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Globular realization and cubical underlying homotopy type of time flow of process algebra

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Abstract. We construct a small realization as flow of every precubical set (modeling for example a process algebra). The realization is small in the sense that the construction does not make use of any cofibrant replacement functor and of any transfinite construction. In particular, if the precubical set is finite, then the corresponding flow has a finite globular decomposition. Two applications are given. The first one presents a realization functor from precubical sets to globular complexes which is characterized up to a natural S-homotopy. The second one proves that, for such flows, the underlying homotopy type is naturally isomorphic to the homotopy type of the standard cubical complex associated with the precubical set.

Contents

1. Introduction

1.1. Presentation of the results. Various topological models of concurrency [Gou03] have been introduced so far. Local pospaces [FGR98] are

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topological spaces equipped with a local partial ordering representing a time flow. D-spaces [Gra03] are topological spaces equipped with a family of continuous paths playing the role of execution paths. A d-space is not necessarily locally partially ordered but their category is complete and cocomplete. This is an advantage of this model on the one of local pospaces. A close framework is the full subcategory of d-spaces which are colimit-generated by a small full subcategory of cubes [FR07]. The interest of the latter category is that it is locally presentable, and that it is therefore possible to construct directed coverings using strict factorization system techniques. Streams [Kri07] are locally preordered topological spaces. The category of streams is also complete and cocomplete. Every d-space and every local pospace can be viewed as a stream. Finally, the globular complexes [GG03] [Gau05a] are topological spaces equipped with a globular decomposition which is the directed analogue of the cellular decomposition of a CW-complex. The globular complexes can be viewed as a subcategory of the categories of local po-spaces, of d-spaces, and of streams. Nevertheless, the category of globular complexes is big enough to contain all examples coming from concurrency [GG03]. All these models start from a topological space representing the underlying state space of the concurrent system. And an additional structure on this topological space models time irreversibility.

In the setting of flows introduced in [Gau03], the nonconstant execution paths are viewed as objects themselves, not as paths of an underlying topological space. The topology of the path space models concurrency and nonconstant execution paths can be composed. So a flow is, by definition, a small category without identity maps enriched over compactly generated topological spaces. The main motivation for introducing this category is the study of the branching and merging homology theories [Gau05b]. Indeed, they impose the functoriality of the mapping associating an object with its set of nonconstant execution paths[1] and also the possibility of composing cubes.[2] None of the other topological models of concurrent systems introduced so far (d-space, local pospace, stream, d-space colimit-generated by cubes) has the first feature since the associated categories contain too many morphisms. More precisely, they contain morphisms contracting nonconstant execution paths. Of course, it is possible to remove the "contracting morphisms" from the categories of d-spaces, local pospaces, streams and d-spaces colimit-generated by cubes. But these models then lose all their interesting properties. These homology theories are expected to be important in the study of higher dimensional bisimulation between concurrent processes by algebraic invariants. Indeed, all these notions are related to the structure of the branching and merging areas of execution paths of a time flow.

[1] See [Gau03] Section 20 for further explanations.

[2] See the introduction and especially Figure 3 of [Gau01] for further explanations.

It is possible to realize every process algebra [WN95] as a precubical set, and as a flow [Gau07c] using a realization functor $|-|_{\mathrm{flow}}$ from precubical sets to flows. This realization functor is complicated to handle since its construction requires the use of the cofibrant replacement functor of the category of flows which is a transfinite construction of length $2^{\aleph_0}$ ([Gau03] Proposition 11.5). The main result of the paper is

Theorem (Theorem 4.2.4 and Corollary 4.2.5)**.** *There exists a small realization functor* $\mathrm{gl}(-)$ *from precubical sets to flows which is colimit-preserving. It is small in the sense that the values on cubes are given by an ordinary nontransfinite induction. It is a realization functor in the sense that it is equivalent to* $|-|_{\mathrm{flow}}$ *up to a natural S-homotopy equivalence of flows.*

Two applications of this result are given in this paper. Other applications will be given in future papers.

The lack of a real underlying topological space in the setting of flows makes some situations very difficult to treat. The first application is:

Theorem (Theorem 5.4.3)**.** *The realization functor* $|-|_{\mathrm{flow}} : \square^{\mathrm{op}}\mathbf{Set} \to \mathbf{flow}$ *from the category of precubical sets to that of flows defined in* [Gau07c] *factors up to a natural S-homotopy equivalence as a composite*

$$\square^{\mathrm{op}}\mathbf{Set} \xrightarrow{\mathrm{gl}^{\mathrm{top}}} \mathbf{glTop} \xrightarrow{\mathrm{cat}} \mathbf{flow}$$

where $\mathbf{glTop}$ *is the category of globular complexes and where* $\mathrm{cat} : \mathbf{glTop} \to \mathbf{flow}$ *is the realization functor from globular complexes to flows defined in* [Gau05a]*. The functor* $\mathrm{gl}^{\mathrm{top}}$ *is unique up to a natural S-homotopy equivalence of globular complexes.*

A notion of underlying homotopy type of flow does exist anyway. The underlying state space of a flow exists, and is unique up to homotopy, not up to homeomorphism [Gau05a]. The fundamental tool to carry out the construction is also the notion of globular complex. This definition enabled us to prove the invariance of the underlying homotopy type of a flow by refinement of observation in [Gau06b]. As a second application of the main result of the paper, or rather, as an application of the first application, the following theorem proposes a simplification of the construction of the underlying homotopy type functor:

Theorem (Theorem 6.2.1)**.** *Let K be a precubical set. The underlying homotopy type of the flow $|K|_{\mathrm{flow}}$ associated with the precubical set K is naturally isomorphic to the homotopy type of the standard cubical complex $|K|_{\mathrm{space}}$ associated with K: i.e., the functor $|-|_{\mathrm{space}}$ is the unique colimit-preserving functor from precubical sets to topological spaces associating the n-cube with the topological n-cube* $[0,1]^n$ *for all* $n \geqslant 0$.

