u(t-s)) - u(t-s)f'(s) - f(s).$$

Then $\varphi_{s,t}$ is tame with constant γ and satisfies the hypotheses of Lemma 2.1. Thus $\varphi_{s,t}$ satisfies a Δ_2-condition with constant γ^3. This implies that

$$u\varphi_{s,t}'(u) \leq \mu\varphi_{s,t}(u) \qquad u > 0$$

where $2^\mu = \gamma^3$. Letting $u = 1$ gives (2.5).

(ii) $\Longrightarrow$ (i). For fixed s,t let

$$\varphi(u) = f(s+ut) + f(s-ut) - 2f(s).$$

Then

$$\begin{aligned} u\varphi'(u) &= ut(f'(s+ut) - f'(s-ut)) \\ &= ut(f'(s+ut) - f'(s)) + ut(f'(s) - f'(s-ut)) \\ &\le \lambda(f(s+ut) - f(s) - utf'(s)) + \lambda(f(s-ut) - f(s) + utf'(s)) \\ &\le \lambda\varphi(u). \end{aligned}$$

Hence φ satisfies a Δ_2-condition with constant 2^λ. □

If f is a tame convex function the optimal constant $\lambda = \lambda(f)$ in (2.5) will be called the *index* of f.

Proposition 2.4. *If f is a tame convex function with index λ then we also have*

$$(2.6)\quad (f'(t)-f'(s))(t-s) \ge \lambda'(f(t)-f(s)-f'(s)(t-s)) \qquad f \in \mathcal{F},\ s,t \in \mathbb{R}$$

where $\lambda' = \lambda/(\lambda-1)$.

Proof. Simply observe that

$$\begin{aligned} &(\lambda-1)(f'(t) - f'(s))(t-s) \\ &\qquad \ge \lambda(f(t) - f(s) + f'(t)(s-t)) + \lambda(f'(t) - f'(s))(t-s) \\ &\qquad \ge \lambda(f(t) - f(s) - f'(s)(t-s)). \end{aligned}$$

□

Remark. This argument is reversible so that λ' is the optimal constant in (2.6).

Let us now give some examples.

Proposition 2.5. *The function $f(t) = |t|^p$ is tame if and only if $p > 1$.*

Proof. Since f satisfies a Δ_2-condition it suffices to check that the convex function $t \to |1+t|^p + |1-t|^p - 2$ also satisfies a Δ_2-condition. This is easily seen to hold if and only if $p > 1$. □

Notice this proof does not provide an estimate for $\lambda(f)$. Of course if $f(t) = t^2$ we have $\lambda(f) = 2$. We will calculate $\lambda(f)$ for $f(t) = t^4$ below but in general it seems too complicated to explicitly estimate the indices for $|t|^p$.

Proposition 2.6. *Let $\mathcal{C}_{2n}$ be the class of all convex polynomials of degree at most $2n$ where $n \in \mathbb{N}$. Then $\mathcal{C}_{2n}$ is uniformly tame.*

Let us denote the polynomials of degree n by $\mathcal{P}_{n-1}$. The proposition is an immediate consequence of the following lemma.

Lemma 2.7. *Let α_n be the largest root of the Legendre polynomial P_n of degree n. Then for any convex polynomial $\varphi \in \mathcal{P}_{2n}$ with $\varphi(0) = \varphi'(0)$ we have*

$$t\varphi'(t) \le \frac{2}{1-\alpha_n}\varphi(t) \qquad 0 < t < \infty$$

and these constants are best possible.

Proof. Let σ_n, μ_n be the optimal constants such that

$$\int_0^1 tf(t)^2\,dt \le \sigma_n \int_0^1 f(t)^2\,dt, \qquad f \in \mathcal{P}_{n-1},$$

and

$$\int_0^1 tf(t)^2\,dt \ge \mu_n \int_0^1 f(t)^2\,dt, \qquad f \in \mathcal{P}_{n-1}.$$

Let us pick a nonzero polynomial $g \in \mathcal{P}_{n-1}$ such that

$$\int_0^1 tg(t)^2\,dt = \sigma_n \int_0^1 g(t)^2\,dt.$$

Then for any polynomial $f \in \mathcal{P}_{n-1}$

$$\int_0^1 t(g(t)+\theta f(t))^2\,dt \le \sigma_n \int_0^1 (g(t)+\theta f(t))^2\,dt \qquad -\infty < \theta < \infty$$

which leads to the fact that

$$\int_0^1 tg(t)f(t)\,dt = \sigma_n \int_0^1 g(t)f(t)\,dt$$

or $(t-\sigma_n)g(t)$ is a polynomial of degree n which is orthogonal to $\mathcal{P}_{n-1}$ in $L_2(0,1)$. Hence $(t-\sigma_n)g(t) = cP_n(2t-1)$ and so σ_n is a root of $P_n(2t-1)=0$. In particular $2\sigma_n - 1 \le \alpha_n$, i.e., $\sigma_n \le \frac{1}{2}(1+\alpha_n)$. On the other hand if we choose $g_0(t) = P_n(2t-1)/(2(t-\alpha_n)-1)$ then by using Gaussian quadrature (see [2] p. 343) to perform the integration it is clear, since $g_0(t)^2, tg_0(t)^2 \in \mathcal{P}_{2n-1}$, that

$$\int_0^1 tg_0(t)^2 = \frac{1}{2}(1+\alpha_n)\int_0^1 g_0(t)^2\,dt.$$

Thus $\sigma_n = \frac{1}{2}(1+\alpha_n)$. Similarly we have $\mu_n = \frac{1}{2}(1-\alpha_n)$. Thus

$$\begin{aligned}\text{(2.7)}\qquad \frac{1-\alpha_n}{2}\int_0^1 f(t)^2\,dt &\le \int_0^1 tf(t)^2\,dt\\ &\le \frac{1+\alpha_n}{2}\int_0^1 f(t)^2\,dt, \qquad f \in \mathcal{P}_{n-1}.\end{aligned}$$

This in turn implies

$$\begin{aligned}\text{(2.8)}\qquad \frac{1-\alpha_n}{2}s\int_0^s f(t)^2\,dt &\le \int_0^s tf(t)^2\,dt\\ &\le \frac{1+\alpha_n}{2}s\int_0^s f(t)^2\,dt, \qquad f \in \mathcal{P}_{n-1},\ s>0.\end{aligned}$$

Now if φ is a convex function in $\mathcal{P}_{2n-1}$ then $\varphi''(t) \ge 0$ for all $t \in \mathbb{R}$ and so we can write $\varphi''(t) = \sum_{j=1}^r f_j(t)^2$ where $f_j \in \mathcal{P}_{n-1}$. If $\varphi(0) = \varphi'(0) = 0$

then if $s > 0$

$$\begin{aligned}
\varphi(s) &= \int_0^s (s-t) \sum_{j=1}^r f_j(t)^2 \, dt \\
&\geq s \int_0^s \sum_{j=1}^r f_j(t)^2 dt - \frac{1+\alpha_n}{2} s \int_0^s \sum_{j=1}^r f_j(t)^2 dt \\
&= \frac{1-\alpha_n}{2} s\varphi'(s).
\end{aligned}$$

Clearly if we define $\varphi(t)$ so that $\varphi''(t) = g_0(t)^2$ as above the estimate is optimal. This gives

$$s\varphi'(s) \leq \frac{2}{1-\alpha_n}\varphi(s), \qquad s > 0. \qquad \square$$

Notice that the lemma gives a more precise estimate of the index of $f \in \mathcal{C}_n$:

Proposition 2.8. *If $f \in \mathcal{C}_n$ then*

$$\lambda(f) \leq \frac{2}{1-\alpha_n}$$

and this estimate is sharp.

