New York Journal of Mathematics

New York J. Math. **15** (2009) 211–217.

A splitting theorem for linear polycyclic groups

Herbert Abels and Roger C. Alperin

Abstract. We prove that an arbitrary polycyclic by finite subgroup of $\mathrm{GL}(n,\overline{\mathbb{Q}})$ is up to conjugation virtually contained in a direct product of a triangular *arithmetic* group and a finitely generated diagonal group.

Contents

1. Introduction

A linear algebraic group defined over a number field K is a subgroup G of $\mathrm{GL}(n,\mathbb{C}), n \in \mathbb{N}$, which is also an affine algebraic set defined by polynomials with coefficients in K in the natural coordinates of $\mathrm{GL}(n,\mathbb{C})$. For a subring R of $\mathbb{C}$ put $G(R) = \mathrm{GL}(n,R) \cap G$. Let $B(n,\mathbb{C})$ and $T(n,\mathbb{C})$ be the ($\mathbb{Q}$-defined linear algebraic) subgroups of $\mathrm{GL}(n,\mathbb{C})$ of upper triangular or diagonal matrices in $\mathrm{GL}(n,\mathbb{C})$ respectively.

Recall that a group Γ is called *polycyclic* if it has a composition series with cyclic factors. Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers in $\mathbb{C}$. Every discrete solvable subgroup of $\mathrm{GL}(n,\mathbb{C})$ is polycyclic (see, e.g., [R]).

1.1. Let $\mathfrak{o}$ denote the ring of integers in the number field K. If H is a solvable K-defined algebraic group then $H(\mathfrak{o})$ is polycyclic (see [S]). Hence every subgroup of a group $H(\mathfrak{o}) \times \Delta$ is polycyclic, if Δ is a finitely generated abelian group.

Received November 15, 2007.
Mathematics Subject Classification. 20H20, 20G20.
Key words and phrases. Polycyclic group, arithmetic group, linear group.
The authors gratefully acknowledge the hospitality received at the Mathematics Department of the University of Chicago during the inception of this work.

ISSN 1076-9803/09

Every polycyclic group is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{Z})$ for some n (see [S]). But not every polycyclic group is arithmetic, i.e., has a subgroup of finite index which is isomorphic to a subgroup of finite index in $H(\mathfrak{o})$ for some K-defined algebraic group H, K a number field, $\mathfrak{o}$ its ring of integers (see [S, Chapter 11] and [GP1]).

A group is said to *virtually* have a property P if there is a subgroup of finite index which has the property P. We prove the following.

Theorem 1.2. *Any virtually polycyclic subgroup of* $\mathrm{GL}(n, \overline{\mathbb{Q}})$ *is up to conjugation virtually contained in a direct product of a triangular arithmetic group and a finitely generated diagonal group.*

One important consequence of our theorem is that the groups described above in 1.1 are the only polycyclic groups contained in $\mathrm{GL}(n, \overline{\mathbb{Q}})$, up to conjugation and passing to subgroups of finite index. Also our result is an important ingredient in the program to prove that certain polycyclic subgroups of finitely generated linear groups in characteristic zero are closed in the profinite topology [AF].

We now make a few further preliminary comments. A group is called *Noetherian* if every subgroup is finitely generated. A group is polycyclic iff it is solvable and Noetherian [S]. Recall that a subgroup Γ of $\mathrm{GL}(n, \mathbb{C})$ is either virtually solvable or contains a nonabelian free subgroup. This is the celebrated Tits alternative [T]. Since a nonabelian free group is not Noetherian then the theorem above holds for every Noetherian subgroup Γ of $\mathrm{GL}(n, \overline{\mathbb{Q}})$.

