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Two-sided averages for which oscillation fails

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Abstract. We show that in any invertible, ergodic, measure-preserving system, the two-sided square function obtained by comparing forward averages with their backwards counterparts, will diverge if the (time) length of the averages grows too slowly. This contrasts with the one-sided case. We also show that for any sequence of times, certain weighted sums of the forward averages diverge. This contrasts with what would happen if the times increased rapidly and two-sided differences were considered.

Contents

1. Introduction

This paper originates with the following elementary but interesting observations relating the usual ergodic averages and the ergodic Hilbert transform. They give the context for our results, which are stated immediately afterward.

We are in the standard ergodic theory setting, with a probability space (X, Σ, m) and an invertible ergodic measure-preserving transformation $T : X \to X$. For an m-integrable function $f : X \to \mathbb{C}$ we define

$$S_n f(x) = \sum_{1}^{n} f(T^k x)\,,\; A_n f(x) = \frac{1}{n} S_n f(x)\,,\;\text{ and }\; H_n^+ f(x) = \sum_{1}^{n} \frac{f(T^k x)}{k}\,.$$

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Applying elementary summation by parts[1] one sees that

$$H_n^+ f(x) = \sum_{k=1}^{n-1} \frac{1}{k+1} \cdot \frac{1}{k} S_k f(x) + A_n f(x) \tag{1.1}$$

$$= \sum_{k=1}^{n-1} \frac{1}{k+1} A_k f(x) + A_n f(x)\,.$$

The astute reader has noted that since T is ergodic, $A_n f(x) \to \int f$ for almost every x and hence, if $\int f \neq 0$, $H_n^+ f(x)$ will diverge almost surely. This of course has been known for some time, but the above calculation gives the simplest proof we know of this fact.

We remark in passing that $H_n^+ f(x)$ will converge almost surely for certain functions with zero integral, for example if $f = g - g \circ T$ where g is bounded. But even in the zero integral case $H_n^+ f(x)$ does not always converge; in fact determining the class of functions f for which $H_n^+ f(x)$ converges is extensively studied, and may depend upon T; see for example [4].

It is also well-known that if we consider instead the symmetric averages

$$H_n f(x) = \sum_{0<|k|\leq n} \frac{f(T^k x)}{k}\,,$$

then $H_n f(x)$ converges almost surely, and the limit Hf is called the ergodic Hilbert transform of f.

We may naively explore what summation by parts reveals in the two-sided case — hoping perhaps for the appearance of a convergent series which might lead to a simple proof of the a.e.-convergence of $Hf(x)$.

Set $A_{-n} f(x) = \dfrac{1}{n} \sum_{k=1}^{n} f(T^{-k} x)$. Then

$$\sum_{0<|k|\leq n} \frac{f(T^k x)}{k} \tag{1.2}$$

$$= \sum_{j=1}^{n-1} \frac{1}{j+1} \left[A_j f(x) - A_{-j} f(x)\right] + \left\{A_n f(x) - A_{-n} f(x)\right\}.$$

Already we may derive something interesting from (1.2), even though it diverts us from our main development.

Since $H_n f(x)$ and the differences $\{A_n f(x) - A_{-n} f(x)\}$ converge almost everywhere, it must be the case that the the first sum on the r.h.s. of (1.2) converges a.e.. On the other hand, if we replace $A_{-n} f(x)$ with its limit $\int f$

[1] For complex sequences (a_k), (b_k) if $S_j = \sum_{k=1}^{j} a_k$ then $\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} S_k (b_k - b_{k+1}) + b_n S_n$.

then that sum becomes

$$\sum_{j=1}^{n-1} \frac{1}{j+1}\left[A_j f(x) - \int f\right] = \sum_{j=1}^{n-1} \frac{1}{j+1} A_j \left(f - \int f\right)(x).$$

We remind the reader of the following result due to Kakutani and Petersen (1981), which gives a precise sense in which there is no general rate of convergence in the pointwise Ergodic Theorem:[2]

