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Heegaard splittings and virtually Haken Dehn filling. II

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Abstract. We use Heegaard splittings to give a criterion for a tunnel number one knot manifold to be nonfibered and to have large cyclic covers. We also show that a knot manifold satisfying the criterion admits infinitely many virtually Haken Dehn fillings. Using a computer, we apply this criterion to the 2 generator, nonfibered knot manifolds in the cusped Snappea census. For each such manifold M, we compute a number $c(M)$, such that, for any $n > c(M)$, the n-fold cyclic cover of M is large.

Contents

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1. Introduction

This paper continues the project, begun in [18], of using Heegaard splittings to construct closed essential surfaces in finite covers of 3-manifolds. The idea, based on the work of Casson and Gordon [5], is to lift a Heegaard splitting of a 3-manifold to a finite cover in which there are disjoint compressing disks on each side. By compressing the lifted Heegaard surface along an appropriate choice of such disks, we hope to arrive at an essential surface.

By a *knot manifold* we mean a connected, compact, orientable 3-manifold whose boundary is a single torus. A *tunnel system* for a knot manifold M is a collection $\{t_1, \ldots, t_n\}$ where the t_i's are disjoint, properly embedded arcs in M, such that $\overline{M - N(\bigcup t_i)}$ is homeomorphic to a handlebody. The *tunnel number* of M, denoted $t(M)$, is the minimal cardinality of a tunnel system for M.

We focus attention on tunnel number one, nonfibered knot manifolds. These are obtained by attaching a single 2-handle to a genus two handlebody. We shall give a condition on the 2-handle which, if satisfied, ensures that in all large enough cyclic covers, the lifted Heegaard splitting can be compressed to obtain an essential surface. There is also a statement about incompressibility after Dehn surgery.

Freedman and Freedman ([9]) have already proved that for *any* nonfibered knot manifold, all but finitely many cyclic covers are large (i.e., contain closed essential surfaces). Cooper and Long [7] then proved a result about virtually Haken Dehn surgery for these manifolds, and also obtained a bound on the number of excluded covers in terms of the genus of the knot.

However, our results provide a computational benefit. We computed, for all of the 453 nonfibered, 2-generator knot manifolds in the SnapPea census, a covering degree past which all cyclic covers are large, and the bounds obtained are typically improvements over known bounds.

For other connections between Heegaard splittings and virtually Haken 3-manifolds, see [13], [14], [15], [17].

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2. Definitions, notation, and statement of results

Let F be a connected, closed, orientable surface of positive genus. Recall that a *compression body* W is a 3-manifold obtained from $(F \times I)$ by first attaching a collection of 2-handles along disjoint curves in one component of $\partial(F \times I)$, and then capping off all resulting 2-sphere boundary components with 3-balls. One of the boundary components of W is homeomorphic to F, and is called the *outer boundary* of W, denoted $\partial_+ W$. The other components of ∂W form the *inner boundary*, denoted $\partial_- W$.

If X is a 3-manifold with boundary, and $\mathcal{S} \subset X$ is a collection of disjoint compression disks for ∂X, we let $X/\mathcal{S} = X - \mathring{N}(\mathcal{S})$. If X is a 3-dimensional submanifold of a 3-manifold Y, and if $\mathcal{S} \subset Y - \mathring{X}$ is a collection of disjoint compression disks for ∂X, then we let $X[\mathcal{S}]$ denote $X \cup N(\mathcal{S})$, where $N(S)$ is a regular neighborhood of S in $Y - \mathring{X}$.

A *disk system* $\mathcal{S}$ for a compression body W is a set of disjoint compressing disks for ∂W of minimal cardinality such that $W/\mathcal{S}$ has incompressible boundary. We shall use some basic facts about compression bodies, which can be found in [2].

A *Heegaard splitting* of a compact 3-manifold is a decomposition

$$M = W_1 \cup_F W_2,$$

where the W_i's are compression bodies with outer boundary homeomorphic to F. The *Heegaard genus* of M, denoted $g(M)$, is the minimal genus of F for all such decompositions. If M has boundary, then a *tunnel system* for M is a collection of properly embedded arcs in M, whose exterior is a handlebody. The *tunnel number* of M, denoted $t(M)$, is the minimal cardinality among all tunnel systems for M. It is an elementary fact that, if M is a knot manifold, then $g(M) = t(M) + 1$.

For the remainder of the paper, M will be a fixed knot manifold, with incompressible boundary, and with $t(M) = 1$ and $b_1(M) = 1$. Thus there is a Heegaard splitting $M = H \cup_F W$, where H is a genus 2 handlebody, and W is a genus 2 compression body. Let $\mathcal{D} = D_1 \cup D_2$ be a disk system for H, and let E be the unique (up to isotopy) nonseparating compression disk for W. We assume that E has been isotoped so that every component of $\partial E - \mathring{N}(\mathcal{D})$ represents an essential arc in $F - \mathring{N}(\mathcal{D})$.

Since $b_1(M) = 1$, there is a unique surjective homomorphism $\phi : \pi_1 M \to \mathbb{Z}$, where $\mathbb{Z}$ is the free factor of $H_1(M)$. Let M_n denote the corresponding n-fold cyclic cover, with M_∞ denoting the infinite cyclic cover. Let H_n, W_n and F_n be the preimages in M_n of H, W and F, respectively. Then

$$M_n = H_n \cup_{F_n} W_n$$

is a Heegaard splitting of M_n of genus $n + 1$.

Let $\alpha_1, \alpha_2 \subset F$ be simple closed curves transverse to $\partial\mathcal{D}$ such that

$$|\alpha_i \cap D_j| = \delta_{ij}$$

(the Kronecker delta function). We also assume that α_1 and α_2 intersect (nontransversely) in a single point p, which will be the base point for $\pi_1 M$, and we assign orientations to α_i and ∂D_i so that the algebraic intersection numbers $I(\alpha_i, \partial D_i)$ are both $+1$. We call such pair of curves $\{\alpha_1, \alpha_2\}$ *dual curves* for the disk system $\{D_1, D_2\}$ of H.

