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# Rigidity of smooth one-sided Bernoulli endomorphisms

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*Dedicated to the memory of Bill Parry.*

Abstract. A measure-preserving endomorphism is one-sided Bernoulli if it is isomorphic to a noninvertible Bernoulli shift. We show that in piecewise smooth settings this property is very strong and far more subtle than the weak Bernoulli property, by extending of results of W. Parry and P. Walters and proving new results based on continuity of the Radon–Nikodym derivative. In particular, we provide tests which work for noninvariant measures if an invariant measure equivalent to a natural measure exists but its density function is not known. Examples of families of interval maps and complex maps on the Riemann sphere illustrate the results.

Contents



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## 1. Introduction

In this paper we give necessary criteria for various smooth and piecewise smooth $n$-to-one maps to be *one-sided Bernoulli*, that is, isomorphic to one-sided $\{p_1, p_2, \ldots, p_n\}$ Bernoulli shifts. Many results exist giving sufficient criteria for the one-sided Bernoulli property, see for example [2, 18, 15, 17, 30, 41], and these apply to a variety of finite measure-preserving $n$-to-one endomorphisms. A one-sided Bernoulli map is a deterministic dynamical system which is as stochastic as possible, and therefore of interest in many settings.

The characterization for uniformly $n$-to-one one-sided Bernoulli maps given in [15] was used to show that with respect to the unique measure of maximal entropy, a rational map of degree $\geq 2$ is one-sided Bernoulli [14]. Special cases of this result were proved using different methods [20], but the result of [14] settled an earlier conjecture of Lyubich [23] and (independently) Mañé [25].

There are many natural examples of smooth noninvertible maps such as interval and toral endomorphisms with Lebesgue measure, and rational maps with conformal measure, that are not measure-preserving. It is of interest to study their Bernoulli properties with respect to some invariant measures known to be equivalent to the given ones, but with unknown density functions. There are many results (see e.g., [1, 21, 22]) showing that such systems can still have a Bernoulli natural extension, see Definition 2.18. In this paper, we show that most of these systems are not one-sided Bernoulli.

The difficulty with proving non-Bernoulli results is that there is little a priori knowledge on the candidate Bernoulli shift (entropy is not a complete invariant for one-sided Bernoulli shifts) or the Bernoulli partition. Our criteria are based on two approaches:

- One approach builds on early papers of Walters [41] and Parry and Walters [30], establishing necessary criteria based on symmetries of the system. We give examples of smooth or piecewise smooth non-invertible maps which are not one-sided Bernoulli. We extend their results to cases where the given measure is not preserved.
- The other approach applies primarily to a one-(real)-dimensional map $T$ preserving a measure $\mu$ equivalent to Lebesgue. We exploit a rigidity result that says certain cohomological equations with measurable solutions must have continuous solutions, often referred to as Livšic

regularity. We show that the one-sided Bernoulli property implies that $T$ is $C^1$ conjugate to a piecewise affine map, see Theorem 3.5 and cohomological equation (3.2) below.

Combining both approaches, we show that many rational maps for which the Hoffman–Heicklen–Rudolph result [14, 15] applies using the measure of maximal entropy, cannot be one-sided Bernoulli with respect to conformal measure (supported on the Julia set).

The paper is organized as follows. In Section 2 we give the basic definitions and assumptions about noninvertible maps; Section 2.4 is an updated review of results from [30] and [41]. We point out the distinctions between one-sided Bernoulli and weak Bernoulli maps because, while in the invertible case they are equivalent, in the noninvertible case they are not.

The main results of this paper are contained in Sections 2.6 and 3. First we extend a classical result about commuting automorphisms of measure-preserving shifts to the case where only an equivalent measure is preserved. This allows us to obtain a more easily checkable condition in smooth settings. In Section 3 we prove some rigidity theorems for piecewise smooth one-sided Bernoulli maps. We provide many new differentiable and holomorphic examples in Section 4, some of which are one-sided Bernoulli and some that are weak Bernoulli but not one-sided Bernoulli, illustrating the results in Sections 2 and 3.

## 2. Noninvertible Bernoulli maps and the Parry–Walters invariants

We first recall the definition of a one-sided Bernoulli shift.

**Definition 2.1.** Fix an integer $n \geq 2$ and let $\mathcal{A} = \{1, \ldots, n\}$ denote a finite state space with the discrete topology. Any vector $p = \{p_1, \ldots, p_n\}$ such that $p_k > 0$ and $\sum p_k = 1$ determines a measure on $\mathcal{A}$, namely $p(\{k\}) = p_k$. Let $\Omega = \prod_{i=0}^{\infty} \mathcal{A}$ be the product space endowed with the product topology and product measure $\rho$ determined by $\mathcal{A}$ and $p$. The map $\sigma$ is the one-sided shift to the left, $(\sigma x)_i = x_{i+1}$. We say $\sigma$ is a *one-sided Bernoulli shift* and denote it by $(\Omega, \mathcal{D}, \rho; \sigma)$, where $\mathcal{D}$ denotes the Borel $\sigma$-algebra generated by the cylinder sets, completed with respect to $\rho$.

**2.1. Nonsingular endomorphisms.** We shall assume throughout that $(X, \mathcal{B}, \mu)$ is a Lebesgue probability space; $\mathcal{B}$ denotes the $\sigma$-algebra of measurable sets and we assume that the measure space is complete. We always assume that $T$ is a surjective *nonsingular endomorphism*; i.e., $T : X \to X$ satisfies: $\mu(A) = 0 \iff \mu(T^{-1}A) = 0$ for every $A \in \mathcal{B}$, and $\mu(T(X) \triangle X) = 0$. If $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$, we say that $T$ is *measure-preserving*, or equivalently $T$ *preserves* $\mu$. We also assume that every point in $X$ has at most countably many preimages under $T$. Without loss of generality we can assume that $T$ is forward measurable and forward nonsingular; i.e., for all measurable sets $A$, $T(A) \in \mathcal{B}$ and $\mu(A) = 0 \iff \mu(TA) = 0$ (see [40]). When we say that a property holds on $X$ ($\mu$ mod 0) or $\mu$ a.e., we mean that there is a set $N \in \mathcal{B}$ with $\mu(N) = 0$, ($N$ is possibly the empty set), such that the property holds for all $x \in X \setminus N$.

**Definition 2.2.** Let $T_1 : (X_1, \mathcal{B}_1, \mu_1) \circlearrowleft$ and $T_2 : (X_2, \mathcal{B}_2, \mu_2) \circlearrowleft$ be two measure-preserving endomorphisms.

A measurable map $\varphi : X_1 \to X_2$ is a *homomorphism* if there exists a set $Y_1 \in \mathcal{B}_1$ of full measure and a set $Y_2 \in \mathcal{B}_2$ of full measure in $X_2$ such that $\varphi$ maps $Y_1$ onto $Y_2$.

If there exists a homomorphism $\varphi$ such that $T_1(Y_1) = Y_1$, $T_2(Y_2) = Y_2$, $\varphi \circ T_1 = T_2 \circ \varphi$ on $Y_1$, and $\mu_2(A) = \mu_1(\varphi^{-1}(A))$ for all $A \in \mathcal{B}_1$, then $T_2$ is called a *factor* of $T_1$ (w.r.t. the measures $\mu_1$ and $\mu_2$), *with factor map* $\varphi$.

If in addition $\varphi$ is injective on $Y_1$ we say it is an *isomorphism.* If $T_2$ is a factor of $T_1$ and $\varphi$ is an isomorphism, then we say that the endomorphisms $T_1$ and $T_2$ are *isomorphic* endomorphisms.

A nonsingular endomorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an *automorphism of* $X$ if there exists $Y \in \mathcal{B}$ of full measure such that the restriction of $T$ to $Y$ is bijective (and $\mu T^{-1} \sim \mu$, but they are not necessarily equal). If an endomorphism $T$ is not an automorphism, then we say $T$ is *noninvertible.*

An $n$-to-one nonsingular endomorphism $T$ on $(X, \mathcal{B}, \mu)$ is called *one-sided Bernoulli* if it is isomorphic to some $n$-state one-sided Bernoulli shift. We usually just say that $T$ is one-sided $p = \{p_1, \ldots, p_n\}$ Bernoulli when we mean it is isomorphic to the noninvertible dynamical system $(\Omega, \mathcal{D}, \rho; \sigma)$. Bernoulli endomorphisms inherit well-known properties of Bernoulli shifts, such as ergodicity and exactness.

**Definition 2.3.** An endomorphism $T$ is *ergodic* if any $B \in \mathcal{B}$ with the property that $B = T^{-1}B$ ($\mu$ mod 0) has either zero or full measure. It is *exact* if any $B \in \mathcal{B}$ with the property that $B = T^{-n}T^nB$ ($\mu$ mod 0) for every $n \geq 0$ has either zero or full measure.

**2.2. Decomposition of a measure with respect to a noninvertible endomorphism.** Since the property of being noninvertible depends on the measure and plays a critical role in what follows, we describe that dependence here. For a nonsingular endomorphism $T$ we consider the sub-$\sigma$-algebra $\mathcal{F} \equiv T^{-1}\mathcal{B} \subseteq \mathcal{B}$; for each set $A \in \mathcal{F}$ there is a set $B \in \mathcal{B}$ such that $A = T^{-1}B$ ($\mu$ mod 0). By a canonical construction given by Rohlin [34], described further in e.g., [6], this sub-$\sigma$-algebra determines, up to sets of $\mu$ measure 0, a unique measurable decomposition of $X$ and $\mu$ as follows. The factor map $\varphi$ maps $X$ onto a Lebesgue factor space $(Y, \mathcal{F})$ with measure $\nu$ defined on $\mathcal{F}$ as the restriction of $\mu$; i.e., $\nu(A) = \mu(A)$ for each $A \in \mathcal{F}$. A point in $Y$ is a collection of points in $X$, namely $y = T^{-1}x$ ($\nu$ mod 0). The commutative diagram illustrates the factor map:

$$\begin{array}{ccc} (X,\mathcal{B},\mu) & \xrightarrow{T} & (X,\mathcal{B},\mu) \\ \downarrow \varphi & & \downarrow \varphi \\ (Y,T^{-1}\mathcal{B},\nu) & \xrightarrow{\mathcal{T}} & (Y,T^{-1}\mathcal{B},\nu) \end{array}$$

with $\nu = \mu|_{T^{-1}\mathcal{B}}$.

By construction of the factor, $T$ gives a well-defined factor transformation on $Y$ $\nu$-a.e.:

$$\mathcal{T}(y) \equiv \varphi(T\,x),$$

so that $\varphi(T\,x) = \mathcal{T}(\varphi\,x)$ for $\mu$-a.e. $x \in X$. We note that $\varphi$ is a measurable isomorphism from $X$ to itself if and only if $T$ is a nonsingular automorphism and in this case $\varphi = T^{-1}$ is the inverse of $T$ on $X$. Since every proper factor map gives a decomposition of $\mu$ over $\nu$ we write for each $B \in \mathcal{B}$,

$$\mu(B) = \int_Y \mu_y(B)d\nu(y) \tag{2.1}$$

where for $\nu$-a.e. $y \in Y$, with $y = \varphi(x)$, $\mu_y \equiv \mu_{T^{-1}x}$ is a measure on $(X,\mathcal{B})$ that is purely atomic (since $T$ is at most countable-to-one), and its support is a subset of the set of points $T^{-1}x$. Also, for each fixed $B \in \mathcal{B}$, the map $y \mapsto \mu_y(B)$ is a measurable function on $Y$.

**Definition 2.4.** For a nonsingular endomorphism $T$, the *index function*, $\mathrm{ind}_T(x)$ is defined ($\mu$ mod 0) to be the cardinality of the support of $\mu_{T^{-1}x} = \mu_{\varphi(x)}$ for $x \in X$.

The function $\mathrm{ind}_T$ is measurable and defined uniquely up to sets of $\mu$ measure 0. It is possible that $\mathrm{ind}_T(x) = +\infty$; however, while the cardinality of the set $T^{-1}x$ provides an upper bound for $\mathrm{ind}_T(x)$, its value depends on the measure $\mu$. The following example, an endomorphism of $[0,1]$ with graph shown in Figure 1, illustrates the dependence of $\mathrm{ind}_T$ on the measure for a given endomorphism.