This paper can be read as a sequel of the papers [Gau05a] and [Gau06b] which study the underlying homotopy type of a flow. Indeed, several results

TABLE 1. Overview of the functors of the paper:

$$
\begin{tikzcd}
\square^{\mathrm{op}}\mathbf{Set} \arrow[r, "\mathrm{gl}^{\mathrm{top}}(-)"] \arrow[d, "|-|_{\mathrm{space}}"'] \arrow[rr, bend left=20, "\mathrm{gl}(-)"] \arrow[rr, bend left=40, "|-|_{\mathrm{flow}}"] & \mathbf{glTop} \arrow[r, "\mathrm{cat}"] \arrow[dl, "|-|"] \arrow[dd, "\gamma_{\mathbf{glTop}}"] & \mathbf{flow} \arrow[dd, "\gamma_{\mathbf{flow}}"'] \arrow[ddll, bend left=70, "\Omega=|-|\circ\overline{\mathrm{cat}}^{-1}\circ\gamma_{\mathbf{flow}}"] \\
\mathbf{top} \arrow[d, "\gamma_{\mathbf{top}}"'] & & \\
\mathbf{Ho}(\mathbf{top}) & \mathbf{glTop}[\mathcal{SH}^{-1}] \cong \mathbf{glTop}/{\sim_S} \arrow[l, "|-|"'] \arrow[r, bend left, "\overline{\mathrm{cat}}"] \arrow[r, phantom, "\simeq"] & \mathbf{Ho}(\mathbf{flow}) \arrow[l, bend left, "\overline{\mathrm{cat}}^{-1}"]
\end{tikzcd}
$$

$\gamma_{\mathbf{top}}, \gamma_{\mathbf{glTop}}, \gamma_{\mathbf{flow}}$	Canonical localization functors
$\vert - \vert_{\mathrm{space}}$	Cubical complex associated with a precubical set
$\vert - \vert_{\mathrm{flow}}$	Realization functor from precubical sets to flows
$\vert - \vert$	Underlying topological space functor
$\mathrm{gl}(-)$	Small realization functor from precubical sets to flows
$\mathrm{gl}^{\mathrm{top}}(-)$	Small realization functor from precubical sets to globular complexes
cat and $\overline{\mathrm{cat}}^{-1}$	Equivalence between globular complexes and flows
Ω	Underlying homotopy type functor

of [Gau05a] and [Gau06b] are used in this work. It can also be read as a continuation of [Gau07c] which initializes the study of flows modeling process algebras. This work is in fact a preparatory work for the study of process algebras up to homotopy.

1.2. Outline of the paper. Section 2 is devoted to the preparatory proofs of some facts about cocubical objects in a simplicial model category. The main Theorem 2.3.3 and Theorem 2.3.4 are used in the proofs of Theorem 4.2.4, of Corollary 4.2.5 and of Theorem 6.2.1. Section 3 constructs the simplicial structure of the model category of flows. This structure is necessary for the application of Theorem 2.3.3 and Theorem 2.3.4 in the proofs of Theorem 4.2.4 and Corollary 4.2.5. This result was not yet available in a published work. Section 4 constructs the small realization functor from precubical sets to flows. Section 5 describes the first application, and Section 6 the second application of the main result of the paper.

1.3. Prerequisites and notations. It is required some familiarity with model category techniques [Hov99] [Hir03], with category theory [ML98] [Bor94] and with simplicial techniques [GJ99]. In a locally small category $\mathcal{C}$, the set of objects from X to Y is denoted by $\mathcal{C}(X, Y)$. The notation $\simeq$ means *weak equivalence* or *equivalence of categories*, the notation $\cong$ means *isomorphism*, the notation $\hookrightarrow$ means *cofibration* and the notation $\twoheadrightarrow$ means *fibration*. Let $\mathcal{C}$ be a cocomplete category. The class of morphisms of $\mathcal{C}$ that are transfinite compositions of pushouts of elements of a set of morphisms K is denoted by $\mathbf{cell}(K)$. An element of $\mathbf{cell}(K)$ is called a *relative K-cell complex*. The category of sets is denoted by $\mathbf{Set}$. The cofibrant replacement functor is denoted by $(-)^{\mathrm{cof}}$. The function complex of a simplicial model category is denoted by $\mathrm{Map}(-, -)$. The initial object is denoted by $\varnothing$. The terminal object is denoted by $\mathbf{1}$. In general, the category of functors from a category $\mathcal{B}$ to a category $\mathcal{M}$ is denoted by $\mathcal{M}^{\mathcal{B}}$. Note that the category $\mathcal{M}^{\mathcal{B}}$ is locally small if and only if the category $\mathcal{B}$ is essentially small [FS95]. The category of simplicial sets is denoted by $\Delta^{\mathrm{op}}\mathbf{Set}$. Δ is the standard category of simplices: the objects are the posets $[n] = \{0 < 1 < \cdots < n\}$ and the maps are the nondecreasing maps (beware of the fact that the notation $[n]$ has a second meaning in this paper). $\Delta[n] = \Delta(-, [n])$ is the standard n-simplex.

