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The L_∞-deformation complex of diagrams of algebras

Yael Frégier, Martin Markl and Donald Yau

Abstract. The deformation complex of an algebra over a colored PROP P is defined in terms of a minimal (or, more generally, cofibrant) model of P. It is shown that it carries the structure of an L_∞-algebra which induces a graded Lie bracket on cohomology.

As an example, the L_∞-algebra structure on the deformation complex of an associative algebra morphism g is constructed. Another example is the deformation complex of a Lie algebra morphism. The last example is the diagram describing two mutually inverse morphisms of vector spaces. Its L_∞-deformation complex has nontrivial l_0-term.

Explicit formulas for the L_∞-operations in the above examples are given. A typical deformation complex of a diagram of algebras is a fully-fledged L_∞-algebra with nontrivial higher operations.

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1. Introduction

In this paper, we construct the deformation complex $(C^*_{\mathsf{P}}(T;T), \delta_{\mathsf{P}})$ of an algebra T over a colored PROP P and observe that it has the structure of an L_∞-algebra. The cochain complex $(C^*_{\mathsf{P}}(T;T), \delta_{\mathsf{P}})$ is so named because its L_∞-structure governs the deformations of T in the form of the *Quantum Master Equation* (13) (Section 4.3).

The existence of an L_∞-structure on the deformation complex of an algebra over an operad was proved in 2002 by van der Laan [51]. Van der Laan's construction was later generalized, in [39], to algebras over properads. The present paper will, however, be based on the approach of the 2004 preprint [36].

Considering colored PROPs is necessary if one is to study L_∞-deformations of, say, morphisms or more general diagrams of algebras over a PROP, module-algebras, modules over an associative algebra, and Yetter–Drinfel'd and Hopf modules over a bialgebra. For example, there is a 2-colored PROP $\mathbf{As}_{\mathtt{B}\to\mathtt{W}}$ whose algebras are of the form $f\colon U \to V$, in which U and V are associative algebras and f is a morphism of associative algebras (Example 2.10). Likewise, there is a 2-colored PROP **ModAlg** whose algebras are of the form (H, A), in which H is a bialgebra and A is an H-module-algebra (Example 2.12). Other examples of colored PROP algebras are given at the end of Section 9.

Let us sketch the construction of the deformation complex $(C^*_{\mathsf{P}}(T;T), \delta_{\mathsf{P}})$, with details given in Section 3. First we take a *minimal model* (Definition 3.4) $\alpha\colon (\mathsf{F}(E), \partial) \to \mathsf{P}$ of the colored PROP P, which should be thought of as a resolution of P. Given a P-algebra $\rho\colon \mathsf{P} \to \mathrm{End}_T$, we define

$$C^*_{\mathsf{P}}(T;T) = \mathrm{Der}(\mathsf{F}(E), \mathcal{E}),$$

in which $\mathcal{E} = \mathrm{End}_T$ is considered an $\mathsf{F}(E)$-module via the morphism $\beta = \rho\alpha$, and $\mathrm{Der}(\mathsf{F}(E), \mathcal{E})$ denotes the vector space of derivations $\mathsf{F}(E) \to \mathcal{E}$. The latter has a natural differential δ that sends $\theta \in \mathrm{Der}(\mathsf{F}(E), \mathcal{E})$ to $\theta\partial$. The reason why we require the minimality of resolutions whenever possible is explained in the first paragraph of Section 3. As we will see in Section 9, the construction sketched out above applies to more general cofibrant resolutions as well.

The L_∞-operations on $C^*_{\mathsf{P}}(T;T)$ are constructed using graph substitutions (Section 4.4). The usefulness of this very explicit construction of the L_∞-operations on $C^*_{\mathsf{P}}(T;T)$ is first illustrated with the example of associative algebra morphisms. For a morphism $g\colon U \to V$ of associative algebras, considered as an algebra over the 2-colored PROP $\mathbf{As}_{\mathtt{B}\to\mathtt{W}}$, we are able to write down explicitly all the L_∞-operations l_k on the deformation complex of g (Theorem 5.5 for $k = 1$, Theorem 6.2 for $k = 2$, and Theorem 6.4 for $k \geq 3$). As expected, the underlying cochain complex of the deformation complex of g is isomorphic to the Gerstenhaber–Schack[1] cochain complex [14, 15, 16] of g (Theorem 5.5). Therefore, the latter also has an explicit L_∞-structure. See Section 5.1 for more discussion about this deformation complex.

A second example is given by the study of the case of Lie algebra morphisms. As in the associative case, there exists a 2-colored PROP $\mathbf{Lie}_{\mathtt{B}\to\mathtt{W}}$ whose 2-colored algebras are morphism of Lie algebras. We obtain then an explicit expression for the L_∞-operations (Theorem 7.5 for $k = 1$, Theorem 8.2 for $k = 2$, and Theorem 8.4 for $k \geq 3$). In particular the first operation l_1 gives a complex isomorphic to the S-cohomology complex [9]. Hence this answers the question left open in [9] of the existence of such an L_∞-structure.

Another example of the L_∞-deformation complex is given in Section 9. There is a 2-colored operad **Iso** (Example 9.1) whose algebras are of the form $F\colon U \leftrightarrows V\colon G$, in which U and V are chain complexes and F and G are mutually inverse chain maps. Using a modification of the results and constructions of earlier sections, we will write down explicitly the L_∞-operations on the deformation complex of a typical **Iso**-algebra T (Example 9.3).

Describing the L_∞-deformation complex requires the knowledge of a resolution of the corresponding colored PROP. So, while it will be interesting to work out the L_∞-deformation complex for morphisms of bialgebras, a description of a resolution of the PROP $\mathbf{Bialg}_{\mathtt{B}\to\mathtt{W}}$ for morphisms of bialgebras is missing, though the minimal model of the bialgebra PROP **Bialg** is known [47].

Acknowledgment. We would like to thank Jim Stasheff and Bruno Vallette for reading the first version of the manuscript and many useful remarks.

2. Preliminaries on colored PROPs

Fix a ground field $\mathbf{k}$, assumed to be of characteristic 0. This assumption is useful in considering models for operads or PROPs since it guarantees the existence of the 'averagization' of a nonequivariant map into an equivariant one. The characteristic zero assumption also simplifies concepts of Lie algebras and their generalizations.

In this section, we review some basic definitions about colored PROPs (and colored operads as their particular instances), their algebras, colored Σ-bimodules, and free colored PROPs. Examples of algebras over colored PROPs can be found at the end of this section.

2.1. Colored $\mathbf{\Sigma}$-bimodule. Let $\mathfrak{C}$ be a nonempty set whose elements are called *colors*. A Σ-*bimodule* is a collection $E = \{E(m,n)\}_{m,n\geq 0}$ of $\mathbf{k}$-modules in which each $E(m,n)$ is equipped with a left Σ_m and a right Σ_n actions that commute with each other.

[1]Be careful with the possible confusion with the complex associated to bialgebras. Both of these complexes share the same name.

A $\mathfrak{C}$-*colored* Σ-*bimodule* is a Σ-bimodule E in which each $E(m,n)$ admits a $\mathfrak{C}$-colored decomposition into submodules,

$$E(m,n) = \bigoplus_{c_i,d_j \in \mathfrak{C}} E\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n}, \tag{1}$$

that is compatible with the Σ_m-Σ_n-actions. Elements of $E(m,n)$ are said to have *biarity* (m,n). A *morphism* of $\mathfrak{C}$-colored Σ-bimodules is a linear biequivariant map that respects the $\mathfrak{C}$-colored decompositions (1).

$\mathfrak{C}$-colored Σ-bimodules and their morphism are examples of $\mathfrak{C}$-colored objects; more examples will follow. If $\mathfrak{C}$ has k elements, we will sometimes call $\mathfrak{C}$-colored objects simply k-*colored* objects.

Definition 2.2. A $\mathfrak{C}$-*colored PROP* ([27, 28], [35, Section 8]) is a $\mathfrak{C}$-colored Σ-bimodule $\mathsf{P} = \{\mathsf{P}(m,n)\}$ (so each $\mathsf{P}(m,n)$ admits a $\mathfrak{C}$-colored decomposition (1)) that comes equipped with two operations: a *horizontal composition*

$$\begin{aligned}\otimes\colon \mathsf{P}\binom{d_{11},\ldots,d_{1m_1}}{c_{11},\ldots,c_{1n_1}} \otimes \cdots \otimes \mathsf{P}\binom{d_{s1},\ldots,d_{sm_s}}{c_{s1},\ldots,c_{sn_s}} &\to \\ \mathsf{P}\binom{d_{11},\ldots,d_{sm_s}}{c_{11},\ldots,c_{sn_s}} &\subseteq \mathsf{P}(m_1+\cdots+m_s, n_1+\cdots+n_s)\end{aligned}$$

and a *vertical composition*

$$\begin{array}{rcl}\circ\colon \mathsf{P}\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n} \otimes \mathsf{P}\binom{b_1,\ldots,b_n}{a_1,\ldots,a_k} & \to & \mathsf{P}\binom{d_1,\ldots,d_m}{a_1,\ldots,a_k} \subseteq \mathsf{P}(m,k), \\ (x,y) & \mapsto & x\circ y.\end{array} \tag{2}$$

These two compositions are required to satisfy some associativity-type axioms. There is also a unit element $1_c \in \mathsf{P}\binom{c}{c}$ for each color c. Moreover, the vertical composition $x \circ y$ in (2) is 0, unless

$$c_i = b_i \quad \text{for} \quad 1 \le i \le n.$$

Morphisms of $\mathfrak{C}$-colored PROPs are unit-preserving morphisms of the underlying Σ-bimodules that commute with both the horizontal and the vertical compositions.

Colored operads are particular cases of colored PROPs having the property that $\mathsf{P}\binom{d_1,\ldots,d_m}{a_1,\ldots,a_k} = 0$ for $m \ge 2$. Note that colored PROPs can also be defined as ordinary (1-colored) PROPs over the semisimple algebra $K = \oplus_{c\in\mathfrak{C}}\mathbf{k}_c$, where each $\mathbf{k}_c$ is a copy of the ground field $\mathbf{k}$ [32, Section 2].

Example 2.3. The $\mathfrak{C}$-*colored endomorphism PROP* $\mathrm{End}_T^{\mathfrak{C}}$ of a $\mathfrak{C}$-graded module $T = \oplus_{c\in\mathfrak{C}} T_c$ is the $\mathfrak{C}$-colored PROP with

$$\mathrm{End}_T^{\mathfrak{C}}\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n} = \mathrm{Hom}_{\mathbf{k}}(T_{c_1}\otimes\cdots\otimes T_{c_n}, T_{d_1}\otimes\cdots\otimes T_{d_m}).$$

The horizontal composition is given by tensor products of $\mathbf{k}$-linear maps. The vertical composition is given by composition of $\mathbf{k}$-linear maps with matching colors.

Definition 2.4. For a $\mathfrak{C}$-colored PROP P, a P-*algebra* is a morphism of $\mathfrak{C}$-colored PROPs

$$\alpha\colon \mathsf{P} \to \mathrm{End}_T^{\mathfrak{C}}$$

for some $\mathfrak{C}$-graded module $T = \oplus_{c\in\mathfrak{C}} T_c$. In this case, we say that T is a P-algebra.

2.5. Pasting scheme for $\mathfrak{C}$-colored PROPs. For natural $m, n \geq 1$, let $\mathsf{UGr}^{\mathfrak{C}}(m,n)$ be the set whose elements are pairs (G,ζ) such that:

(1) $G \in \mathsf{UGr}(m,n)$ is a directed (m,n)-graph [35, p. 38].
(2) For each vertex $v \in \mathrm{Vert}(G)$, the sets $\mathrm{out}(v)$ (outgoing edges from v) and $\mathrm{in}(v)$ (incoming edges to v) are labeled $1,\ldots,q$ and $1,\ldots,p$, respectively, where $\#\mathrm{out}(v) = q$ and $\#\mathrm{in}(v) = p$.
(3) $\zeta\colon \mathrm{edge}(G) \to \mathfrak{C}$ is a function that assigns to each edge in G a color in $\mathfrak{C}$. For any edge $l \in \mathrm{edge}(G)$, $\zeta(l) \in \mathfrak{C}$ is called the *color of* l.

There is a $\mathfrak{C}$-colored decomposition

$$\mathsf{UGr}^{\mathfrak{C}}(m,n) = \coprod_{c_i,d_j\in\mathfrak{C}} \mathsf{UGr}^{\mathfrak{C}}\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n},$$

where $(G,\zeta) \in \mathsf{UGr}^{\mathfrak{C}}\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n}$ if and only if the input legs $\{l_{\mathrm{in}}^1,\ldots l_{\mathrm{in}}^n\}$ of G have colors $c_1,\ldots,c_n$ and the output legs $\{l_{\mathrm{out}}^1,\ldots,l_{\mathrm{out}}^m\}$ of G have colors $d_1,\ldots,d_m$.

As in [35], $\mathsf{UGr}^{\mathfrak{C}}(m,n)$ and $\mathsf{UGr}^{\mathfrak{C}}\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n}$ are the categories with color-respecting isomorphisms as morphisms. Elements in $\mathsf{UGr}^{\mathfrak{C}}(m,n)$ are called *$\mathfrak{C}$-colored directed (m,n)-graphs.*

2.6. Decoration on colored directed graphs. Let E be a $\mathfrak{C}$-colored Σ-bimodule and (G,ζ) be a $\mathfrak{C}$-colored directed (m,n)-graph. Define

$$E(G,\zeta) = \bigotimes_{v\in\mathrm{Vert}(G)} E\binom{\zeta(o_v^1),\ldots,\zeta(o_v^q)}{\zeta(i_1^v),\ldots,\zeta(i_p^v)}, \tag{3}$$

where $\mathrm{in}(v) = \{i_1^v,\ldots,i_p^v\}$ and $\mathrm{out}(v) = \{o_v^1,\ldots,o_v^q\}$. Its elements are called *E-decorated $\mathfrak{C}$-colored directed (m,n)-graphs.*

For an element $\Gamma = \otimes_v e_v \in E(G,\zeta)$, the element $e_v \in E\binom{\zeta(o_v^1),\ldots,\zeta(o_v^q)}{\zeta(i_1^v),\ldots,\zeta(i_p^v)}$ corresponding to the vertex $v \in \mathrm{Vert}(G)$ is called the *decoration of* v.

In other words, $E(G,\zeta)$ is the space of decorations of the vertices of the $\mathfrak{C}$-colored directed (m,n)-graph (G,ζ) with elements of E with matching biarity and colors.

2.7. Free colored PROP. Let E be a $\mathfrak{C}$-colored Σ-bimodule. For $c_1, \ldots, c_n$, $d_1, \ldots, d_m \in \mathfrak{C}$, define the module

$$\mathsf{F}^{\mathfrak{C}}(E)\binom{d_1, \ldots, d_m}{c_1, \ldots, c_n} = \operatorname{colim} E(G, \zeta),$$

where the colimit is taken over the category $\mathsf{UGr}^{\mathfrak{C}}\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n}$. Then

$$\mathsf{F}^{\mathfrak{C}}(E) = \left\{ \mathsf{F}^{\mathfrak{C}}(E)(m,n) = \bigoplus_{c_i, d_j \in \mathfrak{C}} \mathsf{F}^{\mathfrak{C}}(E)\binom{d_1, \ldots, d_m}{c_1, \ldots, c_n} \right\}$$

is a $\mathfrak{C}$-colored PROP, in which the horizontal composition $\otimes$ is given by disjoint union of E-decorated $\mathfrak{C}$-colored directed (m,n)-graphs. The vertical composition $\circ$ in $\mathsf{F}^{\mathfrak{C}}(E)$ is given by grafting of $\mathfrak{C}$-colored legs with matching colors.

Note that there is a natural $\mathbb{Z}_{\geq 0}$-grading,

$$\mathsf{F}^{\mathfrak{C}}(E) = \bigoplus_{k \geq 0} \mathsf{F}^{\mathfrak{C}}_k(E),$$

where $\mathsf{F}^{\mathfrak{C}}_k(E)$ is the submodule generated by the monomials involving k elements in E.