Proposition 2.9. *If $f(t) = t^4$ then $\lambda(f) = 3 + \sqrt{3}$.*

Proof. Note that $\alpha_2 = 1/\sqrt{3}$ and by the proof of Lemma 2.7 if $\varphi''(t) = ((2t-1) - 1/\sqrt{3})^2$ then $\lambda(\varphi) = 3 + \sqrt{3}$. This implies $\lambda(f) = 3 + \sqrt{3}$. $\square$

If $n \geq 3$ it may be shown that $2n < \lambda(t^{2n}) < 2(1-\alpha_n)^{-1}$. It seems that the index for a power function $|t|^p$ for arbitrary p cannot be given by elegant formula.

We conclude this section with some further remarks on tame scalar convex functions. If $f : \mathbb{R} \to \mathbb{R}$ is a convex, its second derivative (as a distribution) is a positive locally finite Borel measure $d^2 f = \mu$. Then $\mu[a, b) = f'_-(b) - f'_+(a)$.

We recall that a measure μ defined on $\mathbb{R}$ is *doubling* if there is a constant C such that $\mu([s-2t, s+2t]) \leq C\mu([s-t, s+t])$ for all $s \in \mathbb{R}$ and $t > 0$.

Proposition 2.10. *If $f : \mathbb{R} \to \mathbb{R}$ is a convex function, then f is tame if and only if $\mu = d^2 f$ is a doubling measure.*

Proof. Let $\varphi_s(t) = f(s+t) + f(s-t) - 2f(s)$. The functions

$$\{\varphi_s : \ -\infty < s < \infty\}$$

satisfy a uniform Δ_2-condition if and only the functions

$$\{\varphi'_{s+} : -\infty < s < \infty\}$$

also satisfy a uniform Δ_2-condition and this is equivalent to the doubling condition for μ. $\square$

Now suppose $F : [0, \infty) \to \mathbb{R}$ is an Orlicz function. We extend F to $\mathbb{R}$ by setting $F(t) = F(-t)$ if $t < 0$. It is easy to see that F (or its extension to $\mathbb{R}$) is then tame if and only if $\mu([s-2t, s+2t]) \le C\mu([s-t, s+t])$ whenever $0 < t < s$. Thus an Orlicz function F is tame if and only if

$$F(t) = \int_0^t (t-s)\, d\mu(s), \qquad t > 0$$

where μ is a doubling measure.

Proposition 2.11. *Let F be a continuously differentiable Orlicz function such that there exist $0 < a < b < \infty$ so that $F'(t)/t^a$ is increasing and $F'(t)/t^b$ is decreasing for $t > 0$. Then F is tame.*

Proof. Note that F' satisfies a Δ_2-condition. Let

$$g_s(\theta) = F'((1+\theta)s) - F'((1-\theta)s), \qquad s > 0, \theta \ge 0.$$

It will be enough to show that the functions $\{g_s : s > 0\}$ satisfy a uniform Δ_2-condition. This follows from the following two estimates. For $\theta \ge 1$ we note that

$$F'(\theta s) \le F'((1+\theta)s) - F'((1-\theta)s) \le 2F'((1+\theta)s) \le 2F'(2\theta s)$$

and so

$$F'(\theta s) \le g_s(\theta) \le 2F'(2\theta s) \le 2^{b+1}F'(\theta s), \qquad \theta \ge 1.$$

On the other hand if $0 < \theta < 1$ then

$$((1+\theta)^a - (1-\theta)^a)F'(s) \le g_s(\theta) \le ((1+\theta)^b - (1-\theta)^b)F'(s),$$

which implies

$$2aF'(s)\theta \le g_s(\theta) \le 2^b F'(s)\theta.$$

□

Remark. The proposition is equivalent to the statement that F' is *quasi-symmetric*; see [8] for the precise definition. Not every tame Orlicz function satisfies the conditions of this proposition. In fact, these conditions imply that $\mu = d^2F$ is absolutely continuous with respect to Lebesgue measure, and not every doubling measure is absolutely continuous (see [8] p. 107 for a discussion).

3. Convex functions on Banach spaces

We now turn to the study of tameness for a continuous convex function on a Banach space X. We will say that a convex function $f : X \to \mathbb{R}$ is *proper* if $\lim_{\|x\|\to\infty} f(x) = \infty$.

The following theorem follows immediately from Theorem 2.3. We refer to [3] for background on differentiability of convex functions.

Theorem 3.1. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a continuous convex function. The following are equivalent:*

(i) *f is tame.*

(ii) *f is Gâteaux differentiable and there exists a constant $\lambda < \infty$ such that*

$$\langle y-x, \nabla f(y)-\nabla f(x)\rangle \leq \lambda(f(y)-\langle \nabla f(x), y-x\rangle - f(x)), \qquad x, y \in X. \tag{3.1}$$

As in the scalar case we define the *index* $\lambda = \lambda(f)$ of a tame continuous convex function to be the optimal constant such that for $x, y \in X$,

$$\langle y-x, \nabla f(y)-\nabla f(x)\rangle \leq \lambda(f(y)-\langle \nabla f(x), y-x\rangle - f(x)).$$

Notice that (3.1) implies the estimate

$$\langle y-x, \nabla f(y)-\nabla f(x)\rangle \geq \lambda'(f(y)-\langle \nabla f(x), y-x\rangle - f(x)), \qquad x, y \in X. \tag{3.2}$$

where as before $\lambda' = \lambda/(\lambda-1)$.

