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Distal actions and ergodic actions on compact groups

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Abstract. Let K be a compact metrizable group and Γ be a group of automorphisms of K. We first show that each $\alpha \in \Gamma$ is distal on K implies Γ itself is distal on K, a local to global correspondence provided Γ is a generalized $\overline{FC}$-group or K is a connected finite-dimensional group. We also prove a connection between distality and ergodicity which is used to show that ergodic actions of nilpotent groups on compact connected finite-dimensional abelian groups contains ergodic automorphisms.

Contents

1.	Introduction	301
2.	Distal and ergodic	305
3.	Compact abelian groups	306
4.	Distal actions	308
5.	Finite-dimensional compact groups	309
6.	Example	316
References		317

1. Introduction

We shall be considering actions on compact groups. By a compact group we shall mean a compact metrizable group and by an automorphism we shall mean a continuous automorphism. For a compact group K, Aut (K) denotes the group of all automorphisms of K. An action of a topological group Γ on a compact metrizable group K by automorphisms is a homomorphism

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$\phi\colon \Gamma \to \text{Aut }(K)$ such that the map $(\alpha, x) \mapsto \phi(\alpha)(x)$ is a continuous map: when only one action is studied or when there is no confusion instead of $\phi(\alpha)(x)$ we may write $\alpha(x)$ for $\alpha \in \Gamma$ and $x \in K$. In such cases, the map ϕ is said to define the action of Γ on K and such actions are called algebraic actions.

We shall assume that a topological group Γ acts on a compact metrizable group K. For each $\alpha \in \Gamma$, $(n, a) \mapsto \alpha^n(a)$ defines a $\mathbb{Z}$-action on K and this action on K is called the $\mathbb{Z}_\alpha$-action. Suppose $K_1 \supset K_2$ are closed Γ-invariant subgroups of K such that K_2 is normal in K_1. By an action of Γ on K_1/K_2, we mean the canonical action of Γ on K_1/K_2 defined by $\alpha(xK_2) = \alpha(x)K_2$ for all $x \in K_1$ and all $\alpha \in \Gamma$.

Suppose Γ acts on the compact groups K and L. We say that K and L are Γ-isomorphic if there exists a continuous isomorphism $\Phi\colon K \to L$ such that $\Phi(\alpha(x)) = \alpha(\Phi(x))$ for all $\alpha \in \Gamma$ and $x \in K$.

It is interesting to find properties of group actions that hold if the property holds for every $\mathbb{Z}_\alpha$-action. We term any such property a local to global correspondence as this property holds for the whole group Γ when it holds locally at every point of Γ. We first state the following well-known classical local to global correspondence for linear actions on vector spaces, a proof of which may be found in [6].

Burnside Theorem. *Let V be a finite-dimensional vector space over the reals and let G be a finitely generated subgroup of* $\mathrm{GL}(V)$*, the group of linear transformations on V. If each element of G has finite order, then G itself is a finite group.*

The main aim of the note is to exhibit such local to global correspondences for algebraic actions on compact groups.

Definition 1.1. We say that the action of Γ on K is distal if for any $x \in K \setminus (e)$, e is not in the closure of the orbit $\Gamma(x) = \{\alpha(x) \mid \alpha \in \Gamma\}$. In such case, we say that Γ is distal (on K).

We now introduce a type of action which is obviously distal.

Definition 1.2. We say that the action of Γ on K is compact (respectively, finite) if the group $\phi(\Gamma)$ is contained in a compact (respectively, finite) subgroup of Aut (K) where ϕ is the map defining the action of Γ on K.

We now see the notion of ergodic action which is hereditarily antithetical to distal action (cf. Proposition 2.1).

Definition 1.3. Let K be a compact group and ω_K be the normalized Haar measure on K. We say that an (algebraic) action of Γ on K is ergodic if any Γ-invariant Borel set A of K satisfies $\omega_K(A) = 0$ or $\omega_K(A) = 1$.

Definition 1.4. Let K be a compact group and α be a continuous automorphism of K. If the action of $\mathbb{Z}_\alpha$ on K is distal (respectively, ergodic), then we say that α is a distal (respectively, ergodic) automorphism of K or α is distal (respectively, ergodic) on K.

We now introduce a class of groups whose action is one of the main studies in this article.

Definition 1.5. A locally compact group G is called a generalized $\overline{FC}$-group if G has a series $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\}$ of closed normal subgroups such that G_i/G_{i+1} is a compactly generated group with relatively compact conjugacy classes for $i = 0, 1, \ldots, n-1$.

It follows from Theorem 2 of [14] that compactly generated locally compact groups of polynomial growth are generalized $\overline{FC}$-groups and any polycyclic group is a generalized $\overline{FC}$-group.

It can easily be seen that the class of generalized $\overline{FC}$-groups is stable under forming continuous homomorphic images and closed subgroups. If H is a compact normal subgroup of a locally compact group G such that G/H is a generalized $\overline{FC}$-group, then it is easy to see that G is also a generalized $\overline{FC}$-group.

It is evident that Γ is distal implies each $\alpha \in \Gamma$ is distal. For actions on connected Lie groups [1] and for certain actions on p-adic Lie groups [17] the local to global correspondence for distality, that is passing from each $\alpha \in \Gamma$ being distal on K to the whole group Γ being distal on K, is valid: the distal notion has a canonical extension to actions on locally compact spaces (cf. [7]). In general each $\alpha \in \Gamma$ is distal need not imply Γ is distal (cf. Example 1, [19]). We will now closely examine a general form of Example 1 of [19].

Example 1.6. Let M be a compact group and Γ be a countably infinite group. Take $K = M^\Gamma$. The (left)-shift action of Γ on K is defined as follows: for $\alpha \in \Gamma$ and $f \in M^\Gamma$, αf is defined to be $\alpha f(\beta) = f(\alpha^{-1}\beta)$ for all $\beta \in \Gamma$. For $x \neq e \in M$, consider $f_x \in M^\Gamma$ defined by $f_x(\alpha) = e$ if $\alpha \neq 1$ and $f_x(\alpha) = x$ if $\alpha = 1$ where 1 is the identity in Γ. Choose a sequence (α_n) is Γ such that $\alpha_n \neq \alpha_m$ whenever $n \neq m$. Then $\alpha_n(f_x) \to e$ in M^Γ which may be seen as follows: for $\alpha \in \Gamma$, $\alpha_n^{-1}\alpha \neq 1$ for large n, so $(\alpha_n f_x)(\alpha) = f_x(\alpha_n^{-1}\alpha) = e$ for large n, hence $\alpha_n(f_x) \to e$ in M^Γ. Thus, the shift action of Γ is not distal. Suppose Γ is a torsion group (for instance, Γ may be the group of all finite permutations). Then each $\alpha \in \Gamma$ is distal but we have seen that Γ is not distal.

If Γ is assumed to be finitely generated nilpotent or finitely generated solvable, then situation as in Example 1.6 does not arise as Γ is torsion implies Γ is finite. Recently Theorem 2.9 of [12] showed that if K is a zero-dimensional compact group and Γ is a generalized $\overline{FC}$-group, then each $\alpha \in \Gamma$ is distal and the whole group Γ is distal are equivalent to the action being equicontinuous (that is, having invariant neighborhoods). Motivated by this, here we prove that each $\alpha \in \Gamma$ is distal on a compact group K if and only if Γ is distal on K provided Γ is a generalized $\overline{FC}$-group: the fact that generalized $\overline{FC}$-groups have a normal series of compactly generated subgroups plays a crucial in the proof of our results.