**Example 2.5.** Let $T : [0,1] \to [0,1]$ be the piecewise linear map defined and shown in Figure 1. Then $T$ is bounded-to-one with respect to Lebesgue measure $m$ (see Section 2.3). The index function takes values 2 on $\left(\frac{1}{2},1\right]$ and 4 on $\left(0,\frac{1}{2}\right)$. Because of the uniform expansion, $T$ preserves a measure $\mu \sim m$ (with $\frac{d\mu}{dm} = \frac{4}{3}$ on $\left[0,\frac{1}{2}\right)$ and $\frac{d\mu}{dm} = \frac{2}{3}$ on $\left(\frac{1}{2},1\right]$, to be precise), but since $\mathrm{ind}_T$ is not constant ($m$ mod 0), $T$ is not one-sided Bernoulli with respect to $m$. In fact, $T$ preserves a measure $\mu \sim m$ with $\mu$ Markov, but not one-sided Bernoulli.

On the other hand, the $\frac{\log 2}{\log 3}$-dimensional Hausdorff measure $\gamma$ supported on the middle thirds Cantor set $C$ is also $T$-invariant. With respect to this measure $T$ is $\left\{\frac{1}{2},\frac{1}{2}\right\}$ Bernoulli, and $\mathrm{ind}_T \equiv 2$ $\gamma$-a.e. Note however, that

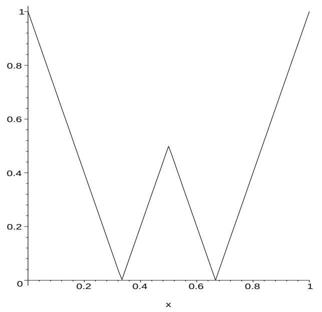


FIGURE 1. The map $T(x) = |\min\{3x-1, 2-3x\}|$ is bounded-to-one w.r.t. $m$, but 2-to-1 w.r.t. Hausdorff measure supported on the middle thirds Cantor set.

$T^{-1}(x)$ contains 4 points for every $x \in \left(0, \frac{1}{2}\right) \cap C$, but only two of these are atoms of $\gamma_{\varphi(x)}$.

The index of an endomorphism is invariant under isomorphism in the sense that if $T_1$ and $T_2$ are isomorphic endomorphisms and $\varphi \circ T_1 = T_2 \circ \varphi$ for some automorphism $\varphi$, then $\mathrm{ind}_{T_1}(x) = \mathrm{ind}_{T_2}(\varphi x)$ ($\mu_1 \bmod 0$) [30]. Therefore, $(X, \mathcal{B}, \mu; T)$ can be isomorphic to a one-sided Bernoulli shift on $n$ states only if the index is the constant function $n$ ($\mu \bmod 0$). However, an invertible two-state Bernoulli shift is isomorphic to a three-state Bernoulli shift if and only if they have the same entropy [29], and this can occur; the index renders the analogous statement false in the case of one-sided shifts, illustrating a subtlety of noninvertible maps.

**2.3. Rohlin partitions and factors.** Assume that $T$ is a nonsingular endomorphism of $(X, \mathcal{B}, \mu)$, not necessarily preserving $\mu$. A partition $\zeta$ is an ordered countable (possibly finite) disjoint collection of nonempty measurable sets, called atoms, whose union is $X$ ($\mu \bmod 0$).

By a result of Rohlin [34] we obtain a partition $\zeta = \{A_1, A_2, A_3, \dots\}$ of $X$ into at most countably many atoms and satisfying:

(1) $\mu(A_i) > 0$ for each $i$.
(2) The restriction of $T$ to each $A_i$, which we will write as $T_i$, is one-to-one ($\mu \bmod 0$).
(3) Each $A_i$ is of maximal measure in $X \setminus \cup_{j<i} A_j$ with respect to (2).
(4) $T_1$ is one-to-one and onto X ($\mu \bmod 0$) by numbering the atoms so that
$$\mu(TA_i) \geq \mu(TA_{i+1})$$
for $i \in \mathbb{N}$.

We call a partition $\zeta$ as defined above a *Rohlin partition* for $T$. When we say that an endomorphism $T$ is *n-to-one*, we mean that every Rohlin partition $\zeta = \{A_1, A_2, A_3, \dots\}$ satisfying (1)–(4) contains precisely $n$ atoms and that $T_i$ is one-to-one and onto $X$ ($\mu$ mod 0) for each $i = 1, \dots, n$. Equivalently, for $\mu$-a.e. $x \in X$, we have $\mathrm{ind}_T(x) = n$ and the set $\{T^{-1}x\}$ contains exactly $n$ points which are atoms of $\mu_{T^{-1}x)}$. If $\zeta$ has $n$ atoms with $1 < n < \infty$, but $T$ does not necessarily map each $A_j$ onto $X$, then we say that $T$ is *bounded-to-one*. Figure 1 shows a bounded-to-one but not $n$-to-one map with respect to $m$. Clearly $T$ is invertible if and only if $\mathrm{ind}_T$ is the constant function 1 ($\mu$ mod 0).

**Definition 2.6.** We fix a nonsingular endomorphism $T$ of $(X, \mathcal{B}, \mu)$ as above. Given any partition $\eta$, we define the *$\sigma$-algebra generated by $\eta$* (under $T$), denoted $\mathcal{F}(\eta)$, to be the smallest sub-$\sigma$-algebra of $\mathcal{B}$ containing:

$$\eta_0^\infty \equiv \bigvee_{i \geq 0} T^{-i}(\eta), \tag{2.2}$$

and complete with respect to $\mu$. A partition $\eta$ is a (one-sided) *generating partition* if $\mathcal{F}(\eta) = \mathcal{B}$ ($\mu$ mod 0).

If $T$ is noninvertible with respect to $\mu$, every Rohlin partition $\zeta$ contains at least two atoms and we consider the sub-$\sigma$-algebra $\mathcal{F}(\zeta) \equiv \mathcal{F}$. Since $T^{-1}\mathcal{F} \subset \mathcal{F}$, we have that each Rohlin partition determines a proper factor map onto a factor space $(Z, \mathcal{F}, \mu|_{\mathcal{F}})$ and $T$ is well-defined on this space. We call this factor *a Rohlin factor*.

Rohlin partitions are not unique; this is easily shown and was known to Parry and Walters [30]. Moreover in Example 2.7 we give an endomorphism such that some Rohlin partitions $\zeta$ are generating and some are not, extending the result to show that the corresponding $\sigma$-algebras $\mathcal{F}(\zeta)$ defined in (2.2) are not unique either.

**Example 2.7.** We consider the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $\mathcal{B}$ the $\sigma$-algebra of Borel sets. If $T : \mathbb{S}^1 \to \mathbb{S}^1$, $T(x) = 2x$ is the angle doubling map, then for any $t \in \left[0, \frac{1}{2}\right)$, the partition $\zeta_t = \{A_0, A_1\}$ with $A_0 = \left[t, t+\frac{1}{2}\right)$, $A_1 = \left[t+\frac{1}{2}, t\right)$ is a Rohlin partition with respect to many measures, some specified below. The partition $\zeta_t$ establishes a coding map $\pi_t : \mathbb{S}^1 \to \{0,1\}^{\mathbb{N}} \equiv \Omega$ given by

$$\pi_t(x)_i = \begin{cases} 0 & \text{if } T^i x \in A_0 \\ 1 & \text{if } T^i x \in A_1, \end{cases}$$

with the following properties:

(1) $\pi_t$ is surjective for every $t \in \left(0, \frac{1}{2}\right)$. (For $t = 0$, there is no point $x$ with $\pi_0(x) = 111\dots$.)
(2) $\pi_t$ is not injective, except for $t = 0$.
(3) For every $t \neq \frac{1}{4}$ and $p \in (0,1)$ and a.e. $\omega \in \Omega$ with respect to $\{p, 1-p\}$ Bernoulli measure, $\pi_t^{-1}(\omega)$ consists of a single point.
(4) If $t = \frac{1}{4}$, $\pi_t$ is a two-to-one map at every point.

(5) Fixing $p = \frac{1}{2}$ and using Lebesgue measure $m$ on $\mathbb{S}^1$, for every $t \in [0, \frac{1}{2})$, $t \neq \frac{1}{4}$, $\pi_t$ is a measure-preserving isomorphism; therefore the Rohlin partitions $\zeta_t$, $t \in [0, \frac{1}{2}) \setminus \{\frac{1}{4}\}$, generate. If $t = \frac{1}{4}$, then (4) implies that $\mathcal{F}(\zeta_{1/4}) \neq \mathcal{B}$.
(6) Despite Statement (4), the original map $T$ on $(\mathbb{S}^1, \mathcal{B}, m)$ and the induced factor map on $(Z, \mathcal{F}(\zeta_{1/4}))$ are isomorphic to each other since each is isomorphic to a one-sided $\{\frac{1}{2}, \frac{1}{2}\}$ Bernoulli shift.

These statements come from descriptions of the quadratic Julia set in terms of symbolic dynamics and quadratic laminations; (1) can be derived from work of Bullett and Sentenac [5]. Items (2) and (3) are described in current work of Bruin and Schleicher, in particular [4, Lemma 8.4] for the $\{\frac{1}{2}, \frac{1}{2}\}$ Bernoulli measure, but similar methods work for the general $\{p, 1-p\}$ Bernoulli measure.

Piecewise affine examples and a proof of Statement (4) using $m$ on $\mathbb{S}^1$ and $\{p, 1-p\}$ Bernoulli measures on $\Omega$ are discussed in Section 4, Example 4.1 of this paper.

**2.4. The Parry Jacobian and Radon–Nikodym derivatives.** Assume $T$ is a bounded-to-one nonsingular endomorphism with Rohlin partition $\zeta$. For $x \in A_i$, define $J_{\mu T_i}(x) = \dfrac{d\mu T_i}{d\mu}(x)$, and for $x \in X$, let

$$J_{\mu T}(x) = \sum_i J_{\mu T_i}(x)\chi_{A_i}(x),$$

where $\chi_A$ denotes the characteristic function of the set $A$; this defines $J_{\mu T}$ ($\mu \bmod 0$). This is the *Jacobian function* for $T$, defined by Parry [30], and is independent of the choice of $\zeta$. To see the independence, let $\mu_{T^{-1}x}$ denote the conditional measure on the set $T^{-1}x$ as given in Section 2.2, then an equivalent characterization given in [30] is:

$$J_{\mu T}(x) = \frac{1}{\mu_{T^{-1}(Tx)}(x)}.$$

Our nonsingularity assumption implies that $J_{\mu T} > 0$ $\mu$-a.e. and we then have the following identities holding $\mu$-a.e. ([8], cf. also [12]):

$$\theta_{\mu T}(x) \equiv \frac{d\mu T^{-1}}{d\mu}(x) = \sum_{y \in T^{-1}x} \frac{1}{J_{\mu T}(y)},$$

$$\omega_{\mu T}(x) \equiv \frac{d\mu}{d\mu T^{-1}}(Tx) = \frac{1}{\theta_{\mu T}(Tx)}.$$

The function $\omega_{\mu T}$ is frequently referred to as the *Radon–Nikodym derivative of* $T$ since when $T$ is one-to-one, $\omega_{\mu T} = \dfrac{d\mu T}{d\mu}$.

In [8] $\omega_{\mu T}$ is characterized ($\mu \bmod 0$) as the unique $T^{-1}\mathcal{B}$-measurable function satisfying:

$$\int_X f \circ T \cdot \omega_{\mu T}\, d\mu = \int_X f\, d\mu \ \text{ for all } \ f \in L^1(X, \mathcal{B}, \mu).$$

Clearly $\omega_{\mu T} = 1$ a.e. if and only if $T$ preserves $\mu$.

If we have an equivalent measure $\nu \sim \mu$, we can write $\frac{d\mu}{d\nu} = g$ with $g > 0$ a.e., and we have

$$J_{\mu T}(x) = \frac{g \circ T}{g}(x) \cdot J_{\nu T}(x) \quad \text{a.e.} \tag{2.3}$$

More generally we use the Jacobian to define the *transfer operator* $\mathcal{L}_{\mu T}$ acting on the space of measurable functions $h : X \to \mathbb{R}$ by

$$\mathcal{L}_{\mu T} h(x) = \sum_{y \in T^{-1}x} \frac{h(y)}{J_{\mu T}(y)}.$$

Since many of our examples are maps on bounded subsets of $X \subset \mathbb{R}^k$ or $\mathbb{C}$, the notation $m_k$ will refer to normalized $k$-dimensional Lebesgue measure on $X$. The notation $m$ is used for Lebesgue measure on $\mathbb{R}$ or any subset of it unless confusion arises.