2. About cocubical objects in a simplicial model category

2.1. Precubical set. A *precubical set* K consists of a family $(K_n)_{n\geqslant 0}$ of sets and of set maps $\partial_i^\alpha : K_n \to K_{n-1}$ with $n \geqslant 1$, $1 \leqslant i \leqslant n$ and $\alpha \in \{0, 1\}$ satisfying the cubical relations $\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha$ for any $\alpha, \beta \in \{0, 1\}$ and for $i < j$ [BH81]. An element of K_n is called a *n-cube*.

A good reference for presheaves is [MLM94]. A precubical set can be viewed as a presheaf over a small category denoted by $\square$ [3] with set of objects $\{[n], n \in \mathbb{N}\}$, generated by the morphisms $\delta_i^\alpha : [n-1] \to [n]$ for $1 \leqslant i \leqslant n$ and $\alpha \in \{0, 1\}$ and satisfying the cocubical relations $\delta_j^\beta \circ \delta_i^\alpha = \delta_i^\alpha \circ \delta_{j-1}^\beta$ for $i < j$ and for all $(\alpha, \beta) \in \{0, 1\}^2$. With the conventions $[0] = \{()\}$, $[n] = \{0, 1\}^n$ for $n \geqslant 1$ and $\{0, 1\}^0 = \{()\}$, the small category $\square$ is the subcategory of the category of sets generated by the set maps $\delta_i^\alpha : [n-1] \to [n]$ for $1 \leqslant i \leqslant n$ and $\alpha \in \{0, 1\}$ defined by

$$\delta_i^\alpha(\epsilon_1, \ldots, \epsilon_{n-1}) = (\epsilon_1, \ldots, \epsilon_{i-1}, \alpha, \epsilon_i, \ldots, \epsilon_{n-1}).$$

The category of precubical sets is denoted by $\square^{\mathrm{op}}\mathbf{Set}$.

[3] All the facts about the small category $\square$ recalled here are used later in the paper.

Let $\square[n] := \square(-,[n])$. This defines a functor, also denoted by $\square$, from $\square$ to $\square^{\mathrm{op}}\mathbf{Set}$. By Yoneda's lemma, one has the natural bijection of sets

$$\square^{\mathrm{op}}\mathbf{Set}(\square[n], K) \cong K_n$$

for every precubical set K. The boundary of $\square[n]$ is the precubical set denoted by $\partial\square[n]$ defined by removing the interior of $\square[n]$: $(\partial\square[n])_k := (\square[n])_k$ for $k < n$ and $(\partial\square[n])_k = \varnothing$ for $k \geqslant n$. In particular, one has $\partial\square[0] = \varnothing$.

Let K be a precubical set. Let $K_{\leqslant n}$ denote the precubical set obtained from K by keeping the p-dimensional cubes of K only for $p \leqslant n$. In particular, $K_{\leqslant 0} = K_0$. Let $\square_n \subset \square$ be the full subcategory of $\square$ whose set of objects is $\{[k], k \leqslant n\}$. A presheaf over $\square_n$ is called a *n-dimensional precubical set*. The category $\square_n^{\mathrm{op}}\mathbf{Set}$ will be identified with the full subcategory of $\square^{\mathrm{op}}\mathbf{Set}$ of precubical sets K such that the inclusion $K_{\leqslant n} \subset K$ is an isomorphism of $\square^{\mathrm{op}}\mathbf{Set}$.

Let K be a precubical set. The *category $\square{\downarrow}K$ of cubes of K* is the small category defined by the pullback of categories

$$\begin{tikzcd} \square{\downarrow}K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \square^{\mathrm{op}}\mathbf{Set}{\downarrow}K \arrow[d] \\ \square \arrow[r] & \square^{\mathrm{op}}\mathbf{Set}. \end{tikzcd}$$

In other terms, an object of $\square{\downarrow}K$ is a morphism $\square[m] \to K$ and a morphism of $\square{\downarrow}K$ is a commutative diagram

$$\begin{tikzcd} \square[m] \arrow[rr] \arrow[dr] & & \square[n] \arrow[dl] \\ & K. & \end{tikzcd}$$

2.2. The category of all small diagrams over a cocomplete category. Let $\mathcal{DM}$ be the category of all small diagrams of objects of a cocomplete category $\mathcal{M}$. The objects are the functors $D : \mathcal{B} \to \mathcal{M}$ where $\mathcal{B}$ is a small category. A morphism from a diagram $D : \mathcal{B} \to \mathcal{M}$ to a diagram $E : \mathcal{C} \to \mathcal{M}$ is a functor $\phi : \mathcal{B} \to \mathcal{C}$ together with a natural transformation $\mu : D \to E \circ \phi$.

Proposition 2.2.1. *Let $\mathcal{M}$ be a cocomplete category. The colimit construction $D \mapsto \varinjlim D$ induces a functor from $\mathcal{DM}$ to $\mathcal{M}$.*