Lemma 2.1. *We may choose a disk system $\mathcal{D}$ so that $\phi(\alpha_1) = 0$ and $\phi(\alpha_2) = 1$.*

Proof. Suppose $\phi(\alpha_i) = n_i$, and that $|n_1| \leq |n_2|$. Let δ be an oriented embedded arc in $\alpha_1 \cup \alpha_2$ such that $\delta \cap (D_1 \cup D_2) = \partial\delta$, and that its orientation agrees with the orientation on α_2, but disagrees with the orientation on α_1. Let D_1' be a properly embedded disk in H obtained by band sum of D_1 and D_2 along the arc δ, and let $D_2' = D_2$. Then obviously we may assume that D_1' and D_2' are disjoint, and see that they form a disk system for H. Choose $\alpha_1' = \alpha_1$, $\alpha_2' = \alpha_2\alpha_1^{-1}$. Then up to an obvious homotopy of α_2' in ∂H, we may consider α_2' as a simple closed curve, and see that $\{\alpha_1', \alpha_2'\}$ form a dual curve pair for the disk system $\mathcal{D}' = \{D_1', D_2'\}$ (with a suitable choice of orientation for $\partial D_1'$). Further we have that $\phi(\alpha_1') = n_1$, and $\phi(\alpha_2') = n_2 - n_1$.

Now we replace the disk system $\mathcal{D}$ with the disk system $\mathcal{D}'$, and repeat the above procedure. Applying the Euclidean Algorithm, we may continue until we have a disk system $\mathcal{D}$ for which $\phi(\alpha_1) = 0$ (say) and $\phi(\alpha_2) = \gcd(n_1, n_2)$. Since $\mathrm{Image}(\phi) = \mathbb{Z}$, n_1 and n_2 are relatively prime, so the resulting disk system satisfies the requirements of Lemma 2.1. □

For the remainder of the paper, we shall assume that the disk system $\mathcal{D}$ has been chosen as in Lemma 2.1.

Let $p \in F - \partial\mathcal{D}$ be a base point for M, and let $p^1, \ldots, p^n$ be the lifts to M_n (where $n \in \mathbb{Z}^+ \cup \{\infty\}$), with the natural indexing. Let δ_1, δ_2 in $F - \partial\mathcal{D}$ be arcs connecting p with ∂D_i. Let $D_i^j \subset M_n$ denote the lift of D_i to M_n corresponding to the lift of δ_i with base point p_j. (In our notation, we have suppressed the dependence of D_i^j on n, trusting the meaning to be clear from context.) Note that D_1^j is between D_2^{j-1} and D_2^j. See Figure 1.

Let $I_{\mathrm{geo}}(\cdot, \cdot)$ be the geometric intersection pairing, and for a loop ℓ in F_∞, define the *width* of ℓ to be:

$$\mathrm{width}(\ell) = \mathrm{Max}(j \mid I_{\mathrm{geo}}(\ell, D_2^j) \neq 0) - \mathrm{Min}(j \mid I_{\mathrm{geo}}(\ell, D_2^j) \neq 0) + 2;$$

in the special case where ℓ is disjoint from all D_2^j's, we define the width to be one. If ℓ is a loop in F which lifts to F_∞, and $\widetilde{\ell}$ and $\widetilde{\ell}'$ are any two lifts to F_∞, then $\mathrm{width}(\widetilde{\ell}) = \mathrm{width}(\widetilde{\ell}')$. Thus for any such loop, we define $\mathrm{width}(\ell) = \mathrm{width}(\widetilde{\ell})$, where $\widetilde{\ell}$ is any lift of ℓ to F_∞.

Since E bounds a disk in W, then ∂E lifts to F_∞, and we set (for the remainder of the paper)

$$k = \mathrm{width}(\partial E).$$

Since ∂M is incompressible, then E intersects D_1 and D_2 nontrivially. If $n \geq k$, let $E^j \subset M_n$ denote the lift of E to M_n which intersects D_1^j, but is disjoint from D_2^j. Then $D_2^1 \cup \bigcup_{j=1}^n D_1^j$ forms a disk system for H_n and $\bigcup_{j=1}^n E^j$ forms a disk system for W_n (cf. Figure 1 for an example with $k = 3$ and $n = 4$).

Set $H_n' = H_n / D_2^1$, set $\mathcal{E}_j = \{E^1, \ldots, E^j\}$, and $\mathcal{E}_j^{(i)} = \mathcal{E}_j - E^i$, $1 \leq i \leq j$. Recall that a collection $\mathcal{C}$ of disjoint simple closed curves in the boundary of a handlebody X is *disk busting* if $\partial X - \mathcal{C}$ is incompressible in X.