Proposition 2.8 (= $\mathfrak{C}$-colored version of Proposition 57 in [35])**.** $\mathsf{F}^{\mathfrak{C}}(E) = \{\mathsf{F}^{\mathfrak{C}}(E)(m,n)\}$ *is the free* $\mathfrak{C}$*-colored PROP generated by the* $\mathfrak{C}$*-colored* Σ*-bimodule* E*. In other words, the functor* $\mathsf{F}^{\mathfrak{C}}$ *is the left adjoint of the forgetful functor from* $\mathfrak{C}$*-colored PROPs to* $\mathfrak{C}$*-colored* Σ*-bimodules.*

In particular, elements in the free $\mathfrak{C}$-colored PROP $\mathsf{F}^{\mathfrak{C}}(E)$ can be written as sums of E-decorated $\mathfrak{C}$-colored directed graphs.

Convention 2.9. From now on, everything will be tacitly assumed to be $\mathfrak{C}$-colored with a suitable set of colors $\mathfrak{C}$. When there is no danger of ambiguity, we will, for brevity, suppress $\mathfrak{C}$ from the notation.

Example 2.10 (Morphisms)**.** Let P be an ordinary PROP (i.e., a 1-colored PROP). Then there is a 2-colored PROP $\mathsf{P}_{\mathtt{B}\to\mathtt{W}}$ whose algebras are of the form $f\colon U \to V$, in which U and V are P-algebras and f is a morphism of P-algebras [31, Example 1]. It can be constructed as the quotient

$$\mathsf{P}_{\mathtt{B}\to\mathtt{W}} = \frac{\mathsf{P}_{\mathtt{B}} * \mathsf{P}_{\mathtt{W}} * \mathsf{F}(f)}{(f^{\otimes m} x_{\mathtt{B}} = x_{\mathtt{W}} f^{\otimes n} \text{ for all } x \in \mathsf{P}(m,n))},$$

where $\mathsf{P}_{\mathtt{B}}$ and $\mathsf{P}_{\mathtt{W}}$ are copies of P concentrated in the colors $\mathtt{B}$ and $\mathtt{W}$, respectively, $x_{\mathtt{B}}$ and $x_{\mathtt{W}}$ are the respective copies of x in $\mathsf{P}_{\mathtt{B}}$ and $\mathsf{P}_{\mathtt{W}}$, and $\mathsf{F}(f)$ is the free 2-colored PROP on the generator $f\colon \mathtt{B} \to \mathtt{W}$. The star $*$ denotes the free product (= the coproduct) of 2-colored PROPs.

In the case that P is the operad $\mathbf{As}$ for associative algebras, cohomology of $\mathbf{As}_{\mathtt{B}\to\mathtt{W}}$-algebras (i.e., associative algebra morphisms) will be discussed in details in Sections 5 and 6.

Example 2.11 (Modules)**.** There is a 2-colored operad **AsMod** whose algebras are of the form (A, M), where A is an associative algebra and M is a left A-module. It can be constructed as the quotient

$$\mathbf{AsMod} = \frac{\mathsf{F}(\mu,\lambda)}{(\mu(\mu\otimes 1_{\mathtt{A}}) - \mu(1_{\mathtt{A}}\otimes\mu), \lambda(\mu\otimes 1_{\mathtt{M}}) - \lambda(1_{\mathtt{A}}\otimes\lambda))}.$$

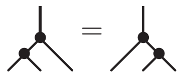
Here $\mathsf{F}(\mu,\lambda)$ is the free 2-colored operad (with $\mathfrak{C} = \{\mathtt{A},\mathtt{M}\}$) on the generators,

$$\mu \in \mathsf{F}(\mu,\lambda)\binom{\mathtt{A}}{\mathtt{A},\mathtt{A}} \quad \text{and} \quad \lambda \in \mathsf{F}(\mu,\lambda)\binom{\mathtt{M}}{\mathtt{A},\mathtt{M}},$$

which encode the multiplication in A and the left A-action on M, respectively.

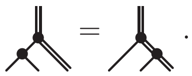
If we depict the multiplication μ as  and the module action λ as , then the associativity of μ is expressed by the diagram

(4)



and the compatibility between the multiplication and the module action by

(5)



The diagrams in the above two displays should be interpreted as elements of the free colored PROP $\mathsf{F}(\mu,\lambda)$, with the $\mathtt{A}$-colored edges of the underlying graph represented by simple lines |, and the $\mathtt{M}$-colored edges by the double lines ‖. We use the convention that the directed edges point upwards, i.e. the composition is performed from the bottom up.

Example 2.12 (Module-algebras)**.** Let $H = (H, \mu_H, \Delta_H)$ be a (co)associative bialgebra. An *H-module-algebra* is an associative algebra (A, μ_A) that is equipped with a left H-module structure such that the multiplication map on A becomes an H-module morphism. In other words, the *module-algebra axiom*

$$x(ab) = \sum_{(x)} (x_{(1)}a)(x_{(2)}b)$$

holds for $x \in H$ and $a, b \in A$, where $\Delta_H(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ using the Sweedler's notation for comultiplication.

This algebraic structure arises often in algebraic topology [2], quantum groups [21], Lie and Hopf algebras theory [7, 40, 49], and group representations [1]. For example, in algebraic topology, the complex cobordism ring $\mathrm{MU}^*(X)$ of a topological space X is an S-module-algebra, where S is the Landweber–Novikov algebra [26, 43] of stable cobordism operations.

Another important example of a module-algebra arises in the theory of Lie algebras. Finite-dimensional simple $sl(2,\mathbb{C})$-modules are, up to isomorphism, the highest weight modules $V(n)$ $(n \geq 0)$ [20, Theorem 7.2]. There is a $U(sl(2,\mathbb{C}))$-module-algebra structure on the polynomial algebra $\mathbb{C}[x, y]$

such that the submodule $\mathbb{C}[x,y]_n$ of homogeneous polynomials of degree n is isomorphic to the highest weight module $V(n)$ [21, Theorem V.6.4]. In other words, all the finite-dimensional simple $sl(2,\mathbb{C})$-modules can be encoded inside a single $U(sl(2,\mathbb{C}))$-module-algebra.

There is a 2-colored PROP **ModAlg** whose algebras are of the form (H, A), where H is a (co)associative bialgebra and A is an H-module-algebra. (The third author first learned about this fact from Bruno Vallette in private correspondence.) It can be constructed as the quotient (with $\mathfrak{C} = \{\mathtt{H}, \mathtt{A}\}$)

$$\mathbf{ModAlg} = \mathsf{F}(\mu_{\mathtt{H}}, \Delta_{\mathtt{H}}, \mu_{\mathtt{A}}, \lambda)/I, \tag{6}$$

where $\mathsf{F} = \mathsf{F}(\mu_{\mathtt{H}}, \Delta_{\mathtt{H}}, \mu_{\mathtt{A}}, \lambda)$ is the free 2-colored PROP on the generators:

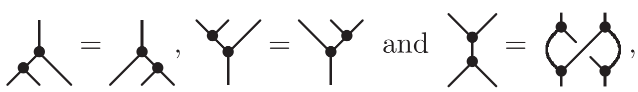
$$\mu_{\mathtt{H}} \in \mathsf{F}\binom{\mathtt{H}}{\mathtt{H},\mathtt{H}},\ \Delta_{\mathtt{H}} \in \mathsf{F}\binom{\mathtt{H},\mathtt{H}}{\mathtt{H}},\ \mu_{\mathtt{A}} \in \mathsf{F}\binom{\mathtt{A}}{\mathtt{A},\mathtt{A}},\ \text{and } \lambda \in \mathsf{F}\binom{\mathtt{A}}{\mathtt{H},\mathtt{A}},$$

which encode the multiplication and comultiplication in H, the multiplication in A, and the H-module structure on A, respectively. The ideal I is generated by the elements:

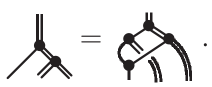
$$\begin{aligned}
&\mu_{\mathtt{H}}(\mu_{\mathtt{H}} \otimes 1_{\mathtt{H}}) - \mu_{\mathtt{H}}(1_{\mathtt{H}} \otimes \mu_{\mathtt{H}}) \quad (\text{associativity of } \mu_{\mathtt{H}}),\\
&(\Delta_{\mathtt{H}} \otimes 1_{\mathtt{H}})\Delta_{\mathtt{H}} - (1_{\mathtt{H}} \otimes \Delta_{\mathtt{H}})\Delta_{\mathtt{H}} \quad (\text{coassociativity of } \Delta_{\mathtt{H}}),\\
&\Delta_{\mathtt{H}}\mu_{\mathtt{H}} - \mu_{\mathtt{H}}^{\otimes 2}(2\ 3)\Delta_{\mathtt{H}}^{\otimes 2} \quad (\text{compatibility of } \mu_{\mathtt{H}} \text{ and } \Delta_{\mathtt{H}}),\\
&\mu_{\mathtt{A}}(\mu_{\mathtt{A}} \otimes 1_{\mathtt{A}}) - \mu_{\mathtt{A}}(1_{\mathtt{A}} \otimes \mu_{\mathtt{A}}) \quad (\text{associativity of } \mu_{\mathtt{A}}),\\
&\lambda(\mu_{\mathtt{H}} \otimes 1_{\mathtt{A}}) - \lambda(1_{\mathtt{H}} \otimes \lambda) \quad (H\text{-module axiom}),\\
&\lambda(1_{\mathtt{H}} \otimes \mu_{\mathtt{A}}) - \mu_{\mathtt{A}}\lambda^{\otimes 2}(2\ 3)(\Delta_{\mathtt{H}} \otimes 1_{\mathtt{A}}^{\otimes 2}) \quad (\text{module-algebra axiom}).
\end{aligned}$$

Here $(2\ 3) \in \Sigma_4$ is the permutation that switches 2 and 3.

If we draw the multiplication μ_H as , the comultiplication Δ_H as , the multiplication μ_A as , and the H-module action λ as , then the bialgebra axioms for H are expressed by



The associativity of μ_H is given by the obvious $\|$-colored version of (4), the H-module axiom by (5), and the module-algebra axiom by

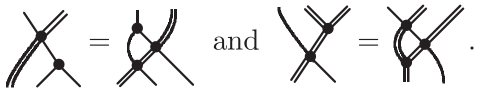


Variants of module-algebras, including module-co/bialgebras and comodule-(co/bi)algebras are algebras over similar 2-colored PROPs. Deformations, in the classical sense [13], of module-algebras and its variants were studied in [53, 54].

Example 2.13 (Entwining structures)**.** An *entwining structure* [4, 6] is a tuple (A, C, ψ), in which $A = (A, \mu)$ is an associative algebra, $C = (C, \Delta)$ is a coassociative coalgebra, and $\psi\colon C \otimes A \to A \otimes C$, such that the following two entwining axioms are satisfied:

$$
\begin{aligned}
\psi(\mathrm{Id}_C \otimes \mu) &= (\mu \otimes \mathrm{Id}_C)(\mathrm{Id}_A \otimes \psi)(\psi \otimes \mathrm{Id}_A), \\
(\mathrm{Id}_A \otimes \Delta)\psi &= (\psi \otimes \mathrm{Id}_C)(C \otimes \psi)(\Delta \otimes \mathrm{Id}_A).
\end{aligned} \tag{7}
$$

If we symbolize μ by , Δ by and ψ by , then the entwining axioms can be written as



This algebraic structure arises in the study of *coalgebra-Galois extension* and its dual notion, *algebra-Galois coextension* [5], generalizing the Hopf-Galois extension of [23].

There is a 2-colored PROP **Ent** whose algebras are entwining structures. It can be constructed as the quotient

$$\mathbf{Ent} = \mathsf{F}(\mu, \Delta, \psi)/I$$

of the free 2-colored PROP $\mathsf{F} = \mathsf{F}(\mu, \Delta, \psi)$ (with $\mathfrak{C} = \{\mathtt{A}, \mathtt{C}\}$) on the generators:

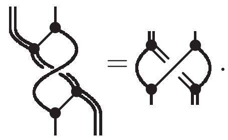
$$\mu \in \mathsf{F}\binom{\mathtt{A}}{\mathtt{A},\mathtt{A}}, \quad \Delta \in \mathsf{F}\binom{\mathtt{C},\mathtt{C}}{\mathtt{C}}, \quad \text{and} \quad \psi \in \mathsf{F}\binom{\mathtt{A},\mathtt{C}}{\mathtt{C},\mathtt{A}}.$$

The ideal I is generated by the elements expressing the associativity of μ, the coassociativity of Δ, and the two entwining axioms (7).

Example 2.14 (Yetter–Drinfel'd modules)**.** A *Yetter–Drinfel'd module* [56] (also called a *crossed bimodule* and *quantum Yang–Baxter module*) over a (co)associative bialgebra (H, μ, Δ) is a vector space M together with a left H-module action $\omega\colon H \otimes M \to M$ and a right H-comodule coaction $\rho\colon M \to M \otimes H$ that satisfy the Yetter–Drinfel'd condition,

$$
\begin{aligned}
&(\mathrm{Id}_M \otimes \mu) \circ (\rho \otimes \mathrm{Id}_H) \circ \tau \circ (\mathrm{Id}_H \otimes \omega) \circ (\Delta \otimes \mathrm{Id}_M) = \\
&\qquad\qquad (\omega \otimes \mu) \circ (\mathrm{Id}_H \otimes \tau \otimes \mathrm{Id}_H) \circ (\Delta \otimes \rho),
\end{aligned} \tag{8}
$$

where τ is the twist isomorphism $H \otimes M \cong M \otimes H$. If we depict μ as , Δ as , ω as , and ρ as , then



Yetter–Drinfel'd modules were introduced by Yetter [56], and are studied further in [25, 44, 45, 46, 48], among others. If the bialgebra H is a finite-dimensional Hopf algebra, then the left-modules over its Drinfel'd double $D(H)$ are exactly the Yetter–Drinfel'd modules over H. These objects play important roles in the theory of quantum groups and mathematical physics. Indeed, a finite-dimensional Yetter–Drinfel'd module M gives rise to a solution of the quantum Yang–Baxter equation [25, 45] (i.e., an R-matrix [21, Chapter VIII]). Conversely, through the so-called *FRT construction* [8, 21], every R-matrix on a finite-dimensional vector space gives rise to a Yetter–Drinfel'd module over some bialgebra. Cohomology for Yetter–Drinfel'd modules and their morphisms over a fixed bialgebra have been studied in [44] and [55], respectively.

There is a 2-colored PROP **YD** whose algebras are of the form (H, M), where H is a bialgebra and M is a Yetter–Drinfel'd module over H. It can be constructed as the quotient

$$\mathbf{YD} = \mathsf{F}(\mu, \Delta, \omega, \rho)/I$$

of the free 2-colored PROP $\mathsf{F} = \mathsf{F}(\mu, \Delta, \omega, \rho)$ (with $\mathfrak{C} = \{\mathtt{H}, \mathtt{M}\}$) on the generators:

$$\mu \in \mathsf{F}\binom{\mathtt{H}}{\mathtt{H},\mathtt{H}}, \ \Delta \in \mathsf{F}\binom{\mathtt{H},\mathtt{H}}{\mathtt{H}}, \ \omega \in \mathsf{F}\binom{\mathtt{M}}{\mathtt{H},\mathtt{M}}, \text{ and } \rho \in \mathsf{F}\binom{\mathtt{M},\mathtt{H}}{\mathtt{M}}.$$

The ideal I is generated by elements expressing the bialgebra axioms for μ and Δ, the left H-module axiom for ω, the right H-comodule axiom for ρ, and the Yetter–Drinfel'd condition (8).

Example 2.15 (Hopf modules)**.** A *Hopf module* over a bialgebra (H, μ, Δ) is a vector space M together with a left H-module action $\omega\colon H \otimes M \to M$ and a right H-comodule coaction $\rho\colon M \to M \otimes H$ that satisfy the Hopf module condition:

$$\rho \circ \omega \; = \; (\omega \otimes \mu) \circ (\mathrm{Id}_H \otimes \tau \otimes \mathrm{Id}_H) \circ (\Delta \otimes \rho). \tag{9}$$

There is a 2-colored PROP **HopfMod** whose algebras are of the form (H, M), in which H is a bialgebra and M is a Hopf module over H. It admits the same construction as **YD**, except that the Yetter–Drinfel'd condition (8) is replaced by the Hopf module condition (9) in the ideal I of relations.

3. Minimal models and cohomology

In this section, we define (i) minimal models for colored PROPs and (ii) cohomology for algebras over a colored PROP based on minimal models. Since minimal models are (at least in most cases) known to be unique up to isomorphism, the cohomology based on minimal models is unique already on the chain level. We will therefore require minimality whenever possible, though an arbitrary cofibrant resolution should give the same cohomology. There are, however, important PROPs that do not have a minimal model, as the colored operad **Iso** considered in Section 9.