Proof. A morphism of diagrams $(\phi, \mu) : D \to E$ gives rise to a morphism $D \to E \circ \phi$ in $\mathcal{M}^{\mathcal{C}}$. For every object W of $\mathcal{M}$, one has the natural set map (where $\mathrm{Diag}_{\mathcal{B}}$ is the constant diagram functor over $\mathcal{B}$ and where $\mathrm{Diag}_{\mathcal{C}}$ is the constant diagram functor over $\mathcal{C}$):

$$\begin{aligned}
&\mathcal{M}(\varinjlim E, W) \\
&\cong \mathcal{M}^{\mathcal{C}}(E, \mathrm{Diag}_{\mathcal{C}}(W)) && \text{by adjunction} \\
&\longrightarrow \mathcal{M}^{\mathcal{B}}(E \circ \phi, \mathrm{Diag}_{\mathcal{C}}(W) \circ \phi) \\
&= \mathcal{M}^{\mathcal{B}}(E \circ \phi, \mathrm{Diag}_{\mathcal{B}}(W)) && \text{since } \mathrm{Diag}_{\mathcal{C}}(W) \circ \phi = \mathrm{Diag}_{\mathcal{B}}(W) \\
&\longrightarrow \mathcal{M}^{\mathcal{B}}(D, \mathrm{Diag}_{\mathcal{B}}(W)) \\
&\cong \mathcal{M}(\varinjlim D, W) && \text{by adjunction.}
\end{aligned}$$

One obtains a map $\varinjlim D \longrightarrow \varinjlim E$ by setting $W = \varinjlim E$. So the colimit construction induces a functor from $\mathcal{DM}$ to $\mathcal{M}$. $\square$

2.3. Cocubical objects in a simplicial model category. Let us consider a simplicial model category $\mathcal{M}$. A *cocubical object* (resp. of dimension $n \geqslant 0$) is a functor from $\square$ (resp. $\square_n$) to $\mathcal{M}$.

Let X be a cocubical object of $\mathcal{M}$. Let $\widetilde{X}_K$ be the functor from the category of cubes $\square{\downarrow}K$ of a precubical set K to $\mathcal{M}$ defined on objects by

$$\widetilde{X}_K(\square[n] \to K) = X([n])$$

and on morphisms by

$$\widetilde{X}_K \begin{pmatrix} \square[m] & \xrightarrow{\square[\delta]} & \square[n] \\ \downarrow & & \downarrow \\ K & = & K \end{pmatrix} = X(\delta).$$

The mapping $X \mapsto \widetilde{X}$ induces a functor from $\mathcal{M}^{\square}$ to $\mathcal{DM}^{\square^{\mathrm{op}}\mathbf{Set}}$. Let $\widehat{(-)}$ be the composite functor

$$\widehat{(-)} : \mathcal{M}^{\square} \xrightarrow{\widetilde{(-)}} \mathcal{DM}^{\square^{\mathrm{op}}\mathbf{Set}} \xrightarrow{\varinjlim} \mathcal{M}^{\square^{\mathrm{op}}\mathbf{Set}}$$

in which the right-hand functor is the functor of Proposition 2.2.1. So one has

$$\boxed{\widehat{X}(K) = \varinjlim_{\square[n] \to K} X([n]) = \varinjlim_{\square\downarrow K} \widetilde{X}_K.}$$

The category $\square$ has a structure of a direct Reedy category with the degree function $d([n]) = n$ for all $n \geqslant 0$. Let us equip the category $\mathcal{M}^{\square}$ of cocubical objects of $\mathcal{M}$ with its Reedy model category structure ([Hir03] Theorem 15.3.4). The following proposition describes the Reedy cofibrations and the Reedy fibrations of cocubical objects.

Proposition 2.3.1. *Let $\mathcal{M}$ be a model category. Then:*

(1) *The Reedy fibrations of cocubical objects are the objectwise fibrations.*
(2) *A cocubical object X of $\mathcal{M}$ is Reedy fibrant if and only if for every $n \geqslant 0$, $X([n])$ is fibrant.*

(3) *A cocubical object X is Reedy cofibrant if and only if for any $n \geqslant 0$, the map $\widehat{X}(\partial\square[n] \subset \square[n])$ is a cofibration of $\mathcal{M}$.*

Proof. A Reedy fibration of $\mathcal{M}^{\square}$ is by definition a map of cocubical objects $X \to Y$ such that for every object $[n]$ of $\square$, the map

$$X([n]) \to M_{[n]}X \times_{M_{[n]}Y} Y([n])$$

is a fibration of $\mathcal{M}$ where $M_{[n]}X$ (resp. $M_{[n]}Y$) is the matching object of X (resp. of Y) at $[n]$. These matching objects are equal to the terminal object $\mathbf{1}$ of $\mathcal{M}$ since the Reedy category $\square$ is direct. So the Reedy fibrations are the objectwise fibrations. Hence the first assertion.

A cocubical object X is therefore Reedy fibrant if and only if for every object $[n]$ of $\square$, the map $X([n]) \to \mathbf{1}$ is a fibration. Hence the second assertion.

A Reedy cofibration of $\mathcal{M}^{\square}$ is by definition a map of cocubical objects $X \to Y$ such that for every object $[n]$ of $\square$, the map

$$L_{[n]}Y \sqcup_{L_{[n]}X} X([n]) \to Y([n])$$

is a cofibration of $\mathcal{M}$ where $L_{[n]}X$ (resp. $L_{[n]}Y$) is the matching object of X (resp. of Y) at $[n]$. The latching category at $\alpha = [n]$, usually denoted by $\partial(\square{\downarrow}\alpha)$, is by definition the full subcategory of the category $\square{\downarrow}\alpha$ of cubes of $\square[n]$ containing the maps $\square[m] \to \square[n]$ different from the identity of $\square[n]$. And the latching object of X at α is by definition

$$L_{[n]}X \cong \varinjlim_{\partial(\square\downarrow\alpha)} X(\square[m]) \cong \widehat{X}(\partial\square[n]).$$

Hence the third assertion. $\square$

Proposition 2.3.2. *The composite functor*

$$\widehat{(-)} : \mathcal{M}^{\square} \xrightarrow{\widetilde{(-)}} \mathcal{D}\mathcal{M}^{\square^{\mathrm{op}}\mathbf{Set}} \xrightarrow{\varinjlim} \mathcal{M}^{\square^{\mathrm{op}}\mathbf{Set}}$$

induces an equivalence of categories

$$\mathcal{M}^{\square} \simeq \mathcal{M}_{\varinjlim}^{\square^{\mathrm{op}}\mathbf{Set}}$$

where $\mathcal{M}_{\varinjlim}^{\square^{\mathrm{op}}\mathbf{Set}}$ is the category of colimit-preserving functors from $\square^{\mathrm{op}}\mathbf{Set}$ to $\mathcal{M}$.

