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# Semisimple ring spectra

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Abstract. We define global dimension and weak dimension for the structured ring spectra that arise in algebraic topology. We provide a partial classification of ring spectra of global dimension zero, the semisimple ring spectra of the title. These ring spectra are closely related to classical rings whose projective modules admit the structure of a triangulated category. Applications to two analogues of the generating hypothesis in algebraic topology are given.

Contents



## 1. Introduction

Classical rings are often studied via their categories of modules or their derived categories. Important invariants such as global and weak dimension are defined in terms of resolutions (by either projective or flat modules) in the module category; such resolutions are examples of exact sequences,

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which make sense in any abelian category. The ring analogues that arise in algebraic topology, on the other hand, do not have abelian module categories. The derived category of modules over a symmetric ring spectrum ([15]) or $S$-algebra ([7]) is a triangulated category, and it therefore has more in common with the derived category of an ordinary ring. Consequently, we use derived categorical formulations of homological dimension that carry over to more general triangulated categories. These ideas have much in common with the theory of projective classes presented in [4]. In this paper, we will use the term 'ring spectrum' to mean either a symmetric ring spectrum or an $S$-algebra. The categories of symmetric ring spectra and $S$-algebras are Quillen equivalent ([21, 0.4], [20, 0.2]), and up to homotopy these ring spectra are the $A_\infty$ ring spectra ([7, 3.4]).

Let $E$ be a ring spectrum, and let $\mathcal{D}_E$ denote the derived category of right $E$-module spectra. Recall that $\mathbf{thick}\langle E\rangle$, the smallest full subcategory of $\mathcal{D}_E$ containing $E$ and closed under suspension, retraction, and cofiber sequences, is exactly the collection of compact $E$-modules. Call an $E$-module *projective* if it is a retract of a coproduct of suspensions of $E$; note that the compact projective modules are exactly the retracts of finite coproducts. Write $\langle E\rangle^0$ for the class of projective modules, and inductively define a module $M$ to belong to $\langle E\rangle^i$ if there exists a cofiber sequence

$$P \longrightarrow Y \longrightarrow \widetilde{M} \longrightarrow \Sigma P$$

with $P$ projective, $Y \in \langle E\rangle^{i-1}$, and $M$ a retract of $\widetilde{M}$. This defines a filtration of $\mathcal{D}_E$ that is not in general exhaustive. Define $\langle E\rangle^i_f$ similarly, where $\langle E\rangle^0_f$ is the class of compact projective modules; this defines an exhaustive filtration of $\mathbf{thick}\langle E\rangle$. The *right global dimension* of $E$, written $\mathrm{r.gl.dim.}\,E$, is the smallest integer $n$ such that $\langle E\rangle^n = \mathcal{D}_E$ (or $\infty$ if no such $n$ exists), and the *right weak dimension* of $E$, written $\mathrm{r.w.dim.}\,E$, is the smallest integer $n$ such that $\langle E\rangle^n_f = \mathbf{thick}\langle E\rangle$ (or $\infty$ if no such $n$ exists). Call $E$ *semisimple* if it has global dimension zero, and call $E$ *von Neumann regular* if it has weak dimension zero (we prove in §2.1 that semisimplicity and von Neumann regularity are right-left symmetric).

**Remark 1.1.** Our definitions are related to at least two others. By [4, 3.2], $\langle E\rangle^0$ is part of a stable projective class with respect to the ideal of ghosts (maps inducing the trivial map on homotopy groups). Hence, in the terminology of [4], $E$ has right global dimension at most $n$ if and only if every $E$-module has length at most $n+1$ with respect to this projective class. It is shown in [4, §7] that the right (and left) global and weak dimensions of the sphere spectrum $S$ are both $\infty$. Note that $\langle E\rangle^0_f$ does not form part of a projective class in $\mathbf{thick}\langle E\rangle$ if there is any compact $E$-module $M$ such that $\pi_*M$ is not finitely generated over $\pi_*E$. But, according to Rouquier ([25, 3.2]), a triangulated category $\mathcal{T}$ has dimension zero if there exists any object $M$ in $\mathcal{T}$ such that the smallest full subcategory of $\mathcal{T}$ containing $M$ and closed under finite coproducts, retracts, and suspensions is $\mathcal{T}$ itself. Hence, if a ring spectrum $E$ is von Neumann regular, then $\mathbf{thick}\langle E\rangle$ is zero-dimensional in this sense. We do not know whether the converse is true.

In Proposition 2.6 we observe that these definitions are consistent with those in classical ring theory, in the sense that the Eilenberg–Mac Lane spectrum $HR$ is a semisimple (von Neumann regular) ring spectrum if and only if $R$ is a semisimple (von Neumann regular) ring. More generally, the main result of [12, §1] shows that the right global dimension of a ring $R$ is equal to the right global dimension of the ring spectrum $HR$. The work in [12, §2] shows that the weak dimension of $R$ is at most the right weak dimension of $HR$, with equality if $R$ is right coherent. We do not know whether the two are equal in general.

The following theorem summarizes the implications of Theorems 3.2, 4.1, and 4.2 for the classification of semisimple and von Neumann regular ring spectra.

**Theorem 1.2.** *Let $E$ be a ring spectrum. If $\pi_*E$ is graded commutative, then:*

(1) *If $E$ is semisimple, then $\pi_*E \cong R_1 \times \cdots \times R_n$, where $R_i$ is either a graded field $k$ or an exterior algebra $k[x]/(x^2)$ over a graded field containing a unit in degree $3|x|+1$ (i.e., $\pi_*E$ is a* graded commutative $\Delta^1$-ring).

(2) *If $E_*$ is countable and $E$ is von Neumann regular, then $(\pi_*E)_{\mathfrak{p}}$ is a graded commutative local $\Delta^1$-ring for every prime ideal $\mathfrak{p}$ of $\pi_*E$.*

*If $E$ is commutative, then:*

(3) *$E$ is semisimple if and only if $\pi_*E$ is a graded commutative $\Delta^1$-ring and for every factor ring of $\pi_*E$ of the form $k[x]/(x^2)$, $x \cdot \pi_*C \neq 0$, where $C$ is the cofiber of $x \cdot E$.*

(4) *If $\pi_*E$ is local or Noetherian, then $E$ is semisimple if and only if $E$ is von Neumann regular.*

Further, any graded commutative $\Delta^1$-ring is the homotopy of some (not necessarily commutative) ring spectrum (see Remark 3.12). Hence, the homotopy of a semisimple ring spectrum need not be a semisimple ring.

This paper is organized as follows. In §2 we present a few general facts about semisimplicity and von Neumann regularity, some in a more general context (triangulated categories) and others, such as Morita invariance, in the context of ring spectra. Whenever $E$ is semisimple, the functor $\pi_*$ induces a triangulation of the projective $\pi_*E$-modules, leading to the strictly algebraic problem of classifying the rings for which the associated category of projective modules admits a triangulation. This is addressed in §3 for commutative rings. In §4, we apply the results in §3 to stable homotopy categories; there is slightly more at issue than whether projective modules admit triangulations. Finally, in §5, we consider two forms of the generating hypothesis. In the stable category of spectra, Freyd's generating hypothesis ([9, §9]) is the conjecture that any map of finite spectra inducing the trivial map of homotopy groups must itself be trivial. The global version of this conjecture — that this is true for all maps of spectra — is easily seen to be false. The present work was motivated in part by the following question: what must be true about a triangulated category in order for it to support a global version of the generating hypothesis?

We would like to thank Sunil Chebolu, who informed us of his work with Benson, Christensen, and Mináč on the generating hypothesis in the stable module category. It is his presentation of the material in [2] that led us to consider certain questions raised in the present paper.

## 2. Semisimplicity and von Neumann regularity

Before turning to the classification of semisimple and von Neumann regular ring spectra, we record a few general facts. Certain results are more usefully stated in a general setting; this framework is established in §2.1. In §2.2, we compare ring spectra to classical rings and show that, up to homotopy, semisimple ring spectra $E$ with $E_*$ local are wedges of suspensions of a Morava $K$-theory $K(n)$. We discuss Morita invariance in §2.3.

**2.1. Triangulated categories.** We assume that the reader is somewhat familiar with triangulated categories. In brief, a triangulated category is an additive category $\mathcal{T}$ together with an automorphism $\Sigma$ of $\mathcal{T}$ called *suspension* and a collection of diagrams called *exact triangles* of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

satisfying several axioms (see [14, A.1.1] or [24] or [28]). Using the suspension functor, one can view the morphism sets in $\mathcal{T}$ as graded groups: define $[X, Y]_k = \operatorname{Hom}_{\mathcal{T}}(\Sigma^k X, Y)$ and

$$[X, Y]_* = \bigoplus_{k \in \mathbb{Z}} [X, Y]_k.$$

Let $S$ be an object in $\mathcal{T}$, and let $\pi_*(-) = [S, -]_*$; we will sometimes write $S_*$ for $\pi_* S$. Recall that a $\pi_* S$-module $N$ is *realizable* if there is an object $X \in \mathcal{T}$ such that $\pi_* X = N$. For any $X \in \mathcal{T}$, the *localizing subcategory generated by* $X$, written $\mathbf{loc}\langle X \rangle$, is the smallest full subcategory of $\mathcal{T}$ containing $X$ that is thick and closed under arbitrary coproducts. If $\mathcal{T} = \mathbf{loc}\langle X \rangle$, then we say that $X$ *generates* $\mathcal{T}$. A cohomology functor $H$ on $\mathcal{T}$ is *representable* if there is an object $Y$ in $\mathcal{T}$ such that the functors $H$ and $[-, Y]_0$ are naturally isomorphic (see [14] for more details). $\mathcal{T}$ is *cocomplete* if arbitrary coproducts exist in $\mathcal{T}$. Call $\mathcal{T}$ a *weak stable homotopy category* if it is a cocomplete triangulated category where every cohomology functor on $\mathcal{T}$ is representable. This definition is 'weak' in the sense that it is obtained from the definition of stable homotopy category given in [14, 1.1.4] by deleting conditions (b) and (c) concerning smash products and generators. Note that $\mathcal{D}_E$ is a weak stable homotopy category with compact generator $E$. Call any object $S$ in a triangulated category $\mathcal{T}$ *semisimple* if $\mathbf{loc}\langle S\rangle = \langle S\rangle^0$ and *von Neumann regular* if $\mathbf{thick}\langle S\rangle = \langle S\rangle^0_f$.