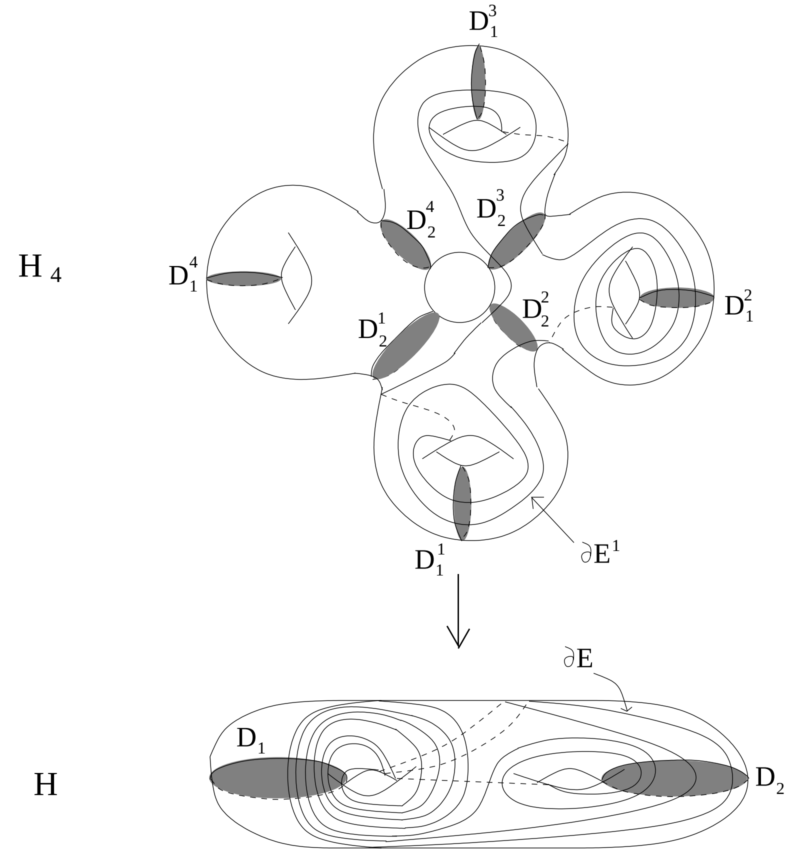

FIGURE 1. An example of a 4-fold cyclic cover

Definition 2.2. Let $m \geq 1$ be an integer. We say that E satisfies the *m-lift condition* if:

(1) The set $\partial\mathcal{E}_m$ is disk-busting in H'_{m+k-1}.
(2) The set $\partial\mathcal{E}_m^{(i)}$ is not disk-busting in H'_{m+k-1}, for all $1 \leq i \leq m$.
(3) For each $1 \leq i \leq m$, there is a compression disk

$$\Delta_i \subset H'_{m+k-1} - \partial\mathcal{E}_m^{(i)}$$

such that $[\partial\Delta_i]$ is linearly independent from $\{[\partial E^{i+1}], \ldots, [\partial E^m]\}$ in $H_1(\partial H'_{m+k-1}; \mathbb{Q})$.

Remark. If $M = H[E]$ is fibered, then it follows from Lemma 4.1 below that condition (1) fails, so E does not satisfy the m-lift condition for any m.

Recall that a 3-manifold M is *large* if it is irreducible, and contains a closed essential surface, i.e., an incompressible surface which is not parallel to a component of ∂M. We prove:

Theorem 2.3. *If E satisfies the m-lift condition, then M_n is large for any $n \geq \mathrm{Max}(m+k-1, 2k-2)$.*

Let λ be a longitude for M (i.e., a simple, closed, essential curve in ∂M which lifts to a loop in M_∞). Fix a meridian μ for M (i.e., a simple closed curve in ∂M which intersects λ exactly once). A slope p/q in ∂M means the pair of homology class $\pm(p[\mu] + q[\lambda])$ and $(p,q) = 1$. We use $M(p/q)$ to denote the closed manifold obtained by Dehn filling of M with slope p/q. Let $b = |\phi(\mu)|$. Then $b > 0$ is a finite integer. Then we have:

Theorem 2.4. *If E satisfies the m-lift condition, then $M(np/q)$ is virtually Haken for any $p \geq 2$, $n \geq \mathrm{Max}\{m+k-1, 2k-2, \mathrm{width}(\lambda)+b\}$, and q with $(pn, q) = 1$.*

Given a 2-generator, 1-relator presentation of a 3-manifold group, there is an algorithm to decide if this presentation corresponds to a genus 2 Heegaard splitting (conjecturally it always does). From the data of such a geometric presentation, it is possible to check if the m-lift condition holds for a given m. Using the computer program GAP, we have shown:

Theorem 2.5. *Every 2-generator, 1-relator 3-manifold M in the SnapPea census of 1-cusped hyperbolic 3-manifolds has a genus 2 Heegaard splitting. Moreover, if $b_1(M) = 1$ and M is nonfibered, then M has a genus 2 Heegaard splitting whose 2-handle satisfies the m-lift condition for some m.*

A complete table of the values of m is available at

www.math.buffalo.edu/~jdmaster.

The first few values are given in Table 1.

To prove these theorems, consider the surface F_n (recall this is the preimage of the Heegaard surface F in M_n). Then F_n is a Heegaard surface for M_n of genus $n+1$, which we shall compress to both sides. On the handlebody side, we compress F_n along a single lift, D_2^1, of D_2; on the compression body side we compress F_n along all the lifts of E which are disjoint from D_2^1. We shall show that if E satisfies the m-lift condition and $n \geq m+k-1$, then the resulting surface is incompressible.

Let

$$\begin{aligned} X_n &= (W_n/\mathcal{E}_{n-k+1})[D_2^1], \\ Y_n &= (H_n/D_2^1)[\mathcal{E}_{n-k+1}], \text{ and} \\ S_n &= \partial Y_n. \end{aligned}$$

Note that $M_n - \mathring{N}(S_n) \cong X_n \amalg Y_n$.

Lemma 2.6. *The surface S_n is connected, has genus $= k-1$, and is not parallel to ∂M_n.*

Proof. To prove that S_n is connected, it is enough to show that

$$[\partial D_2^1], [\partial E^1], \ldots, [\partial E^n]$$

are linearly independent in $H_1(F_n)$.

Let $I(\cdot,\cdot)$ denote the algebraic intersection pairing on the first homology group of a surface. Recall from Lemma 2.1 that $\phi(\alpha_2)=1$ is a generator of $\phi(\pi_1(M))=\mathbb{Z}$. It follows that $I([\partial E],[\partial D_2])=0$ and that $\{[\partial E],[\partial D_2]\}$ are linearly independent. We may complete $\{[\partial E],[\partial D_2]\}$ to a symplectic basis $\{[\partial E],[\partial E]^*,[\partial D_2],[\partial D_2]^*\}$, i.e., a basis satisfying

$$I([\partial E],[\partial E]^*)=I([\partial D_2],[\partial D_2]^*)=1,$$
$$I([\partial E]^*,[\partial D_2])=I([\partial E]^*,[\partial D_2]^*)=I([\partial E],[\partial D_2])=I([\partial E],[\partial D_2]^*)=0.$$