Corollary 3.2. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a tame continuous convex function. Suppose $\lambda = \lambda(f)$ is the index of f. If f attains a minimum at a then there is a constant C so that*

$$f(x) - f(a) \leq C \max\bigl(\|x-a\|^{\lambda}, \|x-a\|^{\lambda'}\bigr).$$

Proof. Let $C = \max\{f(x) - f(a) : \ \|x-a\| = 1\}$. The result follows from the fact that

$$t^{-\lambda'}(f(a+t(x-a)) - f(a))$$

is increasing and

$$t^{-\lambda}(f(a+t(x-a)) - f(a))$$

is decreasing in t for $t > 0$ by Theorem 3.1. ☐

Corollary 3.3. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a tame continuous convex function. Then f is continuously Fréchet differentiable and $f \to \nabla f$ is locally Hölder continuous.*

Proof. For any $a \in X$ the function $g(x) = f(x) - \langle x-a, \nabla f(a)\rangle$ is tame and assumes a minimum at $x = a$. The estimate in Corollary 3.2 then implies Fréchet differentiability. Furthermore for any $u, x \in X$ and $\tau \in \mathbb{R}$, we have

$$\langle \tau u, \nabla f(x) - \nabla f(a)\rangle \leq g(x+\tau u) - g(x) \leq g(x+\tau u) - g(a).$$

If $0 < \|x-a\| < 1/2$ and $\|u\| = 1$ take $\tau = \|x-a\|$; then we have an estimate

$$\langle u, \nabla f(x) - \nabla f(a)\rangle \leq C\|x-a\|^{\lambda'-1}$$

by Corollary 3.2 where $C = C(a, f)$. Since u is arbitrary

$$\|\nabla f(x) - \nabla f(a)\| \leq C\|x-a\|^{\lambda'-1}, \qquad \|x-a\| \leq 1. \qquad ☐$$

Theorem 3.4. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a tame continuous convex function with index $\lambda = \lambda(f)$. If f is proper then f assumes its minimum at a unique point a and there is a constant $c > 0$ so that*

$$c \min\bigl(\|x-a\|^{\lambda}, \|x-a\|^{\lambda'}\bigr) \leq f(x) - f(a).$$

Proof. First we assume f attains its minimum at $x = a$. Pick $R > 0$ so that $\inf\{f(x) : \|x - a\| = R\} = \delta > 0$. Then arguing as in the proof of Corollary 3.2 we obtain

$$f(x) - f(a) \geq \delta \min(\|x - a\|^{\lambda} R^{-\lambda}, \|x - a\|^{\lambda'} R^{-\lambda'}) \qquad x \in X$$

and we also obtain the uniqueness of a.

We now turn to the general case; we show that f attains a minimum. Note that f is uniformly continuous on bounded sets and bounded below. Now let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$ let $X_{\mathcal{U}}$ be the corresponding ultraproduct, i.e., the quotient of $\ell_\infty(X)$ by the subspace $c_{0,\mathcal{U}}(X)$ of all sequences $\xi = (\xi_n)_{n=1}^\infty$ such that $\lim_{\mathcal{U}} \|\xi_n\| = 0$. If we define $f_{\mathcal{U}}$ on $\ell_\infty(X)$ by $f_{\mathcal{U}}(\xi) = \lim_{\mathcal{U}} f(\xi_n)$. Then $f = g \circ q$ where $q : \ell_\infty(X) \to X_{\mathcal{U}}$ is the quotient map and g is easily seen to be a proper tame continuous convex function. Thus g attains a unique minimum.

If f fails to attain a minimum there is a bounded sequence $(\xi_n)_{n=1}^\infty$ so that, for some $\epsilon > 0$, $\|\xi_m - \xi_n\| \geq \epsilon$ for $m \neq n$ and

$$\lim_{n\to\infty} f(\xi_n) = \inf\{f(x) : x \in X\} = \sigma,$$

say. But then

$$f_{\mathcal{U}}(\xi_1, \xi_2, \ldots) = f_{\mathcal{U}}(\xi_2, \xi_3, \ldots) = \sigma$$

so that

$$q(\xi_1, \xi_2, \ldots) = q(\xi_2, \xi_3, \ldots)$$

and hence

$$\lim_{\mathcal{U}} \|\xi_n - \xi_{n+1}\| = 0$$

contrary to hypothesis. □

If $f : X \to \mathbb{R}$ is a tame proper continuous convex function we can define its Fenchel dual $f^* : X^* \to \mathbb{R}$ by

$$f^*(x^*) = \sup_{x\in X}(\langle x, x^*\rangle - f(x)) \qquad x^* \in X^*.$$

Note that by Theorem 3.4 the function $x \to f(x) - \langle x, x^*\rangle$ is also proper and tame. Theorem 3.4 then implies that f^* is well-defined and the supremum is attained uniquely. Furthermore f^* is continuous and convex.

Theorem 3.5. *If $f : X \to \mathbb{R}$ is a tame proper continuous convex function with index $\lambda = \lambda(f)$. Then $f^* : X^* \to \mathbb{R}$ is also a tame proper continuous convex function. Furthermore X is reflexive and $\lambda(f^*) = \lambda$.*

Proof. It is clear that f^* is proper since

$$f^*(x^*) \geq \|x^*\| - \sup_{x\in B_X} f(x).$$

Suppose $x^* \in X^*$. Then there is a unique $x \in X$ such that

$$f(x) + f^*(x^*) = \langle x, x^*\rangle,$$

and then $\nabla f(x) = x^*$. Hence for any $y^* \in X^*$ we have

$$f^*(y^*) - f^*(x^*) - \langle x, y^* - x^* \rangle \geq 0$$

so that x regarded as an element of X^{**} belongs to the subdifferential $\nabla f^*(y^*)$ (which we do not yet know to be single-valued). Next suppose $y^* \in X^*$ and let y be the unique solution of $\langle y, y^* \rangle = f(y) + f^*(y^*)$, so that $y^* = \nabla f(y)$. Thus by Theorem 3.1 we have

$$\begin{aligned} \langle y - x, y^* - x^* \rangle &\leq \lambda(f(x) - \langle x - y, y^* \rangle - f(y)) \\ &= \lambda(\langle x, x^* \rangle - f^*(x^*) - \langle x - y, y^* \rangle - \langle y, y^* \rangle + f^*(y^*)) \\ &= \lambda(f^*(y^*) - f^*(x^*) - \langle x, y^* - x^* \rangle). \end{aligned}$$

Now, for fixed $x^*, u^* \in X^*$, consider the function

$$h(t) = f^*(x^* + tu^*) - f^*(x^*) - t\langle x, u^* \rangle$$

(where as before $\langle x, x^* \rangle = f(x) + f^*(x^*)$. If h is differentiable at some t then setting $y^* = x^* + tu^*$ it is clear that $h'(t) = \langle y - x, u^* \rangle$ where $\langle y, y^* \rangle = f(y) + f^*(y^*)$. Hence $th'(t) \leq \lambda h(t)$ for $-\infty < t < \infty$. Since h is nonnegative, convex and $h(0) = 0$ we deduce that $h(t) + h(-t)$ satisfies a Δ_2-condition with constant 2^λ. Thus f^* is tame and is Gâteaux differentiable everywhere. We deduce that $\nabla f^*(x^*)$ can be identified with $x \in X$ where $f(x) + f^*(x^*) = \langle x, x^* \rangle$. Hence $\lambda(f^*) \leq \lambda$.