First we recall the notions of modules and derivations for colored PROPs.

3.1. Modules. For a $\mathfrak{C}$-colored PROP P and a $\mathfrak{C}$-colored Σ-bimodule U, a P-*module* structure on U [30, p. 203] consists of the following operations:

$$\begin{aligned}
\circ = \circ_l \colon \mathsf{P}\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n} \otimes U\binom{b_1,\ldots,b_n}{a_1,\ldots,a_k} &\to U\binom{d_1,\ldots,d_m}{a_1,\ldots,a_k},\\
\circ = \circ_r \colon U\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n} \otimes \mathsf{P}\binom{b_1,\ldots,b_n}{a_1,\ldots,a_k} &\to U\binom{d_1,\ldots,d_m}{a_1,\ldots,a_k},\\
\otimes = \otimes_l \colon \mathsf{P}\binom{d_1,\ldots,d_{m_1}}{c_1,\ldots,c_{n_1}} \otimes U\binom{b_1,\ldots,b_{m_2}}{a_1,\ldots,a_{n_2}} &\to U\binom{d_1,\ldots,d_{m_1},b_1,\ldots,b_{m_2}}{c_1,\ldots,c_{n_1},a_1,\ldots,a_{n_2}},\\
\otimes = \otimes_r \colon U\binom{d_1,\ldots,d_{m_1}}{c_1,\ldots,c_{n_1}} \otimes \mathsf{P}\binom{b_1,\ldots,b_{m_2}}{a_1,\ldots,a_{n_2}} &\to U\binom{d_1,\ldots,d_{m_1},b_1,\ldots,b_{m_2}}{c_1,\ldots,c_{n_1},a_1,\ldots,a_{n_2}}.
\end{aligned}$$

As usual, the vertical operations $\circ_l$ and $\circ_r$ are trivial unless $b_i = c_i$ for $1 \le i \le n$. The following compatibility axioms are also imposed on the four operations:

$$\begin{gathered}
f \circ (g \circ h) = (f \circ g) \circ h,\\
f \otimes (g \otimes h) = (f \otimes g) \otimes h,\\
(f_1 \circ f_2) \otimes (g_1 \circ g_2) = (f_1 \otimes g_1) \circ (f_2 \otimes g_2).
\end{gathered}$$

Here exactly one of f, g, and h lies in U and the other two lie in P. Likewise, exactly one of f_1, f_2, g_1, and g_2 lies in U and the other three lie in P.

We note that P-modules can also be defined as abelian group objects in the category $\mathtt{PROP}/\mathsf{P}$ of $\mathfrak{C}$-colored PROPs over P.

For example, if $\beta \colon \mathsf{P} \to \mathsf{Q}$ is a morphism of $\mathfrak{C}$-colored PROPs, then Q becomes a P-module via β in the obvious way.

3.2. Derivations. Given a P-module U, a *derivation* $P \to U$ is a $\mathfrak{C}$-colored Σ-bimodule morphism $d \colon P \to U$ that satisfies the usual derivation property with respect to both the vertical operations $\circ$ and the horizontal operations $\otimes$ [30, p. 204]. Denote by $\mathrm{Der}(\mathsf{P}, U)$ the vector space of derivations $\mathsf{P} \to U$.

Proposition 3.3 (= $\mathfrak{C}$-colored version of Proposition 3 in [36])**.** *Let U be an $\mathsf{F}^{\mathfrak{C}}(E)$-module for some $\mathfrak{C}$-colored Σ-bimodule E. Then there is a canonical isomorphism*

$$\mathrm{Der}(\mathsf{F}^{\mathfrak{C}}(E), U) \cong \mathrm{Hom}^{\mathfrak{C}}_{\Sigma}(E, U), \tag{10}$$

where $\mathrm{Hom}^{\mathfrak{C}}_{\Sigma}(E, U)$ denotes the vector space of $\mathfrak{C}$-colored Σ-bimodule morphisms $E \to U$.

In one direction, isomorphism (10) takes a derivation $\theta \in \mathrm{Der}(\mathsf{F}^{\mathfrak{C}}(E), U)$ to its restriction $\theta|_E$ to the space E of generators. In the other direction, it takes a map $\varphi \colon E \to U \in \mathrm{Hom}^{\mathfrak{C}}_{\Sigma}(E, U)$ to its unique extension $Ex(\varphi) \colon \mathsf{F}^{\mathfrak{C}}(E) \to U$ as a derivation such that $Ex(\varphi)|_E = \varphi$.

Following [29, 34], we make the following definition.

Definition 3.4. Let P be a $\mathfrak{C}$-colored PROP. A *minimal model* of P is a differential graded $\mathfrak{C}$-colored PROP $(\mathsf{F}^{\mathfrak{C}}(E), \partial)$ for some $\mathfrak{C}$-colored Σ-bimodule E together with a homology isomorphism

$$\rho\colon (\mathsf{F}^{\mathfrak{C}}(E), \partial) \to (\mathsf{P}, 0)$$

such that the following *minimality condition* is satisfied:

$$\partial(E) \subseteq \bigoplus_{k\geq 2} \mathsf{F}^{\mathfrak{C}}_k(E).$$

In other words, the image of E under ∂ consists of decomposables.

3.5. Cohomology. Here we define cohomology of an algebra over a colored PROP following [30, 36].

Let P be a $\mathfrak{C}$-colored PROP, and let $(\mathsf{F}^{\mathfrak{C}}(E), \partial) \xrightarrow{\rho} (\mathsf{P}, 0)$ be a minimal model of P. Let $\mathsf{P} \xrightarrow{\alpha} \mathrm{End}^{\mathfrak{C}}_T$ be a P-algebra structure on $T = \oplus_{c\in\mathfrak{C}} T_c$. Consider $\mathrm{End}^{\mathfrak{C}}_T$ as an $\mathsf{F}^{\mathfrak{C}}(E)$-module via the morphism

$$\beta = \alpha\rho\colon \mathsf{F}^{\mathfrak{C}}(E) \to \mathrm{End}^{\mathfrak{C}}_T.$$

Then the map

$$\begin{gathered}\mathrm{Der}(\mathsf{F}^{\mathfrak{C}}(E), \mathrm{End}^{\mathfrak{C}}_T) \xrightarrow{\delta} \mathrm{Der}(\mathsf{F}^{\mathfrak{C}}(E), \mathrm{End}^{\mathfrak{C}}_T)\\ \theta \mapsto \theta\partial\end{gathered}$$

is well-defined and is a differential ($\delta^2 = 0$) because $\partial^2 = 0$.

Definition 3.6. In the above setting, define the cochain complex

$$C^*_{\mathsf{P}}(T;T) = \uparrow \mathrm{Der}(\mathsf{F}^{\mathfrak{C}}(E), \mathrm{End}^{\mathfrak{C}}_T)^{-*}, \tag{11}$$

where the degree $+1$ differential δ_{P} is induced by δ, $\uparrow$ denotes suspension, and $-*$ denotes reversed grading. We call $(C^*_{\mathsf{P}}(T;T), \delta_{\mathsf{P}})$ the *deformation complex of* T. Its cohomology,

$$H^*_{\mathsf{P}}(T;T) = H(C^*_{\mathsf{P}}(T;T), \delta_P),$$

is called the *cohomology of* T *with coefficients in itself.*

Note that if $\mathfrak{C} = \{*\}$, i.e., P is an ordinary (1-colored) PROP, then $(C^*_{\mathsf{P}}(T;T), \delta_{\mathsf{P}})$ and $H^*_{\mathsf{P}}(T;T)$ defined above coincide with the definitions in [30, 36].

4. L_∞-structure on $C^*_{\mathsf{P}}(T;T)$ and deformations

In this section, we observe that the deformation complex $(C^*_{\mathsf{P}}(T;T), \delta_{\mathsf{P}})$ (Definition 3.6) of an algebra T over a colored PROP P has the natural structure of an L_∞-algebra (Theorem 4.2). The relationship between this L_∞-algebra and deformations of T is discussed in Section 4.3. An explicit construction of the L_∞-operations l_k in $(C^*_{\mathsf{P}}(T;T), \delta_{\mathsf{P}})$ is given in Section 4.4. This construction will first be applied in Sections 5 and 6 to obtain very explicit formulas for the operations l_k in the deformation complex of an associative algebra morphism.

First we recall the notion of an L_∞-algebra.

Definition 4.1 (Definition 2.1 in [24], Example 3.90 in [37])**.** An L_∞-*structure* on a $\mathbb{Z}$-graded module V consists of a sequence of operations $(\delta = l_1, l_2, l_3, \dots)$ with

$$l_n \colon V^{\otimes n} \to V$$

of degree $2-n$ such that each l_n is anti-symmetric and the condition

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} l_j \left(l_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}, ..., x_{\sigma(n)}\right) = 0 \tag{12}$$

holds for $n \geq 1$. Here σ runs through all the $(i, n-i)$-unshuffles for $i \geq 1$, and

$$\chi(\sigma) = \operatorname{sgn}(\sigma) \cdot \varepsilon(\sigma; x_1, \dots, x_n),$$

where $\varepsilon(\sigma; x_1, \dots, x_n)$ is the Koszul sign given by

$$x_1 \wedge \cdots \wedge x_n = \varepsilon(\sigma; x_1, \dots, x_n) \cdot x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}.$$

In this case, we call $(V, \delta, l_2, l_3, \dots)$ an L_∞-*algebra*. The anti-symmetry of l_n means that

$$l_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \chi(\sigma) l_n(x_1, \dots, x_n)$$

for $\sigma \in \Sigma_n$ and $x_1, \dots, x_n \in V$.

Theorem 4.2. *In the setting of* §3.5*, there exists an L_∞-algebra structure $(\delta_{\mathsf{P}}, l_2, l_3, \dots)$ on $C^*_{\mathsf{P}}(T;T)$ capturing deformations of colored P-algebras in the sense of* §4.3 *below. This L_∞-structure induces a graded Lie algebra structure on the cohomology $H^*_{\mathsf{P}}(T;T)$.*

Proof. This is the $\mathfrak{C}$-colored version of [36, Theorem 1], whose proof, with some very minor modifications, applies to the $\mathfrak{C}$-colored setting as well. In fact, Sections 3 and 4 in [36] (which contain the proof of Theorem 1 in that paper) apply basically verbatim to the $\mathfrak{C}$-colored setting. An explicit "graphical" construction of the operations l_k will be given below (§4.4). $\square$

4.3. Deformations of colored PROP algebras. Section 5 in [36] concerning deformations of algebras over a PROP also applies to the $\mathfrak{C}$-colored setting without change. In particular, deformations of an algebra T over a $\mathfrak{C}$-colored PROP P (i.e., $\mathsf{F}^{\mathfrak{C}}(E)$-algebra structures on T) correspond to elements $\kappa \in C^1_{\mathsf{P}}(T;T)$ that satisfy the *Quantum Master Equation* [36, Eq.(4)]:

$$0 = \delta_{\mathsf{P}}(\kappa) + \frac{1}{2!} l_2(\kappa, \kappa) - \frac{1}{3!} l_3(\kappa, \kappa, \kappa) - \frac{1}{4!} l_4(\kappa, \kappa, \kappa, \kappa) + \cdots . \tag{13}$$

In other words, the L_∞-algebra

$$(C^*_{\mathsf{P}}(T;T), \delta_{\mathsf{P}}, l_2, l_3, \dots)$$

in Theorem 4.2 controls the deformations of T as a P-algebra. As explained in [36, Introduction], this L_∞-algebra is an L_∞-version of the Deligne groupoid [18, 19] governing deformations that are described by the usual *Master Equation* (also known as the *Maurer–Cartan Equation*):

$$0 = d\kappa + \frac{1}{2}[\kappa, \kappa].$$

When P is a properad [50], there is another approach to studying L_∞-deformations of P-algebras due to Merkulov and Vallette [39]. Their approach is based on a generalization of Van der Laan's *homotopy (co)operads* [51] to *homotopy (co)properads.* They show that the deformation complex $(C^*_{\mathsf{P}}(T;T), \delta_{\mathsf{P}})$ inherits a L_∞-algebra structure from a homotopy properad (Theorem 28 of [39]). Vallette recently informed the third author in private correspondence that the paper [39] can also be extended to the colored setting.

4.4. Construction of the operations l_k on $C^*_{\mathsf{P}}(T;T)$. Here we describe how the operations l_k in Theorem 4.2 are constructed, again following [36, Section 2] closely.

Suppose that $F_1, \ldots, F_k \in \mathrm{Hom}^{\mathfrak{C}}_{\Sigma}(E, \mathrm{End}^{\mathfrak{C}}_T)$ and that $\Gamma \in E(G, \zeta)$ is an E-decorated $\mathfrak{C}$-colored directed (m,n)-graph (3) with underlying $\mathfrak{C}$-colored graph $(G,\zeta) \in \mathsf{UGr}^{\mathfrak{C}}(m,n)$. Let $v_1, \ldots, v_k \in \mathrm{Vert}(G)$ be k distinct vertices in G. Consider the $\mathrm{End}^{\mathfrak{C}}_T$-decorated $\mathfrak{C}$-colored directed (m,n)-graph

$$\Gamma^{\{v_1,\ldots,v_k\}}_{\{\beta\}}[F_1, \ldots, F_k] \in \mathrm{End}^{\mathfrak{C}}_T(G, \zeta)$$

obtained from Γ by:

(1) replacing the decoration $e_{v_i} \in E$ of the vertex v_i by $F_i(e_{v_i}) \in \mathrm{End}^{\mathfrak{C}}_T$ for $1 \le i \le k$, and
(2) replacing the decoration $e_v \in E$ of any vertex $v \notin \{v_1, \ldots, v_k\}$ by $\beta(e_v) = \alpha\rho(e_v)$.

The graph $\Gamma^{\{v_1,\ldots,v_k\}}_{\{\beta\}}[F_1, \ldots, F_k]$ is visualized in Figure 1 which is a colored version of a picture taken from [36]. Using the $\mathfrak{C}$-colored PROP structure on $\mathrm{End}^{\mathfrak{C}}_T$ (Example 2.3), the graph $\Gamma^{\{v_1,\ldots,v_k\}}_{\{\beta\}}[F_1, \ldots, F_k]$ produces an element

$$\gamma\left(\Gamma^{\{v_1,\ldots,v_k\}}_{\{\beta\}}[F_1, \ldots, F_k]\right) \in \mathrm{End}^{\mathfrak{C}}_T \begin{pmatrix} \zeta(l^1_{\mathrm{out}}), \ldots, \zeta(l^m_{\mathrm{out}}) \\ \zeta(l^1_{\mathrm{in}}), \ldots, \zeta(l^n_{\mathrm{in}}) \end{pmatrix} \subseteq \mathrm{End}^{\mathfrak{C}}_T(m,n).$$

Here $\{l^1_{\mathrm{out}}, \ldots, l^m_{\mathrm{out}}\}$ and $\{l^1_{\mathrm{in}}, \ldots, l^n_{\mathrm{in}}\}$ are the output and input legs, respectively, of (G,ζ).

Now pick cochains $f_1, \ldots, f_k \in C^*_{\mathsf{P}}(T;T)$ corresponding to $F_1, \ldots, F_k \in \mathrm{Hom}^{\mathfrak{C}}_{\Sigma}(E, \mathrm{End}^{\mathfrak{C}}_T)$ under the isomorphism (10):

$$C^*_{\mathsf{P}}(T;T) = \uparrow \mathrm{Der}(\mathsf{F}^{\mathfrak{C}}(E), \mathrm{End}^{\mathfrak{C}}_T)^{-*} \cong \uparrow \mathrm{Hom}^{\mathfrak{C}}_{\Sigma}(E, \mathrm{End}^{\mathfrak{C}}_T)^{-*}. \tag{14}$$

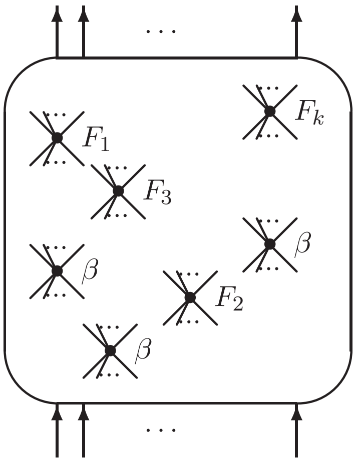


FIGURE 1. $\mathrm{End}_T^{\mathfrak{C}}$-decorated graph $\Gamma_{\{\beta\}}^{\{v_1,\ldots,v_k\}}[F_1,\ldots,F_k]$. Vertices labelled F_i are decorated by $F_i(e_{v_i})$, $1 \le i \le k$, the remaining vertices are decorated by $\beta(e_v)$.