Proof. Consider the functor $F : \mathcal{M}_{\varinjlim}^{\square^{\mathrm{op}}\mathbf{Set}} \to \mathcal{M}^{\square}$ defined by $F(Z) = Z \circ \square$. Then for every cocubical object X of $\mathcal{M}$, one has the natural isomorphisms of $\mathcal{M}$

$$F(\widehat{X})([n]) \cong \widehat{X}(\square[n]) \cong \varinjlim_{\square[m]\to\square[n]} X([m]) \cong X([n])$$

and, since Z is colimit-preserving,

$$\widehat{F(Z)}(K) \cong \varinjlim_{\square[n]\to K} Z(\square[n]) \cong Z(K). \qquad \square$$

By [Hir03] Theorem 15.3.4 again, the Reedy model structure of $\mathcal{M}^{\square}$ is simplicial with the tensor product and cotensor product of a cocubical object X by a simplicial set K defined by the composites $X \otimes K := (- \otimes K) \circ X$ and $X^K := (-)^K \circ X$.

Theorem 2.3.3. *Let $\mathcal{M}$ be a simplicial model category. Let I, X and Y be three cocubical objects of $\mathcal{M}$. Let $p_X : X \to I$ and $p_Y : Y \to I$ be two objectwise trivial fibrations of cocubical objects of $\mathcal{M}$. Assume that for every $n \geqslant 0$, the maps $\widehat{X}(\partial\square[n]) \to \widehat{X}(\square[n])$ and $\widehat{Y}(\partial\square[n]) \to \widehat{Y}(\square[n])$ are cofibrations of $\mathcal{M}$ and $I([n])$ is fibrant. Then:*

- *There exists a natural transformation from $\widehat{X}$ to $\widehat{Y}$ over $\widehat{I}$, i.e., a map $\widehat{X} \to \widehat{Y}$ such that the following diagram commutes:*

$$\begin{tikzcd} \widehat{X} \arrow[rr] \arrow[dr, "\widehat{p_X}"'] & & \widehat{Y} \arrow[dl, "\widehat{p_Y}"] \\ & \widehat{I}. & \end{tikzcd}$$

- *Take two natural transformations $\widehat{\mu} : \widehat{X} \to \widehat{Y}$ and $\widehat{\nu} : \widehat{X} \to \widehat{Y}$ over $\widehat{I}$. Then there exists a simplicial homotopy between $\widehat{\mu}(K)$ and $\widehat{\nu}(K)$ which is natural with respect to K.*
- *For any natural transformation $\widehat{\mu} : \widehat{X} \to \widehat{Y}$ over $\widehat{I}$ and any natural transformation $\widehat{\nu} : \widehat{Y} \to \widehat{X}$ over $\widehat{I}$, the map $\widehat{\mu}(K) \circ \widehat{\nu}(K)$ is naturally simplicially homotopy equivalent to $\mathrm{Id}_{\widehat{Y}(K)}$ and the map $\widehat{\nu}(K) \circ \widehat{\mu}(K)$ is naturally simplicially homotopy equivalent to $\mathrm{Id}_{\widehat{X}(K)}$, natural meaning natural with respect to K.*

Proof. The cocubical object X is Reedy cofibrant by Proposition 2.3.1. The map p_Y is a trivial Reedy fibration by Proposition 2.3.1 as well. Let $A \to B$ be a cofibration of simplicial sets. Consider a commutative diagram of simplicial sets:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Map}(X,Y) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "k"] & \mathrm{Map}(X,I) \end{tikzcd}$$

where the map $(p_Y)_* : \mathrm{Map}(X,Y) \to \mathrm{Map}(X,I)$ is induced by the composition by p_Y. By adjunction, the lift k exists if and only if the lift k' exists in the commutative diagram of cocubical objects of $\mathcal{M}$

$$\begin{tikzcd} X \otimes A \arrow[r] \arrow[d] & Y \arrow[d] \\ X \otimes B \arrow[r] \arrow[ur, dashed, "k'"] & I. \end{tikzcd}$$

The map $X \otimes A \to X \otimes B$ is the pushout product of the Reedy cofibration $\varnothing \to X$ by the cofibration of simplicial sets $A \to B$, and therefore is a

Reedy cofibration. Hence the existence of k' and k. So the simplicial map $(p_Y)_* : \mathrm{Map}(X,Y) \to \mathrm{Map}(X,I)$ is a trivial fibration of simplicial sets. Let F be the fibre over p_X of the simplicial map $(p_Y)_* : \mathrm{Map}(X,Y) \to \mathrm{Map}(X,I)$ defined by the pullback diagram of simplicial sets:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(X,Y) \arrow[d, "\simeq"] \\ \Delta[0] \arrow[r, "p_X"] & \mathrm{Map}(X,I). \end{tikzcd}$$

Since the pullback of a trivial fibration is a trivial fibration, the map $F \to \Delta[0]$ is a trivial fibration. The lift ℓ in the commutative diagram of simplicial sets

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, hook] & F \arrow[d, "\simeq"] \\ \Delta[0] \arrow[r, equal] \arrow[ur, dashed, "\ell"] & \Delta[0] \end{tikzcd}$$

gives $\ell(\mathrm{Id}_{[0]}) \in F_0 \subset \mathrm{Map}(X,Y)_0 = \mathcal{M}^{\square}(X,Y)$. By definition, $\ell(\mathrm{Id}_{[0]}) : X \to Y$ is a morphism of cocubical objects over I. Hence a natural transformation $\widehat{\ell(\mathrm{Id}_{[0]})} : \widehat{X} \to \widehat{Y}$ over $\widehat{I}$ and the first assertion.