One can ask if the polycyclic group Γ itself has the properties described in the theorem, rather than passing to a subgroup Γ_1 of finite index. A simple example shows that this is impossible: consider the dihedral group Γ of order 6, represented as the reflection group in $\mathbb{R}^2$. Then $\Gamma \subset \mathrm{GL}(2, K)$, where $K = \mathbb{Q}(\sqrt{3})$. But observe that the commutator subgroup of the subgroup Γ_1 of finite index in our Theorem 2.1 is unipotent, in particular torsion free.

2. Restatement and proof

We shall prove the following more precise statement of our theorem.

Theorem 2.1. *Let* Γ *be a polycyclic subgroup of* $\mathrm{GL}(n, \overline{\mathbb{Q}})$. *There is a number field* K, *an element* $g \in \mathrm{GL}(n, K)$, *a subgroup* Γ_1 *of finite index in* Γ, *and* K*-defined algebraic groups* $H < B(n, \mathbb{C})$ *and* $D < T(n, \mathbb{C})$ *such that* H *centralizes* D, $H \cdot D$ *is a direct product of* H *and* D, *and* $g\,\Gamma_1\,g^{-1}$ *is contained in* $H(\mathfrak{o}) \times \Delta$, *where* Δ *is a finitely generated subgroup of* $D(K)$.

Note that it suffices to prove the theorem for $\overline{\mathbb{Q}}$ and its subring of algebraic integers instead of K and $\mathfrak{o}$ as in the theorem, because then there is a number field $K \subset \overline{\mathbb{Q}}$ with the following properties. The entries of g and of the H and D-components of a finite generating set of $g\Gamma_1 g^{-1}$ are in K. The groups

H and D are defined over K. Then the theorem is true for K and its ring of integers. Thus changing notation from now on we let $\mathfrak{o}$ be the ring of algebraic integers in $\overline{\mathbb{Q}}$ and $\mathfrak{o}_K = \mathfrak{o} \cap K$ for a number field K.

We say that a set S of elements of $\overline{\mathbb{Q}}$ has bounded denominators if there is an integer $m \in \mathbb{Z}$ such that $m \cdot s \in \mathfrak{o}$ for every $s \in S$. Similarly a set of matrices with entries in $\overline{\mathbb{Q}}$ has bounded denominators if the set of their entries has bounded denominators. Note that if V and W are two finite dimensional vector spaces over $\overline{\mathbb{Q}}$ it makes sense to say that a set of $\overline{\mathbb{Q}}$-linear maps from V to W has bounded denominators, since if the corresponding matrices for one set of bases for V and W do, they do so for every one. Clearly, if a subgroup Γ_1 of finite index in a group $\Gamma < \mathrm{GL}(n, \overline{\mathbb{Q}})$ has bounded denominators, then so does Γ. The following lemma is well known.

Lemma 2.2. *Let K be a number field and let Γ be a subgroup of $\mathrm{GL}(n, K)$ with bounded denominators. Then $\Gamma \cap \mathrm{GL}(n, \mathfrak{o})$ is of finite index in Γ. So Γ is virtually contained in $\mathrm{GL}(n, \mathfrak{o})$.*

Proof. Consider the action of Γ on the set Λ of $\mathfrak{o}_K$-lattices in K^n. If the denominators of all $\gamma \in \Gamma$ are bounded by m, then $\lambda := \sum_{\gamma \in \Gamma} \gamma(m\mathfrak{o}_K^n)$ is a lattice, fixed by Γ and lies between $m\mathfrak{o}_k^n$ and $\mathfrak{o}_K^n$. So Γ acts on the set M of lattices between λ and $m^{-1}\lambda$, a finite set, hence the stabilizer $\Gamma \cap \mathrm{GL}(n, \mathfrak{o})$ of the lattice $\mathfrak{o}_K^n \in M$ is of finite index in Γ. □