Suppose $T = (T_1, \dots, T_k) : \mathbb{R}^k \to \mathbb{R}^k$ is a continuously differentiable map on an open set $\mathcal{O}$. If for every $x \in \mathcal{O}$ the classical Jacobian, $\det\big(\frac{\partial T_i}{\partial x_j}\big)$, is nonzero, then $T$ is a diffeomorphism between $\mathcal{O}$ and $T(\mathcal{O})$ with $J_{m_k T}(x) = \big|\det\big(\frac{\partial T_i}{\partial x_j}\big)(x)\big|$. For example, if $T : \mathbb{R} \to \mathbb{R}$ is $C^1$, the transfer operator becomes

$$\mathcal{L}_{mT} h(x) = \sum_{y \in T^{-1}x} \frac{h(y)}{|T'(y)|}.$$

and if $T : \mathbb{C} \to \mathbb{C}$ is holomorphic the Jacobian with respect to Lebesgue measure on $\mathbb{C}$ is $J_{m_2 T}(z) = |T'(z)|^2$.

For differentiable maps in one real variable or holomorphic maps in one complex variable, a variation of Lebesgue measure, called conformal measure is frequently the "best dynamical measure". For example, for rational maps $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ the interesting dynamics take place on the Julia set, on which Lebesgue measure typically is not useful; however a natural variant of it is (see the examples in Subsection 4.2). Here, and throughout we use the notation $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ to denote the Riemann sphere, and for a rational map $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ the Julia set is denoted by $\mathcal{J}(R)$. A general introduction to complex dynamics can be found for example in [28].

**Notation 2.8.** In order to treat the real and complex case together, we let $Y$ denote any of $I$, $\mathbb{R}$ or $\mathbb{S}^1$ (respectively, $\mathbb{C}$ or $\mathbb{C}_\infty$); $I \subset \mathbb{R}$ is always a compact interval. We endow $Y$ with $\mathcal{B}$, the $\sigma$-algebra of Borel sets. Assume that $T$ is a $C^1$ (respectively, holomorphic) map on $Y$ and simply call $T$

*smooth.* If $X \subset Y$ is such that $T(X) = X$, then $(X, \mathcal{B}_X)$ has the restricted Borel structure on it.

**Definition 2.9.** A measure $m_\alpha$, $\alpha > 0$, on $(X, \mathcal{B}_X)$ is called *$\alpha$-conformal* w.r.t. the smooth map $T$ if the Jacobian $J_{m_\alpha T}(x) = |T'(x)|^\alpha$ $m_\alpha$-a.e. For instance, Lebesgue measure $m$ is 1-conformal w.r.t. piecewise $C^1$ maps on the interval or circle.

In the general setting it is easy to see the following classical identities hold ($\mu \bmod 0$), cf. [12].

**Lemma 2.10.** *Let $T$ be a bounded-to-one endomorphism. Then:*

(1) $\theta_{\mu T} = \frac{d\mu T^{-1}}{d\mu}(x) = \mathcal{L}_{\mu T}1.$
(2) *$T$ preserves $\mu$ if and only if $\mathcal{L}_{\mu T}1 = 1$.*
(3) *$T$ preserves a measure $\nu \sim \mu$ if and only if $\mathcal{L}_{\mu T}g = g$ and $d\nu = gd\mu$.*

The next result was proved in [6] and illustrates the roles of Rohlin partitions in this study.

**Lemma 2.11.** *Let $T$ on $(X, \mathcal{B}, \mu)$ be an $n$-to-one measure preserving endomorphism and $\zeta = \{A_1, \dots A_n\}$ a Rohlin partition. As in* (2.2)*, we denote by $\mathcal{F}$ the associated sub $\sigma$-algebra generated by $\zeta$.*

(1) *Then the induced factor map of $T$ on $\mathcal{F}$, is isomorphic to a measure preserving shift on $n$ states, so with $\mathcal{A} = \{1, 2, , \dots, n\}$ the diagram*

$$\begin{array}{ccc} (X, \mathcal{B}, \mu) & \xrightarrow{T} & (X, \mathcal{B}, \mu) \\ \downarrow \pi & & \downarrow \pi \\ (\mathcal{A}^{\mathbb{N}}, \mathcal{C}, \nu) & \xrightarrow{\sigma} & (\mathcal{A}^{\mathbb{N}}, \mathcal{C}, \nu) \end{array}$$

*commutes, where $\nu$ is the factor measure induced by $\mu$.*

(2) *If in addition there exists a Rohlin partition for $T$ such that the Jacobian $J_{\mu T}(x) = \frac{1}{p_i}$ for all $x \in A_i$, then the induced Rohlin factor $(\mathcal{A}^{\mathbb{N}}, \mathcal{C}, \nu)$ is isomorphic to a one-sided $p = \{p_1, \dots, p_n\}$ Bernoulli shift* (*see Definition* 2.1)*.*

If $\psi : (X_1, \mathcal{B}_1, \mu_1) \to (X_2, \mathcal{B}_2, \mu_2)$ is a measure preserving isomorphism, then $J_{\mu_1 \psi} = 1$, and if $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure preserving endomorphism and $\varphi : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a nonsingular automorphism, the preceding discussion gives these chain rules:

$$\omega_{\mu(T\circ\varphi)} = \omega_{\mu T} \circ \varphi \cdot \omega_{\mu\varphi} = J_{\mu\varphi} (\mu \bmod 0) \tag{2.4}$$

and

$$\omega_{\mu(\varphi\circ T)} = \omega_{\mu\varphi} \circ T \cdot \omega_{\mu T} = J_{\mu\varphi} \circ T (\mu \bmod 0). \tag{2.5}$$

Chain rules (2.4) and (2.5) prove the next well-known result.

**Proposition 2.12.** *If $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic probability measure preserving endomorphism and $\varphi : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, is a non-singular automorphism that commutes with $T$, then $\varphi$ preserves $\mu$.*

**Proof.** The commuting hypothesis gives that (2.4) is equal to (2.5). Therefore $J_{\mu\varphi}$ is a $T$-invariant measurable function, and the ergodicity of $T$ implies it is the constant function $1\,(\mu \bmod 0)$. $\square$

Clearly every Bernoulli shift is a measure preserving $n$-to-one endomorphism. As a corollary to Proposition 2.12 we obtain two necessary conditions for an endomorphism to be one-sided Bernoulli shift when the given measure is preserved [30].

**Corollary 2.13.** *Let $T$ be one-sided $\{p_1, p_2, \ldots, p_n\}$ Bernoulli on $(X, \mathcal{B}, \mu)$ and preserve $\mu$. Then $(\mu \bmod 0)$ we have $\mathrm{ind}_T(x) = n$ and the set of values $\{J_{\mu T}(y)\}_{y \in \mathrm{supp}(\mu_{T^{-1}x})} = \left\{\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\right\}$.*

**Proof.** Using Definition 2.1, the conjugating isomorphism $\varphi : X \to \Omega$, and the statement preceding (2.4), we have $(\mu \bmod 0)$

$$J_{\mu T}(y) = J_{\mu(\varphi^{-1}\sigma\varphi)}(y) = J_\rho(\sigma(\varphi y)) \cdot J_{\rho\sigma}(\varphi y) \cdot J_\mu(\varphi y) = J_{\rho\sigma}(\varphi y).$$

Since $y \in T^{-1}x \Leftrightarrow \varphi(y) \in \sigma^{-1}(\varphi x)$, the result follows. $\square$

While Corollary 2.13 gives a necessary condition for one-sided Bernoulli, it is not sufficient, as Example 4.1 for $t = \frac{1}{4}$ shows. Moreover the next result indicates that the Jacobian condition of Corollary 2.13 is not always possible to verify.

**Corollary 2.14.** *If $T$ on $(X, \mathcal{B}, \mu)$ preserves $\nu \sim \mu$ with $g = d\nu/d\mu$, and is one-sided $\{p_1, p_2, \ldots, p_n\}$ Bernoulli, then for $\mu$-a.e. $x \in X$ and $y \in T^{-1}(x)$, $J_{\mu T}(y) = \dfrac{g(x)}{g(y) \cdot p_k}$ for some $k$.*

**Example 2.15.** Consider the modified Boole map

$$T : \mathbb{R} \to \mathbb{R}, \quad T(x) = \tfrac{1}{2}\left(x - \tfrac{1}{x}\right)$$

with Lebesgue measure $m$. For any $x_0 \neq 0$ testing the Jacobians

$$J_{mT}(y) = \frac{1}{|T'(y)|} = \left|1 \pm \frac{x_0}{\sqrt{1 + x_0^2}}\right| \text{ at } y \in T^{-1}x_0$$

would not indicate whether or not this map is isomorphic to the $\left\{\frac{1}{2}, \frac{1}{2}\right\}$ Bernoulli shift (it is, by [3]), or whether this variation is: $S(x) = a\big(x - \frac{1}{x}\big)$, $a \in (0,1) \setminus \left\{\frac{1}{2}\right\}$ is (it is not, by Theorem 2.21 below). In each case $m$ is not preserved, but equivalent probability measures are, so we are in the setting of Corollary 2.14 and not Corollary 2.13.

**2.5. The weak Bernoulli property.** We recall a condition strictly weaker than one-sided Bernoulli for endomorphisms [10], but equivalent to Bernoulli in the invertible case [9].

**Definition 2.16.** Let $T$ be an endomorphism of $X$ preserving the measure $\mu$. Let $\zeta = \{P_1, P_2, \cdots\}$ and $\eta = \{Q_1, Q_2, \cdots\}$ be partitions. The partition $\zeta$ *is independent of* $\eta$ if

$$\mu(P_i \cap Q_j) = \mu(P_i)\mu(Q_j) \text{ for all } i, j.$$

The partition $\zeta$ *is $\varepsilon$-independent of* $\eta$ if

$$\sum_i \sum_j |\mu(P_i \cap Q_j) - \mu(P_i)\mu(Q_j)| \le \varepsilon.$$

For an ergodic measure-preserving endomorphism $T$ on $(X, \mathcal{B}, \mu)$, (invertible or noninvertible) a partition $\zeta$ is *weak Bernoulli* if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \ge 1$,

$$\bigvee_0^m T^{-i}\zeta \quad \text{is } \varepsilon\text{-independent of} \quad \bigvee_N^{N+m} T^{-i}\zeta.$$

It was proved by Friedman and Ornstein in [9] that for an invertible transformation $T$, if there exists a weak Bernoulli partition $\zeta$ such that:

$$\zeta_{-\infty}^{\infty} \equiv \bigvee_{i=-\infty}^{\infty} T^{-i}(\zeta)$$

generates $\mathcal{B}$ (note that this is a two-sided generator), then $T$ is isomorphic to an (invertible) Bernoulli shift. The first example of a noninvertible endomorphism with a weak Bernoulli generator in the sense of (2.2) is due to Furstenberg [10]. It is clear that a measure-preserving endomorphism $T$ is one-sided Bernoulli if and only if there exists an independent generating partition.

**Definition 2.17.** We say that a noninvertible endomorphism $T$ on $(X, \mathcal{B}, \mu)$ has the *weak Bernoulli property* or that $T$ is *weak Bernoulli* if there exists a weak Bernoulli generating partition $\mathcal{P}$ for $T$ (in the sense of Definition 2.6 (2.2)).

**Definition 2.18.** An automorphism $\widetilde{T}$ is *the natural extension* of the (noninvertible endomorphism) $T$ if $T$ is a measurable factor of $\widetilde{T}$ and any other automorphism $S$ which has $T$ as a factor also has $\widetilde{T}$ as a factor.

It was shown by Rohlin [35] that a unique natural extension exists for every finite measure preserving endomorphism; the construction of the invertible extension leads to a straightforward proof that a weakly Bernoulli endomorphism has a weakly Bernoulli, hence Bernoulli natural extension. This is well-known (see e.g., [1, 21]).

Weakly Bernoulli endomorphisms exhibit many highly mixing properties and have been well-studied; for example weakly Bernoulli toral endomorphisms were described by Adler [1], Smorodinsky [37] and others in the early 70's. Conditions under which piecewise smooth bounded-to-one interval maps are weakly Bernoulli were given by Ledrappier in [21]. Haydn [13] showed that rational maps $R$ on the Riemann sphere are weakly Bernoulli with respect to equilibrium measures of Hölder potentials $\psi$, when the *supremum gap* holds: $P(\psi) > \sup\{\psi(z) : z \in \mathcal{J}\}$, where $P$ is the pressure function and $\mathcal{J}$ the Julia set of $R$. When $\mathcal{J}$ is sufficiently far from the critical points of a hyperbolic $R$, these conditions hold for the potential $\psi = -\log|R'|$; in this case, $R$ is weakly Bernoulli with respect to the $R$-invariant measure $\mu$ that is equivalent to $t$-conformal measure ($\sim$ $t$-dimensional Hausdorff measure) for $t = \dim_H(\mathcal{J})$.

Entropy gives a simple necessary test for one-sided Bernoulli maps. Note that, contrary to two-sided Bernoulli shifts, entropy is not a complete invariant in the setting of one-sided Bernoulli shifts. We give an example showing that many of these weakly Bernoulli maps are not one-sided Bernoulli.