To see X is reflexive, suppose $x^{**} \in X^{**}$. Then $x^* \to \langle x^*, x^{**} \rangle - f^*(x^*)$ attains its minimum at some x^*; but then $x^{**} = \nabla f^*(x^*) \in X$. Now since $f^{**} = f$ we deduce $\lambda(f^*) = \lambda(f)$. □

We conclude this section by showing that the weak dual greedy algorithm can be used to find the minimum of a proper tame continuous convex function.

Theorem 3.6. *Let f be a proper tame continuous convex function on a Banach space X. Then for any dictionary and any initial point, the weak dual greedy algorithm with weakness $0 < c < 1$ yields a sequence converging to the minimizer of f.*

Proof. We suppose a is the unique minimizer of f. Let D be a dictionary and suppose $x_0 \in X$. We define the sequences $(x_n)_{n=0}^\infty \subset X$, $(d_n)_{n=1}^\infty \subset D$ and $(t_n)_{n=1}^\infty \in [0, \infty)$ so that

$$\langle d_n, \nabla f(x_{n-1}) \rangle \geq c \sup_{d \in D} \langle d, \nabla f(x_{n-1}) \rangle \qquad n = 1, 2, \ldots, \tag{3.3}$$

$$f(x_{n-1} - t_n d_n) = \inf_{t \geq 0} f(x_{n-1} - t d_n) \tag{3.4}$$

and

$$x_n = x_{n-1} - t_n d_n. \tag{3.5}$$

First suppose $\sum_{n=1}^{\infty} t_n < \infty$. Then the sequence $(x_n)_{n=1}^{\infty}$ is convergent to some $u \in X$. Then $\nabla f(x_n)$ is also norm convergent to $\nabla f(u)$ by Corollary 3.3. But, since $\langle d_n, \nabla f(x_n)\rangle = 0$,

$$|\langle d_n, \nabla f(u)\rangle| \leq \|\nabla f(u) - \nabla f(x_n)\|$$

and

$$|\langle d_n, \nabla f(u) - \nabla f(x_{n-1})\rangle| \leq \|\nabla f(u) - \nabla f(x_{n-1})\|$$

so that

$$\lim_{n\to\infty} |\langle d_n, \nabla f(x_{n-1})\rangle| = 0$$

which implies

$$\lim_{n\to\infty} \sup_{d\in D} |\langle d, \nabla f(x_n)|\rangle| = 0.$$

Thus

$$\langle d, \nabla f(u)\rangle = 0, \qquad d \in D$$

and this means that $\nabla f(u) = 0$, i.e., $u = a$.

Now let us consider the case when $\sum_{n=1}^{\infty} t_n = \infty$. In this case we must have $t_n > 0$ for all n, since $t_n = 0$ implies $t_j = 0$ for $j > n$.

Now since $\langle d_n, \nabla f(x_n)\rangle = 0$,

$$t_n\langle d_n, \nabla f(x_{n-1})\rangle \leq \lambda(f(x_{n-1}) - f(x_n))$$

and hence by (3.3),

$$\sup_{d\in D} t_n|\langle d, \nabla f(x_{n-1})\rangle| \leq \lambda c^{-1}(f(x_{n-1}) - f(x_n)). \tag{3.6}$$

Notice that the sequence $(f(x_n))_{n=1}^{\infty}$ is monotonically decreasing and bounded below by $f(a)$. If $s_n = t_1 + \cdots + t_n$ then arguing as in [7] we have $\sum t_n/s_n = \infty$ and since $\sum(f(x_{n-1}) - f(x_n)) < \infty$ we may find a subsequence $\mathbb{M}$ of $\mathbb{N}$ so that

$$\lim_{n\in\mathbb{M}} \frac{s_n((f(x_{n-1}) - f(x_n))}{t_n} = 0.$$

Hence by (3.6)

$$\lim_{n\in\mathbb{M}} s_n \sup_{d\in D} |\langle d, \nabla f(x_{n-1})| = 0. \tag{3.7}$$

Let x^* be any weak*-cluster point of the (bounded) sequence $(\nabla f(x_{n-1}))_{n\in\mathbb{M}}$. Then by (3.7) and since $\lim_{n\to\infty} s_n = \infty$ we have $\langle d, x^*\rangle = 0$ for every $d \in D$, which implies that $x^* = 0$. Thus 0 is the only weak*-cluster point of the sequence $(\nabla f(x_{n-1}))_{n\in\mathbb{M}}$. It follows that the sequence $(\nabla f(x_{n-1}))_{n\in\mathbb{M}}$ is weak*-convergent to 0.

Returning to (3.7), we deduce that

$$\lim_{n\in\mathbb{M}} \sum_{j=1}^{n-1} t_j\langle d_j, \nabla f(x_{n-1})\rangle = 0,$$

or

$$\lim_{n\in\mathbb{M}}\langle x_0 - x_{n-1}, \nabla f(x_{n-1})\rangle = 0. \tag{3.8}$$

Since $(\nabla f(x_{n-1}))_{n\in\mathbb{M}}$ is weak*-convergent to 0,

$$\lim_{n\in\mathbb{M}}\langle x_{n-1} - a, \nabla f(x_{n-1})\rangle = 0.$$

Now

$$0 \le f(x_{n-1}) - f(a) \le \langle x_{n-1} - a, \nabla f(x_{n-1})\rangle$$

and so $\lim_{n\in\mathbb{M}} f(x_{n-1}) = f(a)$. By monotonicity this implies

$$\lim_{n\to\infty} f(x_n) = f(a)$$

and by Corollary 3.2, $\lim_{n\to\infty}\|x_n - a\| = 0$. □

4. Property (Γ)

We start by giving an equivalent formulation of property (Γ). We recall the definition of property (Γ) was given in (1.1).