Theorem 1.1 (Kakutani and Petersen [9]). *If $b_k \geq 0$ and $\sum b_k = \infty$ then there exists $f \in L^\infty$ with $\int f = 0$ so that*

$$\sup_L \left|\sum_{k=1}^{L} b_k A_k f(x)\right| = \infty \text{ a.e.}$$

Thus we see that we can always find an f so that the modified sum

$$\sum_{j=1}^{n-1} \frac{1}{j+1} A_j \left(f - \int f\right)(x)$$

diverges to infinity; yet, the original sum

$$\sum_{j=1}^{n-1} \frac{1}{j+1} \left[A_j f(x) - A_{-j} f(x)\right]$$

must converge a.e. for *all* integrable f (by the known a.e. convergence for $H_n f$). This says that somehow $A_{-j} f(x)$ is a better predictor for $A_j f(x)$ than its eventual limit $\int f$. It would be interesting to have a better understanding of why this is so.

Returning to our main development, our wish was to show directly that the r.h.s. of (1.2) converges almost everywhere, which would provide a proof of the a.e. convergence of $H_n f(x)$. The term $\{A_n f(x) - A_{-n} f(x)\}$ on the r.h.s. is certain to converge almost surely, which leaves the task of showing directly that the sum $\sum_{j=1}^{n-1} \frac{1}{j+1} \left[A_j f(x) - A_{-j} f(x)\right]$ converges. If we knew, for example, that[3]

$$\mathcal{S} f(x)^2 = \sum_{j=1}^{\infty} |A_j f(x) - A_{-j} f(x)|^2 < \infty \,(\text{a.e.}),$$

then an application of the Cauchy–Schwartz inequality to the first term on the r.h.s. of (1.2) would give our new proof of the a.e. convergence for the Hilbert transform. The connotation $\mathcal{S} f$ refers to *square* function, and one might be optimistic that the desired convergence would hold, because of previous results on square functions in ergodic theory . More precisely it is

[2]For a related, categorical-type negative result, see Corollary 3.4 of [5].

[3]Note the difference between Sf and $\mathcal{S} f$.

shown in [7] that for any increasing subsequence (n_k) of natural numbers, and any $f \in L^1$, the square function

$$\left(\sum_{k=1}^{\infty} |A_{n_k} f(x) - A_{n_{k+1}} f(x)|^2\right)^{\frac{1}{2}} \tag{1.3}$$

is finite a.e.. This is a one-sided result (only involving positive powers of T), but if we define the two-sided square function induced by (n_k) as

$$\mathcal{S}_{(n_k)} f(x) = \left(\sum_{j=1}^{\infty} \left|A_{n_j} f(x) - A_{-n_j} f(x)\right|^2\right)^{\frac{1}{2}},$$

then it follows from results in [6] that

$$\mathcal{S}_{(2^k)} f(x) = \left(\sum_{n=1}^{\infty} |A_{2^n} f(x) - A_{-2^n} f(x)|^2\right)^{\frac{1}{2}}$$

is finite a.e. for each $f \in L^1$. (In fact $f \to \mathcal{S}_{(2^k)} f$ will satisfy strong (p,p) inequalities for $1 < p < \infty$ and a weak (1,1) inequality, as will the original one-sided square function (1.3)). The proof is based upon transfering the setting to the integers, adding and subtracting a suitable dyadic martingale, getting favorable estimates on the differences, and using known results for square functions on martingales. See [6] for details.

Unfortunately for our original pursuit of a simple, direct proof of convergence for $H_n f$, our results show that there is indeed something special about one-sided averages, and lacunary sequences:

Theorem 1.2. *There exists $f \in L^\infty$ so that $\mathcal{S} f(x) = \infty$ a.e.*

Corollary 1.3. *$f \to \mathcal{S} f$ is unbounded on any L^p, $1 < p < \infty$.*

Actually these results are corollaries of the theorem we prove (Theorem 2.1), which gives sufficient conditions on sequences (n_k) so that $\mathcal{S}_{(n_k)} f$ will diverge. The statement is technical and thus omitted from the introduction, but the main point is relayed by Theorem 1.2 and Corollary 1.3.