If $\mathcal{T}$ is cocomplete, then idempotents split in $\mathcal{T}$ ([3]), so every projective right $\pi_*S$-module is realizable as $\pi_*Y$, where $Y \in \langle S\rangle^0$. Hence, $\pi_*$ induces an equivalence of categories

$$\Phi : \langle S\rangle^0 \longrightarrow \mathcal{P},$$

where $\mathcal{P}$ is the category of projective right $\pi_*S$-modules.

**Proposition 2.1.** *Let $\mathcal{T}$ be a weak stable homotopy category with compact generator $S$. The following are equivalent:*

(1) *$S$ is semisimple.*
(2) *The functor $\pi_*(-)$ is full and faithful.*
(3) *The functor $\pi_*(-)$ is faithful.*
(4) *The realizable modules and the projective modules coincide.*
(5) *The realizable modules and the injective modules coincide.*

*If any one of the above conditions is satisfied, then $\pi_*S$ is a quasi-Frobenius ring.*

A ring is *quasi-Frobenius* if it is right Noetherian and right self-injective. This condition is right-left symmetric, and the quasi-Frobenius rings are exactly the rings for which the collections of projective and injective modules coincide. In fact, a ring is quasi-Frobenius if and only if every projective module is injective, if and only if every injective module is projective. More information on quasi-Frobenius rings may be found in [18, §15].

**Proof.** The implications (1) $\implies$ (2) $\implies$ (3) should be clear; we now prove that (3) $\implies$ (1). Assume $\pi_*$ is faithful. Using a generating set for the $\pi_*S$-module $\pi_*Y$, construct a map $f : X \longrightarrow Y$ such that $X \in \langle S\rangle^0$ and $\pi_*f$ is surjective. This map fits into an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \Sigma X.$$

Since $\pi_*f$ is surjective, $\pi_*g = 0$. Since $\pi_*$ is faithful, $g = 0$, forcing $Y$ to be a retract of $X$. Hence, $\mathcal{T} = \langle S\rangle^0$.

We next prove that (4) and (5) are equivalent. If $I$ is an injective $\pi_*S$-module, then the functor $\mathrm{Hom}^0_{\mathbb{Z}}(\pi_*(-), I)$ is a cohomology functor on $\mathcal{T}$, hence representable by an object $E(I) \in \mathcal{T}$. In this situation, $\pi_*E(I) = I$, so every injective module is realizable. As already observed, every projective module is realizable. If every realizable module is projective, then every injective module is projective, and $\pi_*S$ a quasi-Frobenius ring. If every realizable module is injective, then every projective module is injective, and again $\pi_*S$ is a quasi-Frobenius ring. In either case, the projective, injective, and realizable modules coincide.

It is clear that (1) $\implies$ (4), so it remains to prove that (4) $\implies$ (1). For any $X \in \mathbf{loc}\langle S\rangle$, $\pi_* X = 0$ if and only if $X$ is trivial. Consequently, for any map $f$ in $\mathbf{loc}\langle S\rangle$, $\pi_* f$ is an isomorphism if and only if $f$ is an equivalence. Now fix $X \in \mathbf{loc}\langle S\rangle$. Since $\pi_* X$ is projective, there is a map $f : Y \longrightarrow X$ such that $Y \in \langle S\rangle^0 \subseteq \mathbf{loc}\langle S\rangle$ and $\pi_* f$ is an isomorphism. Hence, $X$ is equivalent to $Y$. Since $\mathcal{T} = \mathbf{loc}\langle S\rangle$, the implication is established. $\square$

We next prove a similar proposition characterizing von Neumann regularity. As above, $\pi_*$ induces an equivalence of categories

$$\Phi_f : \langle S\rangle_f^0 \longrightarrow \mathcal{P}_f,$$

where $\mathcal{P}_f$ denotes the category of finitely generated projective right $\pi_* S$-modules.

**Proposition 2.2.** *Let $\mathcal{T}$ be a cocomplete triangulated category, and let $S \in \mathcal{T}$ be compact. The following are equivalent:*

(1) *$S$ is von Neumann regular.*
(2) *For all $X \in \mathbf{thick}\langle S\rangle$, $\pi_* X$ is projective.*

*If either condition is satisfied, then $\pi_* S$ is a left and right IF-ring.*

A ring is *injective-flat* (an IF-ring) if every injective module is flat. According to [6], this condition is not left-right symmetric, and a ring is a left and right IF-ring if and only if it is right and left coherent and right and left FP-injective. More information on IF-rings may be found in [8, §6].

**Proof.** If $S$ is von Neumann regular, then (2) holds since $\Phi_f$ is an equivalence of categories; the converse is also clear. Now suppose $S$ is von Neumann regular; to show that $\pi_* S$ is a right IF-ring, it suffices to show that every finitely presented module $M$ embeds in a finitely generated projective module ([8, 6.8]). Since $M$ is finitely presented, it is the cokernel of a map $f : A \longrightarrow B$ of finitely generated projective modules. There is a map $\widetilde{f} : \widetilde{A} \longrightarrow \widetilde{B}$ in $\mathbf{thick}\langle S\rangle = \langle S\rangle_f^0$ such that $f = \pi_* \widetilde{f}$ and an exact triangle

$$\widetilde{A} \xrightarrow{\widetilde{f}} \widetilde{B} \longrightarrow \widetilde{C} \longrightarrow \Sigma \widetilde{A}$$

with $\widetilde{C} \in \langle S\rangle_f^0$. Hence, $M$ embeds in the finitely generated projective module $\pi_* \widetilde{C}$, completing the proof. Using a similar argument, one can show that $\pi_* S$ is also a left IF-ring; $\pi_*(-)$ is replaced with the functor $[-, S]_*$, which induces an equivalence of categories from $\langle S\rangle_f^0$ to the category of projective left $\pi_* S$-modules. $\square$

**Remark 2.3.** Assume $\mathcal{T}$ is a monogenic stable homotopy category and Brown category (see [14, 4.1.4]); then, $S$ is von Neumann regular if and only if the flat modules and realizable modules coincide. Indeed, for all $X \in \mathcal{T}$, $\pi_* X$ is the direct limit over a system of modules of the form $\pi_* X_\alpha$, where $X_\alpha \in \mathbf{thick}\langle S\rangle$. Consequently, if $S$ is von Neumann regular, then every realizable module is flat ([18, 4.4]), and every flat module is realizable (since $(-) \otimes_{\pi_* S} F$ is representable for any flat module $F$). On the other hand, representability of cohomology functors implies that arbitrary products of $\pi_* S$ are realizable. If every realizable module is flat, this forces $\pi_* S$ to be left coherent ([18, 4.47]). Now, for all $X \in \mathbf{thick}\langle S \rangle$, $\pi_* X$ is a finitely presented flat module and therefore projective ([18, 4.30]). By Proposition 2.2, $S$ is von Neumann regular.

We next show that if $\pi_* S$ is commutative and $S$ is semisimple, then so are its algebraic localizations. In a Brown category, the same is true of von Neumann regularity. For any $\pi_* S$-module $M$, let $M_{\mathfrak{p}}$ denote the localization of $M$ at the prime ideal $\mathfrak{p}$. Suppose $\mathcal{T}$ is a cocomplete triangulated category with compact generator $S$. The proofs of [14, 2.3.17, 3.3.3, 3.3.7] show that there exists a localization functor $L_{\mathfrak{p}}$ on $\mathcal{T}$ such that $\pi_* L_{\mathfrak{p}} X \cong (\pi_* X)_{\mathfrak{p}}$. We mean 'localization functor' in the sense of definition [14, 3.1.1], omitting the condition involving smash products, since $\mathcal{T}$ may not have a product structure. In particular, the natural map $Y \xrightarrow{\iota_Y} L_{\mathfrak{p}} Y$ induces an isomorphism

$$[L_{\mathfrak{p}} Y, L_{\mathfrak{p}} X]_* \xrightarrow{\cong} [Y, L_{\mathfrak{p}} X]_*. \tag{2.1}$$

Call an object $\mathfrak{p}$-*local* if it lies in the image of $L_{\mathfrak{p}}$, and let $\mathcal{T}_{\mathfrak{p}}$ denote the full subcategory of $\mathfrak{p}$-local objects. Since $S$ is compact, $L_{\mathfrak{p}}$ commutes with coproducts, and $L_{\mathfrak{p}} S$ is compact in $\mathcal{T}_{\mathfrak{p}} = \mathbf{loc}\langle L_{\mathfrak{p}} S \rangle$.