**Lemma 2.19.** *Suppose $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure preserving $n$-to-one endomorphism, and $h_\mu(T) > \log n$. Then $T$ is not isomorphic to a one-sided Bernoulli shift.*

**Proof.** The maximal entropy for an $n$-state one-sided Bernoulli shift is $\log n$. $\square$

**Example 2.20.** The map $A(x, y) = (3x+y, x+y) \pmod 1$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ gives a two-to-one $m_2$-preserving map of $\mathbb{T}^2$ with entropy $\log(2+\sqrt{2}) > \log 2$. By Lemma 2.19, $A$ is not one-sided Bernoulli, despite being weakly Bernoulli [1].

**2.6. Parry–Walters invariant and commuting automorphisms.** Assume $T$ is a measure preserving endomorphism on $(X, \mathcal{B}, \mu)$, and let $\beta_\mu(T) \equiv \beta_\mu$ denote the smallest $\sigma$-algebra with respect to which $J_{\mu T}$ is measurable and such that $T^{-1}\beta_\mu \subset \beta_\mu$.

Our starting point is an observation by Walters [41] that if $p \in (0, 1)$ and $p \neq \frac{1}{2}$, then $\beta_\rho(\sigma) = \beta_\rho = \mathcal{B}$ for the one-sided $\{p, 1-p\}$ Bernoulli shift, and if $p = \frac{1}{2}$, then $\beta_\rho = \{\emptyset, X\}$. From this he showed that there are no measure-preserving automorphisms commuting with a $\{p, 1-p\}$ Bernoulli shift unless $p = \frac{1}{2}$. In this section we push this result further in order to obtain some differentiable non-Bernoulli maps.

We prove a (slightly stronger) version of a theorem from [30] and [41, Theorem 2], starting with the basic ideas laid out in the original work from the 1970's. The results in this section provide us with many new examples in later sections.

**Theorem 2.21.** *Suppose $p \neq \frac{1}{2}$:*

(1) *Let $\sigma$ on $(\Omega, \rho)$ be the one-sided $\{p, 1-p\}$ Bernoulli shift. Then there exists no nontrivial nonsingular automorphism $\varphi : (\Omega, \rho) \to (\Omega, \rho)$ with $\varphi \circ \sigma = \sigma \circ \varphi$ ($\mu \bmod 0$).*

(2) *If $T$ on $(X, \mathcal{B}, \mu)$ is a one-sided $\{p, 1-p\}$ Bernoulli endomorphism, then there is no nontrivial nonsingular commuting automorphism $\varphi : (X, \mu) \to (X, \mu)$.*

**Proof.** Under the assumption that $p \neq \frac{1}{2}$, we have that $\beta_\rho = \mathcal{D}$, the entire $\sigma$-algebra of Borel sets. Since $\sigma$ is ergodic, applying Proposition 2.12 it follows that $\varphi$ preserves $\rho$. We consider the factor map induced by $\sigma$ on $\beta_\rho$. Two points $x \sim_{\beta_\rho} y$ under the factor relation if and only if $J_{\rho\sigma}(x) = J_{\rho\sigma}(y)$, $J_{\rho\sigma}(\sigma x) = J_{\rho\sigma}(\sigma y), \ldots,$ and $J_{\mu\sigma}(\sigma^n x) = J_{\rho\sigma}(\sigma^n y)$ for all $n \in \mathbb{N}$. By assumption and the chain rule on $J_{\rho(\varphi\circ\sigma)}$,

$$J_{\rho(\varphi\circ\sigma)}(x) = J_{\rho\varphi}(\sigma x) \cdot J_{\rho\sigma}(x) = J_{\rho\sigma}(x).$$

Since $\varphi \circ \sigma = \sigma \circ \varphi$, this is equal to

$$J_{\rho(\sigma\circ\varphi)}(x) = J_{\rho\sigma}(\varphi x) \cdot J_{\rho\varphi}(x) = J_{\rho\sigma}(\varphi x).$$

Therefore $x$ and $\varphi(x)$ belong to the same atom of $\beta_\rho = \mathcal{D}$, so $x = \varphi(x)$ under the factor map induced by $\beta_\rho$, which is the identity map. This is a contradiction unless $\varphi$ is the identity map on a set of full measure.

In the case that $T$ preserves $\mu$, and is isomorphic to $\sigma$ on $(\Omega, \rho)$, it follows immediately that there is no nontrivial automorphism $\varphi$ on $X$ preserving $\mu$ because any such automorphism would result in one on $(\Omega, \rho)$. □

The following corollary leads to the construction of examples; the addition here is that we have a checkable condition even if we do not know the precise invariant measure.

**Corollary 2.22.** [30] *Suppose $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure preserving two-to-one endomorphism, and $h_\mu(T) < \log 2$. If there exists a nontrivial nonsingular automorphism $\varphi$ commuting with $T$, then $T$ is not isomorphic to a one-sided $\{p, 1-p\}$ Bernoulli shift.*

**Proof.** Since $h_\mu(T) < \log 2$, we know that $T$ is not isomorphic to the $\left\{\frac{1}{2}, \frac{1}{2}\right\}$ Bernoulli shift. The result follows from Theorem 2.21 (1). □

For a $\{p_1, p_2, \ldots, p_n\}$ one-sided Bernoulli shift, and the resulting product measure $\rho$, in order to extend Theorem 2.21 one must ask when $\beta_\rho(\sigma) = \mathcal{D}$. Of course $\beta_\rho(\sigma) = \{\emptyset, X\}$ if each $p_k = \frac{1}{n}$, but neither case above need to occur. For example, if $p = \left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}$, $\beta_\rho$ is neither trivial nor equal to $\mathcal{B}$, but gives rise to a $\left\{\frac{1}{2}, \frac{1}{2}\right\}$ Bernoulli shift factor.

However, we still obtain results for $n$-to-one Bernoulli shifts and the proof of (1) and (2) is the same as for Theorem 2.21.

**Corollary 2.23.** (1) *If $T$ on $(X, \mathcal{B}, \mu)$ is a one-sided Bernoulli endomorphism isomorphic to $\sigma$ on $(\Omega, \rho)$, and $\beta_\rho = \mathcal{D}$, then there is no nontrivial automorphism $\varphi : (X, \mu) \to (X, \mu)$ preserving any $\sigma$-finite measure equivalent to $\mu$.*

(2) *If $T$ on $(X, \mathcal{B}, \mu)$ is a one-sided Bernoulli endomorphism and $\varphi$ is a $\mu$-preserving automorphism commuting with $T$, then $J_{\mu T}$ is constant on orbits of $\varphi$ ($\mu \bmod 0$).*

(3) *If $T$ on $(X, \mathcal{B}, \mu)$ is measure preserving and $n$-to-one and has a Rohlin factor that is isomorphic to the $\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\}$ one-sided Bernoulli shift, then either $T$ is isomorphic to its Rohlin factor or $T$ is not one-sided Bernoulli.*

**Proof.** Corollary 2.11 shows that the assumption on the Rohlin factor is a shift factor of $T$ such that $h_\nu(\sigma) = \log n$; hence $h_\mu(T) \geq \log n$. By Lemma 2.19, $h_\mu(T) > \log n$ implies the result, and if $h_\mu(T) = \log n$, either $T$ is not Bernoulli or is isomorphic to its Rohlin factor. □

# 3. Rigidity of piecewise smooth noninvertible Bernoulli interval maps

In this section we prove a series of results showing the rigidity of a smooth one-sided Bernoulli interval map. In particular, each such map is conjugate to a piecewise linear map via a conjugating map which is as differentiable as can be hoped for, depending on the setting.

**3.1. Radon–Nikodym derivatives of interval maps.** We begin a version of the so-called Folklore Theorem which we apply to obtain continuity of Radon–Nikodym derivatives, following the exposition given in [24, Theorem III.1.3].

**Theorem 3.1** (Folklore Theorem). *Let $(X, \mathcal{B}, \mu)$ be a probability space equipped with a metric $d$. Suppose the map $T : X \to X$ satisfies:*

(1) *There is a finite or countable partition $\mathcal{P}$ of $X$, i.e., $X = \cup_{P \in \mathcal{P}} P$ $(\bmod\ \mu)$ and $\mu(P \cap Q) = 0$ for distinct $P, Q \in \mathcal{P}$.*

(2) *$T|_P : P \to X$ is one-to-one and onto for each $P \in \mathcal{P}$.*

(3) *There exists $\lambda > 1$ and $N \geq 1$ such that*

$$d(T^N x, T^N y) \geq \lambda\, d(x, y)$$

*if $x, y$ belong to the same atom of $\bigvee_{i=0}^{N-1} T^{-i}(\mathcal{P})$.*

(4) *There are $\gamma, C_0 > 0$, such that distortion of the Jacobian satisfies:*

$$\left| \frac{J_{\mu T}(x)}{J_{\mu T}(y)} - 1 \right| \leq C_0 d(Tx, Ty)^\gamma$$

*for all $x, y$ in the same atom of $\mathcal{P}$.*

*Then there is a $T$-invariant measure $\nu \ll \mu$ and:*

- $\frac{d\nu}{d\mu}$ *is Hölder continuous* (*with exponent* $\gamma$), *bounded and bounded away from* 0.
- $T$ *is* exact *with respect to* $\mu$.
- $\nu(A) = \lim_{n\to\infty} \mu(T^{-n}(A))$ *for every* $A \in \mathcal{B}$.

We next apply the Folklore Theorem to first return maps for $C^2$ interval maps to prove continuity of some Radon–Nikodym derivatives. The notation *acip* stands for absolutely continuous (w.r.t. Lebesgue) invariant probability measure in what follows. Let $\mathrm{orb}(A) = \cup_{n\geq 0} T^n(A)$ be the (*forward*) *orbit* of the set (or point) $A$, and let Crit denote the set of *critical points* of $T$, i.e., the points $c$ where $T'(c) = 0$. The following proposition is useful only when orb(Crit) is not dense. As in 2.8, $I$ is compact.

**Proposition 3.2.** *Let* $T : I \to I$ *be a* $C^2$ *interval map with an acip* $\mu \ll m$. *Then the Radon–Nikodym derivative* $g = \frac{d\mu}{dm}$ *is continuous at every point in* $I \setminus \overline{\mathrm{orb}(\mathrm{Crit})}$.

**Proof.** Every acip has at most #Crit ergodic components, whose supports have pairwise disjoint interiors. So we restrict our attention to a single ergodic component with acip $\mu$; its support is a finite union of intervals $I_k, k = 1, \dots, N$ which are permuted cyclically under the map, and $\mu$-a.e. $x \in \mathrm{supp}(\mu)$ has a dense orbit in $\mathrm{supp}(\mu)$. In particular, $\mathrm{supp}(\mu)$ contains no periodic attractors or periodic intervals of period $\neq N$. Therefore $\mathrm{supp}(\mu)$ has a dense set of periodic orbits. These facts can be found e.g., in [38]; the monograph [26] covers the unimodal case.

Take $t \in \mathrm{supp}(\mu) \setminus \overline{\mathrm{orb}(\mathrm{Crit})}$. We show that $\frac{d\mu}{dm}$ is continuous at $t$. Let $V'$ be the component of $\mathrm{supp}(\mu) \setminus \overline{\mathrm{orb}(\mathrm{Crit})}$ containing $t$ and let $V$ be a neighborhood of $t$, compactly contained in $V'$, such that $\mathrm{orb}(\partial V) \cap V^\circ = \emptyset$. Because the set of periodic points is dense in $\mathrm{supp}(\mu)$, we can find such a $V'$ by giving it periodic boundary points.

For $x \in V$, let $\tau(x) := \min\{n \geq 1 : T^n(x) \in V\}$ be the *first return time*, and let $V_* = \{x \in V : \tau(x) \text{ is well-defined}\}$. Then

$$F : V_* \to V, \ F : x \mapsto T^{\tau(x)}(x).$$

is the *first return map*. If $x \in V_*$, let $U'_x$ be the maximal neighborhood of $x$ on which $T^{\tau(x)}$ is monotone. Since $T^{\tau(x)}(\partial U'_x) \subset \mathrm{orb}(\mathrm{Crit})$, $T^{\tau(x)}(U'_x) \subset V'$ and hence there is a smaller neighborhood $U_x$ such that $T^{\tau(x)}$ maps $U_x$ monotonically onto $V$. Moreover, $T^i(U_x) \cap V = \emptyset$ for every $0 < i < \tau(x)$. Indeed, if $T^i(U_x) \subset V$, then $\tau(x)$ is not the first return time to $V$, and if $T^i(U_x) \cap \partial V \neq \emptyset$, then $V \subset T^{\tau(x)-i}(\partial V)$ contradicting the definition of $V$. Hence $\tau(x) = \tau(y)$ for all $y \in U_x$.