Let $\alpha\subset F$ be an embedded loop representing $[\partial E]^*$, and intersecting ∂E geometrically exactly once (such representative always exists). Then α lifts homeomorphically to loops $\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_n\subset F_n$ (since $I([\alpha],[\partial D_2])=0$). As ∂E also lifts to F_n as $\partial E^1,\dots,\partial E^n$, it is easy to see that $I(\widetilde{\alpha}_i,\partial E^j)=\delta_{ij}$. Also

$$\begin{aligned}&I([\widetilde{\alpha}_i],[\partial D_2^1])\\&=I([\widetilde{\alpha}_i],[\bigcup_j\partial D_2^j]/n)(\text{ since all lifts of }\partial D_2\text{ are homologous in }F_n)\\&=I([\alpha],[\partial D_2])/n\\&=0.\end{aligned}$$

Recall that we have a map $\phi:\pi_1M\to\mathbb{Z}$; let $\beta\subset\partial M$ be a loop with $\phi(\beta)\neq0$, and let β_n be the preimage in M_n. Then $[\beta_n],[\widetilde{\alpha}_1],\dots,[\widetilde{\alpha}_n]$ are dual classes for $[\partial D_2^1],[\partial E^1],\dots,[\partial E^n]$ in $H_1(F_n,\mathbb{Q})$, which proves the linear independence, and completes the proof that S_n is connected.

The linear independence of $[\partial D_2^1],[\partial E^1],\dots,[\partial E^n]$ also allows us to compute:

$$\begin{aligned}\text{genus}(S_n)&=\text{genus}(H_n/D_2^1)-|\mathcal{E}_{n-k+1}|\\&=n-(n-k+1)=k-1.\end{aligned}$$

Finally, note that S_n is not parallel into ∂M_n, since every loop in S_n projects to an element in $\ker(\phi)$ (because S is disjoint from D_2^1), but each component of ∂M_n contains a loop whose projection is not in $\ker(\phi)$. □

To prove Theorems 2.3 and Theorem 2.4, we shall show that when $n\geq\text{Max}\{m+k-1,2k-2\}$, S_n is incompressible in both X_n and Y_n, and that when $n\geq\text{Max}\{m+k-1,2k-2,\text{width}(\lambda)\}$, S_n remains incompressible in an equivariant Dehn filling of M_n along ∂M_n (which may have several components) which is a free cyclic cover of $M(np/q)$.

3. Background on 1-relator groups and 3-manifolds

We will require the following result of Magnus (see [16]). The statement given here is easily seen to be equivalent to the standard statement.

Theorem 3.1 (Freiheitsatz for 1-relator groups)**.** *Let*

$$G = \langle x_1, \dots, x_n \mid w(x_1, \dots, x_n) = 1 \rangle$$

be a 1*-relator group, where w is a freely reduced word. Let $\mathcal{X} = \{x_1, \dots, x_n\}$, let $\mathcal{X}^* \subset \mathcal{X}$, and suppose that some $x_i \in \mathcal{X} - \mathcal{X}^*$ appears in w. Then $\mathcal{X}^*$ freely generates a free subgroup of G.* □

Corollary 3.2. *Let $w(x_1, \dots, x_k)$ be a word in which x_1 and x_k appear nontrivially, and consider the group*

$$G = \langle \dots, x_{-1}, x_0, x_1, \dots \mid w(x_i, \dots, x_{i+k-1}) = 1, \forall i \in \mathbb{Z} \rangle.$$

Then for any i, the set $\{x_i, \dots, x_{i+k-2}\}$ freely generates a free subgroup of G.

Proof. Let $G_i = \langle x_i, \dots, x_{i+k-1} \mid w(x_i, \dots, x_{i+k-1}) \rangle$, and let $J_i = \langle x_{i+1} \rangle * \dots * \langle x_{i+k-1} \rangle$. By repeated applications of Theorem 3.1, the group G has the structure of the following iterated amalgamated free product over free subgroups: $G \cong \dots *_{J_{i-1}} G_i *_{J_i} G_{i+1} *_{J_{i+1}} \cdots$. By Theorem 3.1, each J_i injects into G_i and G_{i+1}. Each G_i thus injects into G, and so we obtain the corollary. □

Suppose $G = \langle x, y \mid w(x, y) \rangle$ is a 1-relator group (where w is a cyclically reduced word) which admits a surjective homomorphism $\psi : G \to \mathbb{Z}$, such that $\psi(y) = 0$. Let $x_i = x^{-i} y x^i$. Then $\ker(\psi)$ is generated by the x_i's, and the relation w lifts to a relation $\widetilde{w}$ on the x_i's, so that $\ker(\psi)$ has a presentation as in the statement of Corollary 3.2. Write $\widetilde{w} = \Pi_j x_{\mu_j}$, and consider the finite integer sequence (μ_j). Then we have:

Theorem 3.3 (Brown)**.** *If (μ_j) has a repeated minimum (or maximum) (i.e., it assumes its minimum (or maximum) value more than once), then $\ker(\psi)$ is not finitely generated.*

The case of a repeated minimum is a special instance of Theorem 4.2 in [4], and the case of a repeated maximum follows from a trivial modification of the proof (which is an application of the Freiheitsatz).

We also need the following, a special case of Corollary 2.2 of [5], which is in turn a slight modification of a theorem proved by Jaco in [12].

Theorem 3.4 (Handle Addition Lemma)**.** *Let M be an irreducible* 3*-manifold with compressible boundary of genus at least* 2*, and suppose $\alpha \subset \partial M$ is a simple closed curve, such that $\partial M - \alpha$ is incompressible in M. Then the* 3*-manifold obtained by adding a* 2*-handle to M along α is irreducible, and has incompressible boundary.*

4. 2-handles in nonfibered manifolds

The results presented in this section are essentially combinations of results due to Brown ([4]) and Bieri–Neumann–Strebel ([1]).