**Proposition 2.4.** *Suppose $\mathcal{T}$ is a cocomplete triangulated category with compact generator $S$, and suppose $\pi_* S$ is commutative. If $S$ is semisimple, then so is $L_{\mathfrak{p}} S$ for all prime ideals $\mathfrak{p}$ in $\pi_* S$. If $\mathcal{T}$ is also a Brown category, then $L_{\mathfrak{p}} S$ is von Neumann regular whenever $S$ is von Neumann regular.*

**Proof.** If $S$ is semisimple, then the fact that $L_{\mathfrak{p}}$ commutes with coproducts implies that $L_{\mathfrak{p}} S$ is semisimple.

Suppose $\mathcal{T}$ is a Brown category. By [14, 4.2.2, 4.2.3], every object $X$ in $\mathcal{T}$ is the minimal weak colimit (see [14, 2.2.1]) of the diagram of compact objects mapping to $X$. Now, let $L_{\mathfrak{p}} Z$ be a compact object in $\mathcal{T}_{\mathfrak{p}}$. Since localization preserves minimal weak colimits ([14, 3.5.1]), $L_{\mathfrak{p}} Z$ is a minimal weak colimit of objects in $L_{\mathfrak{p}}(\mathbf{thick}\langle S \rangle)$. By compactness, the identity map of $L_{\mathfrak{p}} Z$ must factor through an object in $L_{\mathfrak{p}}(\mathbf{thick}\langle S \rangle)$ ([14, 4.2.1]). Consequently, every small object in $\mathcal{T}_{\mathfrak{p}}$ is the retract of an object in $L_{\mathfrak{p}}(\mathbf{thick}\langle S \rangle)$. If $S$ is von Neumann regular, then $L_{\mathfrak{p}}(\mathbf{thick}\langle S \rangle) \subseteq \langle L_{\mathfrak{p}} S \rangle_f^0$, so $\langle L_{\mathfrak{p}} S \rangle_f^0 = \mathbf{thick}\langle L_{\mathfrak{p}} S \rangle$. This completes the proof. □

Note that the proof of Proposition 2.4 is in fact valid for any smashing localization functor (i.e., one that preserves coproducts; see [14, 3.3.2]), not just the algebraic localizations.

**2.2. Ring spectra.** In this section we establish a few basic facts specific to ring spectra. First, we address left-right symmetry.

**Proposition 2.5.** *Let $E$ be a ring spectrum. $E$ is right semisimple (von Neumann regular) if and only if $E$ is left semisimple (von Neumann regular).*

**Proof.** Since the left $E$-modules are the same as the right $E^{\mathrm{op}}$-modules, we need to show that $E$ is right semisimple (von Neumann regular) if and only if $E^{\mathrm{op}}$ is right semisimple (von Neumann regular). As in [7, §III], we will use the notation $\mathcal{M}_E(-,-)$ to denote morphisms in the category $\mathcal{M}_E$ of right $E$-modules and $F_E(-,-)$ for function objects in $\mathcal{M}_E$. Further, we have two duality functors

$$D(-) = F_E(-,E) : \mathcal{M}_E \longrightarrow \mathcal{M}_{E^{\mathrm{op}}}$$

and

$$D^{\mathrm{op}}(-) = F_{E^{\mathrm{op}}}(-,E) : \mathcal{M}_{E^{\mathrm{op}}} \longrightarrow \mathcal{M}_E.$$

Now, let $M$ be a right $E$-module. We have the following sequence of natural bijections:

$$\begin{aligned}\mathcal{M}_E(M, D^{\mathrm{op}}DM) &\cong \mathcal{M}_{E\wedge_S E^{\mathrm{op}}}(M \wedge_S DM, E)\\ &\cong \mathcal{M}_{E^{\mathrm{op}}\wedge_S E}(DM \wedge_S M, E)\\ &\cong \mathcal{M}_{E^{\mathrm{op}}}(DM, DM).\end{aligned}$$

The first and third bijections are applications of [7, III.6.5(i)]. The identity map of $DM$ therefore provides a natural transformation from the identity functor to the functor

$$M \mapsto D^{\mathrm{op}}DM.$$

Since this natural transformation induces an isomorphism when $M = E$, it is an isomorphism for all $M$ in $\mathbf{thick}\langle E\rangle$.

Note that both $D$ and $D^{\mathrm{op}}$ preserve compact projective modules. Hence, if every compact right $E^{\mathrm{op}}$-module is projective, then so is every compact right $E$-module. This proves that $E$ is von Neumann regular if and only if $E^{\mathrm{op}}$ is. Now, if $E$ is right semisimple, then it is right von Neumann regular, and so $E^{\mathrm{op}}$ is right von Neumann regular. As in Remark 2.3, this means that every realizable right $\pi_*E^{\mathrm{op}}$-module is flat. However, since $E$ is right semisimple, $\pi_*E$ is quasi-Frobenius by Proposition 2.1; hence, $\pi_*E^{\mathrm{op}}$ is quasi-Frobenius and every flat module is projective. Therefore $D(E^{\mathrm{op}}) = \langle E^{\mathrm{op}}\rangle^0$ and $E^{\mathrm{op}}$ is semisimple. □

The following proposition gives the correspondence between our definitions and the ones in classical ring theory, where the semisimple rings are those of global dimension zero and the von Neumann regular rings are those of weak dimension zero.

**Proposition 2.6.** *Let $HR$ denote the Eilenberg–Mac Lane spectrum associated to the ring $R$. $HR$ is semisimple if and only if $R$ is semisimple, and $HR$ is von Neumann regular if and only if $R$ is von Neumann regular.*

**Proof.** It is well-known (see, for example, [7, IV.2.4]) that the derived category of $HR$-module spectra is equivalent to the derived category of $R$-modules, $\mathcal{D}(R)$. In [19, §4] it is shown that $\pi_*$ is faithful on $\mathcal{D}(R)$ if and only if $R$ is semisimple, and in [13, 1.3, 1.5] it is shown that $\mathbf{thick}\langle R\rangle = \langle R\rangle_f^0$ if and only if $R$ is von Neumann regular. $\square$

Semisimple ring spectra are also related to field spectra, as defined in [11, 1.7]. A noncontractible ring spectrum $E$ is a *field* if $E_*X = \pi_*E \wedge X$ is a free $E_*$-module for all spectra $X$. According to [11, 1.9], every field has the homotopy type of a wedge of suspensions of the Morava $K$-theory $K(n)$, for some fixed $n$ ($0 \leq n \leq \infty$) and prime $p$. If $E$ is semisimple and $E_*$ is a ring over which every projective module is free, then $E$ is a field. Note, however, that since $E_*$ is quasi-Frobenius by Proposition 2.1, $E_*$ must in fact be local in this case (see [17, 23.6]). We now have

**Proposition 2.7.** *If $E$ is a semisimple ring spectrum and $E_*$ is local, then $E$ has the homotopy type of a wedge of suspensions of $K(n)$ for some fixed $n$ and prime $p$.*

Even if $E_*$ is not local, some $K(n)$ is a summand of $E$. Note that a field spectrum is not necessarily semisimple: for example, consider $E = H\mathbb{Q}[x]$, the polynomial algebra over $H\mathbb{Q}$ generated by $x$ in degree 2. Since $E$ is flat over $H\mathbb{Q}$, $E_*X \cong E_* \otimes_{\mathbb{Q}} H\mathbb{Q}_*X$ is always a free $E_*$-module, hence $E$ is a field. But, the homotopy of the cofiber of multiplication by $x$ on $E$ is $\mathbb{Q}$, a neither free nor projective $E_*$-module, so $E$ is not semisimple. We obtain a better correspondence between semisimplicity and the notion of a skew field object given in [14, 3.7.1(d)]. A *skew field object* in a homotopy category is a ring object $R$ such that *every* $R$-module is isomorphic to a coproduct of suspensions of $R$. Consequently, a ring spectrum $E$ that is a skew field object in $\mathcal{D}_E$ is semisimple if and only if $E$ is a semisimple ring spectrum and $E_*$ is local.

**2.3. Morita invariance.** We should begin by emphasizing that our general definitions of semisimplicity and von Neumann regularity depend on the ambient triangulated category, not just on the object $S$. For example, the Morava $K$-theory ring spectrum $K(n)$ is semisimple as an object in the derived category $\mathcal{D}_{K(n)}$, but it is not von Neumann regular as an object in the stable category of spectra $\mathcal{S}$. For example, the spectrum $L_{K(n)}F$, where $F$ is a finite type $n$ spectrum, is in $\mathbf{thick}\langle K(n)\rangle \subseteq \mathcal{S}$ ([16, 8.12]), but it is not a wedge of suspensions of $K(n)$. The following proposition summarizes how the general definitions of semisimplicity and von Neumann regularity relate to the particular definitions for ring spectra.

**Proposition 2.8.** *A right $E$-module $M$ is von Neumann regular as an object in $\mathcal{D}_E$ if and only if the endomorphism ring spectrum* $\mathrm{End}_E(M)$ *is von Neumann regular as a ring spectrum. If $M$ is compact in $D_E$, then the same is true of semisimplicity.*

This proposition follows easily from the following more general one. Recall that for any object $S$ in a triangulated category $\mathcal{T}$, $\mathbf{thick}\langle S\rangle$ is filtered by the subcategories $\langle S\rangle_f^i$ and $\mathbf{loc}\langle S\rangle$ is filtered by the subcategories $\langle S\rangle^i$. If $\mathcal{T}$ is a triangulated category with filtration $\mathcal{F}_i$ and $\mathcal{U}$ is a triangulated category with filtration $\mathcal{G}_i$, then call an equivalence of triangulated categories $\Phi : \mathcal{T} \longrightarrow \mathcal{U}$ a *filtered equivalence* if the restriction of $\Phi$ to $\mathcal{F}_i$ induces an equivalence $\mathcal{F}_i \xrightarrow{\cong} \mathcal{G}_i$ for all $i$.