It can easily be observed that the compact group K in Example 1.6 can not be a connected finite-dimensional group and so we in fact prove that if K is a compact connected finite-dimensional group, then (with no restriction on Γ) each $\alpha \in \Gamma$ is distal on K if and only if Γ is distal on K.

The study of ergodic actions on compact groups is a key tool in proving the afore-stated results. We first establish a connection between distal actions and nonergodic actions. This makes us to ponder if there is any local to global correspondence for nonergodic actions and leads us to the question of determining K and Γ so that action of Γ on K is ergodic if and only if Γ contains an ergodic automorphism: this is a local to global correspondence for nonergodicity. The following example is useful in determining conditions on K and Γ to obtain a local to global correspondence for nonergodicity.

Example 1.7. Let Γ be a countable infinite group and M be a compact abelian group. Let $K = M^\Gamma$. We consider the shift action of Γ on K defined as in Example 1.6. We now claim that the shift action of Γ on K is ergodic. Let $\hat{M}$ be the (dual) group of characters on M. Then the dual $\hat{K}$ of K consists of functions $f\colon \Gamma \to \hat{M}$ such that $f(b)$ is the trivial character for all but finitely many $b \in \Gamma$ (see Theorem 17 of [15]). Then the dual action of Γ on the dual $\hat{K}$ is given by $af(b) = f(a^{-1}b)$ for all $f \in \hat{K}$ and all $a, b \in \Gamma$. Let $f \in \hat{K}$. Then define $F = \{b \in \Gamma \mid f(b) \neq 1\}$ where 1 is the trivial character on M, the identity in $\hat{M}$. If $af = f$, then $a^{-1}F \subset F$ and hence $aF = F$. Since Γ is infinite, if the orbit $\Gamma(f)$ is finite, then for infinitely many $a \in \Gamma$, $af = f$ and $aF = F$, hence F is empty or infinite. Thus, the orbit $\Gamma(f)$ is infinite for any nontrivial $f \in \hat{K}$. This implies that the action of Γ on K is ergodic. Suppose Γ is a torsion group (one may take $F_n = \prod_{k=1}^{n} \mathbb{Z}/k\mathbb{Z}$ and $\Gamma = \cup F_n$). We get that the action of Γ on K is ergodic. Since any $a \in \Gamma$ has finite order, the action of $\mathbb{Z}_a$ is never ergodic for any $a \in \Gamma$. Using the counterexamples to the Burnside problem we get finitely generated infinite (nonsolvable) torsion groups and so such groups act ergodically but no element of which is ergodic.

If K is a compact connected finite-dimensional (abelian) group, then situation as in Example 1.7 does not arise. In this aspect Berend [2] proved that an ergodic action of commuting epimorphisms on a compact connected finite-dimensional abelian group contains ergodic epimorphisms. Recently [3] proved that certain hereditarily ergodic actions of solvable groups on compact connected finite-dimensional abelian groups contain ergodic automorphisms. In this article we apply our study of distal actions and ergodic actions to prove that an ergodic action of a nilpotent group on a compact connected finite-dimensional abelian group admits ergodic automorphisms and we provide examples to show that this type of result is limited to nilpotent actions (cf. Example 5.16).

Having explained our results, it is easy to see that only $\phi(\Gamma)$ matters and not all of Γ. So, we may assume that Γ is a group of automorphisms of K.

2. Distal and ergodic

We now explore the connection between distal actions and ergodic actions on compact (metrizable) groups using the dual structure of compact groups.

Let K be a compact group and Γ be a group acting on K. Let $\hat{K}$ be the equivalence classes of continuous irreducible unitary representations of K. If π is a continuous irreducible unitary representation of K, then $[\pi] \in \hat{K}$ denotes the set of all continuous irreducible unitary representations of K that are unitarily equivalent to π. We write $\pi_1 \sim \pi_2$ if $\pi_1, \pi_2 \in [\pi]$ for some $[\pi] \in \hat{K}$. For a continuous irreducible unitary representation π of K and $\alpha \in \Gamma$, $\alpha(\pi)$ is defined by

$$\alpha(\pi)(x) = \pi(\alpha^{-1}(x))$$

for all $x \in K$ and it can easily be verified that $\alpha(\pi)$ is also a continuous irreducible unitary representation of K. If $\alpha \in \Gamma$ and $\pi_1, \pi_2 \in [\pi]$, then $\alpha(\pi_1) \sim \alpha(\pi_2)$. Thus, the map $(\alpha,\ [\pi]) \mapsto \alpha[\pi] = [\alpha(\pi)]$ is well-defined and is known as the dual of action of Γ on the dual $\hat{K}$ of K. For $k \geq 1$, let $\mathrm{U}_k(\mathbb{C})$ be the group of unitaries on $\mathbb{C}^k$ and I_k denote the identity matrix in $\mathrm{U}_k(\mathbb{C})$. Then $\mathrm{U}_k(\mathbb{C})$ is a compact group and for each $[\pi] \in \hat{K}$, there exists a $k \geq 1$ such that $\pi(x) \in \mathrm{U}_k(\mathbb{C})$ for all $x \in K$: see [8] for details on representations of compact groups.

Proposition 2.1. *Let K be a compact group and Γ be a group of automorphisms of K. Then the following are equivalent:*

(1) *Γ is distal on K.*
(2) *For each Γ-invariant nontrivial closed subgroup L of K, action of Γ on L is not ergodic.*
(3) *For each Γ-invariant nontrivial closed subgroup L of K, there exists a nontrivial continuous irreducible unitary representation π of L such that the orbit $\Gamma[\pi] = \{\alpha[\pi] \mid \alpha \in \Gamma\}$ is finite in $\hat{L}$.*

Proof. Let L be a nontrivial Γ-invariant closed subgroup of K. If the action of Γ on L is ergodic, then by Theorem 2.1 of [2], $\Gamma(x) = \{\alpha(x) \mid \alpha \in \Gamma\}$ is dense in L for some $x \in L$. Since L is nontrivial, $x \neq e$ and hence e is in the closure of $\Gamma(x)$ for $x \neq e$. Thus, we get that (1) $\Rightarrow$ (2) and that (2) $\Rightarrow$ (3) follows from Theorem 2.1 of [2].

Now assume that (3) holds. Let $x \neq e$ be in K and L be the closed subgroup generated by $\Gamma(x)$. Then L is a nontrivial Γ-invariant closed subgroup of K. Then by assumption there exists a nontrivial $[\pi_1] \in \hat{L}$ such that $\Gamma([\pi_1])$ is finite. Let $\Gamma_0 = \{\alpha \in \Gamma \mid \alpha(\pi_1) \sim \pi_1\}$. Then Γ_0 is a closed subgroup of Γ of finite index. Let $\Gamma_1 = \cap_{\alpha\in\Gamma}\alpha\Gamma_0\alpha^{-1}$. Then Γ_1 is a normal subgroup of Γ of finite index and Γ_1 is contained in Γ_0. Let