Let G be the Zariski closure of Γ, a solvable linear algebraic group defined over $\overline{\mathbb{Q}}$. Then the connected component G^0 of G can be triangularized over $\overline{\mathbb{Q}}$, so there is an element $g \in \mathrm{GL}(n, \overline{\mathbb{Q}})$ such that $g\,G^0 g^{-1} \subset B(n, \mathbb{C})$, by the Lie–Kolchin theorem (see [B1]). By passing to the subgroup $\Gamma \cap G^0$ of Γ of finite index, we may assume that $\Gamma \subset B(n, \overline{\mathbb{Q}})$ and that G is connected with respect to the Zariski topology. Let U be the set of unipotent elements of G. Then $U = G \cap U(n, \mathbb{C})$ is a $\overline{\mathbb{Q}}$-defined normal subgroup of G, where $U(n, \mathbb{C})$ is the group of unipotent upper triangular matrices. There is a $\overline{\mathbb{Q}}$-defined torus $T < G$ such that G is the semidirect product of T and U, by the structure theorem for solvable connected groups ([B1]). Let ρ be the representation of G on the Lie algebra $\mathfrak{u}$ of U obtained by restricting the adjoint representation of G on its Lie algebra $\mathfrak{g}$ to $\mathfrak{u}$.

The main claim of this proof is that $\rho(\Gamma)$ has bounded denominators. We address this next.

2.3. Let σ be the representation of G on the abelianized Lie algebra

$$\mathfrak{v} = \mathfrak{u}^{ab} = \mathfrak{u}/\mathfrak{u}'$$

of $\mathfrak{u}$ deduced from ρ. Since U is contained in the kernel of this representation, the representation σ factors through a representation of $T \cong G/U$ which is $\overline{\mathbb{Q}}$-rational and hence decomposes into weight spaces. Thus if $X(G) \cong X(T)$ is the set of $\overline{\mathbb{Q}}$-defined characters of G, we have

$$\mathfrak{v} = \oplus_{\lambda \in X(T)} \mathfrak{v}^{\lambda}$$

where $\mathfrak{v}^\lambda$ is the vector subspace of weight vectors for λ:

$$\mathfrak{v}^\lambda = \{v \in \mathfrak{v}\ ;\ \sigma(g)v = \lambda(g)v \text{ for every } g \in G\}.$$

Put $\mathfrak{v}^* = \oplus_{\lambda \neq 0} \mathfrak{v}^\lambda$. We thus have a $\overline{\mathbb{Q}}$-decomposition

$$\mathfrak{v} = \mathfrak{v}^0 \oplus \mathfrak{v}^*,$$

where $\mathfrak{v}^0$ is the weight space corresponding to the trivial character,

$$\mathfrak{v}^0 = \{v \in \mathfrak{v}\ ;\ \sigma(g)v = v \text{ for every } g \in G\}.$$

We first claim that $\sigma(\Gamma)$ has bounded denominators. The exponential map $\exp : \mathfrak{v} \to U/U' =: V$ is a $\overline{\mathbb{Q}}$-defined isomorphism of affine algebraic groups and

$$\exp(\sigma(g)v) = \iota(g) \exp v, \tag{2.4}$$

where $\iota(g)$ is the automorphism of V induced by conjugation with $g \in G$. The image Δ of the commutator group $\Gamma' \subset U$ of Γ in $V = U/U'$ is Zariski dense in $G'/U' = \exp(\mathfrak{v}^*)$, so $\log \Delta$ spans $\mathfrak{v}^*$. On the other hand, $\log \Delta$ is a finitely generated subgroup of the group of $\overline{\mathbb{Q}}$-points of $\mathfrak{v}^*$ and is $\sigma(\Gamma)$-invariant by (2.4), so $\sigma(\Gamma)$ has bounded denominators, since $\sigma(\Gamma)\mathfrak{v}^*$ does and $\sigma(\Gamma) \mid \mathfrak{v}^0$ is trivial.