**Proposition 2.9.** *Let $E$ be a ring spectrum, let $M$ be a right $E$-module, and let $K = \operatorname{End}_E(M)$. There is a filtered equivalence of* $\mathbf{thick}\langle M\rangle$ *in $\mathcal{D}_E$ with* $\mathbf{thick}\langle K\rangle$ *in $\mathcal{D}_K$. If $M$ is compact, then there is a filtered equivalence of* $\mathbf{loc}\langle M\rangle$ *with* $\mathbf{loc}\langle K\rangle = \mathcal{D}_K$.

**Proof.** We continue to use the notation in the proof of Proposition 2.5. Let

$$K = \operatorname{End}_E(M) = F_E(M, M).$$

Note that $M$ is a $K$-$E$-bimodule. Consider the adjoint functor pair

$$\begin{tikzcd} \mathcal{D}_E \arrow[r, bend left, "F_E(M,-)"] & \mathcal{D}_K . \arrow[l, bend left, "(-)\wedge_K M"] \end{tikzcd}$$

Adjointness induces two natural transformations

$$\eta \;:\; \mathbf{1}_{\mathcal{D}_E} \longrightarrow F_E(M,-)\wedge_K M$$

and

$$\zeta \;:\; F_E(M, (-)\wedge_K M) \longrightarrow \mathbf{1}_{\mathcal{D}_K}.$$

Since $\eta_M$ is an isomorphism, $\eta$ induces an isomorphism on $\mathbf{thick}\langle M\rangle$. Similarly, $\zeta$ induces an isomorphism on $\mathbf{thick}\langle K\rangle$. Our adjoint functors therefore induce an equivalence between $\mathbf{thick}\langle M\rangle$ and $\mathbf{thick}\langle K\rangle$. If $M$ is compact in $\mathcal{D}_E$, then $F_E(M,-)$ commutes with coproducts, so $\eta$ induces an isomorphism on $\mathbf{loc}\langle M\rangle$ and $\zeta$ induces an isomorphism on $\mathbf{loc}\langle K\rangle = \mathcal{D}_K$. In this situation, we have an equivalence of $\mathbf{loc}\langle M\rangle$ with $\mathcal{D}_K$.

We next show that these inverse equivalences are filtration preserving. The functor $\Phi = (-)\wedge_K M$ preserves coproducts, so $\Phi(\langle K\rangle_f^0) \subseteq \langle M\rangle_f^0$ and $\Phi(\langle K\rangle^0) \subseteq \langle M\rangle^0$. Since $\Phi$ preserves cofiber sequences, $\Phi(\langle K\rangle_f^i) \subseteq \langle M\rangle_f^i$ and $\Phi(\langle K\rangle^i) \subseteq \langle M\rangle^i$ for all $i \geq 0$, by induction on $i$. The same is true of the functor $F_E(M,-)$ for the filtration on $\mathbf{thick}\langle M\rangle$, and also for the filtration on $\mathbf{loc}\langle M\rangle$ when $M$ is compact. This completes the proof. $\square$

In the Morita theory of classical rings, two rings $R$ and $S$ are Morita equivalent if and only if $S \cong \operatorname{End}_R(P)$ for some progenerator $P$ in the category of $R$-modules $\mathbf{Mod}(R)$ ([18, 18.33]). A right $R$-module $P$ is a *progenerator* if and only if it is a finitely generated projective module such that $R$ is a retract of a finite sum of copies of $P$ ([18, 18.8]). If $E$ is a ring spectrum, then $P_*$ is a progenerator of $\mathbf{Mod}(E_*)$ if and only if it is realizable as the homotopy of an $E$-module $P$ such that $\langle P\rangle_f^0 = \langle E\rangle_f^0$. As a consequence of the last proposition, we have the following result on the 'classical' Morita invariance of global and weak dimension.

**Proposition 2.10.** *Let $E$ be a ring spectrum, and let $P$ be a right $E$-module such that $\langle P\rangle_f^0 = \langle E\rangle_f^0$. Then*

$$\text{r. gl. dim.}\, E = \text{r. gl. dim.}\, \mathrm{End}_E(P) \;\; \textit{and} \;\; \text{r. w. dim.}\, E = \text{r. w. dim.}\, \mathrm{End}_E(P).$$

**Proof.** This follows from Proposition 2.9 with $M = P$. $\square$

If $P$ is an arbitrary compact generator of $\mathcal{D}_E$, then we do not know whether global and weak dimension are preserved in the sense of this proposition. For example, suppose r. gl. dim. $E = n$. If $P$ is a compact generator, then $E \in \langle P\rangle_f^k$ for some $k$, and the global dimension of $\mathrm{End}_E(P)$ is at most $k+n$, but we do not know of an example where the dimension of $\mathrm{End}_E(P)$ is actually larger than $n$. Put another way, we do not know whether concepts such as semisimplicity and von Neumann regularity are Morita invariant in the more general sense of [26]. Following [26, 4.20], call a pair of ring spectra $E$ and $F$ *Morita equivalent* if there is a chain of spectral Quillen equivalences between the model categories $\mathcal{M}_E$ and $\mathcal{M}_F$ of right modules. Semisimplicity is Morita invariant in the following sense.

**Proposition 2.11.** *Suppose $E$ and $K$ are Morita equivalent symmetric ring spectra with $\pi_*E$ commutative. If $E$ is semisimple, then $K$ is semisimple.*

Our proof requires an understanding of the relationship between generators in the derived category $\mathcal{D}_E$ and generators in the category of right $E_*$-modules. The next two propositions address this relationship, but our analysis is far from complete when $E_*$ is noncommutative. We do not know whether von Neumann regularity is also Morita invariant.

**Proposition 2.12.** *Let $E$ be a ring spectrum, and let $M$ be a compact projective $E$-module. Then, $M$ is a generator of $\mathcal{D}_E$ if and only if*

$$\mathrm{Hom}_{E_*}(M_*, N) \neq 0$$

*for every nontrivial realizable $E_*$-module $N$.*

**Proof.** Throughout the proof, we use the fact that

$$\mathcal{D}_E(X, Y)_* \cong \mathrm{Hom}_{E_*}(X_*, Y_*)$$

whenever $X$ is a projective $E$-module.

Suppose $M$ is a generator of $\mathcal{D}_E$, and let $N = X_* \neq 0$ be a realizable $E_*$-module. The functor $\mathcal{D}_E(-, X)_*$ vanishes on a localizing subcategory of $\mathcal{D}_E$. If it vanishes on $M$, then it also vanishes on $X$, forcing $X$ to be trivial. Consequently, since $X$ is nontrival, $\mathcal{D}_E(M, X)_* = \mathrm{Hom}_{E_*}(M_*, N) \neq 0$.

Conversely, suppose $\mathrm{Hom}_{E_*}(M_*, N) \neq 0$ for all nontrivial realizable modules $N$. Then, $\mathcal{D}_E(M, X)_*$ must be nontrivial whenever $X$ is nontrivial.

There exists a localization functor $L$ whose kernel is the localizing subcategory generated by the compact object $M$. Now, since $\mathcal{D}_E(M, LX)_* = 0$ for all $X$, the only local object is the trivial object. Hence, the kernel of $L$ must be all of $\mathcal{D}_E$; i.e., $M$ generates $\mathcal{D}_E$. $\square$

We do not know whether the commutativity assumption in the following proposition may be dropped. If semisimplicity is indeed a Morita invariant notion, then the proposition must be true without it (provided $E$ is semisimple). Note that any right (or left) Artinian ring — and hence any quasi-Frobenius ring — is semiperfect ([1, p. 303]).

**Proposition 2.13.** *Let $E$ be a ring spectrum, and let $M$ be a compact projective $E$-module. Then:*

1. *If $M_*$ is a generator of $\mathbf{Mod}(E_*)$, then $M$ is a generator of $\mathcal{D}_E$.*
2. *Suppose $\pi_*E$ is commutative and semiperfect. If $M$ is a generator of $\mathcal{D}_E$, then $M_*$ is a generator of $\mathbf{Mod}(E_*)$.*

**Proof.** Suppose $M_*$ is a generator of $\mathbf{Mod}(E_*)$. By definition of generator, $\mathrm{Hom}_{E_*}(M_*, -)$ acts faithfully on $\mathbf{Mod}(E_*)$; by Proposition 2.12, $M$ is a generator of $\mathcal{D}_E$. This establishes (1).

Assume $E_*$ is commutative and semiperfect, and suppose $M$ is a generator of $\mathcal{D}_E$. Since $E_*$ is semiperfect, it admits a decomposition as a product of rings $E_* = (e_1E_*) \times \cdots \times (e_nE_*)$, where the $e_i$ form a complete set of orthogonal local idempotents ([1, 27.6]). Since $M_*$ is projective, it is a product of copies of these factors ([1, 27.11]); we will show that each factor appears at least once, implying that $M_*$ is a generator of $\mathbf{Mod}(E_*)$ ([18, 18.8]). Since $M$ is a generator of $D(E)$, $\mathrm{Hom}_{E_*}(M_*, e_iE_*) \neq 0$ for all $i$ by Proposition 2.12. This forces $e_iE_*$ to be a factor of $M_*$, as there are no maps from $e_jE_*$ to $e_iE_*$ when $i \neq j$ (this uses commutativity). This establishes (2). $\square$

We now give the proof of Proposition 2.11.