Additionally we prove the following:

Theorem 1.4. *Let (n_k) be any increasing sequence of positive integers. Then there exists an $f \in L^\infty$ with $\int f = 0$ for which*

$$\sum_{k=1}^{\infty} \left|\frac{S_{n_k} f(x)}{n_k \log k}\right|^p = \infty \text{ a.e. for all } 1 < p < \infty.$$

Again, this is interesting because of the fact that if we take (n_k) lacunary and instead of subtracting the expected value, we subtract the backward averages we get the positive result

$$\sum_{k=1}^{\infty}\left|\frac{S_{n_k}f(x) - S_{-n_k}f(x)}{n_k}\right|^2 < \infty \text{ a.e., } f \in L^1,$$

by comparison with a suitable martingale.

2. Proofs of main results

We first state and prove Theorem 2.1, then outline how to obtain Theorem 1.2 and Corollary 1.3 as corollaries. Finally we prove Theorem 1.4.

As in the introduction we have a probability space (X, Σ, m) and an invertible ergodic measure-preserving transformation $T : X \to X$. For an m-integrable function $f : X \to \mathbb{C}$ and $n > 0$ we define

$$S_n f(x) = \sum_{1}^{n} f(T^k x),$$

$$A_n f(x) = \frac{1}{n} S_n f(x),$$

$$A_{-n} f(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(T^{-n}x).$$

Fix a strictly increasing function ϕ on $\mathbb{Z}^+$ with the property that given $B > 0$ there are infinitely many $n \in \mathbb{Z}^+$ such that $\phi^{-1}(4\phi(n)) - \phi^{-1}(2\phi(n)) > B$. Let Ψ be a strictly increasing function on $[0, \infty)$ such that $\Psi(0) = 0$ and $\lim_{x\to\infty}\Psi(x) = \infty$. Define

$$\mathcal{S}_\Psi f(x) = \Psi^{-1}\left(\sum_{n=1}^{\infty}\Psi(\left|A_{\phi(n)}f(x) - A_{-\phi(n)}f(x)\right|)\right).$$

Theorem 2.1. *If Ψ and ϕ are as above then there is a function $f \in L^\infty$ such that $\mathcal{S}_\Psi f(x) = \infty$ almost surely.*

Remark 2.2. A typical example of a ϕ that satisfies the above condition is $\phi(n) = n^p$ for some $p > 0$, and a typical example of a Ψ is $\Psi(x) = x^2$. In particular, if we take $p = 1$ we get a two-sided analogue of the classical one-sided ergodic square function $\left(\sum_{k=1}^{\infty}|A_{n_k}f(x) - A_{n_{k+1}}f(x)|^2\right)^{\frac{1}{2}}$.
As previously mentioned, it is shown in [7] that the one-sided square function converges almost surely regardless of the choice of sequence (n_k), yet here we will see that the two-sided analogue fails to do so for certain choices of (n_k) (determined by $\phi(n)$).

An example of a ϕ that does not satisfy the above condition is $\phi(n) = 2^n$, since then $\phi^{-1}(4\phi(n)) - \phi^{-1}(2\phi(n)) = \log_2(4 \times 2^n) - \log_2(2 \times 2^n) = \log_2 4 - \log_2 2 = 1$, and we see that the required condition for ϕ fails for any $B > 1$.

In general as ϕ increases more rapidly, ϕ^{-1} increases more slowly, so that functions ϕ satisfying the above condition may be thought of as slowly increasing.

Proof of Theorem 2.1. First we show, using a straightforward Rohlin tower construction, that there is a function $f \in L^\infty$ such that $\mathcal{S}_\Psi f(x) = \infty$ on a set of measure at least $\frac{1}{10}$. The general result will follow by choosing the towers independently, as discussed at the end of the proof.