This means that the domain $V_*$ consists of a countable union of disjoint intervals, which we can renumber as $\{U_i\}_{i\in\mathbb{N}}$. We noted above that $\mu$-a.e. $x \in \mathrm{supp}(\mu)$ has a dense orbit in $\mathrm{supp}(\mu)$, so $V_* = \cup_i U_i$ has full Lebesgue measure in $V$, and the same holds for $V_\infty = \bigcap_{i\geq 0} F^{-i}(V_*)$, the set on which all iterates of $F$ are well-defined.

The Koebe Principle [26, Chapter IV.1] guarantees that the distortion condition (4) of the Folklore Theorem 3.1 holds for $C_0 = C_0(V', V)$ and $\gamma = 1$. One can extend these arguments to show that the same distortion bound holds uniformly over all iterates of $F$, and due to the absence of periodic attractors, there is some iterate $F^N$ such that $|(F^N)'(x)| \geq 2$ for all $x \in V_\infty$.

The Folklore Theorem 3.1 gives an $F$-invariant measure $\nu$ on $V$ with $\frac{d\nu}{dm}$ continuous and bounded away from 0. On $I$, the measure defined by

$$\mu_0(A) = \sum_i \sum_{j=0}^{\tau_i - 1} \nu(T^{-j}(A) \cap U_i) \tag{3.1}$$

can be checked to be absolutely continuous, $\sigma$-finite and $T$-invariant. Therefore $\mu_0 \sim \mu$, and since $\mu$ is a probability measure, $d\mu = Cd\mu_0$ for some $C \in (0, \infty)$.

For each $A \subset V$, Equation (3.1) gives that $\mu_0 = \sum_i \nu(A \cap U_i) = \nu(A)$, so $\frac{d\mu}{dm} = C\frac{d\mu_0}{dm} = C\frac{d\nu}{dm}$ is indeed continuous at $t$. □

**Remark 3.3.** The Folklore Theorem implies that $\frac{d\mu}{dm}$ is bounded away from 0 on each $U_i$ above. Moreover, for any fixed $U_i$, there exists an $N$ such that $\bigcup_{k=0}^N T^k(U_i) \supset \text{supp}(\mu)$. From this it is not hard to derive that $\frac{d\mu}{dm}$ is bounded away from 0 on $\text{supp}(\mu)$, cf. [19, Theorem 1(3)].

**3.2. Rigidity of smooth Bernoulli maps.** We now turn to some results regarding one-sided Bernoulli differentiable maps of the interval and circle.

**Definition 3.4.** Suppose we have two interval maps $T, S : I \to I$.

- We say $S$ and $T$ are $C^0$-*conjugate*, or *topologically conjugate* if there exists a homeomorphism $\psi : I \to I$ such that $S = \psi \circ T \circ \psi^{-1}$.
- If $\psi$ above is a $C^k$ diffeomorphism (with a $C^k$ inverse), we say $T$ and $S$ are $C^k$-*conjugate*.

**Theorem 3.5.** *Let $T : I \to I$ be a piecewise $C^2$ $n$-to-1 interval map preserving a probability measure $\mu$ equivalent to Lebesgue measure $m$ such that the Radon–Nikodym derivative $g(x) = \frac{d\mu}{dm}$ is continuous and bounded away from 0. Then $T$ is one-sided Bernoulli on $(I, \mathcal{B}, \mu)$ if and only if $T$ is $C^1$-conjugate to a map $S : I \to I$ whose graph consists of $n$ linear pieces, with slopes $\pm\frac{1}{p_i}$ such that $h_\mu(T) = -\sum_{i=1}^n p_i \log p_i$.*

**Proof.** ($\Leftarrow$): The map $S$ is clearly one-sided Bernoulli with respect to the invariant measure $m$, with $h_m(S) = -\sum_{i=1}^n p_i \log p_i$. Under the assumption that $S$ is $C^1$-conjugate to $T$, this implication follows immediately.

($\Rightarrow$): Define $\psi : I \to I$ by $\psi(x) = \mu([0, x])$. Since $g$ is positive and continuous, $\psi$ is a $C^1$ diffeomorphism and $\psi'(x) = g(x) > 0$, so $\psi^{-1}$ is $C^1$ as well. Furthermore the map $S := \psi \circ T \circ \psi^{-1}$ preserves $m$.

It remains to show that $S$ is piecewise linear or equivalently it suffices to show that $S'$ is piecewise constant. By construction, $S$ is topologically conjugate to $T$, so $h_m(S) = h_\nu(T)$. We differentiate both sides of $S \circ \psi = \psi \circ T$, and write $\varphi := \log \psi'$ to obtain

$$\log |T'| + \varphi \circ T - \varphi = \log |S'| \circ \psi. \tag{3.2}$$

The left-hand side is defined and continuous except at the set $C$ of discontinuities of $T'$. Therefore $S'$ is continuous, except at $\psi(C)$, (where possibly $C = \emptyset$). Let $\pi : I \to \mathcal{A}^{\mathbb{N}}$ be the isomorphism between $(I, \mathcal{B}, \mu; T)$ and a one-sided $p = \{p_1, \dots, p_n\}$ Bernoulli shift $\sigma$. Then $\Psi := \psi \circ \pi^{-1}$ is an isomorphism between $(I, \mathcal{B}, m; S)$ and the Bernoulli shift $(\Omega, \mathcal{D}, \rho; \sigma)$, see Figure 2. It follows that for $m$-a.e. $x$, the Jacobian functions $J_{mS}(y_j)$ at the $n$ preim-

$$\begin{tikzcd}
(\mathcal{A}^{\mathbb{N}}, \rho) \arrow[rrr, "\sigma"] \arrow[dd, "\Psi"'] & & & (\mathcal{A}^{\mathbb{N}}, \rho) \arrow[dd, "\Psi = \psi \circ \pi^{-1}"] \\
& (I, \mu) \arrow[ul, "\pi"'] \arrow[dl, "\psi"] \arrow[r, "T"] & (I, \mu) \arrow[ur, "\pi"] \arrow[dr, "\psi"'] & \\
(I, m) \arrow[rrr, "S"] & & & (I, m)
\end{tikzcd}$$

FIGURE 2. Commutative diagram to construct $\Psi = \psi \circ \pi^{-1}$.

ages $y_1, \dots, y_n$ of $x$ take the values $\frac{1}{p_1}, \dots, \frac{1}{p_n}$ since $J_{mS}(y) = J_{\rho\sigma}(\Psi^{-1} y)$. But since $m$ is Lebesgue measure, the Jacobian of $S$ coincides with the absolute value of the derivative, and the derivative is piecewise continuous. It follows that $S'$ is piecewise constant as asserted. $\square$

**Corollary 3.6.** *Let $T : I \to I$ be a piecewise $C^2$ expanding $n$-to-1 map. Then $T$ is one-sided Bernoulli on $(I, \mathcal{B}, m)$ if and only if $T$ is $C^1$-conjugate to a map $S : I \to I$ whose graph consists of $n$ linear pieces, with slopes $\pm\frac{1}{p_i}$ such that $h_\mu(T) = -\sum_{i=1}^n p_i \log p_i$.*

**Proof.** The Folklore Theorem 3.1 gives an invariant probability measure $\mu$ such that $g = \frac{d\mu}{dm}$ is continuous and bounded away from 0 and $\infty$. Hence Theorem 3.5 applies. $\square$

**Remarks 3.7.** (1) The partition $\mathcal{P} = \{P_1, \dots, P_n\}$ into intervals such that $T : P_i \to (0, 1)$ is $C^2$ onto generates $\mathcal{B}$. The coding map

$$\pi : I \to \Omega = \prod_0^\infty \{1, \dots, n\}$$

gives the isomorphism with the Bernoulli shift $(\Omega, \mathcal{D}, \rho; \sigma)$.

(2) Theorem 3.5 shows that one-sided Bernoulli interval maps are rare, the property is quite rigid, as the $C^1$-conjugacy implies that multipliers of all periodic points of $T$ are the same as the multipliers of the corresponding periodic points of the piecewise linear map $S$, see Corollary 3.14.

**Example 3.8.** Chebyshev polynomials form a well-known collection of one-sided Bernoulli maps. Scaled to $[0,1]$, the Chebyshev polynomial $Ч_n$ of degree $n$ can be defined by $Ч_n = \psi \circ S_n \circ \psi^{-1}$, where $\psi(x) = \frac{2}{\pi} \arcsin \sqrt{x}$ and $S_n : I \to I$ is the continuous map with $n$ linear branches of slope $\pm n$ and $S_n(0) = 0$. For example, $Ч_2(x) = 4x(1-x)$ and $Ч_3(x) = 9x - 24x^2 + 16x^3$. Clearly $(I, \mathcal{B}, m; S_n)$ is $\{\frac{1}{n}, \dots, \frac{1}{n}\}$ Bernoulli, and so is $(I, \mathcal{B}, \mu; Ч_n)$ with invariant measure $\mu = m \circ \psi$. A straightforward computation shows that the Radon–Nikodym derivative

$$g(x) = \frac{d\mu}{dm}(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

for each $n$. Since $g(x) \to \infty$ as $x \to 0$ or $1$, the hypotheses of Theorem 3.5 are not met, and indeed $\psi$ has infinite derivative at these points. The following corollary shows how Theorem 3.5 can be rephrased and extended to include this (and similar) examples.

**Corollary 3.9.** *Let $(I, \mathcal{B}, m; T)$ be as in Theorem 3.5, and assume that the Radon–Nikodym derivative $g(x) = \frac{d\mu}{dm}$ is continuous and positive on $(0,1)$. Then $T$ is one-sided Bernoulli on $(I, \mathcal{B}, m)$ if and only if $T$ is topologically conjugate to a map $S : I \to I$ whose graph consists of $n$ linear pieces, with slopes $\pm\frac{1}{p_i}$ such that $h_\mu(T) = -\sum_{i=1}^n p_i \log p_i$. In this case the conjugacy $\psi$ is $C^1$ on $(0,1)$ and it has a $C^1$ inverse.*

**Proof.** $(\Leftarrow)$: The system $(I, \mathcal{B}, m; S)$ is one-sided Bernoulli as in the proof of Theorem 3.5. As $\psi$ is $C^1$ on $(0,1)$, it is still absolutely continuous on $I$, so $T$ is one-sided Bernoulli as well.

$(\Rightarrow)$: Following the proof of Theorem 3.5, we see that $\psi'(x) = g(x)$, except that $\psi'$ need not be defined (or be infinite) at $x = 0, 1$. The points $0$ and $1$ and $T^{-1}(\{0,1\})$ should be added to the set $C$ because $\varphi$ or $\varphi \circ T$ are undefined at those points. Therefore $\psi(C) \subset \{0,1\} \cup S^{-1}(\{0,1\})$ as well. But this has no further effect on the conclusion that $S'$ is piecewise constant. $\square$

**Corollary 3.10.** *Let $T : I \to I$ be a $C^2$ $n$-to-1 map with $T(\partial I) \subset \partial I$ and whose critical points map into $\partial I$. Assume also that all periodic points of $T$ are repelling and that all critical points are nonflat (i.e., some derivative $D^n T(c) \neq 0$). Then $T$ is one-sided Bernoulli on $(I, \mathcal{B}, m)$ if and only if $T$ is topologically conjugate to a map $S : I \to I$ whose graph consists of $n$ linear* *pieces, with slopes* $\pm\frac{1}{p_i}$ *such that* $h_\mu(T) = -\sum_{i=1}^n p_i \log p_i$. *Moreover the conjugacy is* $C^1$ *on* $(0,1)$.

**Proof.** The conditions (including the nonflatness of critical points) guarantee the existence of an invariant probability measure $\mu \ll m$, see [26, Chapter V.3]. In this case, the Radon–Nikodym derivative is unbounded at $\partial I$, but by Proposition 3.2, $g(x)$ is continuous and positive elsewhere. Thus Corollary 3.9 applies. □

The following corollary gives a version of the Theorem 3.5 in the case of smooth circle maps; related results appear in [36] and [16].

**Corollary 3.11.** *If* $T : \mathbb{S}^1 \to \mathbb{S}^1$ *is an expanding* $C^2$ *degree* $n \geq 2$ *circle map and* $T(0) = 0$*, then* $T$ *is one-sided Bernoulli if and only if it is* $C^1$*-conjugate to* $x \mapsto nx \pmod 1$.