**Proof of Proposition 2.11.** Assume $E$ is semisimple. By [26, 4.20], there exists a $K$-$E$-bimodule $M$ such that $M$ is a compact generator of $\mathcal{D}_E$ and $K \cong F_E(M, M)$. Consider the $E$-$K$-bimodule $Q = F_E(M, E)$. Since $E$ is semisimple, $M$ is projective, and $Q_* = \mathrm{Hom}_{E_*}(M_*, E_*)$. By Proposition 2.13 (2), $M_*$ is a finitely generated projective generator of $\mathbf{Mod}(E_*)$, so $Q_*$ is a finitely generated projective generator of $\mathbf{Mod}(K_*)$ by [18, 18.22]. By Proposition 2.13 (1), $Q$ must be a compact projective generator of $\mathcal{D}_K$.

The map

$$E \longrightarrow F_K(Q, Q),$$

adjoint to the identity map of $Q$ as a $K$-module ([7, III.6.2]), is an equivalence since

$$F_K(Q, Q)_* \cong \mathrm{Hom}_{K_*}(Q_*, Q_*) \cong E_*$$

(the first isomorphism by projectivity of $Q$, the second by [18, 18.17(c)]). If either $E$ or $K$ is flat (a ring spectrum $E$ is *flat* if $E\wedge -$ preserves stable equivalences of symmetric spectra), then by the proof of [26, 4.20], there exists an $E$-$K$-bimodule $Q'$ derived equivalent to $Q$ such that the derived smash product

$$-\wedge_E^L Q' : \mathcal{D}_E \longrightarrow \mathcal{D}_K$$

is an equivalence of categories. Now every object of $\mathcal{D}_K$ is a retract of a coproduct of suspensions of the projective $K$-module $Q'$, and $K$ is semisimple.

If neither $E$ nor $F$ is flat, take a cofibrant replacement $QE$ of $E$. Since $QE$ is flat and Morita equivalent to both $E$ and $F$, two applications of the above argument complete the proof. $\square$

## 3. Projective modules

In section §2.1, we proved that if $E$ is semisimple, then the derived category of $E$-module spectra is equivalent to the category $\mathcal{P}$ of projective right $E_*$-modules. This equivalence endows $\mathcal{P}$ with a triangulation. Similarly, if $E$ is von Neumann regular, then the category $\mathcal{P}_f$ of finitely generated projective right $E_*$-modules admits a triangulation. For any graded module $M$, write $M[n]$ for the shifted module with $M[n]_i = M_{i-n}$. In the induced triangulation on either $\mathcal{P}$ or $\mathcal{P}_f$, the shift functor is determined by the suspension of spectra: $\pi_*\Sigma X = \pi_{*-1}X = \pi_*X[1]$. We have proved:

**Proposition 3.1.** *Let $\mathcal{T}$ be a cocomplete triangulated category with compact generator $S$. If $S$ is semisimple (von Neumann regular), then the category $\mathcal{P}$ (or $\mathcal{P}_f$) admits a triangulation with shift functor $\Sigma = (-)[1]$.*

This proposition implies that if $S$ is semisimple, then $\pi_*S$ must be a product of fields and exterior algebras on one generator by Theorem 3.2 (see §3.1). However, the converse of this statement if false; the triangulation on $\mathcal{P}$ may not be induced by $\pi_*$ (a counter-example is provided in Remark 5.2). The relevant condition is included in Theorem 4.1, presented in §4.

We now characterize the local rings and the commutative rings for which the associated category of projective modules admits a triangulation. Our two main results are Theorems 3.2 and 3.3, stated in the next section.

**3.1. Notation and overview.** Let $R$ be a graded ring. By a module we mean a graded right $R$-module. Let $\mathbf{Mod}(R)$ be the category of all $R$-modules, let $\mathcal{P}$ be the category of projective $R$-modules, and let $\mathcal{P}_f$ denote the category of finitely generated projective $R$-modules. For any $R$-module $M$, write $M[n]$ for the shifted module with $M[n]_i = M_{i-n}$. For any element $m \in M$, denote by $|m|$ the degree of $m$. Write $\mathrm{Hom}_k(M, N)$ for degree $k$ module maps from $M$ to $N$; observe that $\mathrm{Hom}_k(M, N) = \mathrm{Hom}_0(M[k], N)$. If $M$ is an $R$-$R$-bimodule (or, if $R$ is graded commutative), then, for any element $x \in R$ of degree $i$, let $x \cdot M$ denote the right module map from $M[i]$ to $M$ induced by left multiplication by $x$.

In considering triangulations of $\mathcal{P}$ or $\mathcal{P}_f$, it seems natural to consider one of two possibilities for $\Sigma$. If $\Sigma(-) = (-)[1]$, then $[M, N]_*$ is the graded group of graded module maps from $M$ to $N$; this shift functor already appeared in Proposition 3.1. Alternatively, we could consider $\Sigma = \mathbf{1}$, the identity functor. When $R$ is concentrated in degree zero, one then obtains a triangulation of the category of ungraded projective (or finitely generated projective) $R$-modules by identifying this category with the thick subcategory of $\mathcal{P}$ (or $\mathcal{P}_f$) generated by the modules concentrated in degree zero. Usually, we will assume that the suspension functor is of the form $\Sigma = (-)[n]$ for some $n$.

For convenience, we will use the term *$\Delta^n$-ring* to refer to any ring for which $\mathcal{P}$ admits a triangulation with suspension $\Sigma = (-)[n]$ and the term *$\Delta^n_f$-ring* for any ring for which $\mathcal{P}_f$ admits a triangulation with suspension $\Sigma = (-)[n]$. Our goal is to characterize the graded commutative rings $R$ for which the categories $\mathcal{P}$ and $\mathcal{P}_f$ admit triangulations. In §3.2, we prove:

**Theorem 3.2.** *Let $\mathcal{P}$ be the category of projective modules over a graded commutative ring $R$. $\mathcal{P}$ admits a triangulation with suspension $\Sigma = (-)[1]$ if and only if $R \cong R_1 \times \cdots \times R_n$, where each factor ring $R_i$ is either a graded field $k$ or an exterior algebra $k[x]/(x^2)$ over a graded field $k$ containing a unit of degree $3|x| + 1$.*

In the ungraded case, we have:

**Theorem 3.3.** *Let $\mathcal{P}$ be the category of projective modules over a commutative ring $R$. $\mathcal{P}$ admits a triangulation with suspension $\Sigma = \mathbf{1}$ if and only if $R \cong R_1 \times \cdots \times R_n$, where each factor ring $R_i$ is either a field $k$, an exterior algebra $k[x]/(x^2)$ over a field $k$ of characteristic 2, or $T/(4)$, where $T$ is a complete 2-ring.*

A complete 2-ring is a complete local discrete valuation ring of characteristic zero whose maximal ideal is generated by 2 (see [22, p. 223]). In [23], it is shown that the category of finitely generated projective $T/(4)$-modules admits a unique triangulation, and we use their methods to construct triangulations for the rings appearing in the above two theorems.

**3.2. Triangulations of projective modules.** It is in this section that we prove Theorems 3.2 and 3.3. Before we begin, we make a simple observation that we will use frequently. Suppose $\mathcal{P}$ (or $\mathcal{P}_f$) is triangulated. If we apply the functor $[R, -]_0$ to an exact triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

then we must obtain a long exact sequence of $R$-modules. Hence, any exact triangle is exact as a sequence of $R$-modules. ('Exact' at $\Sigma A$ means $\ker(-\Sigma f) = \operatorname{im} h$.)

First we show that if $\mathcal{P}$ admits a triangulation, then $R$ must be a quasi-Frobenius ring, and if $\mathcal{P}_f$ admits a triangulation, then $R$ must be an IF-ring (cf. Propositions 2.1 and 2.2). Note that IF-rings are coherent ([8, 6.9]), so all $\Delta^n$-rings and $\Delta^n_f$-rings are coherent.

**Proposition 3.4.** *If $\mathcal{P}$ admits a triangulation, then $R$ is quasi-Frobenius.*

**Proof.** Let $M$ be an $R$-module. There is a map $f : A \longrightarrow B$ of projective modules whose cokernel is $M$. This map must lie in an exact triangle in $\mathcal{P}$,

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow \Sigma A.$$

Exactness at $B$ implies that $M$ is isomorphic to a submodule of the projective module $C$. If $M$ is injective, then it must be a summand of $C$ and therefore projective. Hence, every injective $R$-module is projective, and $R$ is quasi-Frobenius. □

**Proposition 3.5.** *If $\mathcal{P}_f$ admits a triangulation, then $R$ is a left and right IF-ring.*

**Proof.** In light of the proof of Proposition 3.4, we see that every finitely presented right $R$-module embeds in a finitely generated projective module. By [8, 6.8], $R$ is a right IF-ring. To show that $R$ is also a left IF-ring, use the functor $[-, R]_0$ to show that every finitely presented left $R$-module embeds in a finitely generated projective module. □

**Remark 3.6.** When $R$ is quasi-Frobenius, one can form StMod($R$), the *stable module category* of $R$. The objects of StMod($R$) are $R$-modules and the morphisms are $R$-module maps modulo an equivalence relation: two maps are equivalent if their difference factors through a projective module. StMod($R$) is a triangulated category. If $M$ is an $R$ module, then the *Heller shift* of $M$, written $\Omega M$, is the kernel of a projective cover $P(M) \longrightarrow M$. This descends to a well-defined, invertible endomorphism of the stable module category, and $\Omega^{-1}$ is the suspension functor for the triangulation of StMod($R$). In Proposition 3.4, the exact triangle gives rise to three short exact sequences:

$$0 \longrightarrow M \longrightarrow C \longrightarrow \Sigma \ker f \longrightarrow 0$$

$$0 \longrightarrow \ker f \longrightarrow A \longrightarrow \operatorname{im} f \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im} f \longrightarrow B \longrightarrow M \longrightarrow 0.$$

Together they imply that, if $\mathcal{P}$ admits a triangulation, then $\Omega^3\Sigma M \cong M$. This observation was made by Heller in [10].