Fix any increasing integer sequence (n_j) satisfying $n_1 = 1$ and the following properties:

(P1) $\frac{1}{n_k} \sum_{j=1}^{k-1} \frac{4\phi(n_j)}{10^j} < \frac{1}{10^{k+1}}$.
(P2) $\Psi^{-1}(\frac{1}{2}\frac{1}{10^k}) \cdot \big(\phi^{-1}(4\phi(n_k)) - \phi^{-1}(2\phi(n_k))\big) > k$.

For each $j \in \mathbb{N}$ form a Rohlin tower of height $16\phi(n_j)$ and error less than $\frac{1}{j}$. Let

$$f_j(x) = \begin{cases} \frac{1}{10^j} & \text{if } x \text{ is in the bottom half of the tower;} \\ \frac{-1}{10^j} & \text{if } x \text{ is in the top half of the tower;} \\ 0 & \text{if } x \text{ is in the error set.} \end{cases}$$

Write

$$f(x) = \sum_{j=1}^{\infty} f_j(x).$$

Let $\alpha_k(x) = \sum_{j=1}^{k-1} f_j(x)$ and $\beta_k(x) = \sum_{j=k+1}^{\infty} f_j(x)$.

Clearly $\|f\|_\infty \le \sum_{j=1}^{\infty} \frac{1}{10^j} = \frac{1}{9}$, so $f \in L^\infty$ as required.

For fixed $k \in \mathbb{N}$ define

$$R_k = \{(\ell, n) : 0 < \ell \le \phi(n_k),\ \phi^{-1}(2\phi(n_k)) < n < \phi^{-1}(4\phi(n_k))\},$$

and

$$B_k^+ = \{x : x \text{ is } \ell \text{ steps above the center of the } k\text{th tower, for some } \ell \in R_k\}.$$

(Since the height of the tower is even, $\ell = 1$ means we are on the first level for which $f(x)$ has a negative value.)

Fix $(\ell, n) \in R_k$ and $x \in B_k^+$ at height ℓ. We estimate

$$|A_{\phi(n)} f(x) - A_{-\phi(n)} f(x)|$$

as follows. We have $A_{\phi(n)} f(x) = A_{\phi(n)}\alpha_k(x) + A_{\phi(n)} f_k(x) + A_{\phi(n)}\beta_k(x)$ and similarly for $A_{-\phi(n)} f(x)$. Hence

$$\begin{aligned} &|A_{\phi(n)} f(x) - A_{-\phi(n)} f(x)| \\ &\quad \ge \big|A_{\phi(n)}\alpha_k(x) + A_{\phi(n)} f_k(x) + A_{\phi(n)}\beta_k(x) \\ &\qquad - A_{-\phi(n)}\alpha_k(x) - A_{-\phi(n)} f_k(x) - A_{-\phi(n)}\beta_k(x)\big| \\ &\quad \ge \big|A_{\phi(n)} f_k(x) - A_{-\phi(n)} f_k(x)\big| \\ &\qquad - \big|A_{\phi(n)}\alpha_k(x)\big| - \big|A_{\phi(n)}\beta_k(x)\big| - \big|A_{-\phi(n)}\alpha_k(x)\big| - \big|A_{-\phi(n)}\beta_k(x)\big|. \end{aligned}$$

We will show that (for fixed $(\ell, n) \in R_k$ and $x \in B_k^+$ at height ℓ) the first term is the dominant term, and the others are comparatively small.

The forward average, $A_{\phi(n)} f_k$ is $\frac{-1}{10^k}$ since because of our restriction on (ℓ, n) we only see negative terms. The backward average is

$$\frac{1}{10^k}\frac{(\phi(n)-\ell)-\ell}{\phi(n)} = \frac{1}{10^k}\frac{\phi(n)-2\ell}{\phi(n)} = \frac{1}{10^k}\left(1-\frac{2\ell}{\phi(n)}\right).$$