If $\xi \in E\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n}$, then $\partial(\xi) \in \mathsf{F}^{\mathfrak{C}}(E)\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n}$ can be written as a finite sum

$$\partial(\xi) = \sum_{s \in S_\xi} \Gamma_s,$$

where each $\Gamma_s \in E(G,\zeta)$ for some $(G,\zeta) \in \mathsf{UGr}^{\mathfrak{C}}\binom{d_1,\ldots,d_m}{c_1,\ldots,c_n}$. Define

$$l_k(f_1,\ldots,f_k)(\xi) \in \mathrm{End}_T^{\mathfrak{C}}\begin{pmatrix} d_1,\ldots,d_m \\ c_1,\ldots,c_n \end{pmatrix}$$

to be the element

$$l_k(f_1,\ldots,f_k)(\xi) \overset{\mathrm{def}}{=} (-1)^{\nu(f_1,\ldots,f_k)} \sum_{s \in S_\xi} \sum_{(v_1,\ldots,v_k)} \gamma\left(\Gamma_{s,\{\beta\}}^{\{v_1,\ldots,v_k\}}[F_1,\ldots,F_k]\right), \tag{15}$$

where $(v_1,\ldots,v_k)$ runs through all the k-tuples of distinct vertices in the underlying graph of Γ_s. The sign on the right-hand side of (15) is given by

$$\nu(f_1,\ldots,f_k) \overset{\mathrm{def}}{=} (k-1)|f_1| + (k-2)|f_2| + \cdots + |f_{k-1}|. \tag{16}$$

Since ξ is arbitrary, (15) specifies an element

$$l_k(f_1,\ldots,f_k) \in \mathrm{Hom}_\Sigma^{\mathfrak{C}}(E, \mathrm{End}_T^{\mathfrak{C}}) \cong C_{\mathsf{P}}^*(T;T). \tag{17}$$

The arguments in Sections 3-4 in [36] ensure that (17) is indeed well-defined. We note that the L_∞ axiom (12) for the operations l_k constructed above is a consequence of $\partial^2 = 0$. Also, an obvious modification of the above construction applies to free cofibrant, not necessarily minimal, models as well. We will see an instance of such a generalization in Section 9.

5. Deformation complex of an associative algebra morphism

In this section and Section 6, we illustrate the L_∞-deformation theory of colored PROP algebras (Section 4) in the case of associative algebra morphisms. Let $g\colon U \to V$ be an associative algebra morphism, and set $T = U \oplus V$ as a 2-colored graded module. Let $\mathbf{As}_{\mathtt{B}\to\mathtt{W}}$ denote the 2-colored operad encoding associative algebra morphisms (Example 2.10). The morphism $g\colon U \to V$ can be regarded as an $\mathbf{As}_{\mathtt{B}\to\mathtt{W}}$-algebra structure on T.

The purposes of this section are (i) to express the differential $\delta_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}$ in $C^*_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T)$ (11) in terms of the Hochschild differential (Theorem 5.5), and (ii) to observe that the cochain complex $(C^*_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T), \delta_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}})$ is isomorphic to the Gerstenhaber–Schack cochain complex $(C^{*+1}_{GS}(g;g), d_{GS})$ of the morphism g [14, 15, 16] (Theorem 5.5). This isomorphism allows us to transfer the L_∞-structure on $(C^*_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T), \delta_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}})$ to the Gerstenhaber–Schack cochain complex $(C^{*+1}_{GS}(g;g), d_{GS})$ (Corollary 5.6).

The materials in this section and Section 6 can be easily dualized to obtain an explicit L_∞-structure on the deformation complex of a morphism of coassociative coalgebras. The associated deformation theory of coalgebra morphisms is the one constructed in [52].

5.1. Background. Deformation of an associative algebra morphism g, in the classical sense of Gerstenhaber [13], was studied by Gerstenhaber and Schack in [14, 15, 16]. In the case of a single associative algebra A, the deformation complex is the Hochschild cochain complex $C^*(A;A)$ of A with coefficients in itself, which has the structure of a differential graded Lie algebra [12]. On the other hand, the work of Gerstenhaber and Schack [14, 15, 16] left open the question of what structure the deformation complex $(C^*_{GS}(g;g), d_{GS})$ of g possesses. Borisov answered this question in [3] by showing that $(C^*_{GS}(g;g), d_{GS})$, while not a differential graded Lie algebra, is isomorphic to the underlying cochain complex of an L_∞-algebra.

With our approach based on minimal models, we are able to write down all the L_∞-operations l_k on $C^*_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T)$ explicitly (with l_1 in Theorem 5.5 and l_k $(k \geq 2)$ in Section 6). In particular, all the higher l_k $(k \geq 3)$ can be written in terms of a certain generalized "comp" operation (28), which extends the classical $\circ_i$ operation in the Hochschild cochain complex [12]. As far as we know, these higher operations l_k have never been explicitly written down before. We believe that this example of associative algebra morphisms will serve as a guide for obtaining explicit formulas for the L_∞-operations in the deformation complexes of other kinds of morphisms and general diagrams.

5.2. The Gerstenhaber–Schack complex $(C^*_{GS}(g;g), d_{GS})$. Here we recall the Gerstenhaber–Schack cochain complex $(C^*_{GS}(g;g), d_{GS})$ [14, 15, 16]. Fix a morphism $g\colon U \to V$ of associative algebras. We also consider V

as a U-bimodule via g. Then

$$C^n_{GS}(g;g) \overset{\text{def}}{=} \mathrm{Hom}(U^{\otimes n}, U) \oplus \mathrm{Hom}(V^{\otimes n}, V) \oplus \mathrm{Hom}(U^{\otimes n-1}, V) \tag{18}$$

for $n \geq 1$. A typical element in $C^n_{GS}(g;g)$ is denoted by (x_U, x_V, x_g) with $x_U \in \mathrm{Hom}(U^{\otimes n}, U)$, $x_V \in \mathrm{Hom}(V^{\otimes n}, V)$, and $x_g \in \mathrm{Hom}(U^{\otimes n-1}, V)$. Its differential is defined as

$$d^n_{GS}(x_U, x_V, x_g) \overset{\text{def}}{=} (bx_U, bx_V, gx_U - x_V g^{\otimes n} - bx_g),$$

where b denotes the Hochschild differential in $\mathrm{Hom}(U^{\otimes *}, U)$, $\mathrm{Hom}(V^{\otimes *}, V)$, or $\mathrm{Hom}(U^{\otimes *}, V)$.

5.3. The minimal model of $\mathbf{As}_{\mathtt{B}\to\mathtt{W}}$. Here we recall from [31, 32] the minimal model of the 2-colored operad $\mathbf{As}_{\mathtt{B}\to\mathtt{W}}$ that encodes associative algebra morphisms. The 2-colored operad $\mathbf{As}_{\mathtt{B}\to\mathtt{W}}$ can be represented as (Example 2.10)

$$\mathbf{As}_{\mathtt{B}\to\mathtt{W}} = \frac{\mathbf{As}_{\mathtt{B}} * \mathbf{As}_{\mathtt{W}} * \mathsf{F}(f)}{(f\mu = \nu f^{\otimes 2})},$$

where μ and ν denote the generators in $\mathbf{As}_{\mathtt{B}}(1,2)$ and $\mathbf{As}_{\mathtt{W}}(1,2)$, respectively, which encode the multiplications in the domain and the target.

Let E be the 2-colored Σ-bimodule with the following generators:

$$\begin{aligned} &\mu_n \colon \mathtt{B}^{\otimes n} \to \mathtt{B} \text{ of degree } n-2 \text{ and biarity } (1,n)\ (n \geq 2),\\ &\nu_n \colon \mathtt{W}^{\otimes n} \to \mathtt{W} \text{ of degree } n-2 \text{ and biarity } (1,n)\ (n \geq 2), \text{ and}\\ &f_n \colon \mathtt{B}^{\otimes n} \to \mathtt{W} \text{ of degree } n-1 \text{ and biarity } (1,n)\ (n \geq 1). \end{aligned}$$

Then the minimal model for $\mathbf{As}_{\mathtt{B}\to\mathtt{W}}$ is

$$(\mathsf{F}(E), \partial) \xrightarrow{\alpha} \mathbf{As}_{\mathtt{B}\to\mathtt{W}},$$

where

$$\alpha(\mu_n) = \begin{cases} \mu & \text{if } n = 2,\\ 0 & \text{otherwise,} \end{cases} \qquad \alpha(\nu_n) = \begin{cases} \nu & \text{if } n = 2,\\ 0 & \text{otherwise,} \end{cases}$$

and

$$\alpha(f_n) = \begin{cases} f & \text{if } n = 1,\\ 0 & \text{otherwise.} \end{cases}$$

The differential ∂ is given by:

$$\partial(\mu_n) = \sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{s=0}^{n-j} (-1)^{i+s(j+1)} \mu_i \circ_{s+1} \mu_j, \tag{19a}$$

$$\partial(\nu_n) = \sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{s=0}^{n-j} (-1)^{i+s(j+1)} \nu_i \circ_{s+1} \nu_j, \tag{19b}$$

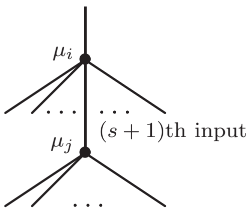


FIGURE 2. The graph corresponding to $\mu_i \circ_{s+1} \mu_j$.

$$\text{(19c)} \qquad \partial(f_n) = -\sum_{l=2}^{n} \sum_{r_1+\cdots+r_l=n} (-1)^{\sum_{1\le i<j\le l} r_i(r_j+1)} \nu_l(f_{r_1} \otimes \cdots \otimes f_{r_l}) - \sum_{\substack{i+j=n+1\\ i\ge 1,\, j\ge 2}} \sum_{s=0}^{n-j} (-1)^{i+s(j+1)} f_i \circ_{s+1} \mu_j.$$

Here

$$\text{(20)} \qquad \mu_i \circ_{s+1} \mu_j \overset{\text{def}}{=} \mu_i \left(1_{\mathtt{B}}^{\otimes s} \otimes \mu_j \otimes 1_{\mathtt{B}}^{\otimes i-s-1}\right),$$

which "plugs" μ_j into the $(s+1)$st input of μ_i (see Figure 2), and similarly for $\nu_i \circ_{s+1} \nu_j$ and $f_i \circ_{s+1} \mu_j$.

5.4. The cochain complex $(C^*_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T), \delta_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}})$. Suppose that $T = U \oplus V$ as a 2-colored graded module and that $g\colon U \to V$ is a morphism of associative algebras represented by the morphism $\rho\colon \mathbf{As}_{\mathtt{B}\to\mathtt{W}} \to \mathrm{End}_T$ of 2-colored operads. Then the canonical isomorphism (14) says in this case,

$$C^*_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T) = \uparrow \mathrm{Der}(\mathsf{F}(E), \mathrm{End}_T)^{-*} \cong \uparrow \mathrm{Hom}_{\Sigma}^{\{\mathtt{B},\mathtt{W}\}}(E, \mathrm{End}_T)^{-*}.$$

Under this isomorphism, an element $\theta \in C^n_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T)$ is uniquely determined by the tuple

$$\text{(21)} \qquad (\theta_U, \theta_V, \theta_g) \overset{\text{def}}{=} (\theta(\mu_{n+1}), \theta(\nu_{n+1}), \theta(f_n)) \in C^{n+1}_{GS}(g;g).$$

This establishes a linear isomorphism

$$C^n_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T) \cong C^{n+1}_{GS}(g;g),$$
$$\theta \leftrightarrow (\theta_U, \theta_V, \theta_g).$$

Denote by δ_{GS} the differential on the graded module $C^*_{GS}(g;g)$ induced by $\delta_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}$. The identification (21) provides an isomorphism

$$(C^*_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T), \delta_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}) \xrightarrow{\cong} (C^{*+1}_{GS}(g;g), \delta_{GS})$$

of cochain complexes.

Theorem 5.5. *For $\theta \in C^{n-1}_{\mathbf{As}_{B\to W}}(T;T)$, we have*

$$\text{(22)} \qquad \delta_{GS}(\theta_U, \theta_V, \theta_g) = \left((-1)^{n+1} b\theta_U, (-1)^{n+1} b\theta_V, g\theta_U - \theta_V g^{\otimes n} - (-1)^n b\theta_g\right),$$

in which b denotes the appropriate Hochschild differential. In particular, there is a cochain complex isomorphism

$$\left(C^{*-1}_{\mathbf{As}_{B\to W}}(T;T), \delta_{\mathbf{As}_{B\to W}}\right) \cong (C^*_{GS}(g;g), d_{GS}) \tag{23}$$

given by

$$\theta \in C^{n-1}_{\mathbf{As}_{B\to W}}(T;T) \mapsto \left((-1)^{\frac{n(n+1)}{2}}\theta_U, (-1)^{\frac{n(n+1)}{2}}\theta_V, (-1)^{\frac{(n-1)n}{2}}\theta_g\right).$$

Since $(C^*_{\mathbf{As}_{B\to W}}(T;T), \delta_{\mathbf{As}_{B\to W}} = l_1, l_2, l_3, \dots)$ is, by Theorem 4.2, an L_∞-algebra, we can use the cochain complex isomorphism (23) to transfer the higher brackets l_k $(k \geq 2)$ to $(C^{*+1}_{GS}(g;g), d_{GS})$.

Corollary 5.6. *There is an L_∞-algebra structure $(d_{GS} = l_1, l_2, l_3, \dots)$ on $C^{*+1}_{GS}(g;g)$ governing deformations of the associative algebra morphism g.*

Proof of Theorem 5.5. Since $\delta_{\mathbf{As}_{B\to W}} = l_1$ in the L_∞-algebra and since the degree of l_1 is $+1$, we have

$$\delta_{GS}(\theta_U, \theta_V, \theta_g) = (l_1(\theta)(\mu_{n+1}), l_1(\theta)(\nu_{n+1}), l_1(\theta)(f_n))$$

by the identification (21). Therefore, to prove (22), it suffices to show:

$$\begin{aligned} l_1(\theta)(\mu_{n+1}) &= (-1)^{n+1} b\theta_U, \\ l_1(\theta)(\nu_{n+1}) &= (-1)^{n+1} b\theta_V, \text{ and} \\ l_1(\theta)(f_n) &= g\theta_U - \theta_V g^{\otimes n} - (-1)^n b\theta_g. \end{aligned} \tag{24}$$

From the description (15) of l_k, the computation of $l_1(\theta)(\mu_{n+1})$ starts with $\partial(\mu_{n+1})$ (19a). As an E-decorated 2-colored directed $(1, n+1)$-graph, the term $\mu_i \circ_{s+1} \mu_j$ in $\partial(\mu_{n+1})$ has two vertices, whose decorations are μ_i and μ_j, see Figure 2. Therefore, the expression (15), when applied to the current situation, gives

$$l_1(\theta)(\mu_{n+1}) = \sum_{\substack{i+j=n+2 \\ i,j\geq 2}} \sum_{s=0}^{n+1-j} (-1)^{i+s(j+1)} \left\{\theta(\mu_i) \circ_{s+1} \beta(\mu_j) + \beta(\mu_i) \circ_{s+1} \theta(\mu_j)\right\}. \tag{25}$$

Note that, since $\theta \in C^{n-1}_{\mathbf{As}_{B\to W}}(T;T)$,

$$\theta(\mu_i) = \begin{cases} 0 & \text{if } i \neq n, \\ \theta_U & \text{if } i = n, \end{cases} \quad \text{and} \quad \beta(\mu_j) = \rho(\alpha(\mu_j)) = \begin{cases} 0 & \text{if } j \neq 2, \\ \mu_U & \text{if } j = 2, \end{cases}$$

where $\mu_U \colon U^{\otimes 2} \to U$ is the multiplication on U. It follows that (25) reduces to

$$\begin{aligned} l_1(\theta)(\mu_{n+1}) &= \sum_{s=0}^{n-1}(-1)^{n+s(2+1)}\theta_U \circ_{s+1} \mu_U + \sum_{s=0}^{1}(-1)^{2+s(n+1)}\mu_U \circ_{s+1} \theta_U \\ &= (-1)^{n+1}\mu_U(-,\theta_U) + \mu_U(\theta_U,-) \\ &\quad + (-1)^{n+1}\sum_{s=1}^{n}(-1)^s\theta_U(\mathrm{Id}_U^{\otimes s-1} \otimes \mu_U \otimes \mathrm{Id}_U^{\otimes n-s}) \\ &= (-1)^{n+1} b\theta_U, \end{aligned}$$

which is the first condition in (24).

