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Conditions for the algebraic determination of the metric from the curvature

Richard Atkins

Abstract. We establish new conditions ensuring that a Riemannian metric may be constructed, up to a conformal factor, from the skew-symmetries of its Riemann curvature tensor.

Contents

1. Introduction

The problem of constructing the metric $g = g_{ij}$, up to a conformal factor, from the Riemann curvature $R = R^i_{jkl}$ has been investigated by Ihrig ([5], [6]). The method hinges on the fact that the lowered curvature tensor R_{ijkl} is skew-symmetric in the indices i and j. Specifically, on an n-dimensional manifold M, the components g_{ij} of the metric may be obtained from the set of linear equations

$$\sum_{s=1}^{n}(g_{is}R^s_{jkl} + g_{js}R^s_{ikl}) = 0 \tag{1}$$

for $1 \leq i, j, k, l \leq n$, provided the solutions $g = g_{ij}$, consisting of symmetric two-tensors, span a vector space of dimension one. The one-dimensionality of solutions is guaranteed by a condition Ihrig terms *total*. The Riemann tensor is total at a point $m \in M$ if

$$\{R(u, v) : u, v \in T_mM\}$$

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generates a vector space of dimension $n(n-1)/2$. This is equivalent to requiring that the dimension of the space generated by the two-forms $R(*,*)^i_j$, defined by the Riemann curvature, is equal to $n(n-1)/2$. This is a rather strong condition to impose; for many examples of interest, equations (1) are sufficient to determine the metric even when the totality requirement is not satisfied. McIntosh and Halford ([8]) have shown that this condition can be weakened for the case of a metric of type (1,3), in that it is sufficient to demand that the dimension of the space generated by the two-forms $R(*,*)^i_j$ be greater than three. A shorter, geometrical proof of this result was obtained by Hall and McIntosh ([4]).

In this paper we provide a sufficient condition of a somewhat different nature for the determination of the metric by equations (1) (up to conformal equivalence). These will appear to be relatively mild in comparison to assumptions based upon the dimension of the space of Riemann two-forms. On the other hand, the latter applies to pseudo-Riemannian metrics of all signatures whereas our approach is valid only for positive definite metrics. In addition, our condition is somewhat more complicated to state.

We shall consider two prerequisites for the Riemann tensor. First, $R(\xi_1,\xi_2)$ must possess distinct eigenvalues for some choice of tangent vectors $\xi_1,\xi_2 \in T_mM$ and second, $R(*,*)$ must be sufficiently nonzero at m, in a sense to be made precise below. It will then follow that the metric can be recovered from the symmetric solutions to (1), up to a conformal scale.

2. Eigenvectors of the curvature

Suppose we are given the Riemann curvature $R(*,*)$ of an unknown positive definite metric g on a manifold M of dimension n. For tangent vectors $\xi_1,\xi_2 \in T_mM$, $R(\xi_1,\xi_2)$ is *nondegenerate* at m if the associated linear transformation

$$R(\xi_1,\xi_2) : T_mM \to T_mM$$

has n distinct (complex) eigenvalues. We shall assume this to be so and denote by $(Z_1,\dots,Z_n)$ an eigenbasis of $R(\xi_1,\xi_2)$, with corresponding distinct eigenvalues $(\lambda_1,\dots,\lambda_n)$, respectively:

$$R(\xi_1,\xi_2)(Z_i) = \lambda_i Z_i$$

for $1 \le i \le n$.

The following lemma is the salient observation.

Lemma 1. (i) *The eigenbasis $(Z_1,\dots,Z_n)$ of $R(\xi_1,\xi_2)$, after a possible reordering, has the form*

$$X_1 + iX_2, X_1 - iX_2, \dots, X_{2k-1} + iX_{2k}, X_{2k-1} - iX_{2k}, (X_n)$$

where the X_i are vectors in T_mM and X_n, the eigenvector corresponding to the zero eigenvalue, is included if n is odd.