$A = \{[\pi] \in \hat{L} \mid \Gamma_1[\pi] = [\pi]\}$. Then A contains π_1. Since Γ_1 is normal in Γ, A is Γ-invariant. Let $L_1 = \cap_{[\pi]\in A}\{g \in L \mid \pi(g) = \pi(e)\}$. Then L_1 is a Γ-invariant closed normal subgroup of L and L_1 is a proper subgroup of L as A is nontrivial. If e is in the closure of $\Gamma(x)$, then since Γ/Γ_1 is finite, e is in the closure of $\Gamma_1(x)$. Let $\alpha_n \in \Gamma_1$ be such that $\alpha_n(x) \to e$ and $[\pi] \in A$. Then there exist $u_n \in \mathrm{U}_k(\mathbb{C})$ (k may depend on π) such that

$$u_n^{-1}\pi(g)u_n = \pi(\alpha_n(g))$$

for all $g \in L$. This implies that

$$u_n^{-1}\pi(x)u_n = \pi(\alpha_n(x)) \to \pi(e) = I_k$$

as $n \to \infty$. Since $\mathrm{U}_k(\mathbb{C})$ is compact, $\pi(x) = I_k$. This implies that $x \in L_1$ which is a contradiction as L_1 is a proper Γ-invariant subgroup of L and L is the closed subgroup generated by $\Gamma(x)$. Thus, e is not in the closure of $\Gamma(x)$. Hence (3) $\Rightarrow$ (1). $\square$

3. Compact abelian groups

We now consider compact abelian groups and prove preliminary results for actions on compact abelian groups using Pontryagin duality of locally compact abelian groups: cf. [15] and [20] for results on duality of locally compact abelian groups and for any unexplained notations.

If G is a group and $A_1, A_2, \ldots, A_n$ are subsets of G, then $\langle A_1, \ldots, A_n\rangle$ is defined to be the subgroup generated by the union of the sets $A_1, A_2, \ldots, A_n$ and if any $A_i = \{g\}$, we may write g instead of $\{g\}$.

Lemma 3.1. *Let K be a compact abelian group and Γ be a group of automorphisms of K. Let α be an automorphism of K such that $\alpha\Gamma\alpha^{-1} = \Gamma$. Suppose the action of Γ is not ergodic on K and for each α-invariant proper closed subgroup L of K, the action of $\mathbb{Z}_\alpha$ on K/L is not ergodic. Then the group generated by Γ and α is not ergodic on K or equivalently there exists a nontrivial character χ on K such that the orbit $\{\beta(\chi) \mid \beta \in \langle\Gamma, \alpha\rangle\}$ is finite.*

Proof. We first note that the assumption on α is equivalent to saying that for any α-invariant nontrivial subgroup A of $\hat{K}$ there exists a nontrivial character $\chi \in A$ such that the orbit $\{\alpha^n(\chi) \mid n \in \mathbb{Z}\}$ is finite.

Let $A = \{\chi \in \hat{K} \mid \Gamma(\chi) \text{ is finite}\}$. Since Γ is not ergodic on K, A is nontrivial. Since $\alpha\Gamma\alpha^{-1} = \Gamma$, A is α-invariant. By assumption on α, there exists a nontrivial χ_0 in A such that $\alpha^k(\chi_0) = \chi_0$ for some $k \geq 1$. Then

$$\Gamma\alpha^n(\chi_0) \subset \cup_{i=1}^{k}\Gamma\alpha^i(\chi_0)$$

for all $n \in \mathbb{Z}$. Since $\chi_0 \in A$ and A is α-invariant, we get that each $\Gamma\alpha^i(\chi_0)$ is finite for $1 \leq i \leq k$ and hence $\{\beta(\chi_0) \mid \beta \in \langle\Gamma, \alpha\rangle\}$ is finite. $\square$

Lemma 3.2. *Let K be a (nontrivial) compact abelian group and Γ be a group of automorphisms of K. Suppose Γ is a generalized $\overline{FC}$-group and for each $\alpha \in \Gamma$ and each α-invariant proper closed subgroup L of K, the* *action of* $\mathbb{Z}_\alpha$ *on* K/L *is not ergodic. Then the action of* Γ *on* K *is not ergodic or equivalently there exists a nontrivial character* χ *on* K *such that the corresponding* Γ*-orbit* $\{\alpha(\chi) \mid \alpha \in \Gamma\}$ *is finite.*

Proof. Since K is compact abelian, Aut (K) is totally disconnected and hence by Proposition 2.8 of [12], Γ contains a compact open normal subgroup Δ such that Γ/Δ contains a polycyclic subgroup of finite index. Let Λ be a closed normal subgroup of Γ of finite index containing Δ such that Λ/Δ is polycylic. Let $\Lambda_0 = \Lambda$ and $\Lambda_i = [\Lambda_{i-1}, \Lambda_{i-1}]$ for $i \geq 1$. Then there exists a $k \geq 0$ such that $\Lambda_k\Delta \neq \Delta$ and $\Lambda_{k+1}\Delta = \Delta$. It can be easily seen that each $\Lambda_i\Delta$ is finitely generated modulo Δ. For $0 \leq i \leq k$, let $\alpha_{i,1}, \dots, \alpha_{i,m}$ be in Λ_i such that $\alpha_{i,1}, \dots, \alpha_{i,m}$ and $\Delta\Lambda_{i+1}$ generate $\Delta\Lambda_i$. It can be easily seen that $\alpha_{i,j}$ normalizes $\langle \alpha_{i,1}, \dots, \alpha_{i,j-1}, \Lambda_{i+1}, \Delta\rangle$ for all i and j with $\alpha_{i,0}$ to be trivial. Then repeated application of Lemma 3.1 yields a nontrivial character $\chi \in \hat{K}$ such that the orbit $\Lambda(\chi)$ is finite. Since Λ is a normal subgroup of finite index in Γ, $\Gamma(\chi)$ is also finite. □

We next prove a lemma which shows that the (global) distal condition in Proposition 2.1, can be relaxed to the local distal condition provided Γ is a generalized $\overline{FC}$-group.

Lemma 3.3. *Let* K *be a (nontrivial) compact abelian group and* Γ *be a group of automorphisms of* K*. Suppose* Γ *is a generalized* $\overline{FC}$*-group and each* $\alpha \in \Gamma$ *is distal on* K*. Then* Γ *is not ergodic on* K *or equivalently there exists a nontrivial character* χ *on* K *such that the corresponding* Γ*-orbit* $\{\alpha(\chi) \mid \alpha \in \Gamma\}$ *is finite.*

Proof. Let $\alpha \in \Gamma$ and L be a α-invariant proper closed subgroup of K. Since α is distal on K, the action $\mathbb{Z}_\alpha$ on K/L is also distal (Corollary 6.10 of [4]). This shows by Proposition 2.1 that the action of $\mathbb{Z}_\alpha$ is not ergodic on K/L. Thus, the result follows from Lemma 3.2. □

We now prove the local to global correspondence for distal actions of generalized $\overline{FC}$-groups on compact abelian groups.

Theorem 3.4. *Let* K *be a compact abelian group and* Γ *be a group of automorphisms of* K*. Suppose* Γ *is a generalized* $\overline{FC}$*-group. Then each* $\alpha \in \Gamma$ *is distal on* K *if and only if* Γ *is distal on* K*.*

Proof. Suppose each $\alpha \in \Gamma$ is distal on K. Let L be a nontrivial Γ-invariant closed subgroup of K. Then each $\alpha \in \Gamma$ is distal on L also. It follows from Lemma 3.3 that Γ is not ergodic on L. Since L is arbitrary Γ-invariant nontrivial closed subgroup, by Proposition 2.1 we get that Γ is distal on K. □

4. Distal actions

We now consider distal actions on compact groups and prove that the distal condition has local to global correspondence for actions on compact groups provided the group of automorphisms is a generalized $\overline{FC}$-group.