Thus (for fixed $(\ell, n) \in R_k$ and $x \in B_k^+$ at height ℓ) we see that

$$A_{-\phi(n)} f_k(x) \geq \frac{1}{10^k}\left(1-\frac{2\ell}{\phi(n)}\right).$$

Since for $(\ell, n) \in R_k$ we have $\phi(n) > 2\ell$ we see that

$$\left|A_{\phi(n)} f_k(x) - A_{-\phi(n)} f_k(x)\right| \geq \frac{1}{10^k}\left|-1-\left(1-\frac{2\ell}{\phi(n)}\right)\right| \geq \frac{1}{10^k}.$$

Now we need estimates on the four "error" terms. We have

$$A_{\phi(n)}\alpha_k(x) \leq \frac{1}{\phi(n)}\sum_{j=1}^{k-1}\frac{8\phi(n_j)}{10^j} \leq \frac{1}{2\phi(n_k)}\sum_{j=1}^{k-1}\frac{8\phi(n_j)}{10^j}$$

and this is less than $\frac{1}{10}\frac{1}{10^k}$ by (P1).

Clearly $A_{-\phi(n)}\alpha_k(x)$ will satisfy the same estimate.

We also have $A_{\phi(n)}\beta_k(x) \leq \frac{1}{n}\sum_{j=k+1}^{\infty}\frac{n}{10^j} \leq \frac{1}{9}\frac{1}{10^k}$, and the same for $A_{-\phi(n)}\beta_k$. Thus

$$\left|A_{\phi(n)} f(x) - A_{-\phi(n)} f(x)\right| \geq \frac{1}{10^k}\left(1-2\frac{1}{10}-2\frac{1}{9}\right) \geq \frac{1}{2}\frac{1}{10^k}.$$

Thus

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Psi(|A_{\phi(n)} f(x) - A_{-\phi(n)} f(x)|)\\
&\geq \sum_{\{n:(\ell,n)\in R_k\}}\Psi\left(\frac{1}{2}\frac{1}{10^k}\right)\\
&\geq \Psi\left(\frac{1}{2}\frac{1}{10^k}\right)\#\{n:(\ell,n)\in R_k\}\\
&= \Psi\left(\frac{1}{2}\frac{1}{10^k}\right)\times\left(\phi^{-1}(4\phi(n_k))-\phi^{-1}(2\phi(n_k))\right)\\
&> k.
\end{aligned}$$

Thus for each $x \in B_k^+$ we have $\mathcal{S}_\Psi f(x) \geq k$. This estimate also will hold on the set

$B_k^- = \{x : x$ is ℓ steps below the center of the kth tower, for some $\ell \in R_k\}$, and thus also on $B_k = B_k^+ \cup B_k^-$, a set of measure $(1-\frac{1}{k})\frac{2\phi(n_k)}{16\phi(n_k)} \geq \frac{1}{10}$ if $k > 5$.

Since k is arbitrary we see that $\mathcal{S}_\Psi f(x) = \infty$ on a set of size at least $\frac{1}{10}$.

We complete the proof as follows. Let L_k (M_j) denote the Lth $(M$th) rung in the kth (jth) tower. If the levels in the distinct towers were (probabilistically) independent, i.e., $m(L_k \cap M_j) = m(L_k)m(M_j)$, then by the second Borel–Cantelli Lemma, $m\{x : x \in B_k \text{ infinitely often }\} = 1$. But in fact such towers may be constructed; see [10], p. 32, especially exercise 166 and the development of that exercise. The exercise states that a tower may be constructed with the levels independent of any given partition of the space. We apply that by considering the partition given by the common refinement of the first j towers, as we construct the $(j+1)^{st}$ tower. □

Proof of Theorem 1.4. Fix the natural number subsequence (n_k), and let b_j denote a to-be-determined nonnegative, nonsummable real sequence. We may suppose $b_j = 0$ unless $j = n_k$ for some k. From the Kakutani–Petersen result (Theorem 1.1) there is an associated $f \in L^\infty$ with $\int f = 0$ so that $\sup_L \left| \sum_{k=1}^{L} b_k A_k f(x) \right| = \infty$ a.e.. By the sparseness of b_j, the only terms that appear in the above sum are those that correspond to averages of length n_k, so we just set $B_k = b_{n_k}$ and have that

$$\sup_L \left| \sum_{k=1}^{L} B_k \frac{S_{n_k} f(x)}{n_k} \right| = \infty \text{ for a.e. } x.$$