Take two natural transformations $\widehat{\mu}$ and $\widehat{\nu}$ from $\widehat{X}$ to $\widehat{Y}$ over $\widehat{I}$. By Proposition 2.3.2, one can suppose that they come from two morphisms of cocubical objects μ and ν from X to Y over I. One obtains the commutative diagram of simplicial sets:

$$\begin{tikzcd} \Delta[0] \sqcup \Delta[0] \arrow[r, "(\mu,\nu)"] \arrow[d, hook] & F \arrow[d, "\simeq"] \\ \Delta[1] \arrow[r] \arrow[ur, dashed] & \Delta[0]. \end{tikzcd}$$

Thus, there exists a simplicial path $\Delta[1] \to F \subset \mathrm{Map}(X,Y)$ between μ and ν, i.e., by adjunction a morphism of cocubical objects $H : X \otimes \Delta[1] \to Y$ such that the two natural transformations $X \rightrightarrows X \otimes \Delta[1] \to Y$ are μ and ν, i.e., H is a simplicial homotopy between μ and ν. One obtains a simplicial homotopy $\widetilde{H}_K \in \mathrm{Map}(\widetilde{X}_K, \widetilde{Y}_K)_1$ between $\widetilde{\mu}_K$ and $\widetilde{\nu}_K$.[4] Since there is an isomorphism

$$\varinjlim(\widetilde{X}_K \otimes \Delta[1]) \cong (\varinjlim \widetilde{X}_K) \otimes \Delta[1]$$

because the functor $- \otimes \Delta[1]$ is colimit-preserving, one obtains a simplicial homotopy $\widehat{H}(K) \in \mathrm{Map}(\widehat{X}(K), \widehat{Y}(K))_1$ between $\widehat{\mu}(K)$ and $\widehat{\nu}(K)$ which is natural with respect to K. Hence the second assertion.

[4]We are working here in the simplicial model category of functors from $\square{\downarrow}K$ to $\mathcal{M}$ which is also equipped with a Reedy structure by setting $d(\square[n] \to K) = n$.

The third assertion is a consequence of the second assertion by noticing that $\mathrm{Id}_{\widehat{X}}$ is a natural transformation from $\widehat{X}$ to itself over $\widehat{I}$ and that $\mathrm{Id}_{\widehat{Y}}$ is a natural transformation from $\widehat{Y}$ to itself over $\widehat{I}$. $\square$

Theorem 2.3.4. *Let $\mathcal{M}$ be a simplicial model category. Let $n \geqslant 0$. Let I, X and Y be three cocubical objects of $\mathcal{M}$ of dimension n. Let $p_X : X \to I$ and $p_Y : Y \to I$ be two objectwise trivial fibrations of cocubical objects of $\mathcal{M}$. Assume that for every $0 \leqslant p \leqslant n$, the maps $\widehat{X}(\partial\square[p]) \to \widehat{X}(\square[p])$ and $\widehat{Y}(\partial\square[p]) \to \widehat{Y}(\square[p])$ are cofibrations of $\mathcal{M}$ and $I([p])$ is fibrant. Then:*

- *There exists a natural transformation from $\widehat{X}$ to $\widehat{Y}$ over $\widehat{I}$.*
- *Take two natural transformations $\widehat{\mu} : \widehat{X} \to \widehat{Y}$ and $\widehat{\nu} : \widehat{X} \to \widehat{Y}$ over $\widehat{I}$. Then there exists a simplicial homotopy between $\widehat{\mu}(K)$ and $\widehat{\nu}(K)$ which is natural with respect to K.*
- *For any natural transformation $\widehat{\mu} : \widehat{X} \to \widehat{Y}$ over $\widehat{I}$ and any natural transformation $\widehat{\nu} : \widehat{Y} \to \widehat{X}$ over $\widehat{I}$, the map $\widehat{\mu}(K) \circ \widehat{\nu}(K)$ is naturally simplicially homotopy equivalent to $\mathrm{Id}_{\widehat{Y}(K)}$ and the map $\widehat{\nu}(K)\circ\widehat{\mu}(K)$ is naturally simplicially homotopy equivalent to $\mathrm{Id}_{\widehat{X}(K)}$, natural meaning natural with respect to the precubical set K of dimension n.*

Proof. Use the Reedy structure of $\square_n$ and the Reedy model structure of $\mathcal{M}^{\square_n}$ in the proof of Theorem 2.3.3. $\square$

3. The weak S-homotopy model category of flows is simplicial

The goal of this section is to prove that the weak S-homotopy model category of **flow** is simplicial (Theorem 3.3.15).