If D is a compressing disk for H_∞, and α is a simple closed curve in H_∞, then let $I_{\mathrm{geo}}(\alpha, D)$ be the geometric intersection number of α and D; in other words, $I_{\mathrm{geo}}(\alpha, D)$ is the minimal cardinality of $\alpha' \cap D$ over all curves α' which is isotopic to α in H_∞.

If $\mathcal{D}$ is a disk system for H, then there are dual simple loops which form a free basis for $\pi_1 H$. If $D \in \mathcal{D}$ corresponds to the generator x_D, and if α is a simple loop in H, then $I_{\mathrm{geo}}(\alpha, D)$ is the number of times the generator x_D appears in a cyclically reduced representative for the conjugacy class of $[\alpha]$ in $\pi_1 H$.

Lemma 4.1. *Suppose M is a knot manifold with $t(M) = 1$, and suppose that E^i and D_i^j are as defined in Section 2.*

(a) *If M is fibered, then $I_{\mathrm{geo}}(\partial E^1, D_1^1) = I_{\mathrm{geo}}(\partial E^1, D_1^k) = 1$ in H_∞.*
(b) *If M is nonfibered, then $I_{\mathrm{geo}}(\partial E^1, D_1^1) \geq 2$ and $I_{\mathrm{geo}}(\partial E^1, D_1^k) \geq 2$ in H_∞.*

Proof. (a) Suppose $I_{\mathrm{geo}}(\partial E^1, D_1^1) \geq 2$ or $I_{\mathrm{geo}}(\partial E^1, D_1^k) \geq 2$. Then by Theorem 3.3 (together with the note in the proceeding paragraph of the present lemma), $\ker(\phi)$ is not finitely generated, and so by [21], M is not fibered.

(b) Suppose E^1 intersects one of the disks, say D_1^k, exactly once. We shall show that in this case M is fibered.

Dual to each D_1^i is a generator x_i for the fundamental group of M_∞. The boundary of the disk E^1 gives a relation among these generators which involves x_k only once; therefore $x_k \in \langle x_1, \ldots, x_{k-1}\rangle \subset \pi_1 M_\infty$. Similarly, using the relation corresponding to the disk E^2, we get that

$$x_{k+1} \in \langle x_1, \ldots, x_k\rangle = \langle x_1, \ldots, x_{k-1}\rangle \subset \pi_1 M_\infty.$$

Continuing in this way, we see that all of the generators x_i with $i \geq k$ can be expressed in terms of $x_1, \ldots, x_{k-1}$.

Let H^* be the component of H_∞ / D_2^1 containing $D_1^1, D_1^2, \ldots$, and let $Q = H^*[E^1, E^2, \ldots]$, which is a submanifold of M_∞. The argument we just gave shows that $\pi_1(Q)$ is finitely generated.

Note that there is a nonseparating incompressible surface S in M with boundary slope λ such that M_∞ is the infinite cover dual to S. Let $\widetilde{S}$ be a lift of S to M_∞ which is disjoint from Q, and let Q^+ be the component of $M_\infty - \mathring{N}(\widetilde{S})$ which contains Q. Then $Q^+ - \mathring{N}(Q)$ is compact, and since $\pi_1 Q$ is finitely generated, $\pi_1 Q^+$ is finitely generated as well.

Let $M_0^- = M - \mathring{N}(S)$, let S_0 and S_1 be the two preimages of S in ∂M_0^-, let $\widetilde{S}_i$ be the preimages of S in M_∞, and let M_i^- be the submanifold of Q bounded by $\widetilde{S}_0$ and $\widetilde{S}_i$. Since $\widetilde{S}_i$ is incompressible, M_i^- is π_1-injective in Q for each i. If neither of the maps $i_* \pi_1 S_j \to \pi_1 M_0$ is onto, then $\{\pi_1 M_i\}$ forms an an infinite sequence of subgroups of $\pi_1 Q$, with $\pi_1 M_{i+1}$ properly containing $\pi_1 M_i$ for each i, which is a contradiction, since $\pi_1 Q$ is finitely generated. Therefore, one of the induced maps $\pi_1 S_j \to \pi_1 M_0$ is onto, and so, as in the proof of Theorem 2 in [21], M is fibered. □

Corollary 4.2. *If M is fibered, then E does not satisfy the m-lift condition for any m.*

Proof. Suppose M is fibered, let $m \geq 1$ be an integer, and consider the cover F_{m+k-1} of F. In F_{m+k-1}, we have that E^j is disjoint from D_1^1 for all $2 \leq j \leq m-k+1$, and by Lemma 4.1(a), $|\partial E^1 \cap \partial D_1^1| = 1$. Therefore there is a compressing disk Δ in H_{m+k-1} (whose boundary is equal to $\partial N(\partial E^1 \cup \partial D_1^1)$) with $\partial\Delta \cap \partial\mathcal{E}_m = \emptyset$, and so E fails condition (2). □

5. Proof of irreducibility of M_n

We shall now begin the proof of Theorem 2.3, which will occupy the next three sections.

Lemma 5.1. *M_n is irreducible for all n.*

Proof. By Theorem 3.4, $M_1 = M$ is irreducible. By [19] (or [8]), the cover of an irreducible manifold is irreducible, so M_i is irreducible for all $i \geq 2$. □

6. Proof that X_n has incompressible boundary

We remark that the m-lift condition is not needed in this case; we only use the assumption that ∂M is incompressible.