Theorem 4.1. *Let K be a compact group and Γ be a group of automorphisms of K. Suppose Γ is a generalized $\overline{FC}$-group. Then the following are equivalent:*

(1) *Each $\alpha \in \Gamma$ is distal on K.*
(2) *The action of Γ on K is distal.*

Proof. Suppose each $\alpha \in \Gamma$ is distal on K. Let $x \in K$ be such that e is in the closure of the orbit $\Gamma(x)$. We now claim that $x = e$.

Suppose K is connected. Let T be a maximal compact connected abelian subgroup of K containing x (see Theorem 9.32 of [11]). Then $\mathrm{Aut}(K) = \mathrm{Inn}(K)\Omega$ where $\Omega = \{\alpha \in \mathrm{Aut}(K) \mid \alpha(T) = T\}$ and $\mathrm{Inn}(K)$ is the group of inner-automorphisms of K (cf. Corollary 9.87 of [11]). Let $\Gamma' = \Gamma\mathrm{Inn}(K)$ and $\Omega' = (\Gamma' \cap \Omega)$. Then Γ' and Ω' are also generalized $\overline{FC}$-groups. Since $\mathrm{Inn}(K)$ is a compact normal subgroup, each $\alpha \in \Gamma'$ is distal on K. Since $\mathrm{Aut}(K) = \mathrm{Inn}(K)\Omega$, $\Gamma' = \mathrm{Inn}(K)\Omega'$. Since e is in the closure of $\Gamma(x)$, e is in the closure of $\Gamma'(x)$. Since Inn (K) is compact, e is in the closure of $\Omega'(x)$. As $x \in T$, applying Theorem 3.4, we get that $x = e$.

We now consider any compact group K. Let K_0 be the connected component of e in K. Then K_0 is Γ-invariant and by Corollary 6.10 of [4], each $\alpha \in \Gamma$ is distal on K/K_0. Since K/K_0 is totally disconnected, by Proposition 2.8 and Lemma 2.3 of [12], K/K_0 has arbitrarily small compact open subgroups invariant under Γ. This shows that $x \in K_0$. Now $x = e$ follows from the connected case. □

Example 1.7 showed that an ergodic action of a general, even a commutative group Γ on a compact abelian group need not imply the existence of a nontrivial subgroup or a nontrivial quotient that admits an ergodic $\mathbb{Z}_\alpha$-action for some $\alpha \in \Gamma$ but we now prove that this can not happen if Γ is assumed to be a generalized $\overline{FC}$-group. Thich may be viewed as an initial result on the existence of ergodic automorphisms for ergodic actions on general compact groups.

Proposition 4.2. *Let K be a (nontrivial) compact group and Γ be a group of automorphisms of K. Suppose Γ is a generalized $\overline{FC}$-group and the action of Γ on K is ergodic. Then we have the following:*

(1) *There exist a $\beta \in \Gamma$ and a β-invariant nontrivial closed subgroup L of K such that the action of $\mathbb{Z}_\beta$ on L is ergodic.*
(2) *In addition if K is abelian, there exist an $\alpha \in \Gamma$ and an α-invariant proper closed subgroup L of K such that the action of $\mathbb{Z}_\alpha$ on K/L is ergodic.*

Proof. Suppose for each $\alpha \in \Gamma$ and each α-invariant nontrivial closed subgroup L of K, the action of $\mathbb{Z}_\alpha$ on L is not ergodic. Then by Proposition 2.1, each $\alpha \in \Gamma$ is distal on K. By Theorem 4.1, the action of Γ on K is distal and hence by Proposition 2.1, the action of Γ on K is not ergodic unless K is trivial. Thus, (1) is proved.

We now assume that K is abelian. Suppose for every $\alpha \in \Gamma$ and for every proper closed α-invariant subgroup L of K, the action of $\mathbb{Z}_\alpha$ on K/L is not ergodic. By Lemma 3.2, the action of Γ on K is not ergodic. Thus, (2) is proved. □

5. Finite-dimensional compact groups

We now consider finite-dimensional compact groups. Let $\mathbb{Q}_d^r$ be the additive group $\mathbb{Q}^r$ with discrete topology ($r > 0$). We may regard $\mathbb{Q}_d^r$ as a finite-dimensional vector space over $\mathbb{Q}$. Let B_r denote the dual of $\mathbb{Q}_d^r$. Then B_r is a compact connected group of finite-dimension and any compact connected finite-dimensional abelian group is a quotient of B_r for some r: see [15].

We first show that distal condition for algebraic actions on B_r has local to global correspondence with no restriction on the acting group Γ. The dual of any automorphism of B_r is a $\mathbb{Q}$-linear transformation of $\mathbb{Q}_d^r$ onto $\mathbb{Q}_d^r$. It can be easily seen that any group of unipotent transformations of $\mathbb{Q}_d^r$ is distal on B_r and the following shows that up to finite extensions these are the only distal actions on B_r which may be proved along the lines of Proposition 2.3 of [13] (with some minor modifications).

Proposition 5.1. *Let Γ be a group of automorphisms of B_r. Suppose Γ is distal on B_r. Then B_r has a series of closed connected Γ-invariant subgroups*

$$B_r = K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_{n-1} \supset K_n = (e)$$

such that the action of Γ on K_i/K_{i+1} is finite for any $i \geq 0$. In particular, Γ is a finite extension of a group of unipotent transformations of $\mathbb{Q}_d^r$.

Theorem 5.2. *Let Γ be group of automorphisms of B_r. Suppose each $\alpha \in \Gamma$ is distal on B_r. Then Γ is distal on B_r. In addition if the dual action of Γ on $\mathbb{Q}_d^r$ is irreducible, then Γ is finite.*

Proof. By considering the dual action of Γ, we may view Γ as a group of linear maps on $\mathbb{Q}_d^r$. Then by Proposition 5.1, eigenvalues of elements of Γ are of absolute value one. Let $V = \mathbb{Q}_d^r \otimes \mathbb{R}$. Then by [5], there exist Γ-invariant $\mathbb{R}$-subspaces $(0) = V_1 \subset V_2 \subset \cdots \subset V_m = V$ such that the action of Γ on V_{i+1}/V_i is isometric. Thus, there exists a Γ-invariant $\mathbb{R}$-subspace W of V such that $W \neq V$ and the action of Γ on V/W is isometric.

Assume that the dual action of Γ on $\mathbb{Q}_d^r$ is irreducible. Since $V = \langle \mathbb{Q}_d^r \rangle$ and $W \neq V$, $W \cap \mathbb{Q}_d^r \neq \mathbb{Q}_d^r$ and is Γ-invariant. Since action of Γ on $\mathbb{Q}_d^r$ is irreducible, $W \cap \mathbb{Q}_d^r = (0)$.

Let $\alpha \in \Gamma$. Then by Proposition 5.1, α^k is unipotent for some $k \geq 1$. This implies that α^k is unipotent and isometric on V/W and hence $\alpha^k(v) \in v+W$ for all $v \in V$. This implies that for $v \in \mathbb{Q}_d^r$, $\alpha^k(v) - v \in W \cap \mathbb{Q}_d^r = (0)$ and hence α^k is identity. Thus, every element of Γ has finite order. It follows from Lemma 4.3 of [2] that Γ is finite. □

We now proceed to show that ergodic action of Γ on a finite-dimensional compact connected abelian group yields an ergodic automorphism in Γ provided Γ is nilpotent.