(ii) $(X_1, \ldots, X_{2k}, (X_n))$ *is an orthogonal basis for* T_mM. *That is,* $g(m)$ *is expressible in the form* $g(m) = \sum_{i=1}^n g_i\theta^i \otimes \theta^i$ *for some constants* $g_i \in \Re^+$, *where*

$$(\theta^1, \ldots, \theta^{2k}, (\theta^n))$$

is the basis of covectors dual to $(X_1, \ldots, X_{2k}, (X_n))$.

Proof. (i) Since ∇ is the Levi-Civita connection of the Riemannian metric g, there exists an orthonormal basis of T_mM, at each $m \in M$, with respect to which $R(\xi_1, \xi_2)$ is represented as a skew-symmetric matrix. Hence the eigenvalues $\lambda_i = \lambda_i(m)$ are purely imaginary with associated eigenvectors $Z_{2l-1} = X_{2l-1} + iX_{2l}$ for λ_{2l-1} and $Z_{2l} = \overline{Z}_{2l-1} = X_{2l-1} - iX_{2l}$ for $\lambda_{2l} = -\lambda_{2l-1}$, $1 \le l \le k$, except for $\lambda_n = 0$ when n is odd, with eigenvector $Z_n = X_n$.

(ii) Let $g = \sum_{i,j=1}^n g^{ij} Z_i \otimes Z_j$ be a parallel metric on M, written contravariantly; thus $R(\xi_1, \xi_2)(g) = 0$. The explicit form of g gives

$$\begin{aligned} R(\xi_1, \xi_2)(g) &= [\nabla_{\xi_1}, \nabla_{\xi_2}](g) - \nabla_{[\xi_1,\xi_2]}(g) \\ &= \sum_{i,j=1}^n g^{ij}(R(\xi_1, \xi_2)(Z_i) \otimes Z_j + Z_i \otimes R(\xi_1, \xi_2)(Z_j)) \\ &= \sum_{i,j=1}^n g^{ij}(\lambda_i Z_i \otimes Z_j + Z_i \otimes \lambda_j Z_j) \\ &= \sum_{i,j=1}^n g^{ij}(\lambda_i + \lambda_j) Z_i \otimes Z_j. \end{aligned}$$

Therefore $g^{ij}(\lambda_i + \lambda_j) = 0$ for all $1 \le i, j \le n$. The eigenvalues λ_i are distinct, by hypothesis, and $\lambda_{2l} = -\lambda_{2l-1}$, $1 \le l \le k$, except for $\lambda_n = 0$ when n is odd. Hence $g^{ij} = 0$, unless $(i, j) = (2l - 1, 2l)$ or $(i, j) = (2l, 2l - 1)$ for some $l \in \{1, \ldots, k\}$, or $i = j = n$ when n is odd. It follows that g^{ij} is block diagonal with 2×2 blocks down the main diagonal and with a single 1×1 block for odd n. In the $(X_1, \ldots, X_n)$-basis the l^{th} 2×2 block has the form

$$\begin{aligned} \sum_{i,j=2l-1}^{2l} g^{ij} Z_i \otimes Z_j &= g^{2l-1,2l} Z_{2l-1} \otimes Z_{2l} + g^{2l,2l-1} Z_{2l} \otimes Z_{2l-1} \\ &= g^{2l-1,2l}(X_{2l-1} + iX_{2l}) \otimes (X_{2l-1} - iX_{2l}) \\ &\quad + g^{2l,2l-1}(X_{2l-1} - iX_{2l}) \otimes (X_{2l-1} + iX_{2l}) \\ &= g_{2l-1} X_{2l-1} \otimes X_{2l-1} + g_{2l} X_{2l} \otimes X_{2l} \end{aligned}$$

where $g_{2l-1} = g_{2l} := g^{2l-1,2l} + g^{2l,2l-1}$. Defining $g_n := g^{nn}$ for odd n,

$$g = \sum_{i=1}^n g_i X_i \otimes X_i. \qquad \square$$

The metric g at m is determined, by the lemma, up to the constants g_i. Denote by

$$h=\sum_{i=1}^{n} h_i\theta^i\otimes\theta^i$$

the tentative form of $g(m)$, where the h_i are positive constants to be ascertained.