Let $W_n' = W_n/\mathcal{E}_{n-k+1}$. By Theorem 3.4 it is enough to prove:

Lemma 6.1. *If $n \geq 2k-2$, then $W_n' - \partial D_2^1$ has incompressible boundary.*

First we need:

Lemma 6.2. *For each n there is a loop $\alpha_n \subset \partial M_n$ such that $I(\alpha_n, D_2^1) \neq 0$.*

Proof. By the exact sequence of the pair, there is a loop $\alpha \in \partial M$ such that $\phi[\alpha] = I(\alpha, D_2^1) \neq 0$. Letting α_n be the preimage of α in ∂M_n, we have $I(\alpha_n, \bigcup_{i=1}^n D_2^i) = nI(\alpha, D_2) \neq 0$. Since $[D_2^i] = [D_2^j] \in H_1(M_n)$ for all i, j, then we have $I(\alpha_n, D_2^1) = I(\alpha, D_2) \neq 0$. □

Proof of Lemma 6.1. Suppose otherwise that there is a compressing disk in $W_n' - \partial D_2^1$. First, if there is a compressing disk, we claim that there must be a nonseparating one. To see this, suppose that Δ is a separating compressing disk. If there are no nonseparating compressing disks, then one of the components of W_n'/Δ is homeomorphic to a surface cross an interval, and the other component is a handlebody containing ∂D_2^1. Every curve in ∂M_n lies on the surface cross interval side, but by Lemma 6.2, there is a curve in ∂M_n which has nontrivial intersection with D_2^1, yielding a contradiction. So we may assume that there is a nonseparating compressing disk Δ in $W_n' - \partial D_2^1$.

Consider the Heegaard surface F_n and the curves ∂D_2^j and ∂E^j, $j = 1, \ldots, n$, in F_n. Note that since $n \geq 2k-2$, $\{\partial E^{n-k+2}, \ldots, \partial E^n\}$ are all disjoint from ∂D_2^{n-k+2} and ∂D_2^k (by considering the definition of $k = \mathrm{width}(E)$). The two simple closed curves ∂D_2^{n-k+2} and ∂D_2^k cut F_n into two components, F_n^1 and F_n^2, one of which, say F_n^1, is disjoint from all $\partial E^{n-k+2}, \ldots, \partial E^n$.

The curves $\partial E^1, \ldots, \partial E^{n-k+1}$ may intersect ∂D_2^{n-k+2} and ∂D_2^k. But by the property of the width k again, any arc component of $F_n^2 \cap (\cup_{j=1}^{n-k+1} \partial E^j)$ has either both endpoints in ∂D_2^{n-k+2} or both endpoints in ∂D_2^k. Further, every arc component of $F_n^2 \cap (\cup_{j=1}^{n-k+1} \partial E^j)$ is disjoint from ∂D_2^1. Let F_n^3 be the subsurface of F_n which is the union of F_n^1 and a small regular neighborhood of the arcs $F_n^2 \cap (\cup_{j=1}^{n-k+1} \partial E^j)$ in F_n^2 (in other words, F_n^3 is F_n^1 with some bands attached, one for each arc component in $F_n^2 \cap (\cup_{j=1}^{n-k+1} \partial E^j)$). Then F_n^3 is a connected subsurface of F_n which contains all $\partial E^1, \ldots, \partial E^{n-k+1}$ but is disjoint from all $\partial E^{n-k+2}, \ldots, \partial E^n$ and ∂D_2^1.

Let F_n^4 be the surface obtained from F_n^3 by surgery along the curves $\partial E^1, \ldots, \partial E^{n-k+1}$ (i.e., cut F_n^3 open along $\{\partial E^1, \ldots, \partial E^{n-k+1}\}$ and fill each of the new boundary circles with a disk), which may not be connected. Note that $\{E^{n-k+2}, \ldots, E^n\}$ is a disk system for the compression body W_n' and $W_n'/\{E^{n-k+2}, \ldots, E^n\}$ is an I-bundle over a surface. As F_n^4 is disjoint from $\{\partial E^{n-k+2}, \ldots, \partial E^n\}$, F_n^4 is contained in the horizontal boundary of the I-bundle. So $\partial F_n^4 \times I$ are vertical annuli of this I-bundle. By standard cut-and-paste operations along arcs and circles of $\Delta \cap (\partial F_n^4 \times I)$, we get a nonseparating compressing disk, still denoted Δ, which is disjoint from the annuli $\partial F_n^4 \times I$. It is easy to see that Δ cannot be contained in $F_n^4 \times I$, so it follows that $\partial\Delta$ is disjoint from all ∂D_2^1, $\partial D_2^k, \ldots$, ∂D_2^{n-k+2}. Hence the width of Δ is strictly less than k.

Let $\delta = \partial\Delta$, and let $\widetilde{\delta}$ be a lift of δ in M_∞, The group $\pi_1 M_\infty$ has the following presentation:

$$\pi_1 M_\infty = \langle \ldots, x_{-1}, x_0, x_1, \ldots \mid w(x_i, \ldots, x_{i+k-1}) = 1, \, \forall i \in \mathbb{Z} \rangle,$$

where the relations correspond to the lifts of E. Since $\mathrm{width}(\delta) < k$, the loop $\widetilde{\delta}$ represents an element in the subgroup of $\pi_1 M_\infty$ generated by the elements $x_i, \ldots, x_{i+k-2}$, for some i. By Corollary 3.2, these elements are a basis for a free subgroup; since $\widetilde{\delta}$ represents a trivial element in $\pi_1 M_\infty$, we see that $\widetilde{\delta}$ represents the trivial word in $x_i, \ldots, x_{i+k-2}$. Thus $\widetilde{\delta}$ bounds a disk in H_∞ by Dehn's lemma, and thus δ bounds a disk in H_n. Thus there is a nonseparating sphere in M_n, contradicting Lemma 5.1. □

Lemma 6.3. *Suppose $n \geq 2k-2$. Then X_n has incompressible boundary.*

Proof. This is a consequence of Lemma 6.1 and Theorem 3.4. □

7. Proof that Y_n has incompressible boundary

In this section, we are under the assumption that $\mathcal{E}$ satisfies the m-lift condition. Recall that $H'_n = H_n/D^1_2$.

Lemma 7.1. *The curve ∂E^{n-k+1} cannot be isotoped in $H'_n[\mathcal{E}^{n-k}]$ to intersect D^n_1 fewer than two times.*

Proof. We have

$$\begin{aligned}\pi_1 H'_n[\mathcal{E}^{n-k}] &= \langle x_1, \dots, x_n \mid w_1, \dots, w_{n-k}\rangle \\ &\cong \langle x_1, \dots, x_{n-1} \mid w_1, \dots, w_{n-k}\rangle * \langle x_n\rangle\end{aligned}$$

where x_j is dual to D^j_1, and the word w_j corresponds to ∂E^j.