The previous paragraph applies verbatim to $l_1(\theta)(\nu_{n+1})$ (with ν_l replacing μ_l everywhere), since the definition of $\partial(\nu_*)$ (19b) admits the same formula as that of $\partial(\mu_*)$. Therefore, it remains to show the last condition in (24).

In $\partial(f_n)$ (19c), the term $\nu_l(f_{r_1} \otimes \cdots \otimes f_{r_l})$ (respectively, $f_i \circ_{s+1} \mu_j$) is an E-decorated 2-colored directed $(1,n)$-graph with $l+1$ (respectively, 2) vertices. Since

$$\beta(f_j) = \rho(\alpha(f_j)) = \begin{cases} g \colon U \to V & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

the same kind of analysis as above gives

$$\begin{aligned} l_1(\theta)(f_n) &= -\theta_V g^{\otimes n} - (-1)^{(n-1)(1+1)}\mu_V(\theta_g \otimes g) - (-1)^{n-1+1}\mu_V(g \otimes \theta_g) \\ &\quad - \sum_{s=0}^{n-2}(-1)^{n-1+s(2+1)}\theta_g \circ_{s+1} \mu_U - (-1)^{1+0} g\theta_U \\ &= g\theta_U - \theta_V g^{\otimes n} \\ &\quad - (-1)^n \left\{ \mu_V(g \otimes \theta_g) + (-1)^n \mu_V(\theta_g \otimes g) + \sum_{s=1}^{n-1}(-1)^s \theta_g \circ_s \mu_U \right\} \\ &= g\theta_U - \theta_V g^{\otimes n} - (-1)^n b\theta_g. \end{aligned}$$

Here $\mu_V \colon V^{\otimes 2} \to V$ denotes the multiplication on V. This establishes the last condition in (24) and finishes the proof of Theorem 5.5. $\square$

6. The higher brackets in $C^*_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T)$

We keep the same setting and notations as in the previous section. The purpose of this section is to describe the L_∞-operations l_k on $C^*_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T)$ for $k \geq 2$. The cases $k = 2$ (Theorem 6.2) and $k \geq 3$ (Theorem 6.4) are treated separately. As an immediate consequence of our explicit formula for l_k ($k \geq 3$), we observe that, when applied to the tensor powers of $C^{\leq q}_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T)$ for some fixed $q \geq 0$, only $\delta_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T) = l_1, l_2, \ldots, l_{q+2}$ can be nontrivial (Corollary 6.5).

6.1. The operation l_2. First we deal with the case $k = 2$. Pick elements $\theta \in C^{n-1}_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T)$ and $\omega \in C^{m-1}_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T)$. Under the identification (21), θ and ω correspond to

$$(\theta_U, \theta_V, \theta_g) \in C^n_{GS}(g;g) \quad \text{and} \quad (\omega_U, \omega_V, \omega_g) \in C^m_{GS}(g;g),$$

respectively.

Since l_2 has degree 0, the element $l_2(\theta, \omega)$ lies in $C^{(n+m-1)-1}_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T)$. Under the identification (21), $l_2(\theta,\omega)$ is uniquely determined by

$$(l_2(\theta,\omega)(\mu_{n+m-1}), l_2(\theta,\omega)(\nu_{n+m-1}), l_2(\theta,\omega)(f_{n+m-2})) \in C^{n+m-1}_{GS}(g;g).$$

Theorem 6.2. *With the notations above, we have*

(26a) $\quad l_2(\theta,\omega)(\mu_{n+m-1}) =$
$$-\sum_{s=1}^{n}(-1)^{(s+1)(m+1)}\theta_U \circ_s \omega_U - (-1)^{n+m}\sum_{s=1}^{m}(-1)^{(s+1)(n+1)}\omega_U \circ_s \theta_U,$$

(26b) $\quad l_2(\theta,\omega)(\nu_{n+m-1}) =$
$$-\sum_{s=1}^{n}(-1)^{(s+1)(m+1)}\theta_V \circ_s \omega_V - (-1)^{n+m}\sum_{s=1}^{m}(-1)^{(s+1)(n+1)}\omega_V \circ_s \theta_V,$$

(26c) $\quad l_2(\theta,\omega)(f_{n+m-2}) =$
$$-\sum_{s=1}^{n-1}(-1)^{(s+1)(m+1)}\theta_g \circ_s \omega_U - (-1)^{n+m}\sum_{s=1}^{m-1}(-1)^{(s+1)(n+1)}\omega_g \circ_s \theta_U$$
$$+(-1)^n\sum_{i=1}^{n}(-1)^{(i-1)m}\theta_V \circ_i \omega_g + \sum_{j=1}^{m}(-1)^{jn}\omega_V \circ_j \theta_g$$
$$+(-1)^{nm+n+m}\theta_g \smile \omega_g + (-1)^{nm}\omega_g \smile \theta_g.$$

In the above theorem, the notation is as in (20) and (21), except that

$$\begin{aligned} \theta_V \circ_i \omega_g &= \theta_V(g^{\otimes i-1} \otimes \omega_g \otimes g^{\otimes n-i}), \\ \omega_V \circ_j \theta_g &= \omega_V(g^{\otimes j-1} \otimes \theta_g \otimes g^{\otimes m-j}), \\ \theta_g \smile \omega_g &= \mu_V(\theta_g \otimes \omega_g), \end{aligned} \tag{27}$$

and similarly for $\omega_g \smile \theta_g$, $\theta_g \circ_s \omega_U$, and $\omega_g \circ_s \theta_U$. In other words, $\theta_V \circ_i \omega_g$ is obtained by plugging ω_g into the ith input of θ_V and g into the other $(n-1)$ inputs of θ_V. Likewise, $\theta_g \smile \omega_g$ is simply the usual cup-product of θ_g and ω_g.

The proof will be given at the end of this section. Note that Gerstenhaber and Schack did construct a bracket $[-,-]$ on their cochain complex $(C^*_{GS}(g;g), d_{GS})$ (see, e.g., the graded commutator bracket of the operation [14, p. 11 (9)] or [16, pp. 158-159]). It is straightforward to check that the linear isomorphism (23) is compatible with $[-,-]$ and l_2 as well.

6.3. The operations l_k for $k \geq 3$. Now consider the cases $k \geq 3$. Pick elements $\theta_s \in C^{n_s-1}_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T)$ $(1 \leq s \leq k)$. Each θ_s corresponds, via the identification (21), to the tuple

$$(\theta_{s,U}, \theta_{s,V}, \theta_{s,g}) = (\theta_s(\mu_{n_s}), \theta_s(\nu_{n_s}), \theta_s(f_{n_s-1})) \in C^{n_s}_{GS}(g;g).$$

Since l_k has degree $2-k$, the element $l_k(\theta_1,\ldots,\theta_k)$ lies in $C^{t-1}_{\mathbf{As}_{\mathtt{B}\to\mathtt{W}}}(T;T)$, where

$$t = 3 - 2k + \sum_{s=1}^{k} n_s.$$

Under the identification (21), $l_k(\theta_1,\ldots,\theta_k)$ is uniquely determined by

$$(l_k(\theta_1,\ldots,\theta_k)(\mu_t), l_k(\theta_1,\ldots,\theta_k)(\nu_t), l_k(\theta_1,\ldots,\theta_k)(f_{t-1})) \in C^t_{GS}(g;g).$$

Now we extend the first $\circ_i$ operation in (27) as follows. Fix $s \in \{1,\ldots,k\}$. Let

$$\mathbf{a} = (a_1,\ldots,\widehat{a_s},\ldots,a_k)$$

be a $(k-1)$-tuple of distinct points in the set $\{1,\ldots,n_s\}$. Then we define

$$\theta_{s,V} \circ_{\mathbf{a}} (\theta_{1,g},\ldots,\widehat{\theta_{s,g}},\ldots,\theta_{k,g}) \in \mathrm{Hom}(U^{\otimes t-1}, V) \tag{28}$$

to be the element obtained by plugging $\theta_{j,g}$ $(1 \leq j \leq k,\ j \neq s)$ into the a_jth input of $\theta_{s,V}$ and g into the other $(n_s - (k-1))$ inputs of $\theta_{s,V}$. Also define the sign

$$(-1)^{\mathbf{a}} = (-1)^{\sum_{1\leq i<j\leq n_s} r_i(r_j+1)},$$

where

$$r_a = \begin{cases} |\theta_j| = n_j - 1 & \text{if } a = a_j \in \{a_1,\ldots,\widehat{a_s},\ldots,a_k\}, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 6.4. *For $k \geq 3$ and notations as above, we have*

$$l_k(\theta_1,\ldots,\theta_k)(\mu_t) = 0, \tag{29a}$$

$$l_k(\theta_1,\ldots,\theta_k)(\nu_t) = 0, \text{ and} \tag{29b}$$

$$l_k(\theta_1,\ldots,\theta_k)(f_{t-1}) = - (-1)^{\nu(\theta_1,\ldots,\theta_k)} \sum_{s=1}^{k} \sum_{\mathbf{a}} (-1)^{\mathbf{a}}\, \theta_{s,V} \circ_{\mathbf{a}} (\theta_{1,g},\ldots,\widehat{\theta_{s,g}},\ldots,\theta_{k,g}). \tag{29c}$$

Here $\nu(\theta_1,\ldots,\theta_k)$ is defined in (16) *and, for each s, $\mathbf{a} = (a_1,\ldots,\widehat{a_s},\ldots,a_k)$ runs through all the $(k-1)$-tuples of distinct points in the set $\{1,\ldots,n_s\}$.*

Corollary 6.5. *Suppose that $k \geq 3$ and that $\theta_s \in C^{n_s-1}_{\mathbf{As}_{B\to W}}(T;T)$, $1 \leq s \leq k$. If*

$$n_s < k-1 \quad \textit{for} \quad 1 \leq s \leq k,$$

then

$$l_k(\theta_1,\ldots,\theta_k) = 0.$$

In other words, for each $q \geq 0$ and any $k \geq q+3$, the operation

$$l_k\colon \left(C^{\leq q}_{\mathbf{As}_{B\to W}}(T;T)\right)^{\otimes k} \to C^*_{\mathbf{As}_{B\to W}}(T;T)$$

is trivial.

Proof of Theorem 6.2. To prove (26a), first note that

$$\partial(\mu_{n+m-1}) = \sum_{\substack{i+j=n+m\\ i,j\geq 2}} \sum_{s=0}^{n+m-1-j} (-1)^{i+s(j+1)} \mu_i \circ_{s+1} \mu_j.$$

Since the E-decorated 2-colored directed $(1, n+m-1)$-graph $\mu_i \circ_{s+1} \mu_j$ has two vertices, we have

$$\begin{aligned}
&l_2(\theta,\omega)(\mu_{n+m-1})\\
&= (-1)^{|\theta|} \sum_{\substack{i+j=n+m\\ i,j\geq 2}} \sum_{s=0}^{n+m-1-j} (-1)^{i+s(j+1)} \{\theta(\mu_i)\circ_{s+1}\omega(\mu_j) + \omega(\mu_i)\circ_{s+1}\theta(\mu_j)\}\\
&= (-1)^{n-1} \left(\sum_{s=0}^{n-1}(-1)^{n+s(m+1)}\theta_U\circ_{s+1}\omega_U + \sum_{s=0}^{m-1}(-1)^{m+s(n+1)}\omega_U\circ_{s+1}\theta_U\right).
\end{aligned}$$

This is exactly (26a) after a shift of the summation indexes.

Since $\partial(\nu_{n+m-1})$ has the same defining formula as $\partial(\mu_{n+m-1})$ (with ν_l replacing μ_l everywhere), the reasoning in the previous paragraph also applies to $l_2(\theta,\omega)(\nu_{n+m-1})$ to establish (26b).

To prove (26c), first note that

$$\begin{aligned}
(30)\quad \partial(f_{n+m-2}) &= -\sum_{l=2}^{n+m-2} \sum_{r_1+\cdots+r_l=n+m-2} (-1)^{\sum_{1\leq i<j\leq l} r_i(r_j+1)} \nu_l(f_{r_1}\otimes\cdots\otimes f_{r_l})\\
&\quad - \sum_{\substack{i+j=n+m-1\\ i\geq 1,\, j\geq 2}} \sum_{s=0}^{n+m-2-j} (-1)^{i+s(j+1)} f_i \circ_{s+1} \mu_j.
\end{aligned}$$

An argument essentially identical to the first paragraph of this proof can be applied to the terms $f_i \circ_{s+1} \mu_j$. This gives rise to the sums

$$(31)\quad -\sum_{s=1}^{n-1}(-1)^{(s+1)(m+1)}\theta_g \circ_s \omega_U - (-1)^{n+m}\sum_{s=1}^{m-1}(-1)^{(s+1)(n+1)}\omega_g \circ_s \theta_U$$

in $l_2(\theta,\omega)(f_{n+m-2})$.

In (30), the E-decorated 2-colored directed $(1, n+m-2)$-graph $\Gamma = \nu_l(f_{r_1}\otimes\cdots\otimes f_{r_l})$ has $l+1$ vertices, say, $v_{\text{top}}, v^1_{\text{bot}}, \ldots, v^l_{\text{bot}}$, with decorations $\nu_l, f_{r_1}, \ldots, f_{r_l}$, respectively. In this graph Γ, the only pairs of distinct vertices are $(v_{\text{top}}, v^*_{\text{bot}})$, $(v^*_{\text{bot}}, v_{\text{top}})$, and $(v^i_{\text{bot}}, v^j_{\text{bot}})$ $(i \neq j)$. The corresponding elements in $l_2(\theta,\omega)(f_{n+m-2})$ (without the signs) are:

(1) $\theta(\nu_l)(\beta(f_{r_1}) \otimes \cdots \otimes \omega(f_{r_i}) \otimes \cdots \beta(f_{r_l}))$ $(1 \le i \le l)$, which is 0 unless $l = n$, $r_i = m-1$, and all the other $r_* = 1$;
(2) $\omega(\nu_l)(\beta(f_{r_1}) \otimes \cdots \otimes \theta(f_{r_j}) \otimes \cdots \otimes \beta(f_{r_l}))$ $(1 \le j \le l)$, which is 0 unless $l = m$, $r_j = n-1$, and all the other $r_* = 1$;
(3) $\beta(\nu_l)(\beta(f_{r_1}) \otimes \cdots \otimes \theta(f_{r_i}) \otimes \cdots \otimes \omega(f_{r_j}) \otimes \cdots \otimes \beta(f_{r_l}))$, which is 0 unless $l = 2$ and $(r_1, r_2) = (n-1, m-1)$;
(4) $\beta(\nu_l)(\beta(f_{r_1}) \otimes \cdots \otimes \omega(f_{r_i}) \otimes \cdots \otimes \theta(f_{r_j}) \otimes \cdots \otimes \beta(f_{r_l}))$, which is 0 unless $l = 2$ and $(r_1, r_2) = (m-1, n-1)$.

Taking all the signs into account, we obtain in $l_2(\theta, \omega)(f_{n+m-2})$ the following sums:

$$\begin{aligned} &- (-1)^{|\theta|} \sum_{i=1}^{n} (-1)^{(i-1)(m-1+1)} \theta_V (g^{\otimes i-1} \otimes \omega_g \otimes g^{\otimes n-i}) \\ &- (-1)^{|\theta|} \sum_{j=1}^{m} (-1)^{(j-1)(n-1+1)} \omega_V (g^{\otimes j-1} \otimes \theta_g \otimes g^{\otimes m-j}) \\ &+ (-1)^{|\theta|} (-1)^{(n-1)(m-1)} \left\{ (-1)^n \mu_V (\theta_g \otimes \omega_g) + (-1)^m \mu_V (\omega_g \otimes \theta_g) \right\}. \end{aligned} \tag{32}$$

The required result (26c) is now obtained by combining (31) and (32). This finishes the proof of Theorem 6.2. □

Proof of Theorem 6.4. The computation of $l_k(\theta_1, \ldots, \theta_k)(\mu_t)$ involves the choice of $k \ge 3$ distinct vertices in the graphs $\mu_i \circ_{s+1} \mu_j$, each of which has only two vertices. It follows that

$$l_k(\theta_1, \ldots, \theta_k)(\mu_t) = 0,$$

which is (29a). The same argument establishes (29b). Moreover, the same reasoning also shows that the terms $f_i \circ_{s+1} \mu_j$ in $\partial(f_{t-1})$ cannot contribute nontrivially to $l_k(\theta_1, \ldots, \theta_k)(f_{t-1})$.