Lemma 5.3. *Let Γ be a group of automorphisms of a compact abelian group K and α be an automorphism of K. Suppose Γ and α are distal on K and $\alpha\Gamma\alpha^{-1} = \Gamma$. Then the group generated by Γ and α is distal on K.*

Proof. Let Δ be the group generated by Γ and α. Let L be a nontrivial closed subgroup of K invariant under Δ. Let $A = \{\chi \in \hat{L} \mid \Gamma(\chi) \text{ is finite}\}$. Since Γ is normalized by α, A is α-invariant. Since α and Γ are distal on K, it follows from Proposition 2.1 that there exists a nontrivial $\chi_0 \in A$ such that $\alpha^k(\chi_0) = \chi_0$ for some $k \geq 1$. Now, $\Gamma\alpha^i(\chi_0) \subset \cup_{j=1}^k \Gamma(\alpha^j(\chi_0))$ for any $i \in \mathbb{Z}$. This implies that the orbit $\Delta(\chi_0)$ is finite. This shows by Proposition 2.1 that Δ is distal on K. □

Lemma 5.4. *Let α be an ergodic automorphism of B_r and L be a closed connected α-invariant subgroup of B_r. Then α is ergodic on L.*

Proof. It can easily be seen that α is ergodic on B_r if and only if no power of α on $\mathbb{Q}_d^r$ has a nonzero fixed point. Let V be the $\mathbb{Q}$-subspace of $\mathbb{Q}_d^r$ such that the dual of L is $\mathbb{Q}_d^r/V$. Since α is ergodic, no power of α on $\mathbb{Q}_d^r$ has a nonzero fixed point and hence no power of α on $\mathbb{Q}_d^r/V$ has a nonzero fixed point. Thus, α is ergodic on L. □

Lemma 5.5. *Let α and β be automorphisms of B_r. Suppose α is contained in a group Γ of automorphisms of B_r such that Γ is distal and β is ergodic and normalizes Γ. Then $\alpha^i\beta^j$ and $\beta^j\alpha^i$ are ergodic for all i and j in $\mathbb{Z}$ with $j \neq 0$.*

Proof. It is enough to show that $\alpha\beta$ and $\beta\alpha$ are ergodic. We first consider the case when Γ is finite. Assume Γ is finite. Let χ be a character such that the orbit $\{(\alpha\beta)^n(\chi) \mid n \in \mathbb{Z}\}$ is finite. Since Γ is finite and Γ is normalized by $\alpha\beta$, the orbit $\widetilde{\Gamma}(\chi)$ is also finite where $\widetilde{\Gamma}$ is the group generated by $\alpha\beta$ and Γ. Since $\beta \in \widetilde{\Gamma}$ and β is ergodic, we get that χ is trivial. Thus, $\alpha\beta$ is ergodic.

We now consider the general case. Let $V = \{\chi \in \mathbb{Q}_d^r \mid \Gamma(\chi) \text{ is finite}\}$. Since Γ is distal, V is a nontrivial $\mathbb{Q}$-subspace and V is invariant under β as Γ is normalized by β. Let L be the closed connected subgroup of B_r such that the dual B_r/L is V. Then L is a proper closed connected subgroup invariant under Γ and β and Γ is finite on B_r/L. Then $\alpha\beta$ is ergodic on

B_r/L. Since the dual of L is $\mathbb{Q}_d^r/V$, $L \simeq B_s$ for some $s < r$. By Lemma 5.4, β is ergodic on L and hence by induction on dimension of B_r, $\alpha\beta$ is ergodic on L. Thus, $\alpha\beta$ is ergodic on B_r. Similarly we may show that $\beta\alpha$ is also ergodic on B_r. □

Lemma 5.6. *Let Δ be a group of automorphisms of B_r. Suppose there exists a Δ-invariant closed connected subgroup K of B_r such that Δ is ergodic on K and Δ is distal on B_r/K. Then K is $N(\Delta)$-invariant where $N(\Delta)$ is the normalizer of Δ in* $\mathrm{Aut}(B_r)$.

Proof. Let V_1 be the $\mathbb{Q}$-subspace of $\mathbb{Q}_d^r$ defined by

$$V_1 = \{\chi \in \mathbb{Q}_d^r \mid \Delta(\chi) \text{ is finite}\}$$

and define V_i inductively by

$$V_i = \{\chi \in \mathbb{Q}_d^r \mid \Delta(\chi) + V_{i-1} \text{ is finite in } \mathbb{Q}_d^r/V_{i-1}\}$$

for any $i > 1$. Then each V_i is a Δ-invariant $\mathbb{Q}$-subspace. Since $\mathbb{Q}_d^r$ has finite-dimension over $\mathbb{Q}$, there exists a n such that $V_n = V_{n+i}$ for all $i \geq 0$ and for any nontrivial $\chi \in \mathbb{Q}_d^r/V_n$, the orbit $\Delta(\chi) + V_n$ is infinite. Let S be a closed subgroup of B_r such that the dual of S is $\mathbb{Q}_d^r/V_n$. Then S is Δ-invariant and connected. The choice of V_n shows that Δ is ergodic on S and Δ is distal on B_r/S. This implies that Δ is distal on $KS/S \simeq K/K\cap S$ and also on $KS/K \simeq S/K\cap S$ but Δ is ergodic on K and also on S. Thus, $S = K\cap S = K$.

If $\beta\Delta = \Delta\beta$, then for any $\chi \in V_1$, $\Delta(\beta(\chi)) = \beta(\Delta(\chi))$ is finite. Thus, V_1 is β-invariant. Since each V_i/V_{i-1} is the space of all characters whose Δ-orbit is finite in $\mathbb{Q}_d^r/V_{i-1}$, we get that V_i is β-invariant for any $i \geq 1$. Thus, K is β-invariant. □

Lemma 5.7. *Let Δ be a nilpotent group of automorphisms of B_r generated by α and a subgroup Γ such that $\alpha\Gamma\alpha^{-1} = \Gamma$. Suppose Γ is distal on B_r. Then there exists a closed connected Δ-invariant subgroup K of B_r such that α is ergodic on K and α is distal on B_r/K.*

Proof. Let V_1 be the $\mathbb{Q}$-subspace of $\mathbb{Q}_d^r$ defined by

$$V_1 = \{\chi \in \mathbb{Q}_d^r \mid (\alpha^n(\chi)) \text{ is finite}\}$$

and define V_i inductively by

$$V_i = \{\chi \in \mathbb{Q}_d^r \mid (\alpha^n(\chi) + V_{i-1}) \text{ is finite in } \mathbb{Q}_d^r/V_{i-1}\}$$

for any $i > 1$. Then each V_i is a α-invariant $\mathbb{Q}$-subspace. Since $\mathbb{Q}_d^r$ has finite-dimension over $\mathbb{Q}$, there exists a n such that $V_n = V_{n+i}$ for all $i \geq 0$ and for any nontrivial $\chi \in \mathbb{Q}_d^r/V_n$, the orbit $(\alpha^n(\chi) + V_n)$ is infinite. Let K be a closed subgroup of B_r such that the dual of K is $\mathbb{Q}_d^r/V_n$. Then K is α-invariant and connected. The choice of V_n shows that α is ergodic on K and α is distal on B_r/K.