Proposition 4.1. *Let X be a Banach with a Gâteaux differentiable norm. Then X has property* (Γ) *if and only if there is a constant β such that*

$$1 - \langle x, Jy\rangle \le \beta(1 - \langle y, Jx\rangle), \qquad \|x\| = \|y\| = 1. \tag{4.1}$$

Proof. Suppose X has property (Γ), i.e., there is a constant C so if $\langle z, Jx\rangle = 0$ then

$$\langle z, J(x+z)\rangle \le C(\|x+z\| - \|x\|).$$

We may assume $C > 1$. Assume $\|x\| = \|y\| = 1$ and let $\langle y, Jx\rangle = \sigma$ and $\langle x, Jy\rangle = \tau$. If $\sigma \le (C-1)/(C+1)$ then since $\tau \ge -1$ we have $(1-\tau) \le (C+1)(1-\sigma)$. If $\sigma > (C-1)/(C+1)$ we have

$$\begin{aligned}(1-\tau) &= (\sigma^{-1} - \tau) - (\sigma^{-1} - 1)\\ &= \langle \sigma^{-1}y - x, Jy\rangle - (\sigma^{-1} - 1)\\ &\le C(\|\sigma^{-1}y\| - 1) - (\sigma^{-1} - 1)\\ &= (C-1)\sigma^{-1}(1-\sigma)\\ &\le (C+1)(1-\sigma).\end{aligned}$$

Thus (4.1) holds with $\beta = C + 1$.

Conversely assume (4.1) holds. Assume that $\|x\| = 1$ and $\langle y, Jx\rangle = 0$. Let $\sigma = \|x+y\|$. Then we have

$$1 - \langle x, J(x+y)\rangle = 1 - \langle x, J(\sigma^{-1}(x+y))\rangle \le \beta(1 - \sigma^{-1}\langle x+y, Jx\rangle).$$

Hence

$$\begin{aligned}\langle y, J(x+y)\rangle &= \sigma - \langle x, J(x+y)\rangle \\ &\le \sigma - 1 + \beta(1-\sigma^{-1}) \\ &\le (\beta+1)(\sigma-1).\end{aligned}$$

Thus (1.1) holds with $C = \beta + 1$. □

Theorem 4.2. *Let X be a Banach space and let $f : X \to [0,\infty)$ be a proper tame continuous function such that $f(0) = 0$ and $f(x) = f(-x)$ for $x \in X$. Let*

$$\|x\|_f = \inf\{\lambda > 0 : \ f(x/\lambda) \le 1\} \qquad x \in X.$$

Then $\|\cdot\|_f$ is an equivalent norm on X with property (Γ).

Proof. Let λ be the index f. Then

$$\min(\|x\|_f^{\lambda'}, \|x\|_f^{\lambda}) \le f(x) \le \max(\|x\|_f^{\lambda'}, \|x\|_f^{\lambda}) \qquad x \in X. \tag{4.2}$$

By Theorem 3.4 this ensures that $\|\cdot\|_f$ is equivalent to the original norm on X. Suppose $x \in X$ and $f(x) = 1$. Then if $\langle y, \nabla f(x)\rangle = 0$ we have

$$\lim_{t\to 0} \frac{f(x+ty) - f(x)}{t} = 0$$

and hence by (4.2)

$$\lim_{t\to 0} \frac{\|x+ty\|_f - 1}{t} = 0.$$

This implies that $\nabla f(x)$ is a multiple of the unique norming functional Jx for $(X, \|\cdot\|_f)$ at x. In particular the norm $\|\cdot\|_f$ is Gâteaux differentiable. It also follows from (4.2) that, if J denotes the duality map for $\|\cdot\|_f$, we have $Jx = \theta(x)^{-1}\nabla f(x)$ whenever $\|x\|_f = 1$, where $\lambda' \le \theta(x) \le \lambda$.

Next suppose $\|x\|_f = \|z\|_f = 1$, i.e., $f(x) = f(z) = 1$. Then

$$\langle z - x, \nabla f(z) - \nabla f(x)\rangle \le \lambda\langle x - z, \nabla f(x)\rangle$$

and so

$$\langle z - x, \nabla f(z)\rangle \le (\lambda - 1)\langle x - z, \nabla f(x)\rangle.$$

From this we obtain

$$\theta(z)(1 - \langle x, Jz\rangle) \le (\lambda-1)\theta(x)(1 - \langle z, Jx\rangle).$$

Using our estimate on $\theta(x), \theta(z)$ this implies

$$(1 - \langle x, Jz\rangle) \le (\lambda-1)^2(1 - \langle z, Jx\rangle). \tag{4.3}$$

An application of Proposition 4.1 now gives the conclusion. □

Theorem 4.3. *Let $(X, \|\cdot\|)$ be a Banach space. Then the following are equivalent:*

(i) *X has property (Γ).*
(ii) *For some (respectively, every) $1 < r < \infty$ the function $f(x) = \|x\|^r$ is tame.*

Proof. (i) $\Longrightarrow$ (ii). Let $x \to Jx$ be the duality map on $X \setminus \{0\}$. Then by assumption there is a constant C so that if $\langle y, Jx\rangle = 0$ then

$$\langle y, J(x+y)\rangle \le C(\|x+y\| - \|x\|).$$

Fix $r > 1$. For any $x, y \in X$ with $\|x\| = \|y\| = 1$ let $\psi = \psi_{x,y}$ be defined by

$$\psi(t) = \|x+ty\|^r - r\lambda t - 1 \qquad t \ge 0$$

where $\lambda = \langle y, Jx\rangle$. Note that

$$x + ty = (1+\lambda t)(x + \frac{t}{1+\lambda t}(y - \lambda x)) \qquad 0 \le t \le \frac{1}{2}.$$

Let

$$\varphi(t) = \|x + t(y - \lambda x)\| - 1 \qquad t \ge 0.$$

Note that

$$t\varphi'(t) = t\langle y - \lambda x, J(x + t(y - \lambda x))\rangle \le C\varphi(t) \qquad t \ge 0.$$

Then

$$\psi(t) = (1+\lambda t)^r(1 + \varphi((1+\lambda t)^{-1}t)) - r\lambda t - 1 \qquad 0 \le t \le \frac{1}{2}.$$

Now

$$\psi(t) = g(t) + h(t) \qquad 0 \le t \le \frac{1}{2}$$

where

$$g(t) = (1+\lambda t)^r - r\lambda t - 1$$

and

$$h(t) = (1+\lambda t)^r \varphi((1+\lambda t)^{-1}t).$$

Here g is convex but h need not be; h is, however, nonnegative for $t > 0$. Since the function $|t|^r$ is tame there is a constant $C_1 = C_1(r)$ so that

$$tg'(t) \le C_1 g(t) \qquad 0 \le t \le \frac{1}{2}.$$

On the other hand

$$h'(t) = r\lambda(1+\lambda t)^{r-1}\varphi((1+\lambda t)^{-1}t) + (1+\lambda t)^{r-2}\varphi'((1+\lambda t)^{-1}t) \qquad 0 \le t \le \frac{1}{2}.$$

Thus

$$th'(t) \le \frac{r\lambda + C}{1+\lambda t} h(t), \qquad 0 \le t \le \frac{1}{2}.$$

Since $|\lambda| \le 1$ this gives a bound

$$th'(t) \le C_2 h(t) \qquad 0 \le t \le \frac{1}{2}$$

where C_2 depends on C and r. Combining we have

$$t\psi'(t) \le C_3\psi(t) \qquad 0 \le t \le \frac{1}{2}$$

where $C_3 = \max(C_1, C_2)$.