The following proposition shows that a triangulation imposes a severe restriction on the local rings that may occur.

**Proposition 3.7.** *Suppose $\mathcal{P}$ admits a triangulation. If $R$ is a local ring with maximal right ideal $\mathfrak{m}$, then $\mathfrak{m}$ is principal and contains no nontrivial proper ideals.*

**Proof.** Let $\mathfrak{m}$ be the maximal right ideal of $R$; since the proposition is clearly true if $\mathfrak{m} = (0)$, assume $\mathfrak{m}$ is nontrivial. Let $S$ be the socle of $R$. Note that $R$ must be quasi-Frobenius by Proposition 3.4; consequently, $S$ is simple and is the minimal nontrivial right (and left) ideal of $R$ ([18, 15.7]). We will argue that $S$ and $\mathfrak{m}$ are equal by showing that they have the same composition length, an analogue of dimension for Artinian rings.

Since $R$ is local, we must in fact have $S \cong R/\mathfrak{m}$; hence, there is a map $f : R \longrightarrow R$ whose image is $S$ and whose kernel is $\mathfrak{m}$. Since $f$ must lie in an exact triangle, there is a free (since $R$ is local) module $F$ and a short exact sequence

$$0 \longrightarrow R/S \longrightarrow F \longrightarrow \Sigma\mathfrak{m} \longrightarrow 0. \tag{3.1}$$

For any $R$-module $M$, let $c(M)$ denote its composition length (see [1, pp. 134–138]). Note that $c(M) = 0$ if and only if $M = 0$ and $c(M) = 1$ if and only if $M$ is simple. By [1, 11.4], if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact, then $c(B) = c(C) + c(A)$. Applied in the current context, we obtain

$$\begin{aligned} c(F) &= c(R/S) + c(\mathfrak{m}) \\ &= (c(R) - c(S)) + (c(R) - c(S)) \\ &< 2c(R). \end{aligned}$$

Hence, we must have $c(F) = c(R)$ and $c(S) = c(\mathfrak{m})$. Since $\mathfrak{m}$ and $S$ have identical composition lengths, they must be equal, and $\mathfrak{m}$ is the only proper nontrivial right ideal of $R$; $\mathfrak{m}$ is principal by uniqueness. □

If $R$ is a graded commutative local ring, then we need only assume that $\mathcal{P}_f$ is triangulated, as the next proposition demonstrates. Note that if $\mathcal{P}$ admits a triangulation, then $\mathbf{thick}\langle R\rangle$ admits a triangulation, and since $R$ must be Noetherian, $\mathbf{thick}\langle R\rangle = \mathcal{P}_f$.

**Proposition 3.8.** *Suppose $\mathcal{P}_f$ admits a triangulation. If $R$ is a graded commutative local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is principal and contains no nontrivial proper ideals.*

**Proof.** First, we observe that, even if $R$ is neither graded commutative nor local, it must satisfy the double annihilator condition $\operatorname{ann}_l \operatorname{ann}_r Rx = Rx$. Let $x \in R$, and let $i = |x|$. We have an exact triangle in $\mathcal{P}$

$$R[i] \xrightarrow{x} R \xrightarrow{\psi} P \xrightarrow{\phi} \Sigma R[i].$$

Observe that $\operatorname{im}\phi = \operatorname{ann}_r Rx$, and if $y \in \operatorname{ann}_l \operatorname{ann}_r Rx$, then $(y \cdot R) \circ \phi = 0$. Exactness at $R[i]$ therefore implies that $\operatorname{ann}_l \operatorname{ann}_r Rx = Rx$.

For the remainder of the proof, we assume $R$ is graded commutative and local. Since $\mathcal{P}_f$ is triangulated, $R$ is coherent; this implies that $\mathrm{ann}(x)$ is finitely generated for all $x \in R$.

Consider $x \in \mathfrak{m}$. We now show that $x^2 = 0$. Since $(x \cdot P) \circ \psi = \psi \circ (x \cdot R) = 0$, we obtain a factorization $x \cdot P = f \circ \phi$, where $f : \mathrm{im}\,\phi \longrightarrow P$. Since $\mathrm{im}\,\phi = \mathrm{ann}(x)$, $(x \cdot P) \circ f = 0$. Hence, $x^2 \cdot P = 0$. Since $R$ is local, $P$ is free ($P$ is nontrivial since $x$ is not a unit). Therefore $x^2 = 0$.

Fix a nonzero element $x \in \mathfrak{m}$. We next show that $\mathrm{ann}(x) = (x)$. Note that $P$ is an extension of finitely generated modules and is therefore finitely generated. Let $\{e_1, \ldots, e_n\}$ be a basis for the free module $P$, let $\phi_i = \phi(e_i)$, and let $\psi(1) = \Sigma e_i t_i$. Since $x$ is nonzero, $\mathrm{im}\,\phi = \mathrm{ann}(x) \subseteq \mathfrak{m}$; hence, $\phi(e_i\phi_i) = \phi_i^2 = 0$, and exactness at $P$ provides an element $z_i \in R$ such that $\psi(z_i) = e_i\phi_i$. This implies that $t_j z_i = 0$ when $i \neq j$, and $t_i z_i = \phi_i$. Since $\mathrm{im}\,\phi = \mathrm{ann}(x)$, the elements $\phi_i$ generate $\mathrm{ann}(x)$. Let $t = t_1 + \cdots + t_n$ and consider $q = \Sigma\phi_i a_i \in \mathrm{ann}(x)$. We have $t(\Sigma z_i a_i) = q$, so $\mathrm{ann}(x) \subseteq (t)$. Since $\psi \circ (x \cdot R) = 0$, we obtain $t_i x = 0$ for all $i$. Hence, $(t) = \mathrm{ann}(x)$. Since $x \neq 0$, $t$ is not a unit, so $(t) \subseteq \mathrm{ann}(t) = \mathrm{ann}\,\mathrm{ann}(x) = (x)$. This proves that $\mathrm{ann}(x) = (t) = (x)$. Further, it is worth remarking that $P$ must be free of rank 1. Suppose $n \geq 2$. It is clear that $e_1 x$ and $e_2 x$ are in the kernel of $\phi$. Hence, for some $a, b \in R$, $e_1 x = \psi(a)$ and $e_2 x = \psi(b)$. This means that $e_1 x b - e_2 x a = 0$, forcing $a, b \in \mathrm{ann}(x) = (x)$. But since $t_i x = 0$ for all $i$, it must be the case that $e_1 x = \psi(a) = \Sigma e_i t_i a = 0$, a contradiction.

Finally, we show that, if $\mathfrak{m} \neq (0)$, it must be the unique, proper, nontrivial, principal ideal of $R$. Suppose $a \in \mathfrak{m}$ and $b \in R$ have the property that $ab \neq 0$. Then, $a \in \mathrm{ann}(ab) = (ab)$. Hence, there is an element $k \in R$ such that $a(1 - bk) = 0$. Since $R$ is local, this forces $a$ to be zero or $b$ to be a unit. Now consider two nonzero elements $x, y \in \mathfrak{m}$. We have $x \in \mathrm{ann}(y) = (y)$ and $y \in \mathrm{ann}(x) = (x)$, so $(x) = (y)$. This forces $\mathfrak{m}$ to be the unique, proper, nontrivial, principal ideal of $R$. □

**Proposition 3.9.** *Let $R$ be a graded commutative local ring with residue field $k$. If $\mathcal{P}$ (or $\mathcal{P}_f$) admits a triangulation with suspension $\Sigma = (-)[n]$, then $R$ is either*

(1) *the graded field $k$,*
(2) *an exterior algebra $k[x]/(x^2)$ with a unit in degree $3|x| + n$, where $\mathrm{char}\,k = 2$ if $n = 0$, or*
(3) *$n$ is even and $R \cong T/(4)$, where $T$ is the unique (up to isomorphism) complete 2-ring with residue field $k$ (of characteristic 2) containing a unit in degree $n$.*

**Proof.** In the proof of Proposition 3.8, we showed that the cofiber of $x \cdot R$ is free of rank 1 for any nontrivial element $x$ in the maximal ideal $\mathfrak{m}$. Hence, any such $x$ fits into an exact triangle of the form

$$R[i] \xrightarrow{x} R \xrightarrow{vx} R[j] \xrightarrow{wx} R[i+n], \tag{3.2}$$

where $i = |x|$ and $v$ and $w$ are units. One can check that $|(vw)^{-1}| = 3i + n$.

Suppose $R$ contains a field. As such, $R$ is a ring of equal characteristic; since it is complete, it contains a field isomorphic to $k = R/\mathfrak{m}$ ([22, 28.3]). So either $\mathfrak{m} = 0$ and $R$ is a graded field, or $\mathfrak{m} = (x)$ for some $x \neq 0$ (by Proposition 3.8) and $R$ is isomorphic to the exterior algebra $k[x]/(x^2)$. In the latter case, we just observed that $k$ must contain a unit of degree $3i+n$.