Let $\Delta_0 = \Delta$ and $\Delta_i = [\Delta, \Delta_{i-1}]$ for $i > 0$. Since Δ is nilpotent, there exists a $k \geq 0$ such that Δ_k is nontrivial and Δ_{k+1} is trivial. Since Δ/Γ is abelian, $\Delta_i \subset \Gamma$ for $i \geq 1$, hence each Δ_i is distal on B_r ($i \geq 1$). Now α commutes with elements of Δ_k and hence V_1 as well as V_i is Δ_k-invariant. Thus, K is Δ_k-invariant. Since α normalizes Δ_k, it follows from Lemma 5.3 that $\langle \alpha, \Delta_k \rangle$ is distal on B_r/K and ergodic on K. Now for $\beta \in \Delta_{k-1}$, $\alpha^{-1}\beta^{-1}\alpha\beta \in \Delta_k$ and hence $\beta^{-1}\alpha\beta \in \langle \alpha, \Delta_k \rangle$. By Lemma 5.6, K is Δ_{k-1}-invariant. Repeating this argument we can get that K is Δ-invariant. □

Lemma 5.8. *Let α be an automorphism of B_r. Then there exists a compact connected subgroup K of B_r isomorphic to B_s for some $s > 0$ such that α is ergodic on K and α is distal on B_r/K. Moreover, if Γ is a nilpotent group of automorphisms of B_r with $\Gamma = \Gamma_0$ and $\Gamma_k = [\Gamma, \Gamma_{k-1}]$ for $k \geq 1$ and if $\alpha \in \Gamma_i \setminus \Gamma_{i+1}$ and Γ_{i+1} is distal on B_r, then K is Γ-invariant.*

Proof. Suppose Γ is a nilpotent group containing α with $\alpha \in \Gamma_i \setminus \Gamma_{i+1}$ and Γ_{i+1} is distal on B_r. Let Δ be the group generated by Γ_{i+1} and α. By Lemma 5.7, B_r contains a closed connected Δ-invariant subgroup K such that α is ergodic on K and α is distal on B_r/K. Since Γ_{i+1} is distal and α normalizes Γ_{i+1}, Δ is distal on B_r/K (cf. Lemma 5.3). Since $\alpha \in \Delta$, Δ is ergodic on K. Since Δ is a normal subgroup, it follows from Lemma 5.6 that K is Γ-invariant. □

Lemma 5.9. *Let Γ be a nilpotent group of automorphisms of B_r and $\alpha, \beta \in \Gamma$. Let $\Gamma_0 = \Gamma$ and $\Gamma_i = [\Gamma, \Gamma_{i-1}]$ for $i \geq 1$. Let $k \geq 1$ be such that $\alpha \in \Gamma_{k-1} \setminus \Gamma_k$. Suppose α is ergodic on B_r and Γ_k is distal on B_r. Then there exists a $i \geq 0$ such that $\alpha^i\beta$ is ergodic on B_r.*

Proof. We prove the result by induction on the dimension of B_r. If $r = 1$, then we have nothing to prove. So, we may assume that $r > 1$. If $\alpha\beta$ is ergodic on B_r, then we are done. Hence we may assume that $\alpha\beta$ is not ergodic on B_r. Let Δ be the group generated by $\alpha\beta$ and Γ_k. Then Δ is a nilpotent group and $[\Delta, \Delta] \subset \Gamma_k$. So, we may assume by Lemma 5.7 that there exists a closed connected Δ-invariant subgroup K of B_r such that $\alpha\beta$ is ergodic on K and $\alpha\beta$ is distal on B_r/K. Since $\alpha\beta$ is not ergodic on B_r, $K \neq B_r$. Then by Lemma 5.3, Δ is distal on B_r/K. Since α and β commute modulo Γ_k, we get that α normalizes Δ. By Lemma 5.6, K is α-invariant and hence by Lemma 5.5, $\alpha^i\beta$ is ergodic on B_r/K for all $i \geq 2$. Since α is ergodic on K, induction hypothesis applied to K in place of B_r and $\alpha^2\beta$ in place of β, we get that $\alpha^j\beta$ is ergodic on K for some $j \geq 2$. Thus, $\alpha^j\beta$ is ergodic on B_r for some $j \geq 2$. □

Lemma 5.10. *Let Γ be a nilpotent group of automorphisms of B_r and $\alpha, \beta \in \Gamma$. Let $\Gamma_0 = \Gamma$ and $\Gamma_i = [\Gamma, \Gamma_{i-1}]$ for $i \geq 1$ and $k \geq 1$ be such that $\alpha \in \Gamma_{k-1} \setminus \Gamma_k$. Let K be a closed Γ-invariant subgroup of B_r isomorphic to B_s for some $s \geq 0$ such that α is ergodic on K and α is distal on B_r/K. If β*

is ergodic on B_r/K and Γ_k is distal on B_r, then there exists $j \geq 0$ such that $\alpha^j\beta$ is ergodic on B_r.

Proof. Since α normalizes Γ_k, by Lemma 5.3, the group generated by α and Γ_k is distal on B_r/K. Since β centralizes α modulo Γ_k, it follows from Lemma 5.5 that $\alpha^i\beta$ is ergodic on B_r/K for all $i \geq 0$. By Lemma 5.9, $\alpha^j\beta$ is ergodic on K for some $j \geq 0$. This shows that for some $j \geq 0$, $\alpha^j\beta$ is ergodic on B_r. □

Lemma 5.11. *Let Γ be a nilpotent group of automorphisms of B_r. Let $\Gamma_0 = \Gamma$ and $\Gamma_i = [\Gamma, \Gamma_{i-1}]$ for $i \geq 1$. Suppose that the action of Γ on B_r is ergodic. Then there exist a series*

$$(e) = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{m-1} \subset K_m = B_r$$

of closed connected Γ-invariant subgroups with each $K_i \simeq B_{r_i}$ for some $r_i \geq 0$ and automorphisms $\alpha_1, \alpha_2, \ldots, \alpha_m$ in Γ with the following properties for each $i = 1, 2, \ldots, m$:

(1) *If k_i is the smallest integer k for which $\alpha_i \notin \Gamma_k$, then the action of Γ_{k_i} on B_r/K_{i-1} is distal.*
(2) *The action of $\mathbb{Z}_{\alpha_i}$ on K_i/K_{i-1} is ergodic.*
(3) *The action of $\mathbb{Z}_{\alpha_i}$ on B_r/K_i is distal.*

Proof. For each $\alpha \in \Gamma$, if the action of $\mathbb{Z}_\alpha$ is distal on B_r, then by Theorem 5.2, the action of Γ is distal. This is a contradiction to the ergodicity of Γ by Proposition 2.1. Thus, the action of $\mathbb{Z}_\alpha$ is not distal for some $\alpha \in \Gamma$.

Since Γ is nilpotent, there exists a k such that $\Gamma_k \neq (e)$ and $\Gamma_{k+1} = (e)$. Now, choose $\alpha_1 \in \Gamma_{k_1-1} \setminus \Gamma_{k_1}$ such that the action of $\mathbb{Z}_{\alpha_1}$ is not distal on B_r but the action of Γ_{k_1} is distal on B_r. By Lemma 5.8, there exists a nontrivial Γ-invariant closed connected subgroup K_1 of B_r isomorphic to B_{r_1} for some $r_1 > 0$ such that α_1 is ergodic on K_1 and α_1 is distal on B_r/K_1.