2.5. To show that $\rho(\Gamma)$ has bounded denominators, first look at the successive quotients $\mathfrak{v}_i := \mathfrak{u}^i/\mathfrak{u}^{i+1}$ of the descending central series $\mathfrak{u}^i$ of $\mathfrak{u}$. Then ρ induces $\overline{\mathbb{Q}}$-rational representations ρ_i on each $\mathfrak{v}_i$ and the Lie bracket induces a surjective linear map $\mathfrak{v}_1 \otimes \mathfrak{v}_i \to \mathfrak{v}_{i+1}$ of representation spaces for every $i \geq 1$. Starting from $\rho_1 = \sigma$, it follows by induction on i, that $\rho_i(\Gamma)$ has bounded denominators for each i. Choose a basis $\mathfrak{B}_i$ for each $\mathfrak{v}_i(\overline{\mathbb{Q}})$. We thus may assume that the representing matrices of $\rho_i(\Gamma)$ are in $\mathrm{GL}(\dim \mathfrak{v}_i, K)$ for an appropriate number field K and have entries in $\mathfrak{o}_K$ by passing to a subgroup of finite index in Γ, if necessary, by Lemma 2.2.

Now choose a set $\mathfrak{B}'_i \subset \mathfrak{u}^i(\overline{\mathbb{Q}})$ which projects onto

$$\mathfrak{B}_i \subset \mathfrak{v}_i(\overline{\mathbb{Q}}) = \mathfrak{u}^i(\overline{\mathbb{Q}})/\mathfrak{u}^{i+1}(\overline{\mathbb{Q}}).$$

The set $\bigcup_{i \geq 1} \mathfrak{B}'_i$ is a basis of $\mathfrak{u}(\overline{\mathbb{Q}})$. Every element of $\rho(\Gamma) \in \mathrm{GL}(\dim \mathfrak{u}, \overline{\mathbb{Q}})$ has block triangular form with respect to this basis with blocks

$$\rho_i(\Gamma) \subset \mathrm{GL}(\dim \mathfrak{v}_i, \mathfrak{o}_K)$$

along the diagonal. Take a finite set S of generators of Γ. Then the off-diagonal blocks of $\rho(\gamma)$ for $\gamma \in S$ lie in some number field $L \supset K$ and have bounded denominators, say bounded by $m \in \mathbb{Z}$. So $\rho(S)$ lies in the group of those matrices of $\mathrm{GL}(\dim \mathfrak{u}, L)$, which have the same block triangular form and whose blocks along the diagonal have entries in $\mathfrak{o}_L$ and whose off-diagonal blocks A_{ij} have entries in L with denominators bounded below by m^{i-j}. Hence $\rho(\Gamma)$ lies in this group and hence has bounded denominators, our claim in 2.3. — One can also rescale the bases $\mathfrak{B}'_i$ inductively so that the entries of all the A_{ij} are in $\mathfrak{o}_L$.

2.6. We claim that $\ker\rho = \ker(\rho \mid T) \times Z(U)$, where $Z(U)$ is the center of U and that $\ker(\rho \mid T) = \bigcap \ker(\lambda \mid T)$, where λ runs through the weights of σ, where σ is the representation of G on $\mathfrak{v} = \mathfrak{u}^{ab}$ of 2.3.

Clearly, $A := \ker(\rho \mid T) \subset \ker\rho, Z := Z(U) \subset \ker\rho$ and A centralizes U, hence $Z(U)$, so $A \times Z(U) \subset \ker\rho$. Recall that the exponential map

$$\exp : \mathfrak{u} \to U$$

is a $\overline{\mathbb{Q}}$-defined isomorphism of affine algebraic varieties and