Form the lowered Riemann curvature by contracting $R(*,*)=R(*,*)^i_j$ with h:

$$R(*,*)_{ij}:=h_iR(*,*)^i_j$$

where we have expressed the tensor indices in terms of the basis

$$(X_1,\dots,X_{2k},(X_n)).$$

The skew-symmetry of the lowered Riemann curvature gives

$$h_iR(*,*)^i_j=-h_jR(*,*)^j_i \tag{2}$$

for all i,j, from which it follows that $R(*,*)^i_j$ is a nonzero two-form if and only if $R(*,*)^j_i$ is nonzero. Also, $R(*,*)^i_i=0$ for all i.

Let $\sim$ denote the equivalence relation on the set $\{1,\dots,n\}$ generated by

$$i\sim j\qquad\text{if and only if}\qquad R(*,*)^i_j\neq 0$$

for all $i\neq j$. Thus $i\sim j$ if and only if $i=j$ or there exists a sequence $i=i_1,\dots,i_r=j$ such that $i_1\neq i_2$, $i_2\neq i_3$, $i_3\neq i_4$,..., $i_{r-1}\neq i_r$ and

$$R(*,*)^{i_l}_{i_{l+1}}\neq 0$$

for $l=1,\dots,r-1$. If i and j are related by such a sequence, h_i may be expressed in terms of h_j by means of the equations

$$\begin{aligned}
h_{i_1}R(*,*)^{i_1}_{i_2}&=-h_{i_2}R(*,*)^{i_2}_{i_1}\\
h_{i_2}R(*,*)^{i_2}_{i_3}&=-h_{i_3}R(*,*)^{i_3}_{i_2}\\
h_{i_3}R(*,*)^{i_3}_{i_4}&=-h_{i_4}R(*,*)^{i_4}_{i_3}\\
&\vdots\\
h_{i_{r-1}}R(*,*)^{i_{r-1}}_{i_r}&=-h_{i_r}R(*,*)^{i_r}_{i_{r-1}}.
\end{aligned}$$

We shall say that the Riemann curvature possesses property P at $m\in M$ if there exist $\xi_1,\xi_2\in T_mM$ such that $R(\xi_1,\xi_2)$ is nondegenerate and there is only one equivalence class: the entire set $\{1,\dots,n\}$. In this case, all the constants $h_1,\dots,h_n$ may be determined from just one constant, say h_1. This gives g up to a conformal factor.

Theorem 2. *A positive definite metric may be constructed algebraically, up to conformal equivalence, from its Riemann curvature by means of the equations*

$$\sum_{s=1}^{n}(g_{is}R^s_{jkl} + g_{js}R^s_{ikl}) = 0$$

for $1 \leq i, j, k, l \leq n$, *if the Riemann curvature satisfies property P.*

Next, we provide examples that illustrate properties P and totality. In particular, we show that neither property generalizes the other.

Example 1. The 4-sphere: total but doesn't satisfy P.

Consider any pair of linearly independent vectors (ξ_1, ξ_2) belonging to the same tangent space at some point of the unit 4-sphere. Apply the Gramm–Schmidt process to ξ_1 and ξ_2 to obtain vectors Y_1 and Y_2. Now extend Y_1 and Y_2 to an orthonormal basis (Y_1, Y_2, Y_3, Y_4) of the tangent space, and denote by $(\omega^1, \omega^2, \omega^3, \omega^4)$ the dual basis. With respect to this basis, the curvature form of the unit 4-sphere is

$$R(*,*) = \begin{pmatrix} 0 & \omega^1 \wedge \omega^2 & \omega^1 \wedge \omega^3 & \omega^1 \wedge \omega^4 \\ \omega^2 \wedge \omega^1 & 0 & \omega^2 \wedge \omega^3 & \omega^2 \wedge \omega^4 \\ \omega^3 \wedge \omega^1 & \omega^3 \wedge \omega^2 & 0 & \omega^3 \wedge \omega^4 \\ \omega^4 \wedge \omega^1 & \omega^4 \wedge \omega^2 & \omega^4 \wedge \omega^3 & 0 \end{pmatrix}$$

from which it may be seen that the curvature is total. Furthermore, $R(\xi_1, \xi_2)$ has rank two. Therefore $\lambda = 0$ is a degenerate eigenvalue of $R(\xi_1, \xi_2)$ and so property P fails to hold.