Let $L = B_r/K_1$. Then the action of Γ on L is ergodic and $L \simeq B_{s_1}$ for $s_1 < r$ as K_1 is nontrivial. By applying induction on the dimension of B_r, we get Γ-invariant closed connected subgroups $(e) = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{m-1} \subset K_m = B_r$ and automorphisms $\alpha_2, \ldots, \alpha_m$ satisfying (1)–(3) for $2 \leq i \leq n$. □

Theorem 5.12. *Let Γ be a nilpotent group of automorphisms of B_r. If Γ is ergodic on B_r, then Γ contains ergodic automorphisms of B_r.*

Proof. We now prove the result by induction on r. If $r = 1$, we are done. By Lemma 5.11, there are Γ-invariant closed connected subgroups $(e) = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{m-1} \subset K_m = B_r$ with each $K_i \simeq B_{r_i}$ for some $r_i \geq 0$ and automorphisms $\alpha_1, \alpha_2, \ldots, \alpha_m$ in Γ satisfying (1)–(3) of Lemma 5.11. We may assume that $K_i \neq K_{i-1}$ for $1 \leq i \leq m$. Induction hypothesis applied to the action of Γ on $B_r/K_1 \simeq B_{r-r_1}$ yields $\beta \in \Gamma$ such that β is ergodic on B_r/K_1. By Lemma 5.10, there exists $\alpha \in \Gamma$ such that α is ergodic on B_r. □

We now consider compact connected finite-dimensional abelian groups. Let K be a compact connected finite-dimensional abelian group. Then

$$\mathbb{Z}^r \subset \hat{K} \subset \mathbb{Q}_d^r$$

and K is a quotient of B_r for some $r \geq 1$. Let α be an automorphism of K. Then α is an automorphism of $\hat{K}$. Since $\mathbb{Z}^r \subset \hat{K}$, α has a canonical extension to an invertible $\mathbb{Q}$-linear map on $\mathbb{Q}_d^r$, say $\widetilde{\alpha}$. Thus, any automorphism α of K can be lifted to a unique automorphism $\widetilde{\alpha}$ of B_r. Let Γ be a group of automorphisms of K and $\widetilde{\Gamma}$ be the group consisting of lifts $\widetilde{\alpha}$ of automorphisms $\alpha \in \Gamma$. We consider Γ and $\widetilde{\Gamma}$ as topological groups with their respective compact-open topologies as automorphism groups of K and B_r. By looking at the dual action, we can see that the topological groups Γ and $\widetilde{\Gamma}$ are isomorphic. If $\phi\colon B_r \to K$ is the canonical quotient map, then for $\alpha \in \Gamma$ and $x \in B_r$, we have

$$\phi(\widetilde{\alpha}(x)) = \alpha(\phi(x))$$

where $\widetilde{\alpha}$ is the lift of α on B_r.

Proposition 5.13. *Let K be a compact connected finite-dimensional abelian group. Let Γ be a group of automorphisms of K and $\widetilde{\Gamma}$ be the corresponding group of automorphisms of B_r. Then Γ is distal (respectively, ergodic) on K if and only if $\widetilde{\Gamma}$ is distal (respectively, ergodic) on B_r.*

Proof. For $\chi \in \mathbb{Q}_d^r$, there exists $n \geq 1$ such that $n\chi \in \hat{K}$ and since $\hat{K} \subset \mathbb{Q}_d^r$, Γ is ergodic on K if and only if $\widetilde{\Gamma}$ is ergodic on B_r. Since K is a quotient of B_r, $\widetilde{\Gamma}$ is distal on B_r implies Γ is distal on K (see [4], Corollary 6.10).

Suppose Γ is distal on K. By Proposition 2.1, there exists a nontrivial character χ_1 in $\hat{K} \subset \mathbb{Q}_d^r$ such that $\Gamma(\chi_1)$ is finite. Let

$$V_1 = \{\chi \in \mathbb{Q}_d^r \mid \widetilde{\Gamma}(\chi) \text{ is finite}\}.$$

Then V_1 is a nontrivial $\widetilde{\Gamma}$-invariant $\mathbb{Q}$-subspace as $\chi_1 \in V_1$. Let M be a closed subgroup of B_r such that the dual of M is $\mathbb{Q}_d^r/V_1$. Then M is $\widetilde{\Gamma}$-invariant and $M \simeq B_s$ for $s < r$. Let $A = V_1 \cap \hat{K}$ and L be a closed subgroup of K such that the dual of L is $\hat{K}/A$. Then L is Γ-invariant. Since $\hat{K}/A \subset \mathbb{Q}_d^r/V_1$, $\hat{K}/A$ has no element of finite order and hence L is connected (see Theorem 30 of [15]). It can be verified that the dimension of L is same as the dimension of M. Hence by induction on the dimension of K we get that $\widetilde{\Gamma}$ is distal on M. Since the action of $\widetilde{\Gamma}$ on B_r/M is finite, $\widetilde{\Gamma}$ is distal on B_r. □

Theorem 5.14. *Let K be a compact connected finite-dimensional abelian group and Γ be a nilpotent group of automorphisms of K. Suppose Γ is ergodic on K. Then there exists an $\alpha \in \Gamma$ such that α is ergodic on K.*

Proof. Now let K be a compact connected finite-dimensional abelian group and r be the dimension of K. Then K is a quotient of B_r. Let $\widetilde{\Gamma}$ be the group of lifts of automorphisms of Γ. Then by Proposition 5.13, $\widetilde{\Gamma}$ is ergodic on B_r. It follows from Theorem 5.12 that there exists $\alpha \in \Gamma$ such that the lift $\widetilde{\alpha}$ of α is ergodic on B_r. Another application of Proposition 5.13 shows that α itself is ergodic on K. □

We now show that the distal condition for algebraic actions on connected finite-dimensional compact groups has a local to global correspondence with no restriction on the acting group Γ.

Theorem 5.15. *Let Γ be a group of automorphisms of a compact connected finite-dimensional group K. Suppose each $\alpha \in \Gamma$ is distal on K. Then the action of Γ on K is distal.*

Proof. If K is abelian, then the result follows from Proposition 5.13 and Theorem 5.2. Suppose K is any finite-dimensional compact connected group. Let $x \in K$ and (α_n) be a sequence in Γ. Suppose $\alpha_n(x) \to e$.

Let T be a maximal compact connected abelian subgroup of K containing x: cf. Theorem 9.32 of [11] for existence of such T. Since K is a connected group, $\mathrm{Aut}(K) = \mathrm{Inn}(K)\Omega$ where $\Omega = \{\alpha \in \mathrm{Aut}(K) \mid \alpha(T) = T\}$ and $\mathrm{Inn}(K)$ is the group of inner automorphisms of K (see Corollary 9.87 of [11]). Let $\Lambda = \mathrm{Inn}(K)\Gamma$. Since $\mathrm{Inn}(K)$ is a compact normal subgroup, each $\alpha \in \Lambda$ is distal on K. Let $\alpha_n = a_n\beta_n$ where $a_n \in \mathrm{Inn}(K)$ and $\beta_n \in \Omega \cap \Lambda$ for all $n \geq 1$. Since $\mathrm{Inn}(K)$ is compact, by passing to a subsequence, if necessary, we may assume that $\beta_n(x) \to e$. Since T is closed in K which is of finite-dimension, T is also of finite-dimension ([16]). It follows from the abelian case that $\Omega \cap \Lambda$ is distal on T and hence $x = e$ as $x \in T$ and $\beta_n \in \Omega \cap \Lambda$. Thus, the action of Γ is distal on K. □

We now provide an example to show that the nilpotency assumption on the acting group Γ in Theorem 5.14 can not be relaxed: it may be noted that Theorem 5.14 is true with no restriction on the acting group Γ if the compact group K is the two-dimensional torus.