3.1. Topological space.

All topological spaces are compactly generated, i.e., are weak Hausdorff k-spaces. Further details about these topological spaces are available in [Bro88] [May99], the appendix of [Lew78] and also the preliminaries of [Gau03]. All compact spaces are Hausdorff. The category of *compactly generated topological spaces* together with the continuous maps is denoted by **top**. The category **top** is equipped with the usual model structure having the weak homotopy equivalences as weak equivalences and having the Serre fibrations as fibrations. This model structure is simplicial and any topological space is fibrant. The homotopy category, i.e., the localization of **top** by the weak homotopy equivalences, is denoted by $\mathbf{Ho}(\mathbf{top})$. The functor $\gamma_{\mathbf{top}} : \mathbf{top} \to \mathbf{Ho}(\mathbf{top})$ is the canonical functor which is the identity on objects. The set of continuous maps $\mathbf{top}(X, Y)$ from X to Y equipped with the Kelleyfication of the compact-open topology is denoted by $\mathbf{TOP}(\mathrm{X}, \mathrm{Y})$. The latter topological space is the internal hom of the cartesian closed category **top**. So one has the natural homeomorphism $\mathbf{TOP}(\mathrm{X} \times \mathrm{Y}, \mathrm{Z}) \cong \mathbf{TOP}(\mathrm{X}, \mathbf{TOP}(\mathrm{Y}, \mathrm{Z}))$. Moreover the covariant functor $\mathbf{TOP}(\mathrm{X}, -) : \mathbf{top} \to \mathbf{top}$ and the contravariant functor $\mathbf{TOP}(-, \mathrm{X}) : \mathbf{top}^{\mathrm{op}} \to \mathbf{top}$ preserve limits [Kel05].

3.2. Flow. A *flow* X is a small category without identity maps enriched over the category of compactly generated topological spaces. The set of objects is denoted by X^0. The space of morphisms from α to β is denoted by $\mathbb{P}_{\alpha,\beta}X$.[5] A morphism of flows $f : X \to Y$ is a set map $f^0 : X^0 \to Y^0$ together with a continuous map $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ preserving the structure. The corresponding category is denoted by **flow**. If for all $\alpha \in X^0$, the space $\mathbb{P}_{\alpha,\alpha}X$ is empty, then X is called a *loopless flow*. The composition law of a flow is denoted by $*$.

Any poset P, and in particular any set viewed as a trivial poset ($x \leqslant y$ if and only if $x = y$), can be viewed as a loopless flow with a nonidentity morphism from α to β if and only if $\alpha < \beta$. This yields a functor from the category of posets with strictly increasing maps to that of flows.

Let Z be a topological space. The flow $\mathrm{Glob}(Z)$ is defined by:

- $\mathrm{Glob}(Z)^0 = \{\widehat{0}, \widehat{1}\}$,
- $\mathbb{P}\mathrm{Glob}(Z) = \mathbb{P}_{\widehat{0},\widehat{1}}\mathrm{Glob}(Z) = Z$,
- $s = \widehat{0}$, $t = \widehat{1}$ and a trivial composition law.

It is called the *globe* of the space Z.

The weak S-homotopy model structure of **flow** is characterized as follows (cf. [Gau03] for further details):

- The weak equivalences are the *weak S-homotopy equivalences*, i.e., the morphisms of flows $f : X \longrightarrow Y$ such that $f^0 : X^0 \longrightarrow Y^0$ is a bijection of sets and such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a weak homotopy equivalence.
- The fibrations are the morphisms of flows $f : X \longrightarrow Y$ such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a Serre fibration.

Note that any flow is fibrant. The homotopy category is denoted $\mathbf{Ho}(\mathbf{flow})$. This model structure is cofibrantly generated. The set of generating cofibrations is the set

$$I^{\mathrm{gl}}_{+} = I^{\mathrm{gl}} \cup \{R : \{0,1\} \longrightarrow \{0\}, C : \varnothing \longrightarrow \{0\}\}$$

with $I^{\mathrm{gl}} = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\}$ where $\mathbf{D}^n$ is the n-dimensional disk and $\mathbf{S}^{n-1}$ the $(n-1)$-dimensional sphere. By convention, the (-1)-dimensional sphere is the empty space. The set of generating trivial cofibrations is

$$J^{\mathrm{gl}} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \subset \mathrm{Glob}(\mathbf{D}^n \times [0,1]), n \geqslant 0\}.$$

[5] Sometimes, an object of a flow is called a state and a morphism a (nonconstant) execution path.

Table 2

Notations of this paper	Notations of [Gau03]
$X \otimes K$	$\lvert K\rvert \boxtimes X$
X^K	$\{\lvert K\rvert, X\}_S$

Let X and U be two flows. Let $\mathbf{FLOW}(X,U)$ be the set $\mathbf{flow}(X,U)$ equipped with the Kelleyfication of the compact-open topology. Let $f,g : X \rightrightarrows U$ be two morphisms of flows. Then a *S-homotopy* is a continuous map $H : [0,1] \to \mathbf{FLOW}(X,U)$ with $H_0 = f$ and $H_1 = g$. This situation is denoted by $f \sim_S g$. The S-homotopy relation defines a congruence on the category **flow**. If there exists a map $f' : U \to X$ with $f \circ f' \sim_S \mathrm{Id}_U$ and $f' \circ f \sim_S \mathrm{Id}_X$, then f is called a *S-homotopy equivalence*.