Now write $B_k = \alpha_k \beta_k$ where $\sum_{k=1}^{\infty} \beta_k^q < \infty$ for all $q > 1$. Then we have

$$\begin{aligned}
\sup_L \left| \sum_{k=1}^{L} B_k \frac{S_{n_k} f(x)}{n_k} \right| &= \sup_L \left| \sum_{k=1}^{L} \beta_k \alpha_k \frac{S_{n_k} f(x)}{n_k} \right| \\
&\leq \sup_L \left(\sum_{k=1}^{L} \beta_k^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{L} \left| \alpha_k \frac{S_{n_k} f(x)}{n_k} \right|^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{k=1}^{\infty} \beta_k^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{\infty} \left| \alpha_k \frac{S_{n_k} f(x)}{n_k} \right|^p \right)^{\frac{1}{p}} \\
&\leq C \left(\sum_{k=1}^{\infty} \left| \alpha_k \frac{S_{n_k} f(x)}{n_k} \right|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

We know that for a.e. x the left-hand side is infinite, and consequently the same is true for the right-hand side. As an example, take $\beta_k = \frac{1}{k}$ and $\alpha_k = \frac{1}{\log k}$. □

This result has the advantage of an interesting conclusion with a relatively easy proof, given the Kakutani–Petersen result. Using a more complicated argument, with a tower construction similar to the proof of Theorem 2.1, we can prove the following stronger result when $p = 2$:

Theorem 2.3. *Let (n_k) be any increasing sequence of positive integers. Then there exists an $f \in L^\infty$ with $\int f = 0$ for which*

$$\sum_{k=1}^{\infty} \left| \frac{S_{n_k} f(x)}{n_k \sqrt{k}} \right|^2 = \infty \, a.e..$$

We thank Mátě Wierdl for pointing this out to us.

3. Closing remarks

Theorems 1.4 and 2.3 give quantitative descriptions of how the backwards averages $A_{-n_k} f(x)$, are a better predictor of the forward averages $A_{n_k} f(x)$ than the eventual limit $\int f$, at least when (n_k) is rapidly increasing. Theorem 1.2 shows that no advantage is gained in the case of slowly increasing (n_k). It would be interesting to have a qualitative explanation for this.

Also, while the Kakutani–Petersen result is a negative statement about speed of convergence for the averages $A_j f(x)$, the averages do not oscillate very much. For example, the following may be found found in [6].

Let g_n be a sequence of L^p functions, say. For a sequence $n_1 < n_2 < \dots$ of natural numbers define the transformation $O = O_{n_k}$ by

$$O(x) = \left(\sum_{k=1}^{\infty} \sup_{n_k \le n < n_{k+1}} |g_n(x) - g_{n_k}(x)|^2 \right)^{\frac{1}{2}} . \tag{3.1}$$

This is known as the *oscillation operator*, and for a given x, the finiteness of $O(x)$ (for all subsequences (n_j) implies, for example, the convergence of $g_n(x)$ $(n \to \infty)$. On the other hand, it is easy to construct examples of sequences of functions $g_n(x)$ which converge to 0 a.e., yet for which there is a subsequence (n_k) for which $O_{n_k}(x)$ is infinite a.e..

However, in the usual ergodic theory setting there is the following positive result:

Theorem 3.1 ([6])**.** *Let T be any measure-preserving transformation on any probability space $(X, \mathcal{B}, m)$. For $f \in L^p$, set $g_n(x) = A_n f(x)$ in* (3.1)*, and write $O(x) = Of(x)$. Then the map $f \to Of(x)$ is weak-type* $(1,1)$*, type* (p,p) *for $1 < p < \infty$, and maps L^∞ to* BMO.

In particular the oscillation is finite a.e., for *every* $f \in L^\infty$. Thus, while the rate of convergence is slow, the aim is fairly true.

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