The remaining statement (29c) is now proved by an argument very similar to the last paragraph in the proof of Theorem 6.2. There is one major difference: In order for the term $\nu_l(f_{r_1} \otimes \cdots \otimes f_{r_l})$ in $\partial(f_{t-1})$ to contribute nontrivially to $l_k(\theta_1, \ldots, \theta_k)(f_{t-1})$, the vertex v_{top} (with decoration ν_l) must be chosen as one of the k distinct vertices because $k \ge 3$ and $\beta(\nu_l) = 0$ for $l \ge 3$. It follows that each nontrivial term in $l_k(\theta_1, \ldots, \theta_k)(f_{t-1})$ has the form (28), except for the sign, which is

$$-(-1)^{\nu(\theta_1, \ldots, \theta_k)} (-1)^{\mathbf{a}}.$$

The desired condition (29c) now follows. □

7. Deformation complex of a Lie algebra morphism

In this section and Section 8, we give a second illustration of the L_∞-deformation theory of colored PROP algebras (Section 4) in the case of Lie algebra morphisms. The parallelism of the analysis in the associative and Lie cases shows the unifying character of this approach.

Let $g\colon U \to V$ be a Lie algebra morphism, and set $T = U \oplus V$ as a 2-colored graded module. Let $\mathbf{Lie}_{B\to W}$ denote the 2-colored operad encoding Lie algebra morphisms. The purposes of this section are (i) to express the differential $\delta_{\mathbf{Lie}_{B\to W}}$ in $C^*_{\mathbf{Lie}_{B\to W}}(T;T)$ (11) in terms of the Chevalley–Eilenberg differential (Theorem 7.5), and (ii) to observe that the cochain complex $(C^*_{\mathbf{Lie}_{B\to W}}(T;T), \delta_{\mathbf{Lie}_{B\to W}})$ is isomorphic to the S-cohomology cochain complex $(\Lambda^*(U,V), \Delta^*)$ of the morphism g [9, 17] (Corollary 7.6). This isomorphism allows us to transfer the L_∞-structure on $(C^*_{\mathbf{Lie}_{B\to W}}(T;T), \delta_{\mathbf{Lie}_{B\to W}})$ to the S-cohomology cochain complex $(\Lambda^*(U,V), \Delta^*)$ (Corollary 7.7).

7.1. Background. The question of deformation of morphisms of Lie algebras was treated for the first time by Nijenhuis and Richardson in [42]. The approach chosen was not the classical method of Gerstenhaber [13], but the use of the formalization of deformation theory in terms of graded algebras on the space of cochains developed by Nijenhuis and Richardson in [41]. The starting point was then the graded Lie algebra on cochains and the differential which were guessed, the deformation theory being only a corollary. As drawbacks, the algebras were not allowed to be deformed and the notion of equivalent deformations was not natural. In order to cure these two problems, the first author reexamined this problem from the classical point of view of Gerstenhaber and introduced in [9] the S-cohomology, concluding his work by addressing the question of the description of a structure for its deformation complex. Later, in [17], Gerstenhaber, Giaquinto and Schack showed that this construction is completely parallel to the one given in [14] which leads to Diagram cohomology of associative algebras, and hence gave the diagrammatic description of S-cohomology.

7.2. The S-Cohomology complex $(\Lambda^n(U,V), \Delta^n)$. Here we recall the S-cochain complex $(\Lambda^*(U,V), \Delta^*)$ [9]. We modify slightly the notations from [9] to be coherent with the present notations.

Fix a morphism $g\colon U \to V$ of Lie algebras. We also consider V as a left U-module via g. Then

$$\Lambda^n(U,V) \stackrel{\text{def}}{=} \mathrm{Hom}(U^{\wedge n}, U) \oplus \mathrm{Hom}(V^{\wedge n}, V) \oplus \mathrm{Hom}(U^{\wedge n-1}, V)$$

for $n \geq 1$. We will also denote a typical element in $\Lambda^n(U,V)$ by (x_U, x_V, x_g) with $x_U \in \mathrm{Hom}(U^{\wedge n}, U)$, $x_V \in \mathrm{Hom}(V^{\wedge n}, V)$, and $x_g \in \mathrm{Hom}(U^{\wedge n-1}, V)$. Its differential is defined as

$$\Delta^n(x_U, x_V, x_g) \stackrel{\text{def}}{=} (bx_U, bx_V, (-1)^{n-1} g x_U - (-1)^{n-1} x_V g^{\otimes n} + bx_g), \tag{33}$$

where b denotes the Chevalley–Eilenberg differential in one of the spaces $\mathrm{Hom}(U^{\wedge *}, U)$, $\mathrm{Hom}(V^{\wedge *}, V)$, or $\mathrm{Hom}(U^{\wedge *}, V)$.

7.3. The minimal model of $\mathbf{Lie}_{B\to W}$. Here we construct the minimal model of the 2-colored operad $\mathbf{Lie}_{B\to W}$ that encodes Lie algebra morphisms. This definition is very similar to the one in the associative category, except for the definition of the differential which differs slightly. Moreover one has to be careful with respect to the symmetry which is a new feature of the Lie category.

The 2-colored operad $\mathbf{Lie}_{B\to W}$ can be represented as

$$\mathbf{Lie}_{B\to W} = \frac{\mathbf{Lie}_B * \mathbf{Lie}_W * \mathsf{F}(f)}{(f\mu = \nu f^{\otimes 2})},$$

where μ and ν denote the generators in $\mathbf{Lie}_B(1,2)$ and $\mathbf{Lie}_W(1,2)$, respectively, which encode the multiplications in the domain and the target.

Let E be the 2-colored Σ-bimodule with the following skew symmetric generators:

$$\begin{aligned} &\mu_n\colon B^{\otimes n} \to B \text{ of degree } n-2 \text{ and biarity } (1,n)\ (n\geq 2)\\ &\nu_n\colon W^{\otimes n} \to W \text{ of degree } n-2 \text{ and biarity } (1,n)\ (n\geq 2), \text{ and}\\ &f_n\colon B^{\otimes n} \to W \text{ of degree } n-1 \text{ and biarity } (1,n)\ (n\geq 1). \end{aligned}$$

Then the minimal model for $\mathbf{Lie}_{B\to W}$ is

$$(\mathsf{F}(E),\partial) \xrightarrow{\alpha} \mathbf{Lie}_{B\to W},$$

where

$$\alpha(\mu_n) = \begin{cases} \mu & \text{if } n=2,\\ 0 & \text{otherwise,}\end{cases} \qquad \alpha(\nu_n) = \begin{cases} \nu & \text{if } n=2,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\alpha(f_n) = \begin{cases} f & \text{if } n=1,\\ 0 & \text{otherwise.}\end{cases}$$

The differential ∂ is given by:

$$\partial(\mu_n) = \sum_{\substack{i+j=n+1\\ i,j\geq 2}} (-1)^{j(i-1)} \sum_{\sigma\in S_{j,i-1}} \operatorname{sgn}(\sigma)\mu_i \circ (\mu_j \otimes \mathrm{Id}^{\otimes^{i-1}})\circ\sigma, \tag{34a}$$

$$\partial(\nu_n) = \sum_{\substack{i+j=n+1\\ i,j\geq 2}} (-1)^{j(i-1)} \sum_{\sigma\in S_{j,i-1}} \operatorname{sgn}(\sigma)\nu_i \circ (\nu_j \otimes \mathrm{Id}^{\otimes^{i-1}})\circ\sigma, \tag{34b}$$

$$\begin{aligned}\partial(f_n) = &\sum_{l=2}^{n} \sum_{\substack{r_1+\cdots+r_l=n\\ r_1\leq\cdots\leq r_l}} (-1)^{\varepsilon} \sum_{\sigma\in S^{<}_{r_1,\dots,r_l}} \operatorname{sgn}(\sigma)\nu_l(f_{r_1}\otimes\cdots\otimes f_{r_l})\circ\sigma\\ &- \sum_{\substack{i+j=n+1\\ i\geq 1, j\geq 2}} (-1)^{j(i-1)} \sum_{\sigma\in S_{j,i-1}} \operatorname{sgn}(\sigma) f_i \circ (\mu_j \otimes \mathrm{Id}^{\otimes^{i-1}})\circ\sigma,\end{aligned} \tag{34c}$$

where $S_{j,i-1}$ denotes the set of $j, i-1$ unshuffles and $S^{<}_{r_1,\dots,r_l}$ the set of $r_1, \dots, r_l$-unshuffles satisfying $\sigma(r_1 + \cdots + r_{i-1} + 1) < \sigma(r_1 + \cdots + r_i + 1)$ if $r_i = r_{i+1}$. The sign in the first line of (34c) is given by

$$\varepsilon = \varepsilon(r_1, \dots, r_l) := \frac{l(l-1)}{2} + \sum_{i=1}^{l-1} r_i(l-i). \tag{35}$$

It is also assumed in this notation that $r_i \le r_{i+1}$. We refer to [10] for the proof that it is a minimal model.

One may alternatively write the above formulas with the summations running over the entire symmetric groups, with coefficients involving factorials. This would reflect the convention in describing the morphism of L_∞-algebras used for instance in [22]. The graded anti-symmetry of the structure operations allows one to bring these formulas into the above 'reduced' form.

7.4. The cochain complex $(C^*_{\mathbf{Lie}_{B\to W}}(T;T), \delta_{\mathbf{Lie}_{B\to W}})$. Suppose that $T = U \oplus V$ as a 2-colored graded module and that $g\colon U \to V$ is a morphism of Lie algebras represented by the morphism $\rho\colon \mathbf{Lie}_{B\to W} \to \mathrm{End}_T$ of 2-colored operads. Then the canonical isomorphism (14) says in this case,

$$C^*_{\mathbf{Lie}_{B\to W}}(T;T) = \uparrow \mathrm{Der}(\mathsf{F}(E), \mathrm{End}_T)^{-*} \cong \uparrow \mathrm{Hom}^{\{B,W\}}_{\Sigma}(E, \mathrm{End}_T)^{-*}.$$

Under this isomorphism, an element $\theta \in C^n_{\mathbf{Lie}_{B\to W}}(T;T)$ is determined by the tuple

$$(\theta_U, \theta_V, \theta_g) \overset{\mathrm{def}}{=} (\theta(\mu_{n+1}), \theta(\nu_{n+1}), \theta(f_n)) \in \Lambda^{n+1}(U,V). \tag{36}$$

This establishes a linear isomorphism

$$\begin{aligned} C^n_{\mathbf{Lie}_{B\to W}}(T;T) &\cong \Lambda^{n+1}(U,V), \\ \theta &\leftrightarrow (\theta_U, \theta_V, \theta_g). \end{aligned}$$

Denote by δ the differential on the graded module $\Lambda^*(U,V)$ induced by $\delta_{\mathbf{Lie}_{B\to W}}$. The identification (36) provides an isomorphism

$$(C^*_{\mathbf{Lie}_{B\to W}}, \delta_{\mathbf{Lie}_{B\to W}}) \xrightarrow{\cong} (\Lambda^{*+1}(U,V), \delta) \tag{37}$$

of cochain complexes.

Theorem 7.5. *For $\theta \in C^{n-1}_{\mathbf{Lie}_{B\to W}}(T;T)$, we have*

$$\delta\,(\theta_U, \theta_V, \theta_g) = \left(b\theta_U, b\theta_V, -b\theta_g + \theta_V g^{\otimes n} - g\theta_U\right), \tag{38}$$

in which b denotes the appropriate Chevalley–Eilenberg differential.

One can then compare this complex (38) with the S-cohomology (33).

Corollary 7.6. *There is a cochain complex isomorphism*

$$\pi = (\pi^*, \pi^*, \widetilde{\pi}^*)\colon (\Lambda^*(U,V), \delta) \xrightarrow{\cong} (\Lambda^*(U,V), \Delta)$$

given by

$$\begin{cases} \pi^n &= \mathrm{Id}, \\ \widetilde{\pi}^n &= (-1)^{n-1}\,\mathrm{Id}\,. \end{cases}$$

Combined with (37)*, we obtain an isomorphism*

$$(C^*_{\mathbf{Lie}_{B\to W}}(T;T), \delta_{\mathbf{Lie}_{B\to W}}) \xrightarrow{\cong} (\Lambda^*(U,V), \Delta) \tag{39}$$

of cochain complexes.

Since $(C^*_{\mathbf{Lie}_{B\to W}}(T;T), \delta_{\mathbf{Lie}_{B\to W}} = l_1, l_2, l_3, \dots)$ is an L_∞-algebra (Theorem 4.2), we can use the cochain complex isomorphism (39) to transfer the higher brackets l_k $(k \geq 2)$ to $(\Lambda^*(U,V), \Delta)$.

Corollary 7.7. *There is an L_∞-algebra structure $(\Delta = l_1, l_2, l_3, \dots)$ on $\Lambda^*(U,V)$ capturing deformations of the Lie algebra morphism g.*

Proof of Theorem 7.5. Since $\delta = l_1$ in the L_∞-algebra and since the degree of l_1 is $+1$, we have

$$\delta\,(\theta_U, \theta_V, \theta_g) = (l_1(\theta)(\mu_{n+1}), l_1(\theta)(\nu_{n+1}), l_1(\theta)(f_n))$$

by the identification (36). Therefore, to prove (38), it suffices to show:

$$\begin{aligned} l_1(\theta)(\mu_{n+1}) &= b\theta_U, \\ l_1(\theta)(\nu_{n+1}) &= b\theta_V, \text{ and} \\ l_1(\theta)(f_n) &= -b\theta_g + \theta_V g^{\otimes n} - g\theta_U. \end{aligned} \tag{40}$$

From the description (15) of the operation l_k, the computation of the term $l_1(\theta)(\mu_{n+1})$ starts with $\partial(\mu_{n+1})$ (34a). As an E-decorated 2-colored directed $(1, n+1)$-graph, the term $\mu_i \circ_{s+1} \mu_j$ in $\partial(\mu_{n+1})$ has two vertices, whose decorations are μ_i and μ_j. Therefore, the expression (15), when applied to the current situation and using notation (20), gives

$$l_1(\theta)(\mu_{n+1}) = \sum_{\substack{i+j=n+2 \\ i,j\geq 2,\ \sigma\in S_{j,i-1}}} (-1)^{j(i-1)} \mathrm{sgn}(\sigma)\,\{\theta(\mu_i)\circ_1 \beta(\mu_j) + \beta(\mu_i)\circ_1 \theta(\mu_j)\} \circ \sigma. \tag{41}$$

Note that, since $\theta \in C^{n-1}_{\mathbf{Lie}_{B\to W}}(T;T)$,

$$\theta(\mu_i) = \begin{cases} 0 & \text{if } i \neq n, \\ \theta_U & \text{if } i = n, \end{cases} \quad \text{and} \quad \beta(\mu_j) = \rho(\alpha(\mu_j)) = \begin{cases} 0 & \text{if } j \neq 2, \\ \mu_U & \text{if } j = 2, \end{cases}$$

where $\mu_U\colon U^{\wedge 2} \to U$ is the multiplication on U. It follows that (41) reduces to

$$l_1(\theta)(\mu_{n+1}) = \underbrace{(-1)^{2(n-1)} \sum_{\sigma \in S_{2,n-1}} \operatorname{sgn}(\sigma)(\theta_U \circ_1 \mu_U) \circ \sigma}_{(a)}$$
$$+ \underbrace{(-1)^{n(2-1)} \sum_{\sigma \in S_{n,1}} \operatorname{sgn}(\sigma)(\mu_U \circ_1 \theta_U) \circ \sigma}_{(b)}.$$

In particular, applied to elements of U, (a) and (b) give:

$$(a)(x_1, ..., x_{n+1}) = (-1)^{s+t-1} \sum_{1 \le s < t \le n+1} \theta_U(\mu_U(x_s, x_t), x_1, ..., \hat{x_s}, ..., \hat{x_t}, ..., x_{n+1}),$$
$$(b)(x_1, ..., x_{n+1}) = (-1)^{n(2-1)} \sum_{1 \le s \le n+1} (-1)^{n+1-s} \mu_U(\theta_U(x_1, ..., \hat{x_s}, ..., x_{n+1}), x_s)$$
$$= (-1)^{-s} \sum_{1 \le s \le n+1} \mu_U(x_s, \theta_U(x_1, ..., \hat{x_s}, ..., x_{n+1})).$$

Therefore, by the definition of the Chevalley–Eilenberg differential, we have

$$l_1(\theta)(\mu_{n+1}) = b\theta_U,$$

which is the first condition in (40).