If $p$ is a unit for all primes $p$, then $R$ contains the field $\mathbb{Q}$. If $p$ is not a unit for some prime, then either $R$ has characteristic $p$ and contains a field, or $p$ is nonzero and is in the maximal ideal. If $p$ is in the maximal ideal $\mathfrak{m}$, then by Proposition 3.8, we must have $\mathfrak{m} = (p)$. If $n$ is odd, this is not possible: since $p$ has degree zero, we saw above that there must be a unit of degree $3 \cdot 0 + n$, which is odd. But if $s$ is an odd unit, then $2s^2 = 0$ by graded commutativity; this forces $2 = 0$, contradicting $\mathfrak{m} = (p)$.

It remains to consider the case where $n$ is even and $\mathfrak{m} = (p)$ for some prime $p$. Let $T$ be the unique (up to isomorphism) complete $p$-ring with residue field $k$ (see [22, §29]). By the discussion on p. 225 of [22], since $R$ is a complete local ring of unequal characteristic, it contains a coefficient ring $A \cong T/(p^2)$. Further, it is observed in the proof of [22, 29.4] that every element can be expanded as a power series in the generators of the maximal ideal with coefficients in $A$. Since the maximal ideal is generated by $p \in A$, we obtain $R \cong A$. To see why $p = 2$, consider the rotation of triangle (3.2),

$$R \xrightarrow{vp} R[j] \xrightarrow{wp} R[n] \xrightarrow{-p} R[n].$$

The map $\kappa$ in the diagram

$$\begin{tikzcd} R \arrow[r, "p"] \arrow[d, equal] & R \arrow[r, "vp"] \arrow[d, "v"] & R[j] \arrow[r, "wp"] \arrow[d, dashed, "\kappa"] & R[n] \arrow[d, equal] \\ R \arrow[r, "vp"] & R[j] \arrow[r, "wp"] & R[n] \arrow[r, "-p"] & R[n] \end{tikzcd}$$

must exist. Hence, $\kappa vp = wpv = -p\kappa v$, so $2p = 0$. This forces $p = 2$.

When $n = 0$ and $R = k[x]/(x^2)$, an argument similar to the one just given shows that $\operatorname{char} k = 2$. □

We next show that, for $n = 0$ and $n = 1$, the rings in the conclusion of Proposition 3.9 are $\Delta^n$-rings.

**Proposition 3.10** ([23])**.** *If $T$ is a complete 2-ring, then $T/(4)$ is a $\Delta^0$-ring.*

**Proof.** In [23], a triangulation of $\mathcal{P}_f$ is constructed for any commutative local ring of characteristic 4 with maximal ideal (2), but the proof in fact shows that $\mathcal{P}$ admits a triangulation. These rings are exactly the rings of the form $T/(4)$ (where $T$ is a complete 2-ring). Certainly, any ring of the form $T/(4)$ is of the type discussed in [23], and in Proposition 3.9 it is shown that any commutative local ring of characteristic 4 with maximal ideal (2) is of the form $T/(4)$. □

**Proposition 3.11.** *Every graded field $k$ is a $\Delta^n$-ring for all $n$. Every exterior algebra $k[x]/(x^2)$ with a unit in degree $3|x|+n$ is a $\Delta^n$-ring, provided $n$ and $|x|$ are not both even when* $\operatorname{char} k \neq 2$.

**Proof.** We will use a graded version of the construction presented in [23]. Fix $n \geq 0$. We will work in the category of differential graded modules over a differential graded algebra $A$, where the degree of all derivations is $-n$. So if $x, y \in A$,

$$d(xy) = d(x)y + (-1)^{n|x|}xd(y).$$

This category admits a triangulation with suspension functor $(-)[n]$. For example, if $u \in A$ has degree $i$, then there is an exact triangle

$$A[i] \xrightarrow{u} A \longrightarrow A \oplus A[i+n] \longrightarrow A[i+n],$$

where the differential on $A[j]$ is $(-1)^j d$ and the differential $D$ on $A \oplus A[i+n]$ is

$$D(a,b) = (da + ub, (-1)^{i+n}db).$$

Let $A = k$ with zero differential. Trivially, the homology of any differential graded $A$-module is projective, so homology induces an equivalence of categories from $D(A)$ (the derived category of $A$) to $\mathcal{P} = \mathbf{Mod}(k)$ by Proposition 2.1.

For the exterior algebra $k[x]/(x^2)$ with a unit $v$ in degree $3|x|+n$, we use the differential graded algebra constructed in [23]. Let $i = |x|$, and let $a$ and $u$ be symbols with degrees $|a| = 2i + n$ and $|u| = i$. Let $A = k\langle a, u\rangle/I$, where $I$ is the two sided ideal generated by the homogeneous elements $a^2$ and $au + ua + v$ (here we see why the existence of the unit $v$ is necessary). Define the differential on $A$ by $da = u^2$ and $du = 0$. One can check that this differential is well-defined; if $i$ and $n$ have opposite parity, then $v$ has odd degree, and so $\operatorname{char} k = 2$ by graded commutativity and signs do not matter. If $i$ and $n$ are both odd, then the signs work out independent of the characteristic of $k$. The differential is not well-defined if $i$ and $n$ are both even and $\operatorname{char} k \neq 2$; fortunately, if $n = 0$, $\operatorname{char} k = 2$ is forced (see Proposition 3.9).

As in [23], it is straightforward to check that $H_*A \cong k[x]/(x^2)$, where $x$ is the homology class of $u$, and for any differential graded $A$-module $M$, $H_*M$ is projective. Again we see that homology induces an equivalence of categories from $D(A)$ to $\mathcal{P}$. $\square$

**Remark 3.12.** According to [27], every simplicial, cofibrantly generated, proper, stable model category with a compact generator $P$ is Quillen equivalent to the module category of a certain endomorphism ring spectrum $\operatorname{End}(P)$. This is true in particular for the derived category of a differential graded algebra. In light of the proof of Proposition 3.11, we see that every graded commutative $\Delta^1$-ring arises as $\pi_*E$ for some (not necessarily commutative) semisimple ring spectrum $E$. Because of Theorem 1.2 (3), however, $\pi_* E$ being a graded commutative $\Delta^1$-ring is not sufficient to conclude that $E$ is semisimple (see Remark 5.2).

Observe that, for any integer $n$, every $\Delta^n$-ring is a $\Delta^n_f$-ring. For if $\mathcal{P}$ admits a triangulation, then $\mathbf{thick}\langle R\rangle$ admits a triangulation. Since $R$ must be Noetherian, $\mathbf{thick}\langle R\rangle = \mathcal{P}_f$. Propositions 3.10 and 3.11 now imply:

**Proposition 3.13.** *Let $n \in \{0, 1\}$. Every ring $R$ in the conclusion of Proposition 3.9 is a $\Delta^n$-ring. Hence, the classes of graded commutative local $\Delta^n$-rings and graded commutative local $\Delta^n_f$-rings coincide.*

According to [18, 15.27], a commutative ring is quasi-Frobenius if and only if it is a finite product of local Artinian rings with simple socle. The following proposition allows us to restrict our attention to the local case. Combined with the above characterization of commutative local $\Delta^n$-rings for $n \in \{0, 1\}$, this completes the proofs of Theorems 3.2 and 3.3.

**Proposition 3.14.** *Suppose $R \cong A \times B$. The category of projective $R$-modules admits a triangulation if and only if the categories of projective $A$-modules and projective $B$-modules each admit a triangulation. The same is true for finitely generated projective modules.*

**Proof.** This is a consequence of the fact that $\mathbf{Mod}(R) \cong \mathbf{Mod}(A) \times \mathbf{Mod}(B)$. □

As a final note, we broaden the scope of the second statement of Proposition 3.13.

**Proposition 3.15.** *Let $n \in \{0, 1\}$. The classes of graded commutative Noetherian $\Delta^n$-rings and graded commutative Noetherian $\Delta^n_f$-rings coincide.*

**Proof.** It suffices to check that every commutative Noetherian $\Delta^n_f$-ring $R$ is a $\Delta^n$-ring. Since $R$ must be an IF-ring (Proposition 3.5), it must be a commutative Noetherian self-injective ring ([8, 6.9]). This makes $R$ quasi-Frobenius by definition, and therefore a product of local rings, each of which must be a $\Delta^n$-ring by Proposition 3.13 (since each local ring is a $\Delta^n_f$-ring). Hence, $R$ is a $\Delta^n$-ring. □

## 4. Stable homotopy categories

In this section, we assume that $\mathcal{T}$ is a monogenic stable homotopy category with sphere object $S$ (see [14]). Certain conclusions can be drawn with weaker hypotheses; this should be clear in the proofs. In particular, Theorems 4.1 and 4.2 also apply to the derived category $\mathcal{D}_E$ of right $E$-module spectra, provided $E_*$ is commutative. We have the following corollary to Theorem 3.2.

**Theorem 4.1.** *Let $\mathcal{T}$ be a monogenic stable homotopy category with unit object $S$. $S$ is semisimple if and only if the following two conditions hold:*

(1) $\pi_*S \cong R_1 \times \cdots \times R_n$, *where* $R_i$ *is either a graded field* $k$ *or an exterior algebra* $k[x]/(x^2)$ *over a graded field containing a unit in degree* $3|x|+1$ ($\pi_*S$ *is a* $\Delta^1$*-ring*).

(2) *For every factor ring of* $\pi_*S$ *of the form* $k[x]/(x^2)$, $x \cdot \pi_*C \neq 0$, *where* $C$ *is the cofiber of* $x \cdot S$.