Now consider the function

$$\rho(t) = \psi_{x,y}(t) + \psi_{x,-y}(t) = \|x+ty\|^r + \|x-ty\|^r - 2 \qquad t \geq 0.$$

According to the above calculation we have

$$\rho'(t) \leq C_3\rho(t) \qquad t \leq \frac{1}{2}.$$

Note that

$$\rho(\tfrac{1}{2}) \geq (3/2)^r + (1/2)^r - 2 > 0.$$

For $t \geq 2$ we have

$$2(t^r - 1) \leq \rho(t) \leq 2((t+1)^r - 1).$$

Combining these estimates it is clear that ρ satisfies a Δ_2-condition with constant γ independent of the choice of x, y with $\|x\| = \|y\| = 1$. Together with the fact that $|t|^r$ is a tame function we conclude by homogeneity that $\|x\|^r$ is itself tame.

The converse follows from Theorem 4.2. □

We recall that a Banach space X is *superreflexive* if every ultraproduct of X is reflexive and this is equivalent to the existence of an equivalent uniformly convex norm on X (see [6] and [13]).

Theorem 4.4. *Let X be a Banach space with property* (Γ). *Then X has a Fréchet differentiable norm and is both uniformly convex and uniformly smooth* (*hence X is superreflexive*). *Furthermore X^* also has property* (Γ).

Proof. Fréchet differentiability follows from Corollary 3.3.

Since $\frac{1}{2}\|x\|^2$ is tame with index λ, say, if $\|x\| = \|y\| = 1$ we have an estimate

$$\|x+ty\|^2 + \|x-ty\|^2 - 2 \leq t^{\lambda'}(\|x+y\|^2 + \|x-y\|^2 - 2) \leq 2t^{\lambda'} \qquad 0 \leq t \leq 1.$$

Similarly

$$\|x+ty\|^2 + \|x-ty\|^2 - 2 \geq 2(t/2)^{\lambda} \qquad 0 \leq t \leq 1.$$

These estimates imply that X is uniformly smooth and uniformly convex.

The function $\frac{1}{2}\|x\|^2$ is tame and hence so is its Fenchel dual $\frac{1}{2}\|x^*\|^2$ on X^* by Theorem 3.5. Hence by Theorem 4.3 X^* also has (Γ). □

Remark. The fact that property (Γ) implies uniform convexity and uniform smoothness was independently obtained by S. Gogyan and P. Wojtaszczyk.

Corollary 4.5. *If X has property* (Γ) *and E is a subspace of a quotient of X, then E also has property* (Γ).

Remark. This is also proved in [7].

Corollary 4.6. *Let X be a Banach space such that there is a proper tame continuous convex function $f : X \to \mathbb{R}$. Then X is superreflexive.*

Proof. If f is proper tame convex function then so is $\frac{1}{2}(f(x)+f(-x))$. Then we can apply Theorem 4.2 to show that X has an equivalent norm with property Γ. If X is a complex Banach space then we may use instead $(2\pi)^{-1}\int_0^{2\pi} f(e^{i\theta}x)d\theta$. $\square$

5. Spaces with property (Γ)

If F is an Orlicz function, we recall that F is tame if $t \to F(|t|)$ is a tame function on $\mathbb{R}$.

Theorem 5.1. *Let F be an Orlicz function. Then $L_F(0,\infty)$ has property (Γ) for the Orlicz norm (respectively the Luxemburg norm) if and only if the Orlicz function F is tame.*

Proof. Suppose F is tame; then F satisfies the Δ_2 condition and the Δ_2^*-condition. The functional

$$f(x)=\int_0^\infty F(|x(t)|)dt$$

is continuous on L_F and is also clearly tame. Hence L_F has property (Γ) for the Luxemburg norm by Theorem 4.2. If F^* is the Fenchel dual of F then L_{F^*} also has property (Γ) for the Luxemburg norm. However $L_{F^*}^* = L_F$ with the Orlicz norm; now we can use Theorem 4.4 to deduce that L_F has property (Γ) for the Orlicz norm.

Conversely suppose $L_F(0,\infty)$ has property (Γ) for the Luxemburg norm. Then L_F is superreflexive and so F satisfies a Δ_2 and a Δ_2^*-condition. This implies the existence of $1<p\le q<\infty$ so that

$$\min(\sigma^p,\sigma^q)F(t)\le F(\sigma t)\le \max(\sigma^p,\sigma^q)F(t), \qquad 0<t<\infty$$

and hence

$$\min(\|x\|^p,\|x\|^q)\le \int_0^\infty F(|x(t)|)\,dt \le \max(\|x\|^p,\|x\|^q), \qquad x\in L_F.$$

Now fix $0<s<\infty$ and define

$$y_t=(s+t)\chi_{(0,\frac{1}{2}(F(s)^{-1})}+(s-t)\chi_{(\frac{1}{2}(F(s)^{-1},F(s)^{-1})} \qquad -\infty<t<\infty.$$

Let

$$g_s(t)=\int_0^\infty F(|y_t(u)|)\,du-1, \qquad 0\le t<\infty$$

and

$$h_s(t)=\|y_t\|^2-1=\frac{1}{2}(\|y_t\|^2+\|y_{-t}\|^2-1), \qquad 0<t<\infty.$$

Then h_s obeys a uniform Δ_2-condition for $0<s<\infty$ with constant C_0, say.

For $t\ge s$ we have

$$g_s(2t)/g_s(t)\le 2F(3t)/F(2t)\le C_1$$

where C_1 is independent of s.

For $t \leq s$ we have

$$g_s(2t) \leq (1 + h_s(2t))^{q/2} - 1, \quad g_s(t) \geq (1 + h_s(t))^{p/2} - 1$$

so that

$$\frac{g_s(2t)}{g_s(t)} \leq \frac{(1 + h_s(2t))^{q/2} - 1}{(1 + h_s(t))^{p/2} - 1} \leq \max_{0 \leq u \leq 1} \frac{(1 + C_0 u)^{q/2} - 1}{(1 + u)^{p/2} - 1} = C_2,$$

say. Thus the functions g_s satisfy a uniform Δ_2-condition. However

$$g_s(t) = \frac{1}{2F(s)}(F(s+t) + F(s-t) - 2F(s))$$

so we deduce that F is tame.