The previous paragraph applies verbatim to $l_1(\theta)(\nu_{n+1})$ (with ν_l replacing μ_l everywhere), since the definition of $\partial(\nu_*)$ (34b) admits the same formula as that of $\partial(\mu_*)$. Therefore, it remains to show the last condition in (40).

In $\partial(f_n)$, the term $\nu_l(f_{r_1} \otimes \cdots \otimes f_{r_l})$ (respectively, $f_i \circ_1 \mu_j$) is an E-decorated 2-colored directed $(1,n)$-graph with $l+1$ (respectively, 2) vertices. Since

$$\beta(f_j) = \rho(\alpha(f_j)) = \begin{cases} g\colon U \to V & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

the same kind of analysis as above gives

$$\begin{aligned} l_1(\theta)(f_n) = {} & \underbrace{\sum_{\sigma \in S^{<}_{1,\ldots,1}} \operatorname{sgn}(\sigma)\theta_V(g^{\otimes n}) \circ \sigma}_{(s_1)} + \underbrace{\sum_{\sigma \in S_{1,n-1}} \operatorname{sgn}(\sigma)\mu_V(g \otimes \theta_g) \circ \sigma}_{(s_2)} \\ & - (-1)^{2(n-1)} \underbrace{\sum_{\sigma \in S_{2,n-2}} \operatorname{sgn}(\sigma)\theta_g(\mu_U \otimes \mathrm{Id}^{\otimes n-2}) \circ \sigma}_{(s_3)} \\ & - (-1)^{n(1-1)} \underbrace{\sum_{\sigma \in S_{n,0}} \operatorname{sgn}(\sigma) g\theta_U \circ \sigma}_{(s_4)}. \end{aligned} \tag{42}$$

We now apply the middle summands of (42) on elements and rearrange them in order to recognize the Chevalley–Eilenberg differential. We have

$$(s_2)(x_1 \otimes \cdots \otimes x_n) = \sum_{1 \le s \le n} (-1)^{s-1} \mu_V(g \otimes \theta_g)(x_s \otimes x_1 \otimes \cdots \otimes \hat{x_s} \otimes \cdots \otimes x_n),$$

so

$$\begin{aligned}((s_2) + (s_3))(x_1 \otimes \cdots \otimes x_n) = \\ &+ \sum_{1 \le s \le n} (-1)^{s-1} \mu_V(g(x_s), \theta_g(x_1, \ldots, \hat{x_s}, \ldots, x_n)) \\ &- \sum_{1 \le s < t \le n} (-1)^{s-1+t-2} \theta_g(\mu_U(x_s, x_t), x_1, \ldots, \hat{x_s}, \ldots, \hat{x_t}, \ldots, x_n) \\ &= -b\theta_g(x_1 \otimes \cdots \otimes x_n).\end{aligned}$$

Considering the fact that both $S_{n,0}$ and $S^{<}_{1,\ldots,1}$ consist of a single element, the trivial permutation, one finally gets

$$l_1(\theta)(f_n) = -b\theta_g + \theta_V g^{\otimes n} - g\theta_U,$$

which ends the proof of Theorem 7.5. □

8. The higher brackets in $C^*_{\mathbf{Lie}_{B\to W}}(T;T)$

We keep the same setting and notations as in the previous section. The purpose of this section is to make the L_∞-operations l_k on $C^*_{\mathbf{Lie}_{B\to W}}(T;T)$ explicit for $k \ge 2$. The cases $k = 2$ (Theorem 8.2) and $k \ge 3$ (Theorem 8.4) are treated separately. As an immediate consequence of our explicit formula for l_k ($k \ge 3$), we observe that, when applied to the tensor powers of $C^{\le q}_{\mathbf{Lie}_{B\to W}}(T;T)$ for some fixed $q \ge 0$, only $\delta_{\mathbf{Lie}_{B\to W}}(T;T) = l_1, l_2, \ldots, l_{q+2}$ can be nontrivial (Corollary 8.5).

8.1. The operation l_2. First we deal with the case $k = 2$. Pick elements $\theta \in C^n_{\mathbf{Lie}_{B\to W}}(T;T)$ and $\omega \in C^m_{\mathbf{Lie}_{B\to W}}(T;T)$. Under the identification (36), θ and ω correspond to

$$(\theta_U, \theta_V, \theta_g) \in \Lambda^n(U,V) \quad \text{and} \quad (\omega_U, \omega_V, \omega_g) \in \Lambda^m(U,V),$$

respectively.

Since l_2 has degree 0, the element $l_2(\theta,\omega)$ lies in $C^{n+m}_{\mathbf{Lie}_{B\to W}}(T;T)$. Under the identification (36), $l_2(\theta,\omega)$ is determined by

$$(l_2(\theta,\omega)(\mu_{n+m+1}), l_2(\theta,\omega)(\nu_{n+m+1}), l_2(\theta,\omega)(f_{n+m})) \in \Lambda^{n+m}(U,V).$$

Theorem 8.2. *With the notations above, we have*

$$\text{(43a)}\quad l_2(\theta,\omega)(\mu_{n+m+1}) = (-1)^{mn}\Bigg(\sum_{\sigma \in S_{m+1,n}} \operatorname{sgn}(\sigma)\theta_U \circ_1 \omega_U \circ \sigma$$

$$+(-1)^{m+n}\sum_{\sigma\in S_{n+1,m}}\operatorname{sgn}(\sigma)\omega_U\circ_1\theta_U\circ\sigma\Bigg),$$

$$\text{(43b)}\quad l_2(\theta,\omega)(\nu_{n+m+1})=(-1)^{mn}\Bigg(\sum_{\sigma\in S_{m+1,n}}\operatorname{sgn}(\sigma)\theta_V\circ_1\omega_V\circ\sigma$$

$$+(-1)^{m+n}\sum_{\sigma\in S_{n+1,m}}\operatorname{sgn}(\sigma)\omega_V\circ_1\theta_V\circ\sigma\Bigg),$$

$$\text{(43c)}\quad l_2(\theta,\omega)(f_{n+m})=$$

$$(-1)^{m(n-1)}\Bigg(\sum_{\sigma\in S_{m+1,n-1}}\operatorname{sgn}(\sigma)\theta_g\circ_1\omega_U\circ\sigma+\sum_{\sigma\in S_{n+1,m-1}}\operatorname{sgn}(\sigma)\omega_g\circ_1\theta_U\circ\sigma\Bigg)$$

$$(-1)^n\Bigg(\sum_{\sigma\in S^<_{1,\dots,1,m}}\operatorname{sgn}(\sigma)\theta_V(g^{\otimes n}\otimes\omega_g)\circ\sigma+\sum_{\sigma\in S^<_{1,\dots,1,n}}\operatorname{sgn}(\sigma)\omega_V(g^{\otimes m}\otimes\theta_g)\circ\sigma\Bigg)$$

$$-\sum_{\sigma\in S^<_{n,m}}\operatorname{sgn}(\sigma)\mu_V(\theta_g\otimes\omega_g)\circ\sigma-(-1)^{n+m}\sum_{\sigma\in S^<_{m,n}}\operatorname{sgn}(\sigma)\mu_V(\omega_g\otimes\theta_g)\circ\sigma.$$

In the above theorem, we use the notation of (20) and of $S^<$ as defined after (34c). One should remark that depending on whether $m<n$ or $n<m$, the first or the second summand in the last bracketed expression above is zero. The proof of the theorem will be given at the end of this section.

8.3. The operations l_k for $k\geq 3$. Now consider the cases $k\geq 3$. Pick elements $\theta_s\in C^{n_s}_{\mathbf{Lie}_{B\to W}}(T;T)$ $(1\leq s\leq k)$. Each θ_s corresponds, via the identification (36), to the tuple

$$(\theta_{s,U},\theta_{s,V},\theta_{s,g})=(\theta_s(\mu_{n_s+1}),\theta_s(\nu_{n_s+1}),\theta_s(f_{n_s}))\in\bigwedge^{n_s}(U;V).$$

Since l_k has degree $2-k$, the element $l_k(\theta_1,\dots,\theta_k)$ lies in $C^t_{\mathbf{Lie}_{B\to W}}(T;T)$, where

$$t=-k+2+\sum_{s=1}^k n_s.$$

Under the identification (36), $l_k(\theta_1,\dots,\theta_k)$ is determined by

$$(l_k(\theta_1,\dots,\theta_k)(\mu_{t+1}),l_k(\theta_1,\dots,\theta_k)(\nu_{t+1}),l_k(\theta_1,\dots,\theta_k)(f_t))\in\bigwedge^t(U;V).$$

Now we extend the $\circ_i$ operation as follows. Fix $s\in\{1,\dots,k\}$. Let

$$\mathbf{a}'=(a_1,\dots,\widehat{a_s},\dots,a_k)$$

be a $(k-1)$-tuple of distinct points in the set $\{1,\dots,n_s+1\}$. Then we define

$$\theta_{s,V}\circ_{\mathbf{a}'}(\theta_{1,g},\dots,\widehat{\theta_{s,g}},\dots,\theta_{k,g})\in\operatorname{Hom}(U^{\otimes t},V)$$

to be the element obtained by plugging $\theta_{j,g}$ ($1 \leq j \leq k$, $j \neq s$) into the a_jth input of $\theta_{s,V}$ and g into the other $(n_s + 2 - k)$ inputs of $\theta_{s,V}$. Also define the coefficient

$$(-1)^{\mathbf{a}'} = (-1)^{\frac{(n_s+1)(n_s)}{2} + \sum_{i=n_s+1} r_i(n_s+1-i)},$$

where

$$r_a = \begin{cases} |\theta_j| = n_j & \text{if } a = a_j \in \{a_1, \dots, \widehat{a_s}, \dots, a_k\}, \\ 1 & \text{otherwise.} \end{cases}$$

Let us remark that the set $\{r_1, \dots, r_{n_s+1}\}$ satisfies $r_1 + \cdots + r_{n_s+1} = t$. One says that $\mathbf{a}'$ is *admissible* if this set also satisfies $r_1 \leq \cdots \leq r_{n_s+1}$. One denotes by A the set of admissible $\mathbf{a}'$.

Theorem 8.4. *For $k \geq 3$ and notations as above, we have*

$$l_k(\theta_1, \dots, \theta_k)(\mu_{t+1}) = 0,$$
$$l_k(\theta_1, \dots, \theta_k)(\nu_{t+1}) = 0, \text{ and}$$
$$l_k(\theta_1, \dots, \theta_k)(f_t) =$$
$$(-1)^{\nu(\theta_1,\dots,\theta_k)} \sum_{s=1}^{k} \sum_{\mathbf{a}' \in A} \sum_{\sigma \in S^{<}_{r_1,\dots,r_{n_s+1}}} \operatorname{sgn}(\sigma)(-1)^{\mathbf{a}'} \theta_{s,V} \circ_{\mathbf{a}'} (\theta_{1,g}, \dots, \widehat{\theta_{s,g}}, \dots, \theta_{k,g}) \circ \sigma,$$

with $\nu(\theta_1, \dots, \theta_k)$ defined as in (16), *and $S^{<}$ defined after* (34c).

Corollary 8.5. *Suppose $k \geq 3$ and $\theta_s \in C^{n_s}_{\mathbf{Lie}_{B\to W}}(T;T)$ $(1 \leq s \leq k)$. If*

$$n_s < k - 1 \quad \text{for} \quad 1 \leq s \leq k,$$

then

$$l_k(\theta_1, \dots, \theta_k) = 0.$$

In other words, for each $q \geq 0$ and any $k \geq q + 3$, the operation

$$l_k \colon \left(C^{\leq q}_{\mathbf{Lie}_{B\to W}}(T;T)\right)^{\otimes k} \to C^{*}_{\mathbf{Lie}_{B\to W}}(T;T)$$

is trivial.

Proof of Theorem 8.2. To prove (43a), first note that

$$\partial(\mu_{n+m+1}) = \sum_{\substack{i+j=n+m+2 \\ i,j \geq 2}} (-1)^{j(i-1)} \sum_{\sigma \in S_{j,i-1}} \operatorname{sgn}(\sigma) \mu_i \circ (\mu_j \otimes \operatorname{Id}^{i-1}) \circ \sigma.$$

Since the E-decorated 2-colored directed $(1, n+m+1)$-graph $\mu_i \circ (\mu_j \otimes \mathrm{Id}^{i-1})$ has two vertices, we have

$$
\begin{aligned}
&l_2(\theta,\omega)(\mu_{n+m+1}) = \\
&\quad (-1)^{|\theta|} \sum_{\substack{i+j=n+m+2\\ i,j\geq 2}} (-1)^{j(i-1)} \sum_{\sigma\in S_{j,i-1}} \mathrm{sgn}(\sigma) \\
&\qquad\qquad \Big\{\theta(\mu_i)\circ(\omega(\mu_j)\otimes g^{\otimes^{i-1}}) + \omega(\mu_i)\circ(\theta(\mu_j)\otimes g^{\otimes^{i-1}})\Big\}\circ\sigma \\
&\quad = (-1)^{mn} \sum_{\sigma\in S_{m+1,n}} \mathrm{sgn}(\sigma)\theta_U\circ(\omega_U\otimes g^{\otimes^n})\circ\sigma \\
&\quad\ +(-1)^{(n+1)m+n} \sum_{\sigma\in S_{n+1,m}} \mathrm{sgn}(\sigma)\omega_U\circ(\theta_U\otimes g^{\otimes^m})\circ\sigma.
\end{aligned}
$$

Since $\partial(\nu_{n+m+1})$ has the same defining formula as $\partial(\mu_{n+m+1})$ (with ν_l replacing μ_l everywhere), the reasoning in the previous paragraph also applies to $l_2(\theta,\omega)(\nu_{n+m+1})$ to establish (43b).

To prove (43c), first note that

$$
\begin{aligned}
\partial(f_{n+m}) = &\sum_{l=2}^{n+m} \sum_{\substack{r_1+\cdots+r_l=n+m\\ r_1\leq\cdots\leq r_l}} (-1)^{\varepsilon} \sum_{\sigma\in S^{<}_{r_1,\ldots,r_l}} \mathrm{sgn}(\sigma)\nu_l(f_{r_1}\otimes\cdots\otimes f_{r_l})\circ\sigma \\
&- \sum_{\substack{i+j=n+m+1\\ i\geq 1, j\geq 2}} (-1)^{j(i-1)} \sum_{\sigma\in S_{j,i-1}} \mathrm{sgn}(\sigma) f_i\circ(\mu_j\otimes \mathrm{Id}^{\otimes^{i-1}})\circ\sigma,
\end{aligned}
$$

where ε is as in (35).

An argument essentially identical to the first paragraph of this proof can be applied to the terms $f_i \circ (\mu_j \otimes \mathrm{Id}^{i-1})$. This gives rise to the sums

$$
\begin{aligned}
&(-1)^{m(n-1)} \sum_{\sigma\in S_{m+1,n-1}} \mathrm{sgn}(\sigma)\theta_g\circ(\omega_U\otimes g^{\otimes^{n-1}})\circ\sigma \\
&+ (-1)^{m(n-1)} \sum_{\sigma\in S_{n+1,m-1}} \mathrm{sgn}(\sigma)\omega_g\circ(\theta_U\otimes g^{\otimes^{m-1}})\circ\sigma
\end{aligned}
\tag{45}
$$

in $l_2(\theta,\omega)(f_{n+m})$.