$$\exp(\rho(g)v) = g\exp(v)g^{-1}$$

for every $g \in G$ and $v \in \mathfrak{u}$. Also $A' := \bigcap \ker(\lambda \mid T)$, λ the weights of σ, contains A. Conversely, $A' \subset \ker\rho_i$, since $\mathfrak{v} \otimes \mathfrak{v}_i \to \mathfrak{v}_{i+1}$ is a surjective map of representation spaces. Here we use the notation of 2.5. So the diagonal blocks of $\rho(t)$ are identity matrices for $t \in A'$, if we choose bases $\mathfrak{B}'_i$ as above. Since $\rho(T)$ is diagonalizable over $\overline{\mathbb{Q}}$, we can choose the bases $\mathfrak{B}'_i$ furthermore in such a way, that every element of every $\mathfrak{B}'_i$ is contained in a weight space of $\rho(T)$ in $\mathfrak{u}^i$. Then $\rho(t)$, $t \in T$, is represented by diagonal matrices and hence $\rho(A') = \mathbf{1}$. It follows that $A' \subset A$ and hence $A = A'$. Furthermore, $\rho(u)$, $u \in U$ is represented by matrices with diagonal blocks $\rho_i(u) = \mathbf{1}$. It follows that for $g = t \cdot u$, $t \in T$, $u \in U$, the representing matrix $\rho(g) = \rho(t) \cdot \rho(u)$ is the identity matrix $\mathbf{1}$ only if $\rho(t) = \mathbf{1}$ and $\rho(u) = \mathbf{1}$. This proves our claims.

We now define the groups H and D of the theorem. Let $D = A^\circ$ be the connected component of

$$A = \ker\sigma \mid T = \ker\rho \mid T = \bigcap \ker\lambda,$$

λ the weights of T on $\mathfrak{u}$ or $\mathfrak{v}$. Then $D = \cap \ker\lambda$, where λ runs through the $\overline{\mathbb{Q}}$-characters of T contained in the vector subspace over $\mathbb{Q}$ in $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ spanned by the weights of $\sigma \mid T$. Let B be a complementary torus of D in T, defined over $\overline{\mathbb{Q}}$. So multiplication $B \times D \to T$ is a $\overline{\mathbb{Q}}$-defined isomorphism of tori. Put $H = B \ltimes U$. Then:

- D is central in G.
- $G = H \times D$.
- $\rho(G) = \rho(H)$.

We can assume that D is diagonal and H remains triangular. D is diagonalizable over $\overline{\mathbb{Q}}$, so the vector space $W = \overline{\mathbb{Q}}^n$ decomposes into weight spaces W^λ, $\lambda \in X(D)$, $W = \bigoplus W^\lambda$. The group D is central in G, so G maps every W^λ to itself. We thus get $\overline{\mathbb{Q}}$-representations α_λ of the solvable connected group G on each of the W^λ, so α_λ can be triangularized over $\overline{\mathbb{Q}}$. Taken these bases of the W^λ's together gives a basis of W for which G is triangular and D is diagonal.

To finish the proof recall the following theorem of Borel [B2, § 5 Theorem 6]. If $\rho : H \to \rho(H)$ is a surjective K-homomorphism of K-groups, K a number field, then $\rho(H(\mathfrak{o}_K))$ contains a subgroup of finite index in

$\rho(H)(\mathfrak{o}_K)$. Recall that we may assume that $\rho(\Gamma) \subset \rho(H)(\mathfrak{o}_K)$. We now may assume, by Borel's theorem, that $\rho(\Gamma) \subset \rho(H(\mathfrak{o}_K))$. Choose for every γ in a finite generating set of Γ an element $h \in H(\mathfrak{o}_K)$ such that $\rho(\gamma) = \rho(h)$. Then $h^{-1}\gamma \in \ker\rho$, hence $h^{-1}\gamma = a(\gamma) \cdot z(\gamma)$ with $a(\gamma) \in A(\overline{\mathbb{Q}})$ and $z(\gamma) \in Z(U)(\overline{\mathbb{Q}})$, by 2.6. So there is a finitely generated (abelian) subgroup Δ of $A(\overline{\mathbb{Q}})$ such that $\Delta \cdot Z(U)\left(\frac{1}{m}\mathfrak{o}_L\right) \cdot H(\mathfrak{o}_L)$ contains Γ where L is an appropriate number field. Note that $H(\mathfrak{o}_L)$ normalizes $\Delta \cdot Z(U)\left(\frac{1}{m}\mathfrak{o}_L\right)$. For an appropriate number N we have that $N \cdot \Delta \cdot Z(U)\left(\frac{1}{m}\mathfrak{o}_L\right) \subset Z(U)(\mathfrak{o}_L) \times D(\overline{\mathbb{Q}})$, since A/D is finite. Then $N\Delta$ is a characteristic subgroup of Δ of finite index, so $N\Delta \cdot H(\mathfrak{o}_L)$ is of finite index in $\Delta \cdot H(\mathfrak{o}_L)$ and contains a subgroup of finite index in Γ. This finishes our proof.