If we assume L_F has property (Γ) for the Orlicz norm then we can argue that F^* is tame by the above reasoning and hence F is also tame. $\square$

If X is a Banach lattice we recall that X is said to be *p-convex* (where $p > 1$) with constant M if we have

$$\|(|x_1|^p + \cdots + |x_n|^p)^{1/p}\| \leq M(\|x_1\|^p + \cdots + \|x_n\|^p)^{1/p}, \qquad x_1, \ldots, x_n \in X$$

and *q-concave* (where $q < \infty$) with constant M if we have

$$(\|x_1\|^q + \cdots + \|x_n\|^q)^{1/q} \leq M\|(|x_1|^q + \cdots + |x_n|^q)^{1/q}\|, \qquad x_1, \ldots, x_n \in X.$$

We refer to [12] pp. 40ff for a discussion of these concepts. If X is p-convex and q-concave then it can always be renormed so that the respective constants are both one ([12] p. 54). Furthermore X is superreflexive if and only if X is p-convex and q-concave for some $1 < p \leq q < \infty$ (combine Theorem 1.f.1 p. 80 and Corollary 1.f.13 p. 92 of [12].

Theorem 5.2. *Let X be a Banach lattice which is p-convex with constant one and q-concave with constant one, where $1 < p < q < \infty$. Then X has property (Γ).*

Proof. First note that

$$(1 + t)^p - 1 \leq \frac{p}{q}((1 + t)^q - 1), \qquad -1 \leq t < \infty, \tag{5.1}$$

and

$$(1 + t^p)^{q/p} - 1 \leq 2^{q/p} t^p, \qquad 0 \leq t \leq 1. \tag{5.2}$$

We next observe that there is a constant $\kappa \geq 2$ such that

$$\frac{|1 + 2t|^q + |1 - 2t|^q}{2} - 1 \leq \kappa\left(\left(\frac{|1 + t|^p + |1 - t|^p}{2}\right)^{q/p} - 1\right) \qquad 0 < t < \infty. \tag{5.3}$$

Thus, using (5.3)

$$\frac{|1 + 2t|^q}{2\kappa} + \frac{|1 - 2t|^q}{2\kappa} + \frac{\kappa - 1}{\kappa} \leq \left(\frac{|1 + t|^p + |1 - t|^p}{2}\right)^{q/p}.$$

Hence if $x, y \in X$ we have

$$\left(\frac{|x+2y|^q}{2\kappa}+\frac{|x-2y|^q}{2\kappa}+\frac{\kappa-1}{\kappa}|x|^q\right)^{1/q} \le \left(\frac{|x+y|^p+|x-y|^p}{2}\right)^{1/p}.$$

Using q-concavity and p-convexity we have

$$\left(\frac{\|x+2y\|^q}{2\kappa}+\frac{\|x-2y\|^q}{2\kappa}+\frac{\kappa-1}{\kappa}\|x\|^q\right)^{1/q} \le \left(\frac{\|x+y\|^p+\|x-y\|^p}{2}\right)^{1/p}.$$

Hence

$$\begin{aligned}(5.4)\qquad \frac{\|x+2y\|^q+\|x-2y\|^q}{2} &- \|x\|^q \\ &\le \kappa\left(\left(\frac{\|x+y\|^p+\|x-y\|^p}{2}\right)^{q/p} - \|x\|^q\right).\end{aligned}$$

Now we show that $x \to \|x\|^p$ is tame. Thus we need show that all functions of the form

$$\varphi(t) = \frac{1}{2}(\|x+ty\|^p + \|x-ty\|^p) - 1, \qquad t \ge 0,$$

where $\|x\| = \|y\| = 1$, satisfy a uniform Δ_2-condition. For $t \ge 1$ we have an estimate $ct^p \le \varphi(t) \le Ct^p$ for uniform constants c, C. Hence we need only consider the case $t \le 1$. In this case, by (5.1), we have

$$\varphi(t) \le \frac{p}{q}\left(\frac{\|x+ty\|^q+\|x-ty\|^q}{2} - 1\right)$$

and by (5.2) we have

$$\left(\left(\frac{\|x+ty\|^p+\|x-ty\|^p}{2}\right)^{q/p} - 1\right) \le 2^{q/p}\varphi(t).$$

Hence, combining with (5.4),

$$\begin{aligned}\varphi(2t) &\le \frac{p}{q}\left(\frac{\|x+2ty\|^q+\|x-2ty\|^q}{2} - 1\right) \\ &\le \frac{\kappa p}{q}\left(\left(\frac{\|x+ty\|^p+\|x-ty\|^p}{2}\right)^{q/p} - 1\right) \\ &\le \frac{\kappa p 2^{q/p}}{q}\varphi(t).\end{aligned}$$

This then completes the proof. □

Remark. If $X = L_F(0,\infty)$ is an Orlicz space then the hypotheses of Theorem 5.2 hold if and only $F(x^{1/p})$ is convex and $F(x^{1/q})$ is concave and this implies that $F'(x)/x^{p-1}$ is increasing and $F'(x)/x^{q-1}$ is decreasing, i.e., we have the hypotheses of Proposition 2.11. Thus as remarked after Proposition 2.11 there are Orlicz spaces with property (Γ) which fail to be p-convex and q-concave with constants one where $1 < p \leq q < \infty$.

Corollary 5.3. *A Banach lattice has an equivalent norm with property* (Γ) *if and only if it is superreflexive.*

Problem. Does every superreflexive space have a renorming with property (Γ)?

Theorem 5.4. *Let X be a Banach space with property* (Γ). *Then $L_r(\mathbb{R}; X)$ has property* (Γ) *whenever $1 < r < \infty$.*

Proof. It is trivial to observe that $\|\cdot\|^r$ is tame on $L_r(\mathbb{R}; X)$ since $\|\cdot\|_X^r$ is tame. ☐

An even easier proof, which we omit, gives:

Theorem 5.5. *Suppose X, Y have property* (Γ). *Then $X \oplus_r Y$ has property* (Γ) *whenever $1 < r < \infty$.*

Theorem 5.6. *Suppose X is a Banach space such that for some $n \in \mathbb{N}$, $\|x + ty\|^{2n}$ is a polynomial of degree $2n$ in t for all $x, y \in X$. Then X has property* (Γ).