In (45), the E-decorated 2-colored directed $(1, n+m)$-graph

$$\Gamma = \nu_l(f_{r_1}\otimes\cdots\otimes f_{r_l})$$

has $l+1$ vertices, say, $v_{\mathrm{top}}, v^1_{\mathrm{bot}}, \ldots, v^l_{\mathrm{bot}}$, with decorations $\nu_l, f_{r_1}, \ldots, f_{r_l}$, respectively. In this graph Γ, the only pairs of distinct vertices are $(v_{\mathrm{top}}, v^*_{\mathrm{bot}})$, $(v^*_{\mathrm{bot}}, v_{\mathrm{top}})$, and $(v^i_{\mathrm{bot}}, v^j_{\mathrm{bot}})$ $(i\neq j)$. The corresponding elements (without the signs) in $l_2(\theta,\omega)(f_{n+m})$ are:

(1) $\theta(\nu_l)(\beta(f_{r_1})\otimes\cdots\otimes\omega(f_{r_i})\otimes\cdots\beta(f_{r_l}))$ $(1\leq i\leq l)$, which is 0 unless $l = n+1$, $r_{n+1} = m$, and all the other $r_* = 1$;

(2) $\omega(\nu_l)(\beta(f_{r_1})\otimes\cdots\otimes\theta(f_{r_j})\otimes\cdots\otimes\beta(f_{r_l}))$ $(1\le j\le l)$, which is 0 unless $l=m+1$, $r_{m+1}=n$, and all the other $r_*=1$;
(3) $\beta(\nu_l)(\beta(f_{r_1})\otimes\cdots\otimes\theta(f_{r_i})\otimes\cdots\otimes\omega(f_{r_j})\otimes\cdots\otimes\beta(f_{r_l}))$, which is 0 unless $l=2$ and $(r_1,r_2)=(n,m)$;
(4) $\beta(\nu_l)(\beta(f_{r_1})\otimes\cdots\otimes\omega(f_{r_i})\otimes\cdots\otimes\theta(f_{r_j})\otimes\cdots\otimes\beta(f_{r_l}))$, which is 0 unless $l=2$ and $(r_1,r_2)=(m,n)$.

Taking all the signs into account, we also obtain the following sums in $l_2(\theta,\omega)(f_{n+m})$:

$$\begin{aligned}
&(-1)^{|\theta|}\sum_{\sigma\in S^<_{1,\ldots,1,m}}\operatorname{sgn}(\sigma)\theta_V(g^{\otimes n}\otimes\omega_g)\circ\sigma,\\
&(-1)^{|\theta|}\sum_{\sigma\in S^<_{1,\ldots,1,n}}\operatorname{sgn}(\sigma)\omega_V(g^{\otimes m}\otimes\theta_g)\circ\sigma,\\
&(-1)^{|\theta|}(-1)^{1+n}\sum_{\sigma\in S^<_{n,m}}\operatorname{sgn}(\sigma)\mu_V(\theta_g\otimes\omega_g)\circ\sigma,\\
&(-1)^{|\theta|}(-1)^{1+m}\sum_{\sigma\in S^<_{m,n}}\operatorname{sgn}(\sigma)\mu_V(\omega_g\otimes\theta_g)\circ\sigma.
\end{aligned}\tag{46}$$

The required result (43c) is now obtained by combining (45) and (46). This finishes the proof of Theorem 8.2. □

Proof of Theorem 8.4. This proof is identical to the proof of Theorem 6.4 if one shifts the indices, replacing t by $t+1$, $s+1$ by 1, and $(-1)^{\mathbf{a}}$ by $-(-1)^{\mathbf{a}'}$ □

9. Deformations of diagrams

Recall that a *diagram* in a category $\mathcal{C}$ is a functor $\mathcal{F}:\mathcal{D}\to\mathcal{C}$ from a small category $\mathcal{D}$ to $\mathcal{C}$; the category $\mathcal{D}$ is called the *shape* of the diagram $\mathcal{F}$. Diagrams of shape $\mathcal{D}$ can equivalently be described as algebras over an $\mathrm{Ob}(\mathcal{D})$-colored operad $\mathbf{D}$ which has only elements of arity 1 (one input, one output) and

$$\mathbf{D}\binom{d}{c}:=\mathrm{Mor}_{\mathcal{D}}(c,d),\quad\text{for } c,d\in\mathrm{Ob}(\mathcal{D}).$$

The operadic composition in $\mathbf{D}$ equals the categorial composition of $\mathcal{D}$. It is clear that $\mathcal{D}$-diagrams in $\mathcal{C}$ are precisely $\mathrm{Ob}(\mathcal{D})$-colored $\mathbf{D}$-algebras in $\mathcal{C}$.

The $\mathrm{Ob}(\mathcal{D})$-colored operad $\mathbf{D}$ as defined above lives in the category of sets. Since we will be primarily interested in diagrams in the category of $\mathbf{k}$-vectors spaces, we may as well consider the $\mathbf{k}$-linear operad generated by $\mathbf{D}$ or assume from the very beginning that $\mathbf{D}$ is given by

$$\mathbf{D}\binom{d}{c}:=\mathrm{Span}_{\mathbf{k}}\left(\mathrm{Mor}_{\mathcal{D}}(c,d)\right),\quad\text{for } c,d\in\mathrm{Ob}(\mathcal{D}),$$

where $\mathrm{Span}_{\mathbf{k}}(-)$ denotes the $\mathbf{k}$-linear span. We will call colored operads of the above form *diagram operads*.

Example 9.1. In this example we describe the diagram operad **Iso** associated to the category $\mathcal{I}so$ consisting of two objects and two mutually inverse maps between these objects. Let $f : \mathtt{B} \to \mathtt{W}$, $g : \mathtt{W} \to \mathtt{B}$ be two degree-zero generators. Then

$$\mathbf{Iso} := \frac{\mathsf{F}(f,g)}{(fg = 1_{\mathtt{W}},\ gf = 1_{\mathtt{B}})},$$

where $\mathsf{F}(f,g)$ denotes the free $\{\mathtt{B},\mathtt{W}\}$-colored operad on the set $\{f,g\}$ and $(fg = 1_{\mathtt{W}},\ gf = 1_{\mathtt{B}})$ the operadic ideal generated by $fg - 1_{\mathtt{W}}$ and $gf - 1_{\mathtt{B}}$.

Algebras over **Iso** consist of two mutually inverse degree zero chain maps $F : U \to V$ and $G : V \to U$. In other words, **Iso**-algebras are diagrams

$$\begin{tikzcd} U \arrow[r, bend left, "F"] & V \arrow[l, bend left, "G"] \end{tikzcd},\ FG = 1 \ \text{ and } \ GF = 1. \tag{47}$$

A typical diagram operad $\mathbf{D}$ does not admit a minimal model. For instance, a hypothetical minimal model of the operad **Iso** from Example 9.1 shall have generators f_0, g_0 for f and g, but also a generator, say f_1, whose boundary kills the difference $f_0 g_0 - 1_{\mathtt{W}}$, i.e. satisfying

$$f_0 g_0 - 1_{\mathtt{W}} = \partial f_1.$$

The "constant" $1_{\mathtt{W}}$ however defies any thinkable notion of minimality.

This phenomenon is related to the fact that a typical diagram operad $\mathbf{D}$, such as **Iso**, is not augmented, by which we mean that it does not admit an operad morphism $\mathbf{D} \to \mathbf{i}$ to the terminal $\mathrm{Ob}(\mathcal{D})$-colored operad $\mathbf{i}$. In the next example we will see that sometimes there still exists a cofibrant resolution whose size is that of a minimal model.

Example 9.2. A small cofibrant resolution of **Iso** was described in [33, Theorem 9]. It is a graded colored differential operad

$$\mathcal{R}_{\mathrm{iso}} := (\mathsf{F}(f_0, f_1, \dots; g_0, g_1, \dots), d),$$

with generators of two types,

$$\text{(i) generators } \{f_n\}_{n\geq 0},\ \deg(f_n) = n, \begin{cases} f_n : \mathtt{B} \to \mathtt{W} \text{ if } n \text{ is even,} \\ f_n : \mathtt{B} \to \mathtt{B} \text{ if } n \text{ is odd,} \end{cases}$$

$$\text{(ii) generators } \{g_n\}_{n\geq 0},\ \deg(g_n) = n, \begin{cases} g_n : \mathtt{W} \to \mathtt{B} \text{ if } n \text{ is even,} \\ g_n : \mathtt{W} \to \mathtt{W} \text{ if } n \text{ is odd.} \end{cases}$$

The differential ∂ is given by

$$\begin{aligned} \partial f_0 &:= 0, & \partial g_0 &:= 0, \\ \partial f_1 &:= g_0 f_0 - 1, & \partial g_1 &:= f_0 g_0 - 1 \end{aligned}$$

and, on remaining generators, by the formula

$$\begin{aligned}
\partial f_{2m} &:= \sum_{0\le i<m} (f_{2i}f_{2(m-i)-1} - g_{2(m-i)-1}f_{2i}),\ m\ge 0,\\
\partial f_{2m+1} &:= \sum_{0\le j\le m} g_{2j}f_{2(m-j)} - \sum_{0\le j<m} f_{2j+1}f_{2(m-j)-1},\ m\ge 1,\\
\partial g_{2m} &:= \sum_{0\le i<m} (g_{2i}g_{2(m-i)-1} - f_{2(m-i)-1}g_{2i}),\ m\ge 0,\\
\partial g_{2m+1} &:= \sum_{0\le j\le m} f_{2j}g_{2(m-j)} - \sum_{0\le j<m} g_{2j+1}g_{2(m-j)-1},\ m\ge 1.
\end{aligned}$$

The above resolution is "minimal' in the following sense. Consider a one-parametric family $\mathbf{Iso}_\varepsilon$ of $\{\mathtt{B},\mathtt{W}\}$-colored operads defined by

$$\mathbf{Iso}_\varepsilon := \frac{\mathsf{F}(f,g)}{(fg=\varepsilon\cdot 1_{\mathtt{W}}, gf=\varepsilon\cdot 1_{\mathtt{B}})},$$

where ε is a formal parameter. The operad $\mathbf{Iso}_0$ clearly describes couples (F,G) of maps $F:U\to V$ and $G:V\to U$ such that $FG=0$ and $GF=0$, while $\mathbf{Iso}_\varepsilon$ is, at a generic ε, isomorphic to the operad $\mathbf{Iso}$. In other words, $\mathbf{Iso}$ is a deformation of $\mathbf{Iso}_0$. It turns out that $\mathbf{Iso}_0$ is an augmented colored operad that admits a minimal model whose generators are the same as the generators of $\mathcal{R}_{\mathrm{iso}}$; see [33, Theorem 10].

An obvious generalization of the machinery developed in the previous sections applies verbatim to resolutions of diagram operads. One typically gets an L_∞-algebra with a nontrivial 'curvature' l_0; see [36, Section 5] for the terminology and definitions. The corresponding Maurer–Cartan equation then involves the l_0-term.

Example 9.3. In this example we describe the L_∞-deformation complex $(C^*_{\mathbf{Iso}}(T,T), l_0, l_1, l_2, \dots)$ for a diagram T as in (47). As we already observed, this deformation complex has, as a consequence of the presence of 1 in the formulas for ∂f_1 and ∂g_1 in $\mathcal{R}_{\mathrm{iso}}$, a nontrivial l_0. On the other hand, since the differential ∂ on the generators of $\mathcal{R}_{\mathrm{iso}}$ does not have higher than quadratic terms, all l_k's are trivial, for $k\ge 3$.

Formula (11) applied to the resolution $\mathcal{R}_{\mathrm{iso}}$ from Example 9.2 gives the underlying cochain complex

$$C^n_{\mathbf{Iso}}(T,T) = \begin{cases} \mathrm{Hom}(U,V)\oplus\mathrm{Hom}(V,U) & \text{for } n\ge 1 \text{ odd, and}\\ \mathrm{Hom}(U,U)\oplus\mathrm{Hom}(V,V) & \text{for } n\ge 1 \text{ even.}\end{cases}$$

Formula (15) makes sense also for $k=0$ and describes $l_0\in C^2_{\mathbf{Iso}}(T,T)$ as the direct sum of the identity maps $\mathrm{Id}_U\oplus\mathrm{Id}_V\in\mathrm{Hom}(U,U)\oplus\mathrm{Hom}(V,V)$.

Likewise, one obtains the following formulas for the operation

$$l_1 : C^*_{\mathbf{Iso}}(T,T)\to C^{*+1}_{\mathbf{Iso}}(T,T).$$

If $\alpha \oplus \beta \in C^n_{\mathbf{Iso}}(T,T)$, $n \geq 1$ odd, then

$$l_1(\alpha \oplus \beta) = (G\alpha + \beta F) \oplus (\alpha G + F\beta) \in C^{n+1}_{\mathbf{Iso}}(T,T).$$

For $\gamma \oplus \delta \in C^n_{\mathbf{Iso}}(T,T)$, $n \geq 1$ even, we have

$$l_1(\gamma \oplus \delta) = (F\gamma - \delta F) \oplus (G\delta - \gamma G) \in C^{n+1}_{\mathbf{Iso}}(T,T).$$

The bracket

$$l_2 : C^m_{\mathbf{Iso}}(T,T) \otimes C^n_{\mathbf{Iso}}(T,T) \to C^{m+n}_{\mathbf{Iso}}(T,T)$$

is given as follows. For $\alpha' \oplus \beta' \in C^m_{\mathbf{Iso}}(T,T)$, $\alpha'' \oplus \beta'' \in C^n_{\mathbf{Iso}}(T,T)$, m,n odd, we have

$$l_2(\alpha' \oplus \beta', \alpha'' \oplus \beta'') = (\beta'\alpha'' + \beta''\alpha') \oplus (\alpha'\beta'' + \alpha''\beta') \in C^{m+n}_{\mathbf{Iso}}(T,T).$$

For $\alpha \oplus \beta \in C^m_{\mathbf{Iso}}(T,T)$, $\gamma \oplus \delta \in C^n_{\mathbf{Iso}}(T,T)$, m odd, n even, we have

$$l_2(\alpha \oplus \beta, \gamma \oplus \delta) = -l_2(\gamma \oplus \delta, \alpha \oplus \beta) = (\alpha\gamma - \delta\alpha) \oplus (\beta\delta - \gamma\beta) \in C^{m+n}_{\mathbf{Iso}}(T,T).$$

Finally, for $\gamma' \oplus \delta' \in C^m_{\mathbf{Iso}}(T,T)$, $\gamma'' \oplus \delta'' \in C^n_{\mathbf{Iso}}(T,T)$, m,n even, we have

$$\begin{aligned} l_2(\gamma' \oplus \delta', \gamma'' \oplus \delta'') = -l_2(\gamma'' \oplus \delta'', \gamma' \oplus \delta') = \\ (\gamma'\gamma'' - \gamma''\gamma') \oplus (\delta'\delta'' - \delta''\delta') \in C^{m+n}_{\mathbf{Iso}}(T,T). \end{aligned}$$

The higher l_k's, $k \geq 3$, are trivial.

Observe that, for each $w \in C^*_{\mathbf{Iso}}(T,T)$, $l_2(l_0, w) = 0$; therefore, by [36, Section 5], $l_1^2 = 0$. In other words, l_1 is a differential and the standard analysis of deformation theory applies. For instance, there exists the canonical element $\chi := F \oplus G \in C^1_{\mathbf{Iso}}(T,T)$ such that

$$l_1(w) = l_2(\chi, w), \ w \in C^*_{\mathbf{Iso}}(T,T).$$

As expected [36, 51], **Iso**-algebras are solutions of the Maurer–Cartan equation in the L_∞-complex of the trivial **D**-algebra $T_\emptyset$ with $F = 0$, $G = 0$. Indeed, if $\kappa = \Phi \oplus \Psi \in C^1_{\mathbf{Iso}}(T,T)$, then the Maurer–Cartan equation

$$-l_0 + \frac{1}{2} l_2(\kappa, \kappa) = 0$$

expands into

$$-(\mathrm{Id}_U \oplus \mathrm{Id}_V) + \frac{1}{2}(2\Psi\Phi \oplus 2\Phi\Psi) = 0,$$

which says precisely that Φ and Ψ are mutually inverse isomorphisms.

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Mathematics Research Unit, 162A, avenue de la faiencerie L-1511, Luxembourg, Grand duchy of Luxembourg
yael.fregier@uni.lu

Mathematical Institute of the Academy, Žitná 25, 115 67 Prague 1, The Czech Republic
markl@math.cas.cz

Department of Mathematics, The Ohio State University at Newark, 1179 University Drive, Newark, OH 43055, USA
dyau@math.ohio-state.edu

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