**Proof.** ($\Leftarrow$): As in Theorem 3.5.

($\Rightarrow$): $T$ is an expanding $C^2$ $n$-to-one (degree $n$) circle map if and only if there is a $C^2$ covering map $\widetilde{T} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{T}(x+1) = T(x) + n$ and $\widetilde{T}'(x) = T'(x) > 1$ for $x \in [0,1)$. Therefore $T$ has exactly $n-1$ fixed points. Let 0 be one of them. We cut the circle at 0 to obtain an interval, and then $T$ can be extended to a $C^2$ $n$-to-one on this interval. Now the proof of Theorem 3.5 applies; it actually gives that the set $C = \emptyset$, because $T'$ has no discontinuities. Therefore $S'$ is constant, and after gluing the interval back to a circle, the map $S$ becomes a linear degree $n$ circle map. There is only one such map: $x \mapsto nx \pmod 1$. □

**Corollary 3.12.** *There are no* $C^2$ *expanding* $n$*-to-one one-sided Bernoulli maps on* $(\mathbb{S}^1, \mathcal{B}, m)$ *that are* $\{p_1, \ldots, p_n\}$ *Bernoulli unless* $p_i = \frac{1}{n}$ *for each* $i$.

**Proof.** If $p_i \neq \frac{1}{n}$ for some $i$, then $S'$ would have to have at least one discontinuity which is impossible as shown in the proof of Corollary 3.11. □

The next two results on conformal measures show how the one-sided Bernoulli property imposes rigidity results on the multipliers of periodic points. Let $Y$ and $X \subset Y$, $T$ be as in Section 2.9 and let $m_\alpha$ denote a conformal measure on $X$. If $f : X \to \mathbb{R}$ is *piecewise constant* then $X = \cup_{j=1}^k F_j$ such that the $F_j$'s are disjoint, open and closed sets, and $f$ is constant on $F_j$, $j = 1, \ldots, k$.

**Proposition 3.13.** *Let* $T$ *be a* $C^1$ $n$*-to-one map on* $(X, \mathcal{B}_X, m_\alpha)$ *preserving a measure* $\mu \sim m_\alpha$ *and assume that the Radon–Nikodym derivative* $g(x) := \frac{d\mu}{dm_\alpha}$ *is continuous and bounded away from* 0*. If* $T$ *has no critical points on* $X$ *and is one-sided* $\{p_1, \ldots, p_n\}$ *Bernoulli, then* $J_{\mu T}$ *is continuous everywhere on* $X$*. Moreover* $J_{\mu T}$ *is constant on components of* $X$*; if* $X$ *is connected,* $J_{\mu T}$ *is constant on* $X$*.*

**Proof.** As $T$ is one-sided $\{p_1, \dots, p_n\}$ Bernoulli, $J_{\mu T}(x) \in \{\frac{1}{p_1}, \dots, \frac{1}{p_n}\}$ for $\mu$-a.e. $x$. By (2.3) we have for each $x \in X$

$$J_{\mu T}(x) = \frac{d\mu}{dm_\alpha}(Tx) \cdot J_{m_\alpha T}(x) \cdot \left(\frac{d\mu}{dm_\alpha}(x)\right)^{-1} = \frac{g(Tx)}{g(x)}\, |T'(x)|^\alpha; \tag{3.3}$$

the hypotheses imply that the right-hand side of the equality is the product of continuous nonzero functions and hence $J_{\mu T}$ is continuous and finite valued. If $X$ is connected, the only finite-valued continuous functions are constants. Therefore $J_{\mu T}$ must be piecewise constant with $F_j = J_{\mu T}^{-1}(p_k)$; each component of $X$ must be contained in a single $F_j$ by continuity. □

**Corollary 3.14.** *Suppose $T$ is a $C^1$ $n$-to-one map on $(X, \mathcal{B}_X, m_\alpha)$ preserving $\mu \sim m_\alpha$ with $g(x) := \frac{d\mu}{dm_\alpha}$ continuous and bounded away from 0. Assume $T$ has no critical points on $X$ and is one-sided Bernoulli. Then for every $k$-periodic point $q$, the multiplier $|(T^k)'(q)|^\alpha$ is a $k$-fold product of numbers $\frac{1}{p_i}$.*

**Proof.** Apply (3.3) to the $k$-fold Jacobian $J_{\mu T^k}(q)$ to see that

$$\begin{aligned} J_{\mu T^k}(q) &= \frac{g(T^k q)}{g(T^{k-1}q)}\, |T'(T^{k-1}q)|^\alpha \cdot \frac{g(T^{k-1}q)}{g(T^{k-2}q)}\, |T'(T^{k-2}q)|^\alpha \\ &\quad \cdots \frac{g(Tq)}{g(q)}\, |T'(q)|^\alpha \\ &= |(T^k)'(q)|^\alpha. \end{aligned}$$

By Proposition 3.13 the left-hand side is a product of numbers $\frac{1}{p_i}$. □

# 4. Examples of non-Bernoulli $n$-to-one maps

**4.1. Maps in one real dimension.** We first give a basic example illustrating Theorem 2.21.

**Example 4.1** (Piecewise affine maps)**.** Fix $p, q \in (0,1)$ with $p + q = 1$. A dynamical system well-known to be isomorphic to the one-sided $\{p, q\}$ Bernoulli shift is $(\mathbb{S}^1, \mathcal{B}, m; T_{p,0})$, where

$$T_{p,0}(x) = \begin{cases} \frac{1}{q}\, x & \text{for } x \in [0, q) = A_2 \\ \frac{1}{p}\, (x-1) + 1 & \text{for } x \in [q, 1) = A_1, \end{cases}$$

and $\mathcal{B}$ is the $\sigma$-algebra of Lebesgue measurable sets. The coding map with respect to the partition $\zeta = \{A_1, A_2\}$ is the isomorphism.

The following variant is no longer one-sided Bernoulli (unless $p = q = \frac{1}{2}$): Let $T_{p,\frac{1}{4}} : \mathbb{S}^1 \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be given by

$$T_{p,\frac{1}{4}}(x) = \begin{cases} \frac{1}{p}\,x & \text{if } x \in \left[0, \frac{p}{2}\right) \\ \frac{1}{q}\left(x - \frac{p}{2}\right) + \frac{1}{2} & \text{if } x \in \left[\frac{p}{2}, \frac{1}{2}\right) \\ \frac{1}{q}\left(x - \frac{1}{2}\right) & \text{if } x \in \left[\frac{1}{2}, \frac{1+q}{2}\right) \\ \frac{1}{p}\left(x - \frac{1+q}{2}\right) + \frac{1}{2} & \text{if } x \in \left[\frac{1+q}{2}, 1\right) \end{cases}$$

(see Figure 3). The map $T_{p,\frac{1}{4}}$ also preserves Lebesgue measure $m$ and com-

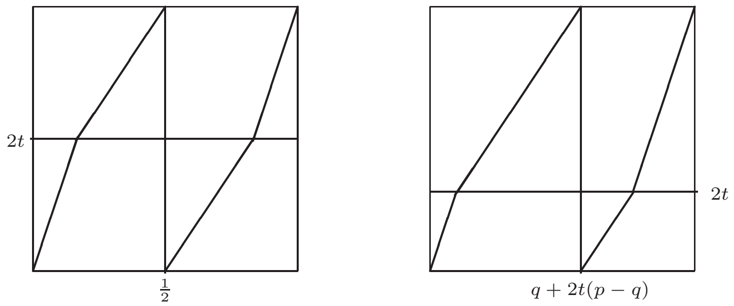


FIGURE 3. The map $T_{p,t}$ is not one-sided Bernoulli for $t = \frac{1}{4}$ (left) but it is for e.g., $t = \frac{3}{20}$ (right).

mutes with the automorphism $\varphi(x) = 1 - x$. Theorem 2.21 implies that $T_{p,\frac{1}{4}}$ cannot be one-sided Bernoulli. Let $A_1 = \left[0, \frac{p}{2}\right) \cup \left[\frac{1+q}{2}, 1\right)$ and $A_2 = \left[\frac{p}{2}, \frac{1+q}{2}\right)$. Then $\zeta = \{A_1, A_2\}$ is a Rohlin partition, and $\varphi$ interchanges $A_i$, $i = 1, 2$. Because of this, the coding map $\pi : \mathbb{S}^1 \to \{1, 2\}^{\mathbb{N}}$ with respect to $\zeta = \{A_1, A_2\}$ is 2-to-1; so $\zeta$ is not generating. If $x \neq y \neq \varphi(x)$, then $\pi(x) \neq \pi(y)$. This can be seen by the fact that $T_{p,\frac{1}{4}}$ is expanding, and the atoms in $\zeta_n := \bigvee_{i=0}^{n-1} T_{p,\frac{1}{4}}^{-i}(\zeta)$ are unions of two intervals (symmetric under $\varphi$), and each of these intervals has length $\frac{1}{2}p^k q^{n-k}$, where $k$ denotes the number of 1's in the first $n$ entries of the coding under $\pi$.

Since $m$ is Lebesgue measure, $J_{mT} = |T'|$, hence $J_{mT}|_{A_1} = \frac{1}{p}$ and $J_{\mu T}|_{A_2} = \frac{1}{q}$. It follows from Lemma 2.11 that $\pi$ is a 2-to-1 factor map onto the $\{p, 1-p\}$ Bernoulli shift. This proves statement (4) in Example 2.7.

The above examples are special cases (namely $t = 0$ and $t = \frac{1}{4}$) of the family $T_{p,t} : \mathbb{S}^1 \to \mathbb{S}^1$ with $t \in \left[0, \frac{1}{2}\right)$, $p + q = 1$ and

$$T_{p,t}(x) = \begin{cases} \frac{1}{p}\,x & \text{if } x \in [0, 2tp) \\ \frac{1}{q}(x - 2tp) + 2t & \text{if } x \in [2tp, q + 2t(p-q)) \\ \frac{1}{q}(x - q - 2t(p-q)) & \text{if } x \in [q + 2t(p-q), q + 2tp) \\ \frac{1}{p}(x - 1) + 1 & \text{if } x \in [q + 2tp, 1). \end{cases}$$

The maps $T_{p,t}$ are piecewise affine, and the derivative has two discontinuities, namely at $T_{p,t}^{-1}(2t)$, see Figure 3. Again, Lebesgue measure is $T_{p,t}$-invariant. This family can be thought of as a "$\{p,q\}$ version" of Example 2.7 in the sense that for every $p \in (0,1)$ and $t \in \left[0,\frac{1}{2}\right)$, the system $(\mathbb{S}^1,\mathcal{B},m;T_{p,t})$ is isomorphic to (or if $t=\frac{1}{4}$, is a two-point extension of) the $\{p,q\}$ Bernoulli shift, and the coding map with respect to the partition $\zeta_{p,t}$ defined below implements the isomorphism (respectively, factor map).

Let $\zeta_{p,t}=\{A_1,A_2\}$ for $A_1=[0,2tp)\cup[q+2tp,1)$ and $A_2=[2tp,q+2tp)$; it is obviously a Rohlin partition. The map $T_{p,t}$ maps both $A_1$ and $A_2$ onto $\mathbb{S}^1$. From this it follows that the coding map $\pi_{p,t}$ is surjective unless $t=0$, when there is no point $x$ with $\pi_{p,0}(x)=111\ldots$. Furthermore, $\pi_{p,t}$ is not injective unless $t=0$. However, unless $t=\frac{1}{4}$, $\zeta_{p,t}$ is generating (so the coding map is injective $m$-a.e.). One can show by induction that for any $n$, the atoms of $\bigvee_{i=0}^{n-1}T^{-i}\zeta_{p,t}$ are unions of intervals whose combined Lebesgue measure is $p^kq^{n-k}$, where $k$ denotes the number of 1's in the first $n$ entries of the coding under $\pi_{p,t}$. By the Kolmogorov extension theorem, this means that $\pi_{p,t}$ is an isomorphism.

Although the Jacobian $J_{mT_{p,t}}$ takes values $\frac{1}{p}$ or $\frac{1}{q}$ at every point except two, and in particular at every periodic point, the Jacobian may fail to be invariant under the isomorphism on a set of $m$ measure 0, which could include some periodic points.

For example, if $t$ is close to $\frac{1}{4}$, then there is a periodic point

$$x=\frac{1}{p}\left(\frac{q}{1+q}+2t(p-q)\right)$$

whose period 2 orbit belongs to $A_2$, so $J_{mT_{p,t}}(x)=J_{mT_{p,t}}(T_{p,t}x)=\frac{1}{q}$. Thus $\pi_{p,t}(x)$ is fixed (not period 2) under $\sigma$; moreover the period 2 point $y=212121\ldots$ of the Bernoulli shift has $J_{\rho\sigma}(y)=\frac{1}{q}$ and $J_{\rho\sigma}(\sigma(y))=\frac{1}{p}$.