Example 5.16. Let Γ be a subgroup of $\mathrm{GL}(n, \mathbb{Q})$. Let Γ^+ be the semidirect product of Γ and $\mathbb{Q}_d^n$ with the canonical action of Γ on $\mathbb{Q}_d^n$. We define an action of Γ^+ on $\mathbb{Q}_d^{n+1}$ by

$$(\alpha, w)(q_1, \ldots, q_n, q_{n+1}) = \alpha(q_1, \ldots, q_n) + wq_{n+1} + (0, \ldots, 0, q_{n+1})$$

for all $(\alpha, w) \in \Gamma^+$ and $(q_1, \ldots, q_n, q_{n+1}) \in \mathbb{Q}_d^{n+1}$: $\mathbb{Q}_d^n$ is identified as a subset of $\mathbb{Q}_d^{n+1}$ via the canonical map $(q_1, \ldots, q_n) \mapsto (q_1, \ldots, q_n, 0)$. It may be useful to note that (α, w) has the following matrix form

$$\begin{pmatrix} \alpha & w^T \\ 0 & 1 \end{pmatrix}$$

where w^T is the transpose of w. Considering the dual action, we get that $\Gamma^+ \subset \mathrm{Aut}(B_{n+1})$. For $z \in \mathbb{Q}_d^n$, $\Gamma^+(z) = \Gamma(z)$ and for $z \in \mathbb{Q}_d^{n+1} \setminus \mathbb{Q}_d^n$, $\Gamma^+(z)$ can easily be seen to be infinite. Thus, Γ is ergodic on B_n if and only if Γ^+

is ergodic on B_{n+1}. For any $\Gamma \subset \mathrm{GL}(n, \mathbb{Q})$, no $(\alpha, w) \in \Gamma^+$ is ergodic on B_{n+1}. For $n \geq 1$, take Γ to be the group generated by $\alpha \in \mathrm{GL}(n, \mathbb{Q})$ that is ergodic on B_n. Then Γ^+ is a solvable group and is ergodic on B_{n+1} but no automorphism in Γ^+ is ergodic on B_{n+1}.

6. Example

We now provide an example to show that the existence of a finite sequence as in Proposition 5.1 need not be true for connected infinite-dimensional compact abelian groups. We first state a general form of Proposition 5.1 which can be proved as in Proposition 2.3 of [13].

Proposition 6.1. *Let K be a compact abelian group and Γ be a group of automorphisms of K. Suppose Γ is distal. Then there exists a collection (K_i) of Γ-invariant closed subgroups of K such that:*

(1) *$K_0 = K$.*
(2) *For $i \geq 0$ either $K_{i+1} = (e)$ or K_{i+1} is a proper subgroup of K_i.*
(3) *The action of Γ on the dual of K_i/K_{i+1} has only finite orbits for any $i \geq 0$.*

In contrast to the finite-dimensional case we now show by an example that the sequence (K_i) in Proposition 6.1 need not be finite. Let T_k be the k-dimensional torus, a product of k copies of the circle group. Let α_k be the automorphism of T_k defined by

$$\alpha_k(x_1, x_2, \ldots, x_k) = (x_1x_2 \ldots x_k, x_2x_3 \ldots x_k, \ldots, x_{k-1}x_k, x_k)$$

for all $(x_1, x_2, \ldots, x_k) \in T_k$. For $0 \leq j \leq k$, let

$$M_{k,j} = \{(x_1, x_2, \ldots, x_{k-j}, e, \ldots, e) \in T_k\}.$$

Then each $M_{k,j}$ is α_k-invariant and α_k is trivial on $M_{k,j}/M_{k,j+1}$ for $j \geq 0$.

We first prove the following fact about T_k and α_k.

Lemma 6.2. *Let T_k and α_k be as above. Suppose there exists a series*

$$T_k = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = (e)$$

of α_k-invariant closed subgroups such that for $i \geq 0$, the action of $\mathbb{Z}_{\alpha_k}$ on M_i/M_{i+1} is finite and M_i/M_{i+1} is not finite. Then $n = k$.

Proof. Let V be the Lie algebra of T_k. We first show that M_{n-1} is one-dimensional. For $0 \leq i < n$, let V_i be the Lie subalgebra of V corresponding to the Lie subgroup M_i. Now, there exists a m such that α_k^m is trivial on M_{n-1}. Suppose $(u_1, u_2, \ldots, u_k) \in V_{n-1}$. Then $\alpha_k^m(u_1, u_2, \ldots, u_k) = (u_1, u_2, \ldots, u_k)$. This implies that for $1 \leq i \leq k-1$, $u_i = u_i + \sum_{j>i}^k m_{i,j}u_j$ where $m_{i,j} \in \mathbb{N}$. For $i = k-1$, $u_{k-1} = u_{k-1} + m_{k-1,k}u_k$ and hence $u_k = 0$. If $u_p = 0$ for all $p > q > 1$, then for $i = q-1$,

$$u_{q-1} = u_{q-1} + \sum_{j \geq q} m_{q-1,j}u_j = u_{q-1} + m_{q-1,q}u_q$$

and hence $u_q = 0$. Thus, V_{n-1} is at most one-dimensional. Since $M_{n-1}/M_n = M_{n-1}$ is not finite, M_{n-1} has dimension one and

$$V_{n-1} = \{(u_1, 0, \ldots, 0) \mid u_1 \in \mathbb{R}\}.$$

It can be seen that $T_k/M_{n-1} \simeq T_{k-1}$ and the action of $\mathbb{Z}_{\alpha_k}$ on T_k/M_{n-1} is same as the action of $\mathbb{Z}_{\alpha_{k-1}}$ on T_{k-1}. Moreover, $M_{i+1}/M_{n-1} \subset M_i/M_{n-1}$ and $\frac{M_i/M_{n-1}}{M_{i+1}/M_{n-1}} \simeq M_i/M_{i+1}$ for $0 \leq i < n-1$ with $M_0/M_{n-1} = T_k/M_{n-1}$ and $M_{n-1}/M_{n-1} = (e)$. By induction on k, we get that $n-1 = k-1$. $\square$

Let $K = \prod_{k\in\mathbb{N}} T_k$. Let $\alpha\colon K \to K$ be the automorphism defined by $\alpha(f)(k) = \alpha_k(f(k))$ for all $f \in K$ and all $k \in \mathbb{N}$. Then α is a continuous automorphism and the $\mathbb{Z}$-action defined by α is distal on K.

If there is a finite sequence

$$(e) = K_n \subset K_{n-1} \subset \cdots \subset K_1 \subset K_0 = K$$

of α-invariant closed subgroups such that the action of $\mathbb{Z}_\alpha$ on K_i/K_{i+1} is finite for $i \geq 0$. This implies that each T_k has a finite series

$$(e) = K_{n,k} \subset K_{n-1,k} \subset \cdots \subset K_{1,k} \subset K_{0,k} = T_k$$

of α_k-invariant closed subgroups such that the action of $\mathbb{Z}_{\alpha_k}$ on $K_{i,k}/K_{i+1,k}$ is finite for $i \geq 0$.

It follows from Lemma 6.2 that $k \leq n$. Since $k \geq 1$ is arbitrary, this is a contradiction. Thus, the sequence (K_i) of closed subgroups as in Proposition 6.1 for K and α is not finite.

Addendum. Recently [18] proved Theorem 5.14 for $\mathbb{Z}^d$-actions on compact groups K provided the $Z(\mathbb{Z}^d)$, centralizer of $\mathbb{Z}^d$ in Aut (K) has DCC, that is any decreasing sequence of $Z(\mathbb{Z}^d)$-invariant closed subgroups is finite.

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