Proof. This follows from Proposition 2.6. ☐

Theorem 5.7. *Let (X_0, X_1) be a compatible pair of complex Banach spaces each with property* (Γ). *Then the complex interpolation spaces $[X_0, X_1]_\theta$ have* (Γ) *for $0 < \theta < 1$.*

Proof. The space $[X_0, X_1]_\theta$ is isometric to a subspace of a quotient of $L_2(\mathbb{R}; X_0) \oplus_2 L_2(\mathbb{R}; X_1)$ (see [1] p. 450). The conclusion follows from Theorems 5.4 and 5.5. ☐

If $\mathcal{H}$ is a separable Hilbert space then, for $1 \leq p < \infty$, the Schatten ideal $\mathcal{S}_p$ consists of all compact operators $T : \mathcal{H} \to \mathcal{H}$ whose singular values $(s_n(T))_{n=1}^\infty$ satisfy

$$\|T\|_{\mathcal{S}_p} = \left(\sum_{n=1}^\infty s_n(T)^p\right)^{1/p} < \infty.$$

Theorem 5.8. *The Schatten ideals $\mathcal{S}_p$ have property* (Γ) *when $1 < p < \infty$.*

Proof. By Theorem 5.6 the spaces $\mathcal{S}_{2n}$ have property (Γ) as long as $n \in \mathbb{N}$. Hence by Theorem 4.4 so do the spaces $\mathcal{S}_{2n/(2n-1)}$. The result then follows by complex interpolation (Theorem 5.7). ☐

Remark. It seems natural to ask if every two-dimensional *real* subspace of S_p embeds isometrically into L_p, which would of course give an alternate approach to such a result. This is true if $p = 1$ (since every two-dimensional real Banach space embeds into L_1, see e.g., [11]), $p = 2$ and $p = 4$ (by a result of Reznick [15] that every two-dimensional space such that $\|x\|^4$ is a polynomial embeds isometrically into L_4 or even ℓ_4^3.).

Theorem 5.9. *Let (X_0, X_1) be a compatible pair of real Banach spaces each with property (Γ). Then the real interpolation spaces $(X_0, X_1)_{\theta,p}$ for $0 < \theta < 1$ and $1 < p < \infty$ each have an equivalent norm with property (Γ).*

Proof. We may define a norm on $(X_0, X_1)_{\theta,p}$ by

$$\|x\| = \left(\int_0^\infty t^{\theta p - 1} K_2(t; x)^p \, dt \right)^{1/p}$$

where

$$K_2(t; x)^2 = \inf\{\|x_0\|_{X_0}^2 + t^2 \|x_1\|_{X_1}^2 \,:\, x = x_0 + x_1\}.$$

It is then clear that the functions $K_2(t; x)^p$ are uniformly tame on $X_0 + X_1$. Indeed $(X_0 + X_1, K_2(t, \cdot))$ is isometric to a quotient of $X_0 \oplus_2 X_1$ which has property (Γ) by Theorem 5.5. Hence $\|x\|^p$ is also tame as a function on $(X_0, X_1)_{\theta,p}$. □

References

[1] Benyamini, Yoav; Lindenstrauss, Joram. Geometric nonlinear functional analysis. Vol. 1. American Mathematical Society Colloquium Publications, 48. *American Mathematical Society, Providence, RI*, 2000. xii+488 pp. ISBN: 0-8218-0835-4. MR1727673 (2001b:46001), Zbl 0946.46002.

[2] Davis, Philip J. Interpolation and approximation. *Blaisdell Publishing Co., Ginn and Co., New York-Toronto-London*, 1963. xiv+393 pp. MR0157156 (28 #393), Zbl 0111.06003.

[3] Deville, Robert; Godefroy, Gilles; Zizler, Václav. Smoothness and renormings in Banach spaces. Pitman Monographs and Surveys in Pure and Applied Mathematics, 64. *Longman Scientific & Technical, Harlow*; copublished in the United States with *John Wiley & Sons, Inc., New York*, 1993. xii+376 pp. ISBN: 0-582-07250-6. MR1211634 (94d:46012), Zbl 0782.46019.

[4] Dilworth, S. J.; Kutzarova, Denka; Temlyakov, Vladimir N. Convergence of some greedy algorithms in Banach spaces. *J. Fourier Anal. Appl.* **8** (2002) 489–505. MR1921984 (2003f:41054), Zbl 1025.41021.

[5] Dilworth, S. J.; Kutzarova, Denka; Shuman, K. L.; Temlyakov, Vladimir N.; Wojtaszczyk, P. Weak convergence of greedy algorithms in Banach spaces. To appear.

[6] Enflo, Per. Banach spaces which can be given an equivalent uniformly convex norm. Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972). *Israel J. Math.* **13** (1972) 281–288. MR0336297 (49 #1073), Zbl 0259.46012.

[7] Ganichev, M.; Kalton, N. J. Convergence of the weak dual greedy algorithm in L_p-spaces. *J. Approx. Theory* **124** (2003) 89–95. MR2010781 (2005d:41043), Zbl 1041.46007.

[8] Heinonen, Juha. Lectures on analysis on metric spaces. Universitext. *Springer-Verlag, New York*, 2001. x+140 pp. ISBN: 0-387-95104-0. MR1800917 (2002c:30028), Zbl 0985.46008.

[9] Huber, Peter J. Projection pursuit. *Ann. Statist.* **13** (1985) 435–525. MR0790553 (88b:62118), Zbl 0595.62059.

[10] Jones, Lee K. On a conjecture of Huber concerning the convergence of projection pursuit regression. *Ann. Statist.* **15** (1987) 880–882. MR0888447 (88e:62178), Zbl 0664.62061.

[11] Lindenstrauss, Joram. On the extension of operators with a finite-dimensional range. *Illinois J. Math.* **8** (1964) 488–499. MR0169062 (29 #6317), Zbl 0132.09803.

[12] Lindenstrauss, Joram; Tzafriri, Lior. Classical Banach spaces. II. Function spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, 97. *Springer-Verlag, Berlin-New York*, 1979. x+243 pp. ISBN: 3-540-08888-1. MR0540367 (81c:46001), Zbl 0403.46022.

[13] Pisier, Gilles. Martingales with values in uniformly convex spaces. *Israel J. Math.* **20** (1975) 326–350. MR0394135 (52 #14940), Zbl 0344.46030.

[14] Rejtő, Lidia; Walter, Gilbert G. Remarks on projection pursuit regression and density estimation. *Stochastic Anal. Appl.* **10** (1992) 213–222. MR1154536 (93f:62086), Zbl 0763.62034.

[15] Reznick, Bruce Banach spaces with polynomial norms. *Pacific J. Math.* **82** (1979) 223–235. MR0549846 (83c:46007), Zbl 0413.46013.

[16] Temlyakov, Vladimir N. Nonlinear methods of approximation. *Found. Comput. Math.* **3** (2003) 33–107. MR1951502 (2003j:41029), Zbl 1039.41012.

[17] Temlyakov, Vladimir N. Greedy algorithms with prescribed coefficients. *J. Fourier Anal. Appl.* **13** (2007) 71–86. MR2296728 (2008a:41021), Zbl 1123.41019.

2335 Alexandria Pike, 63, Southgate, KY 41071
ganichev_m@yahoo.com

Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211
nigel@math.missouri.edu

This paper is available via http://nyjm.albany.edu/j/2009/15-4.html.