The word w_{n-k+1} can be cyclically permuted to have the form

$$w_{n-k+1} = \mathcal{W}_1 x_n^{\ell_1} \mathcal{W}_2 x_n^{\ell_2} \dots \mathcal{W}_t x_n^{\ell_t},$$

where the $\mathcal{W}$'s are freely reduced words involving only x_j's with $n-k+1 \le j \le n-1$, and each $\mathcal{W}_j$ represents a nontrivial element in the group $\langle x_1, \dots, x_n \mid w_1, \dots, w_{n-k}\rangle$.

Suppose ∂E^{n-k+1} can be isotoped to be disjoint from D^n_1 in $H'_n[\mathcal{E}^{n-k}]$. Then, using the relations w_j, $j \le n-k$, the word w_{n-k+1} can be rewritten entirely in terms of x_j's, $j \le n-1$, and this would imply that one of the $\mathcal{W}_j$'s must be trivial in $\pi_1 H'_n[\mathcal{E}^{n-k}]$. However, $\mathcal{W}_j$ is a freely reduced word on $x_{n-k+1}, \dots, x_{n-1}$, which by Corollary 3.2 freely generate a free subgroup, for a contradiction.

Suppose ∂E^{n-k+1} can be isotoped to intersect D^n_1 exactly once. Then, as in the proof of Lemma 4.1, M is fibered, and so by Corollary 4.2, $\mathcal{E}$ does not satisfy the m-lift condition, for a contradiction. □

Lemma 7.2. *If $n \ge m+k-1$, then Y_n has incompressible boundary.*

Proof. Let $n_0 = m+k-1$. We first prove the result in the case where $n = n_0$.

Claim. *The manifold Y_{n_0} has incompressible boundary.*

Proof. Recall $H'_{n_0} = H_{n_0}/D^1_2$. By condition (2) of Definition 2.2, the manifold

$$H'_{n_0} - \partial\mathcal{E}^{(m)}_m = H'_{n_0} - \partial\mathcal{E}_{m-1}$$

has compressible boundary, and by condition (1), $H'_{n_0} - \partial\mathcal{E}_m$ has incompressible boundary; therefore, by Theorem 3.4, the manifold $(H'_{n_0} - \partial\mathcal{E}_{m-1})[E^m]$ has incompressible boundary.

Suppose for some $i \in [1, m]$, that $H'_{n_0}[\overline{\mathcal{E}}_i] - \partial\mathcal{E}_i$ has incompressible boundary, where $\overline{\mathcal{E}}_i = \mathcal{E}_m - \mathcal{E}_i$. By condition (2) of Definition 2.2, there is a compression disk for $H'_{n_0} - \partial\mathcal{E}^{(i)}_m$ which is also a compression disk for $H'_{n_0}[\overline{\mathcal{E}}_i] - \partial\mathcal{E}_{i-1}$. Therefore, by Theorem 3.4, $H'_{n_0}[\overline{\mathcal{E}}_{i-1}] - \partial\mathcal{E}_{i-1}$ has incompressible boundary.

By induction on i, it follows that $Y_{n_0} = H'_{n_0}[\mathcal{E}_m]$ has incompressible boundary. $\square$

Now, suppose $n > n_0$. and proceed by induction on n. Suppose Y_n has incompressible boundary. The manifold Y_{n+1} is obtained from Y_n by adding a 1-handle Z to H'_n, and then attaching a 2-handle E^{n-k+2}. We claim that $(Y_n \cup Z) - \partial E^{n-k+2}$ has incompressible boundary.

Suppose otherwise, and let Δ be a compressing disk. Since Y_n has incompressible boundary, the maximal compression body for $\partial(Y_n \cup Z)$ has a unique disk system consisting only of D_1^{n+1}. Therefore if Δ is nonseparating, it is isotopic to D_1^{n+1}; then ∂E^{n-k+2} can be isotoped off of ∂D_1^{n+1} in $Y_n \cup Z$, contradicting Lemma 7.1.

Suppose Δ is separating, so it separates off a solid torus $V \subset H'_{n+1}$, with $V \supset D_1^{n+1}$. Since $\partial E^{n-k+2} \cap D_1^{n+1} \neq \emptyset$, we have $\partial E^{n-k+2} \subset V$.

So Y_{n+1} contains the punctured lens space $V[E^{n-k+2}]$. By Lemma 7.1, this lens space cannot be B^3. So Y_n is not irreducible, contradicting Lemma 5.1. This completes the proof that $(Y_n \cup Z) - \partial E^{n-k+2}$ has incompressible boundary. Then by Theorem 3.4, Y_{n+1} has incompressible boundary. $\square$

8. Proof of Theorem 2.3 and Theorem 2.4

Proof of Theorem 2.3. Assume M satisfies the hypotheses of Theorem 2.3. Then by Lemmas 6.3 and 7.2, M_n contains an incompressible closed surface S_n, which is not parallel into ∂M_n by Lemma 2.6. $\square$

Proof of Theorem 2.4. Suppose the hypotheses of Theorem 2.4 are satisfied. Let b_n be the number of boundary components of M_n. Then it is easy to see that b_n is equal to the largest common divisor of n and b (recall from the proceeding paragraph of Theorem 2.4 that $b = |\phi(\mu)|$ is a finite integer). Let each boundary component of ∂M_n have the coordinate basis induced from the basis $\{\mu, \lambda\}$ of ∂M. Let $M_n(b_n p/q)$ denote the closed manifold obtained by Dehn filling each component of ∂M_n with slope $b_n p/q$. Then it is easy to check that $M(np/q)$ is cyclically covered by $M_n(b_n p/q)$. We have shown that M_n contains an incompressible surface S_n. When $n \geq \text{width}(\lambda) + b$ as well, there are b successive lifts of λ contained in $S_n = \partial Y_n$, which implies that S_n has an annular compression to each component of ∂M_n, with slope 0. Since $p > 1$, then by repeatedly applying Theorem 2.4.3 of [6] b_n times, we see that S_n remains incompressible in $M_n(b_n p/q)$. Also, by [20], $M_n(b_n p/q)$ is irreducible. Hence $M(np/q)$ is virtually Haken. $\square$

9. Brief description of algorithm

A. Checking that SnapPea presentations are geometric. Given an algebraic word on x and y, we need to know if it can be represented by a simple closed curve in the boundary of a genus 2 handlebody. To do this, we attempt to draw a Heegaard diagram. So we start with four disks in the plane, corresponding to x, x^{-1}, y and y^{-1}. The word w is represented by a collection of edges connecting these disks, and we need to find a representation which embeds in the plane.