This phenomenon needs to be taken into account in the proofs of Propositions 4.5 and 4.10. It also demonstrates the importance of the smoothness assumption in Theorem 3.5 because the map $T_{p,t}$ is in general not $C^1$-conjugate to $T_{p,0}$; the multipliers at corresponding periodic points do not agree.

**Example 4.2.** Using Corollaries 3.11 and 3.12 one can construct a $C^\infty$ expanding two-to-one non-Bernoulli circle map. Define

$$T(x)=2x+\varepsilon\sin 4\pi x$$

for $|\varepsilon|<\frac{1}{4\pi}$. For each such $\varepsilon$, $T$ is a $C^\infty$ two-to-one expanding map of $\mathbb{S}^1=\mathbb{R}/\mathbb{Z}$, it is topologically conjugate to $S(x)=2x \pmod 1$ [36], and there is an invariant probability measure $\mu\sim m$ (with $\mu\neq m$ for $\varepsilon\neq 0$). However, the derivative at the fixed point 0 is $2+4\pi\varepsilon$ and not 2, and therefore $T$ is not one-sided Bernoulli by Corollary 3.6 and the subsequent remarks. By [21], $T$ is weakly Bernoulli with weak Bernoulli partition $\left\{\left[0,\frac{1}{2}\right),\left[\frac{1}{2},1\right)\right\}$.

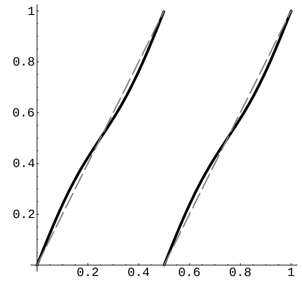


FIGURE 4. Non-Bernoulli example and the one-sided $\left\{\frac{1}{2}, \frac{1}{2}\right\}$ Bernoulli (dotted line).

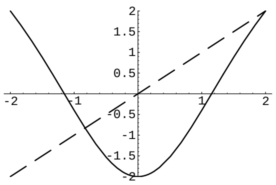


FIGURE 5. The graph of a non-Bernoulli unimodal map $R_a$.

**Example 4.3.** *Blaschke products on the circle.* In [3] the following family of degree two rational maps is studied: for $a \in (0, 1)$, define:

$$B_a(z) = \frac{z(2\sqrt{a} - 1 + z)}{1 + (2\sqrt{a} - 1)z},$$

with $a \in (0, 1)$.

Each of these maps takes the unit disk to itself and $\mathcal{J}(B_a) = \mathbb{S}^1$; Moreover each is a degree 2 expanding map and $a = \frac{1}{4}$ if and only if $B_a$ is isomorphic to the $\left\{\frac{1}{2}, \frac{1}{2}\right\}$ Bernoulli shift [3]. Then by Corollary 3.12 we have that for all other values of $a \in (0, 1)$, $B_a : \mathbb{S}^1 \to \mathbb{S}^1$ is weakly Bernoulli but not one-sided Bernoulli with respect to $m$.

**Example 4.4.** We consider the following family of unimodal maps on $I = [-2, 2]$, for $a \in (0, 1)$:

$$R_a(x) = \frac{-8 + (2 + 8a)x^2}{4 + (-1 + 4a)x^2}.$$

We obtain the Chebyshev polynomial when $a = \frac{1}{4}$. Otherwise there are no polynomial maps in the family. From [3] we have the following easily shown properties and we see the graph of a typical map in Figure 5.

- $R_a(-2) = R_a(2) = 2$, $R_a(0) = -2$, and $R_a$ has one critical point at $x = 0$, is strictly decreasing on $[-2, 0)$ and strictly increasing on $(0, 2]$.
- $R_a[-2, 2] = R_a^{-1}[-2, 2] = [-2, 2]$.

- $R'_a(-2) = -\frac{1}{a}$ and $R'_a(2) = \frac{1}{a}$, so $x = 2$ is a repelling fixed point.
- Each $R_a$ is finite postcritical (has a finite forward critical orbit).
- There is one other fixed point in $[-2, 2]$, namely
$$p = \frac{-2}{1 + 2\sqrt{a}} \in \left(-2, -\tfrac{2}{3}\right),$$
with derivative $R'(p) = -(1 + 2\sqrt{a})$. That is, $p$ is always repelling.
- The Schwarzian derivative of $R_a$,
$$S(R_a) := \frac{R'''_a}{R'_a} - \frac{3}{2}\left(\frac{R''_a}{R'_a}\right)^2 = \frac{-3}{2z^2}.$$

These properties imply the existence of an invariant probability measure $\mu_a \sim m$, and by [21] these maps are weakly Bernoulli. We show that except for the $\left\{\frac{1}{2}, \frac{1}{2}\right\}$ Bernoulli Chebyshev case, $R_a$ cannot be one-sided Bernoulli.

**Proposition 4.5.** *The map $R_a$ is not one-sided Bernoulli w.r.t. $m$ except if $a = \frac{1}{4}$, the Chebyshev polynomial.*

**Proof.** Using Corollary 3.14 if $R_a$ were $\{p, 1-p\}$ Bernoulli, then at each fixed point $x_0$ not on a critical orbit, $|R'(x_0)| = \frac{1}{p}$ or $\frac{1}{1-p}$ and at each period two cycle $\{y_1, y_2\}$, we have that $|(R_a^2)'(y_i)| = \frac{1}{p^2}, \frac{1}{p(1-p)}$, or $\frac{1}{(1-p)^2}$.

We calculate directly using the conformally conjugate map:
$$f_a(x) = a\left(x + \tfrac{1}{x} + 2\right) = a\frac{(x+1)^2}{x}$$
(cf. [3]). The unique fixed point not in a critical orbit is $x_0 = -1 + \frac{1}{1+\sqrt{a}}$ with derivative $-1 - 2\sqrt{a}$, so $J_{mf_a}(x_0) = 1 + 2\sqrt{a} > 1$. Then $p = \frac{1}{1+2\sqrt{a}}$, so $1 - p = \frac{2\sqrt{a}}{1+2\sqrt{a}}$ with associated Jacobian $\frac{1+2\sqrt{a}}{2\sqrt{a}} > 1$.

Moving to period two points, setting $y_1 = \frac{-\sqrt{1+a}-1}{\sqrt{1+a}}$, and $y_2 = \frac{-\sqrt{1+a}+1}{\sqrt{1+a}}$, we first note that $y_1 y_2 = \frac{a}{1+a}$ and that $(y_i + 1)^2 = \frac{1}{1+a}$. Hence
$$f_a(y_1) = \frac{a(y_1+1)^2}{y_1} = \frac{a}{(a+1)y_1} = \frac{y_1 y_2}{y_1} = y_2,$$
and similarly $f_a(y_2) = y_1$. Using that $f'_a(x) = a\left(1 - \frac{1}{x^2}\right)$, we compute that $|(f_a^2)'(y_i)| = 3 + 4a$, $i = 1, 2$.

Setting $\frac{1}{p^2} = 3 + 4a$ gives
$$(1 + 2\sqrt{a})^2 = 1 + 4\sqrt{a} + 4a = 3 + 4a,$$
which implies that $\sqrt{a} = \frac{1}{2}$, giving only the solution $a = \frac{1}{4}$ in the interval. Similarly, if $\frac{1}{(1-p)^2} = 3 + 4a$, we have
$$\frac{1 + 4\sqrt{a} + 4a}{4a} = 3 + 4a,$$

or equivalently, $16a^2 + 8a - 4\sqrt{a} - 1 = 0$. Treating this as a degree 4 polynomial in $u = \sqrt{a}$ and factoring out the known solution $u = \frac{1}{2}$, we again obtain using basic calculus on the cubic polynomial $1 + 6u + 4u^2 + 8u^3$ that $u = \frac{1}{2}$ or $a = \frac{1}{4}$ is the only solution in the interval $(0, 1)$.

Finally we suppose that

$$\frac{1}{p} \cdot \frac{1}{1-p} = \frac{(1+2\sqrt{a})^2}{2\sqrt{a}} = 3 + 4a.$$

This yields the polynomial equation $8a^{\frac{3}{2}} - 4a + 2a^{\frac{1}{2}} + 1 = 0$, and basic calculus as above gives that $a = \frac{1}{4}$ is the only solution. $\square$

**Remark 4.6.** Although $R_a$ is a symmetric map (i.e., $R_a(-x) = R_a(x)$) with a symmetric critical orbit, $a = \frac{1}{4}$ is the only parameter for which the Radon–Nikodym derivative $\frac{d\mu_a}{dm}$ is symmetric. At $a = \frac{1}{4}$, $R_a$ reduces to $x \mapsto x^2 - 2$, with $\frac{d\mu_{1/4}}{dm} = \frac{1}{\pi\sqrt{4-x^2}}$. For general $a \in (0, 1)$, $\frac{d\mu_a}{dm}$ is continuous on $(-2, 2)$ by Proposition 3.2, and by Lemma 2.10 (3) it is a fixed point of the transfer operator

$$\mathcal{L}_{mR_a} h(x) = \sum_{y \in R_a^{-1}(x)} \frac{h(y)}{|R_a'(y)|}.$$

Inserting $h = \frac{d\mu_a}{dm}$ at the fixed point $p$, we find $h(p) = \frac{h(p)+h(-p)}{1+2\sqrt{a}}$. If $h$ were symmetric (and hence $h(p) = h(-p)$), then this would reduce to $1+2\sqrt{a} = 2$, so $a = \frac{1}{4}$.

**4.2. Examples on the Riemann sphere.** Recall that $\mathbb{C}_\infty$ denotes the Riemann sphere. Sullivan showed in [39, Theorem 3] that every rational map $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of degree $d \geq 2$ has a conformal measure for some $\alpha \in (0, 2]$, but the question as to whether $m_\alpha$ is unique and/or nonatomic is not entirely resolved. Under certain conditions (see [11] for extensive results in this direction), there is a minimal $\alpha_* \in (0, 2]$, for which the Julia set supports a unique ergodic nonatomic $\alpha_*$-conformal measure with respect to which $R$ is $d$-to-one, whereas for $\alpha \in (\alpha_*, 2]$, only atomic $\alpha$-conformal measures (supported on backward orbits of critical and neutral periodic points) exist. Moreover, $\alpha_*$ is the Hausdorff dimension of the Julia set.

**Definition 4.7.** Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map of degree $d \geq 2$. $R$ is *hyperbolic* if the closure of the orbits of all critical points is disjoint from Julia set $\mathcal{J}(R)$ and *subhyperbolic* if all critical points in $\mathcal{J}(R)$ have finite forward orbits and the closure of the set of orbits of all critical points in the Fatou set is disjoint from the Julia set $\mathcal{J}(R)$.

Sullivan [39, Theorem 4] showed that for hyperbolic maps, $\alpha_*$ is the Hausdorff dimension of the Julia set, and the normalized $\alpha_*$-dimensional Hausdorff measure is $\alpha_*$-conformal; the corresponding result for subhyperbolic maps was proved in [7] and this covers most of the examples in this subsection.

It is known that for hyperbolic and subhyperbolic maps there is an invariant probability measure $\mu$ equivalent to the $\alpha_*$-conformal measure above [7, 11] and it exhibits highly chaotic behavior (e.g., ergodic, exact, positive Lyapunov exponents). The next result gives a large class of examples for which the measure $\mu$ cannot be one-sided Bernoulli.

**Theorem 4.8.** *If $R$ is a hyperbolic rational map of degree $d \geq 2$ on $\mathbb{C}_\infty$, and if $\mathcal{J}(R)$ is connected, then $R$ is not one-sided Bernoulli with respect to $\mu \sim m_\alpha$ unless $R$ conformally conjugate to $z \mapsto z^{\pm d}$.*

**Proof.** The hypotheses imply that $R$ is uniformly expanding on $\mathcal{J}(R)$ with respect to a smooth Riemannian metric. Therefore by the Folklore Theorem 3.1, we see that if $g = \frac{d\mu}{dm_\alpha}$, then $g$ is continuous on $\mathcal{J}(R)$.