Note added in proof. The (anonymous) referee asked the following interesting question: "Can Theorem 1.2 be made unique, perhaps by taking a maximal arithmetic subgroup and a complement?" We do not see a way to make it unique. We do have a certain degree of uniqueness for Δ, as follows. Assuming that the Zariski closure G of Γ is connected one can choose the group Δ, the "nonarithmetic part", in Theorem 2.1 in the following subgroup D_1 of D. The group $N := D \ltimes U$ (see 2.3 and 2.6) is the nilradical of G. So D is the set of semisimple elements of N, hence a characteristic subgroup of G. Let X_1 be the set of all characters $\chi : G \to \mathrm{GL}_1$ such that $\chi(\Gamma) \subset \mathfrak{o}^*$ and let N_1 be the connected component of the identity in $\bigcap_{\chi \in X_1} \ker\chi$. Then $N_1 \subset N$, since the weights λ of G on $\mathfrak{v}$ are in X_1, by 2.3. So the set D_1 of semisimple elements of N_1 is an algebraic subgroup of D, defined over any splitting field of G. Then one can choose a complementary torus B_1 of D_1 in T and define $H_1 = B_1 \ltimes U$. Then Theorem 2.1 holds for H_1 and D_1 replacing H and D.

References

[AF] Alperin, Roger C.; Farb, Benson. Separability of solvable subgroups in linear groups. Preprint.

[B1] Borel, Armand. Linear algebraic groups. Second edition. Graduate Texts in Mathematics, 126. *Springer-Verlag, New York*, 1991. xii+288 pp. ISBN: 0-387-97370-2. MR1102012 (92d:20001), Zbl 0726.20030.

[B2] Borel, Armand. Density and maximality of arithmetic subgroups. *J. Reine Angew. Math.* **224** (1966) 78–89. MR0205999 (34 #5824), Zbl 0158.03105.

[GP1] Grunewald, Fritz; Platonov, Vladimir. Non-arithmetic polycyclic groups. *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique* **326** (1998) 1359–1364. MR1649174 (99i:20064), Zbl 0974.20035.

[GP2] Grunewald, Fritz; Platonov, Vladimir. Solvable arithmetic groups and arithmeticity problems. *Internat. J. Math.* **10** (1999) 327–366. MR1688145 (2000d:20066), Zbl 1039.20026.

[R] Raghunathan, M. S. Discrete subgroups of Lie groups. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68, *Springer-Verlag, New York*, 1972. ix+227. MR0507234 (58 #22394a), Zbl 0254.22005.

[S] Segal, Daniel. Polycyclic groups. Cambridge Tracts in Mathematics, 82. *Cambridge University Press, Cambridge*, 1983. xiv+289 pp. ISBN: 0-521-24146-4. MR0713786 (85h:20003), Zbl 0516.20001.

[T] Tits, Jacques. Free subgroups in linear groups. *J. Algebra* **20** (1972) 250–270. MR0286898 (44 #4105), Zbl 0236.20032.

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, GERMANY
abels@mathematik.uni-bielefeld.de

Department of Mathematics, San Jose State University, San Jose, CA 95192, USA
alperin@math.sjsu.edu

This paper is available via http://nyjm.albany.edu/j/2009/15-11.html.