To program this on the computer, we start placing edges on the graph one by one, as indicated by the word w. When placing an edge, the intial point is determined from the previous step, but there may be a choice of terminal point. However, it is possible to keep track of the choices which are made, and if an impossible situation is arrived at, we retreat to the previous choice, and change it. In this way, the computer found for every given word, a geometric representation.

B. Checking that the m-lift condition holds. The only nonelementary step in checking the m-lift condition algorithmically is to find, for a given collection of loops in the fundamental group of a handlebody H, a specific compressing disk for H which is disjoint from the loops. An algorithm for this was given by Whitehead ([23], or see [22]), which we implemented on GAP.

Note that Whitehead's algorithm allows one to determine the existence of a compressing disk in polynomial time in the length of the word; however, to construct the disk explicitly requires exponential time. For our application, we are saved this difficulty, since for each compressing disk Δ we only need to compute $[\partial\Delta] \in H_1(\partial H)$. This allows the algorithm to run in polynomial time.

TABLE 1. Data on SnapPea census manifolds

manifold name:	width of 2-handle:	satisfies m-lift for:	n-fold cyclic cover large for:
m006	$k = 3$	$m = 2$	$n \geq 4$
m007	$k = 3$	$m = 2$	$n \geq 4$
m015	$k = 4$	$m = 2$	$n \geq 6$
m017	$k = 3$	$m = 1$	$n \geq 4$
m029	$k = 3$	$m = 2$	$n \geq 4$
m030	$k = 3$	$m = 2$	$n \geq 4$
m032	$k = 3$	$m = 2$	$n \geq 4$
m033	$k = 3$	$m = 2$	$n \geq 4$
m035	$k = 3$	$m = 1$	$n \geq 4$
m037	$k = 3$	$m = 1$	$n \geq 4$
m045	$k = 3$	$m = 1$	$n \geq 4$
m046	$k = 3$	$m = 1$	$n \geq 4$
m053	$k = 3$	$m = 1$	$n \geq 4$
m054	$k = 3$	$m = 2$	$n \geq 4$
m058	$k = 3$	$m = 1$	$n \geq 4$

m059	$k = 3$	$m = 1$	$n \geq 4$
m061	$k = 3$	$m = 2$	$n \geq 4$
m062	$k = 3$	$m = 2$	$n \geq 4$
m066	$k = 3$	$m = 1$	$n \geq 4$
m067	$k = 3$	$m = 1$	$n \geq 4$
m073	$k = 3$	$m = 1$	$n \geq 4$
m074	$k = 3$	$m = 2$	$n \geq 4$
m076	$k = 3$	$m = 1$	$n \geq 4$
m077	$k = 3$	$m = 1$	$n \geq 4$
m079	$k = 3$	$m = 1$	$n \geq 4$
m080	$k = 3$	$m = 1$	$n \geq 4$
m084	$k = 3$	$m = 1$	$n \geq 4$
m085	$k = 3$	$m = 1$	$n \geq 4$
m089	$k = 3$	$m = 1$	$n \geq 4$
m090	$k = 3$	$m = 1$	$n \geq 4$
m093	$k = 3$	$m = 2$	$n \geq 4$
m094	$k = 3$	$m = 2$	$n \geq 4$
m104	$k = 3$	$m = 1$	$n \geq 4$
m105	$k = 3$	$m = 3$	$n \geq 5$
m110	$k = 3$	$m = 1$	$n \geq 4$
m111	$k = 3$	$m = 1$	$n \geq 4$
m137	$k = 3$	$m = 2$	$n \geq 4$
m139	$k = 4$	$m = 3$	$n \geq 6$
m148	$k = 3$	$m = 1$	$n \geq 4$
m149	$k = 3$	$m = 1$	$n \geq 4$
m202	$k = 4$	$m = 2$	$n \geq 6$
m203	$k = 4$	$m = 1$	$n \geq 6$
m208	$k = 4$	$m = 1$	$n \geq 6$
m249	$k = 5$	$m = 4$	$n \geq 8$
m259	$k = 5$	$m = 3$	$n \geq 8$
m260	$k = 5$	$m = 3$	$n \geq 8$
m261	$k = 3$	$m = 1$	$n \geq 4$
m262	$k = 3$	$m = 1$	$n \geq 4$
m285	$k = 3$	$m = 1$	$n \geq 4$
m286	$k = 3$	$m = 1$	$n \geq 4$
m287	$k = 5$	$m = 5$	$n \geq 9$
m288	$k = 5$	$m = 3$	$n \geq 8$
m292	$k = 5$	$m = 3$	$n \geq 8$
m319	$k = 3$	$m = 1$	$n \geq 4$

m320	$k = 3$	$m = 1$	$n \geq 4$
m328	$k = 4$	$m = 1$	$n \geq 6$
m329	$k = 4$	$m = 2$	$n \geq 6$
m340	$k = 5$	$m = 1$	$n \geq 8$
m357	$k = 4$	$m = 2$	$n \geq 6$
m366	$k = 4$	$m = 1$	$n \geq 6$
m388	$k = 4$	$m = 1$	$n \geq 6$

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