If $R$ is one-sided $\{p_1, \ldots, p_n\}$ Bernoulli, $J_{\mu R}(z) \in \left\{\frac{1}{p_1}, \ldots, \frac{1}{p_n}\right\}$ for $\mu$-a.e. $z$. By (2.3) we have

$$J_{\mu R}(z) = g(Tz) \cdot J_{m_\alpha T}(z) \cdot g(z)^{-1} = \frac{g(Tz)}{g(z)}\,|R'(z)|^\alpha,$$

so $J_{\mu R}$ is continuous everywhere on $\mathcal{J}(R)$ as $|R'(z)|^\alpha$ and $g(z)$ are defined and nonzero. A continuous function taking values from a finite set is constant on a connected space. Therefore $J_{\mu R}(z)$ must take the value $\frac{1}{d}$ everywhere by Lemma 2.10, and hence $\mu$ is the unique measure of maximal entropy, which is only equivalent to conformal measure $m_\alpha$ for maps conformally conjugate to $z \mapsto z^{\pm d}$ when $R$ is hyperbolic by [42]. $\square$

**Example 4.9.** Consider quadratic polynomials of the form $f_c(z) = z^2 + c$. If $c$ belongs to the interior of a hyperbolic component of the Mandelbrot set $\mathcal{M}$ (which is the same as the interior of $\mathcal{M}$ provided the conjecture that $\mathcal{M}$ is locally connected is true), then $f_c$ has a connected hyperbolic Julia set $\mathcal{J}_c$. Hence Theorem 4.8 implies that $(\mathcal{J}_c, \mu_c; f_c)$ is not one-sided Bernoulli.

We now assume that $c$ does not belong to the Mandelbrot set $\mathcal{M}$, so $f_c$ has a hyperbolic Cantor Julia set $\mathcal{J}_c \subset \mathbb{C}$, supporting an $\alpha$-conformal measure $m_\alpha$ for $\alpha = \dim_H(\mathcal{J}_c)$. Clearly, there is a Rohlin partition of $\mathcal{J}_c$ into two atoms such that $f_c$ maps each atom onto $\mathcal{J}_c$. Moreover, $f_c^N$ is uniformly expanding on $\mathcal{J}_c$ for some $N \geq 1$, hence by the Folklore Theorem 3.1, $\mathcal{J}_c$ supports an invariant measure $\mu_c \sim m_\alpha$, and its Radon–Nikodym derivative $g = \frac{d\mu_c}{dm_\alpha}$ is continuous, bounded and bounded away from 0.

**Proposition 4.10.** *Let $c \in \mathbb{C} \setminus \mathcal{M}$. If in addition*

(i) $c \notin (\frac{1}{4}, \infty)$,

(ii) $\operatorname{Re} c \neq -\frac{1}{2}$,

(iii) $2|1+c| \neq \left|1 - 2c \pm \sqrt{1-4c}\right|$,

*then $(\mathcal{J}_c, \mu_c; f_c)$ is not one-sided Bernoulli.*

**Proof.** The map $f_c$ has two fixed points

$$x_\pm = \frac{1 \pm \sqrt{1-4c}}{2} \quad \text{with} \quad f_c'(x_\pm) = 1 \pm \sqrt{1-4c}.$$

and one periodic orbit of period 2 consisting of the points

$$y_\pm = \frac{-1 \pm \sqrt{1-4(1+c)}}{2}.$$

Assume by contradiction that $f_c$ is one-sided $\{p, 1-p\}$ Bernoulli. Apply Corollary 3.14 to the fixed points to get

$$|f_c'(x_\pm)|^\alpha = |1 \pm \sqrt{1-4c}|^\alpha \in \left\{ \frac{1}{p}, \frac{1}{1-p} \right\}.$$

Condition (i) implies that $|f_c'(x_+)| \neq |f_c'(x_-)|$. So we can assume that $|f_c'(x_+)|^\alpha = \frac{1}{p}$ and $|f_c'(x_-)|^\alpha = \frac{1}{1-p}$. Multiplying these two and taking logarithms gives

$$\alpha \log |f_c'(x_+) \cdot f_c'(x_-)| = -\log p - \log(1-p). \tag{4.1}$$

Corollary 3.14 applied to the period 2 orbit gives

$$|(f_c^2)'(y_\pm)|^\alpha = |4(1+c)|^\alpha \in \left\{ \frac{1}{p(1-p)}, \frac{1}{p^2}, \frac{1}{(1-p)^2} \right\}. \tag{4.2}$$

If the left-hand side is $\frac{1}{p(1-p)}$, we can combine it with (4.1) to derive

$$\alpha \log |f_c'(y_+) \cdot f_c'(y_-)| = \alpha \log |f_c'(x_+) \cdot f_c'(x_-)|,$$

whence $\log|4c| = \log|4(1+c)|$. But this is only true when $\operatorname{Re} c = -\frac{1}{2}$, violating condition (ii). If the left-hand side in (4.2) is $\frac{1}{p^2}$, then we get

$$|4(1+c)|^\alpha = |(f_c^2)'(y_\pm)|^\alpha = |(f_c')(x_+)|^{2\alpha} = |2 - 4c + 2\sqrt{1-4c}|,$$

so $2|1+c| = |1 - 2c + \sqrt{1-4c}|$, contrary to condition (iii). Finally, the left-hand side in (4.2) equal to $\frac{1}{(1-p)^2}$ leads to $2|1+c| = |1-2c-\sqrt{1-4c}|$ which is again excluded by condition (iii). $\square$

We continue with some examples of rational maps with commuting automorphisms to which Theorem 2.21 applies.

**Example 4.11.** We define a family of quadratic rational maps of the Riemann sphere by:

$$R_\lambda(z) = \lambda\big(z + \tfrac{1}{z}\big),$$

for any nonzero $\lambda \in \mathbb{C}$. It is easy to see that the critical points are $c = \pm 1$, and the forward orbits of the two critical values are negatives of each other so there is basically one critical orbit to follow. It is well-known from the general theory of complex dynamics that $h_{top}(R_\lambda) = \log 2$ and that there exists a unique measure of maximal entropy for each value of $\lambda$. In [20] it is shown that if $\lambda = \pm\frac{i}{2}$, then $\mathcal{J}(R_\lambda) = \mathbb{C}_\infty$ and $R_\lambda$ is isomorphic to the $\{\frac{1}{2}, \frac{1}{2}\}$ Bernoulli shift. When $\lambda = \frac{1}{2}$, we have the degree two Chebyshev polynomial which is also one-sided Bernoulli. In all other cases, it was shown in [42] that the measure of maximal entropy is singular with respect to conformal measure on $\mathbb{C}_\infty$.

The automorphism $\varphi(z) = -z$ commutes with $R_\lambda$ for each $\lambda$. Endowed with the standard Riemannian volume form $m_2$ (locally two-dimensional Lebesgue measure or standard surface area on the sphere), it is clear that $\varphi$ preserves $m_2$.

There are many examples of critically finite maps in this family which then preserve a probability measure equivalent to $\nu \sim m_2$ and $\mathcal{J}(R_\lambda) = \mathbb{C}_\infty$. However, since $h_\nu < \log 2$, such maps are not one-sided Bernoulli. For example, setting $\lambda = \pm\frac{1}{2}\sqrt{-1 \pm 2i}$ gives four values with the property that

$$\pm 1 \mapsto \pm 2\lambda \mapsto \pm i \mapsto 0 \mapsto \infty,$$

and $\infty$ is a repelling fixed point with multiplier $\frac{1}{\lambda}$ of modulus $\frac{2}{5^{1/4}} > 1$.

Since $R_\lambda(i) = 0$ for every value of $\lambda$, any critical orbit landing on $i$ leads to a postcritically finite map, which is $m_2$ ergodic, preserves a measure equivalent to $m_2$ but is not one-sided Bernoulli (apart from $\lambda = \pm\frac{i}{2}$ discussed above).

Applying a result of Rees [33], it can be shown that there is a set of parameters of positive measure (no longer critically finite) with these properties, obtained by pushing off from the postcritically finite parameters. It was shown by Milnor ([27, Theorem 5.1]) that this family of maps is the only quadratic family of maps with nontrivial commuting automorphisms up to conformal conjugacy so this is the only family of quadratic maps to which Theorem 2.21 applies.

If $\lambda = -\frac{1}{2}$ then the automorphism group $G$ of $R_\lambda$ is nonabelian of order 6. To see this, one can show that $R_{-1/2}$ is conformally conjugate to the map $z \mapsto \frac{1}{z^2}$, and this map commutes with the automorphisms generated by $z \mapsto -z$, $z \mapsto \overline{z}$, and $z \mapsto e^{2\pi i/3}z$. The Julia set of $R_{-1/2}$ is the imaginary axis and the two critical points, $\pm 1$ form a superattracting period 2 orbit. From this we deduce it is not a parabolic example, so it is non-Bernoulli with respect to the invariant probability measure equivalent to one-dimensional Lebesgue measure $m$ on $\mathcal{J}_{R_{-1/2}}$.

We summarize this family of examples with the following result.

**Proposition 4.12.** *For quadratic rational maps of the form*

$$R_\lambda(z) = \lambda\big(z + \tfrac{1}{z}\big), \quad \lambda \in \mathbb{C} \setminus \{0\},$$

*we have:*

(1) *For all $|\lambda| > 1$, and whenever $|\lambda| < 1$ corresponds to a hyperbolic or subhyperbolic map with respect to conformal measure $m_\alpha$, $R_\lambda$ is not one-sided Bernoulli.*
(2) *There exists a set $\mathcal{E} \subset \{\lambda : |\lambda| < 1\}$, $m_2(\mathcal{E}) > 0$, yielding (nonhyperbolic) maps with $\mathcal{J}(R_\lambda) = \mathbb{C}_\infty$, such that with respect to $m_2$, $R_\lambda$ is not one-sided Bernoulli.*

**Proof.** When $|\lambda| > 1$, the map is well-known to be hyperbolic with a Cantor Julia set (see e.g., [27]). For each map $R_\lambda$, $\lambda \neq 0$, we have a nontrivial group of automorphisms commuting with $R_\lambda$ and leaving the corresponding Jacobian for the conformal measure invariant. Applying Theorem 2.21 we have the result for (1). In the second case, using the parameters coming from the results of Rees [33], we have the result for $m_2$. □

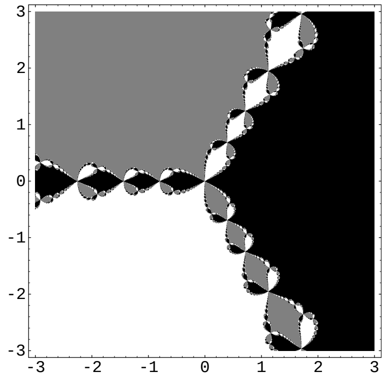


FIGURE 6. The Julia set separating basins of super-attracting fixed points for the rational function of Newton's root-finding algorithm for $z^3 - 1$.

**Example 4.13.** Let $N_d : \mathbb{C} \to \mathbb{C}$ be the rational map associated the Newton algorithm for finding the roots of the equation $z^d - 1 = 0$. The root basins and $\mathcal{J}(N_d) \equiv \mathcal{J}$ are shown in Figure 6, and $N_d$ is given by:

$$N_d(z) = z - \frac{z^d - 1}{dz^{d-1}} = \frac{(d-1)z^d + 1}{dz^{d-1}}.$$

The map $N_d$ has $d$ super-attracting fixed points at the $d$-th roots of unity, as well as a critical point of order $d-1$ at the origin. This point maps to $\infty$ which is a repelling fixed point with derivative $\frac{d}{d-1}$ in the spherical metric.

The point 0 is the only critical point in $\mathcal{J}$, and $N_d(0) = \infty = N_d(\infty)$. Therefore $N_d$ is subhyperbolic and preserves a measure $\mu \sim m_\alpha$, but does not satisfy the hypotheses of Theorem 4.8. However one can show that $g = \frac{d\mu}{dm_\alpha}$ is continuous and bounded away from 0 on $\mathcal{J}\backslash\{\infty\}$. Since $\mathcal{J}\backslash\{0,\infty\}$ consists of $d$ connected components, say $A_1, \ldots, A_d$, it follows as in Theorem 4.8 that $J_{\mu N_d}$ is constant on each $A_i$. Moreover one can show that the sets $A_1, \ldots, A_d$ form a Rohlin partition for $N_d$ ($\mu$ mod 0).

The dihedral group $\mathcal{G}$ generated by $z \mapsto e^{2\pi i/d}z$ and $z \mapsto \bar{z}$ is the group of symmetries of $\mathcal{J}$, and $\mathcal{G}$ transitively permutes the sets $A_1, \ldots, A_d$. More precisely, $N_d \circ \varphi = \varphi \circ N_d$ for each $\varphi \in \mathcal{G}$ and clearly $\varphi \in \mathcal{G}$ is also nonsingular respect to the invariant measure $\mu$.

The proof of Theorem 2.21 shows that $J_{\mu N_d}(z) = J_{\mu N_d}(\varphi z)$ for all $\varphi \in \mathcal{G}$, so $J_{\mu N_d}$ is constant $\mu$-a.e. On the other hand, it follows from [42] and [7] that $\mu$ is not the measure of maximal entropy. Therefore Corollary 2.23 (2) implies that $(J, \mathcal{B}, \mu; N_d)$ is not one-sided Bernoulli.

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