Example 2. A conformally flat metric in 3-space: not total but satisfies P.

Consider the metric $g = e^{2xy}(dx^2 + dy^2 + dz^2)$ on

$$M := \Re^3 - \{(x, y, z) : 2xy = 1\}.$$

In the $(\partial_x, \partial_y, \partial_z)$-frame the curvature form is given by

$$R(*,*) = \begin{pmatrix} 0 & 0 & \theta \\ 0 & 0 & \psi \\ -\theta & -\psi & 0 \end{pmatrix}$$

where

$$\theta = -x^2 dx \wedge dz + (-1 + xy) dy \wedge dz$$
$$\psi = (-1 + xy) dx \wedge dz - y^2 dy \wedge dz$$

whence it is apparent that the totality condition fails to hold. We shall see, nevertheless, that property P satisfied. Choose $\xi_1 := \partial_x$ and $\xi_2 := \partial_z$. Then

$$R(\xi_1, \xi_2) = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{pmatrix}$$

which has distinct eigenvalues

$$\lambda_1 = ci, \quad \lambda_2 = -ci \quad \text{and} \quad \lambda_3 = 0$$

where $a := -x^2$, $b := -1 + xy$ and $c := \sqrt{a^2 + b^2}$. Corresponding eigenvectors are

$$Z_1 := \begin{pmatrix} -ai/c \\ -bi/c \\ 1 \end{pmatrix} \qquad Z_2 := \begin{pmatrix} ai/c \\ bi/c \\ 1 \end{pmatrix} \qquad \text{and} \qquad Z_3 := \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}.$$

This gives the orthogonal frame

$$X_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad X_2 := \begin{pmatrix} -a/c \\ -b/c \\ 0 \end{pmatrix} \qquad \text{and} \qquad X_3 := \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}.$$

The curvature form in the (X_1, X_2, X_3)-frame is

$$R(*,*) = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta/c^2 & 0 & 0 \end{pmatrix}$$

where

$$\alpha = (a\theta + b\psi)/c$$
$$\beta = b\theta - a\psi.$$

The two-form α is always nonzero and β is not a scalar multiple of α when $2xy \neq 1$. It follows that P holds on the manifold M.

Example 3. The 3-sphere: total and satisfies P.

Choose a local orthonormal frame (Y_1, Y_2, Y_3) of the 3-sphere and let $(\omega^1, \omega^2, \omega^3)$ be the dual coframe. The curvature form with respect to this frame is

$$R(*,*) = \begin{pmatrix} 0 & \omega^1 \wedge \omega^2 & \omega^1 \wedge \omega^3 \\ \omega^2 \wedge \omega^1 & 0 & \omega^2 \wedge \omega^3 \\ \omega^3 \wedge \omega^1 & \omega^3 \wedge \omega^2 & 0 \end{pmatrix}.$$

It is evident that the curvature is total. By choosing $\xi_1 = X_1$ and $\xi_2 = X_2$ at a point on the manifold we obtain

$$R(\xi_1, \xi_2) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = -i$, $\lambda_2 = i$ and $\lambda_3 = 0$, with corresponding eigenvectors

$$Z_1 := \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \qquad Z_2 := \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad Z_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In the associated orthogonal frame (X_1, X_2, X_3) where

$$X_1 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad X_2 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad X_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

the curvature form is given by

$$R(*,*) = \begin{pmatrix} 0 & \omega^2 \wedge \omega^1 & \omega^2 \wedge \omega^3 \\ \omega^1 \wedge \omega^2 & 0 & \omega^1 \wedge \omega^3 \\ \omega^3 \wedge \omega^2 & \omega^3 \wedge \omega^1 & 0 \end{pmatrix}$$

and so P is satisfied.

Finally, we note that a flat space of dimension greater than one clearly is neither total nor satisfies property P.

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DEPARTMENT OF MATHEMATICS, TRINITY WESTERN UNIVERSITY, 7600 GLOVER ROAD, LANGLEY, BC, V2Y 1Y1 CANADA
richard.atkins@twu.ca

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