**Proof.** First, note that every stable homotopy category has a symmetric monoidal structure (for which $S$ is the unit) compatible with the triangulation. This forces $R = \pi_*S$ to be graded commutative ([14, A.2.1]). Further, we may use the monoidal product to take any element $x \in \pi_*S$ and obtain a map $x \cdot X$ for any $X \in \mathcal{T}$. For example, if $e \in \pi_*S$ is idempotent, then $e \cdot X$ is an idempotent endomorphism of $X$. Since idempotents split, there is a decomposition $X \simeq Y \vee Z$ such that $\pi_*Y \cong e\pi_*X$ and $\pi_*Z = (1-e)\pi_*X$ (see [14, 1.4.8]).

Suppose $S$ is semisimple. By Proposition 3.1, the category $\mathcal{P}$ of projective modules over $\pi_*S$ admits a triangulation with suspension functor $(-)[1]$. By Theorem 3.2, condition (1) is satisfied, and $\pi_*S \cong R_1 \times \cdots \times R_n$, where $R_i$ is either a graded field $k$ or an exterior algebra $k[x]/(x^2)$. Since idempotents split in $\mathcal{T}$, $S \cong A_1 \vee \cdots \vee A_n$, where $\pi_*A_i = R_i$. The cofiber $C_i$ of $x \cdot A_i$ is a summand of $C$, the cofiber of $x \cdot S$. Since $\pi_*C_i$ must be a nontrivial free $R_i$-module, $x \cdot C_i \neq 0$. This proves (2).

Conversely, if condition (1) holds, then $R$ and $S$ admit decompositions as above. We now show that it suffices to assume $R$ is local. Let $\mathcal{T}_i = \mathbf{loc}\langle A_i \rangle$; $\mathbf{loc}\langle A_i \rangle$ is an ideal by [14, 1.4.6]. Every $X \in \mathcal{T}$ admits a decomposition $X \cong X_1 \vee \cdots \vee X_n$, where $X_i \in \mathcal{T}_i$ (take $X_i = X \wedge A_i$). We claim that this decomposition is orthogonal, in that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_i, \mathcal{T}_j) = 0$ whenever $i \neq j$ (i.e., every map from an object in $\mathcal{T}_i$ to an object in $\mathcal{T}_j$ is trivial). First, observe that if $i \neq j$, then any map $f : A_i \longrightarrow A_j$ is trivial, as follows. It suffices to show that the induced map $f_* : R_i \longrightarrow R_j$ is zero. Let $e_k \in \pi_*S$ be the idempotent corresponding to $R_k$. We now have $f_*(x) = f_*(e_i x) = e_i f_*(x) = 0$, for all $x \in R_i$. So indeed, $[A_i, A_j]_* = 0$. Now, $[A_i, -]_*$ vanishes on $\mathbf{loc}\langle A_j \rangle$ since $A_i$ is compact; therefore $[-, X]_*$ vanishes on $\mathbf{loc}\langle A_i \rangle$ for any $X \in \mathbf{loc}\langle A_j \rangle$. In summary, there is an orthogonal decomposition

$$\mathcal{T} = \mathbf{loc}\langle S \rangle \cong \mathcal{T}_1 \vee \cdots \vee \mathcal{T}_n.$$

We now see that $\pi_*$ is faithful on $\mathcal{T}$ if and only if the functors $[A_i, -]_*$ are faithful on $\mathcal{T}_i$ for $i = 1, \ldots, n$. It therefore suffices to assume that $R$ is a local ring (see Proposition 2.1 (3)).

We must now show that if $R = \pi_*S$ is local and satisfies conditions (1) and (2), then $S$ is semisimple. By Proposition 2.1 (4), we need only check that $\pi_*X$ is always projective (or free, since $R$ is local). If $R$ is a graded field, then this is trivially true. If $R \cong k[x]/(x^2)$, then condition (2) tells us that $\pi_*C$ is free, where $C$ is the cofiber of $x \cdot S$. Since every map of free $k[x]/(x^2)$-modules is the coproduct of a trivial map, an isomorphism, and multiplication by $x$, it is easy to check that $\pi_*X$ is free for all $X \in \mathbf{thick}\langle S \rangle$.

For arbitrary $X \in \mathcal{T}$, $\pi_* X$ is the direct limit of a system of modules of the form $\pi_* X_\alpha$, where $X_\alpha \in \mathbf{thick}\langle S\rangle$ ([14, 2.3.11]). Since any direct limit of flat modules is flat, $\pi_* X$ is flat. Over $k[x]/(x^2)$, flat implies free. This completes the proof. □

Using localization, we also have the following corollary to Theorem 3.2.

**Theorem 4.2.** *Let $\mathcal{T}$ be a monogenic stable homotopy category with unit object $S$. If $\pi_* S$ is local or Noetherian, then $S$ is semisimple if and only if it is von Neumann regular. If $\mathcal{T}$ is also a Brown category, and $S$ is von Neumann regular, then for any prime ideal $\mathfrak{p}$, $(\pi_* S)_{\mathfrak{p}}$ is either a graded field $k$ or an exterior algebra $k[x]/(x^2)$ with a unit in degree $3|x|+1$.*

The derived category $\mathcal{D}_E$ is a Brown category if $E_*$ is countable, but there are ring spectra $E$ where $\mathcal{D}_E$ is not Brown (see [5]).

**Proof.** Since $\mathcal{T}$ is a monogenic stable homotopy category, the unit object $S$ is a compact generator and $R = \pi_* S$ is commutative.

Certainly, semisimplicity implies von Neumann regularity. Assume $S$ is von Neumann regular. By Proposition 3.1, $R$ is a $\Delta^1_f$-ring. If $R$ is local or Noetherian, then it is also a $\Delta^1$-ring by Proposition 3.13 or 3.15. We now wish to invoke Theorem 4.1. To do so, we must verify condition (2). Arguing as in the proof of Theorem 4.1, it suffices to assume that $R \cong k[x]/(x^2)$. For this ring, condition (2) is implied by the fact that $\pi_* C$ must be free since $S$ is von Neumann regular. This proves the first implication.

For the second, assume $\mathcal{T}$ is a Brown category and $S$ is von Neumann regular. By Theorem 3.2 and Propositions 2.4 and 3.1, $R_{\mathfrak{p}}$ is a $\Delta^1_f$-ring. Since it is a graded commutative local ring, it must be $\Delta^1$-ring by Proposition 3.13. By Proposition 3.9, $R$ is either a graded field $k$ or an exterior algebra $k[x]/(x^2)$ over a field with a unit in degree $3|x|+1$. □

## 5. The generating hypothesis

Note that $\mathcal{D}_E$ is almost an example of a stable homotopy category in the sense of [14] (it may not admit a symmetric monoidal product), where $E$ plays the role of the sphere. In any stable homotopy category $\mathcal{C}$ with sphere object $S$, the functor $\pi_*(-) = [S, -]_*$ is of particular interest, and it is natural to ask how different the homotopy category $\mathcal{C}$ is from the algebraic category of graded modules over $\pi_* S$. For example, one might like to know whether $\pi_*(-)$ acts faithfully. More concretely, let $\mathcal{T}$ be a triangulated category, and let $S \in \mathcal{T}$ be a distinguished object. Write $\pi_*(-)$ for the functor $[S, -]_*$. We say that $\mathcal{T}$ *satisfies the global generating hypothesis* if $\pi_*$ is a faithful functor from $\mathcal{T}$ to the category of graded right modules over $\pi_* S$. We say that $\mathcal{T}$ *satisfies the strong generating hypothesis* if, for any map $f : X \longrightarrow Y$ with $X \in \mathbf{thick}\langle S\rangle$ and $Y \in \mathcal{T}$, $\pi_* f = 0$ implies $f = 0$. The following proposition follows from Propositions 2.1 and 2.2.

**Proposition 5.1.** *Let $\mathcal{T}$ be a weak stable homotopy category with compact generator $S$. $\mathcal{T}$ satisfies the global generating hypothesis if and only if $S$ is semisimple, and $\mathcal{T}$ satisfies the strong generating hypothesis if and only if $S$ is von Neumann regular.*

**Proof.** The first assertion is true by definition of the global generating hypothesis and Proposition 2.1. If $S$ is von Neumann regular, then $\mathcal{T}$ trivially satisfies the strong generating hypothesis since $\mathbf{thick}\langle S\rangle = \langle S\rangle_f^0$. Conversely, if $\mathcal{T}$ satisfies the strong generating hypothesis, then one can argue that $\mathbf{thick}\langle S\rangle \subseteq \langle S\rangle_f^0$ (cf. the proof of (4) $\implies$ (1) in Proposition 2.1). □

**Remark 5.2.** As promised in §3, we give an example of a stable homotopy category where the sphere $S$ is not semisimple even though $\pi_*S$ is a $\Delta^1$-ring. Condition (2) in Theorem 4.1 is therefore necessary. Consider the stable module category $\mathrm{StMod}(k[G])$ associated to a finite $p$-group $G$ and field $k$ of characteristic $p$. In $\mathrm{StMod}(k[G])$, $\pi_*S = \hat{H}(G;k)$, the Tate cohomology of $G$. For $n \geq 1$, $\hat{H}(\mathbb{Z}/(3^n);\mathbb{F}_3) \cong \mathbb{F}_3[y,y^{-1}][x]/(x^2)$, where $|x| = 1$ and $|y| = 2$. However, it is shown in [2] that $\mathrm{StMod}(\mathbb{F}_3[\mathbb{Z}/(3)])$ satisfies the global generating hypothesis, though $\mathrm{StMod}(\mathbb{F}_3[\mathbb{Z}/(3^n)])$ does not for $n \geq 2$.

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