3.3. Simplicial structure of the model category of flows. Let K be a nonempty connected simplicial set. Let X be a flow. Then one has the isomorphism of topological spaces

$$\mathbf{TOP}(|\mathrm{K}|, \mathbb{P}\mathrm{X}) \cong \bigsqcup_{(\alpha,\beta)\in \mathrm{X}^0\times \mathrm{X}^0} \mathbf{TOP}(|\mathrm{K}|, \mathbb{P}_{\alpha,\beta}\mathrm{X})$$

where the topological space $|K|$ is the geometric realization of the simplicial set K. Note the latter isomorphism is false if K is empty or not connected. The associative composition law $* : \mathbb{P}X \times_{X^0} \mathbb{P}X \to \mathbb{P}X$ gives rise to a continuous map

$$\begin{aligned} * : \mathbf{TOP}(|\mathrm{K}|, \mathbb{P}\mathrm{X}) \times_{\mathrm{X}^0} \mathbf{TOP}(|\mathrm{K}|, \mathbb{P}\mathrm{X}) &\cong \mathbf{TOP}(|\mathrm{K}|, \mathbb{P}\mathrm{X} \times_{\mathrm{X}^0} \mathbb{P}\mathrm{X}) \\ &\to \mathbf{TOP}(|\mathrm{K}|, \mathbb{P}\mathrm{X}). \end{aligned}$$

The homeomorphism

$$\mathbf{TOP}(|\mathrm{K}|, \mathbb{P}\mathrm{X}) \times_{\mathrm{X}^0} \mathbf{TOP}(|\mathrm{K}|, \mathbb{P}\mathrm{X}) \cong \mathbf{TOP}(|\mathrm{K}|, \mathbb{P}\mathrm{X} \times_{\mathrm{X}^0} \mathbb{P}\mathrm{X})$$

holds since the functor $\mathbf{TOP}(|\mathrm{K}|, -)$ is limit-preserving. Hence the following definition:

Definition 3.3.1. Let K be a nonempty connected simplicial set. Let X be an object of **flow**. Let X^K be the flow defined by:

- $(X^K)^0 = X^0$,
- $\mathbb{P}_{\alpha,\beta}(X^K) = \mathbf{TOP}(|\mathrm{K}|, \mathbb{P}_{\alpha,\beta}\mathrm{X})$ for all $(\alpha,\beta) \in X^0 \times X^0$, and
- the above composition law.

Several theorems of [Gau03] are going to be used. Therefore, it is helpful for the reader to give the correspondence between the notations of this paper (left column of Table 2) and of [Gau03] (right column of Table 2) for a flow X and a nonempty connected simplicial set K.

Proposition 3.3.2. *Let K be a nonempty connected simplicial set. The mapping $X \mapsto X^K$ gives rise to an endofunctor of* **flow**. *The functor $(-)^K$ is a right adjoint.*

Proof. Consequence of [Gau03] Theorem 7.8. ☐

Notation 3.3.3. Let K be a nonempty connected simplicial set. Let us denote by $- \otimes K$ the left adjoint of $(-)^K$.

Definition 3.3.4. Let K be a nonempty simplicial set. Let $(K_i)_{i\in I}$ be its set of nonempty connected components. Let:

- $X \otimes K := \bigsqcup_{i\in I} X \otimes K_i$,
- $X^K := \prod_{i\in I} X^{K_i}$.

And let $X \otimes \varnothing = \varnothing$ and $X^{\varnothing} = \mathbf{1}$.

Proposition 3.3.5. *Let K be a simplicial set. The pair of functors $(-\otimes K)$:* **flow** $\leftrightarrows$ **flow** *: $(-)^K$ is a categorical adjunction.*

Proof. If $K = \varnothing$, then one has the isomorphisms

$$\mathbf{flow}(X \otimes \varnothing, Y) \cong \mathbf{flow}(\varnothing, Y) \cong \mathbf{1} \cong \mathbf{flow}(X, Y^{\varnothing}).$$

Now let K be a nonempty simplicial set. Let $(K_i)_{i\in I}$ be its set of nonempty connected components. Then one has

$$\begin{aligned}
\mathbf{flow}(X \otimes K, Y) &\cong \mathbf{flow}(\bigsqcup_{i\in I}(X \otimes K_i), Y) && \text{by definition of } X \otimes K \\
&\cong \prod_{i\in I} \mathbf{flow}(X \otimes K_i, Y) \\
&\cong \prod_{i\in I} \mathbf{flow}(X, Y^{K_i}) && \text{by Proposition 3.3.2} \\
&\cong \mathbf{flow}(X, \prod_{i\in I} Y^{K_i}) \\
&\cong \mathbf{flow}(X, Y^K) && \text{by definition of } X^K.
\end{aligned}$$

Hence the adjunction. ☐

Definition 3.3.6. Let X and Y be two objects of **flow**. Let

$$\operatorname{Map}(X, Y) := \mathbf{flow}(X \otimes \Delta[*], Y).$$

It is called the function complex from X to Y.

Proposition 3.3.7. *Let X and Y be two flows. Then one has the natural isomorphism of simplicial sets*

$$\operatorname{Map}(X, Y) \cong \operatorname{Sing}(\mathbf{FLOW}(X, Y))$$

where Sing *is the singular nerve functor.*

Proof. Let $n \geqslant 0$. Since the topological space $|\Delta[n]|$ is nonempty and connected, one has $\mathrm{Sing}(\mathbf{FLOW}(X,Y))_n = \mathbf{top}(|\Delta[n]|, \mathbf{FLOW}(X,Y)) \cong \mathbf{flow}(X \otimes \Delta[n], Y)$ by [Gau03] Theorem 7.9. $\square$

Proposition 3.3.8. *Let $\mathcal{B}$ be a small category. Let $X : \mathcal{B} \to$ **flow** be a functor. Then one has the natural isomorphisms of simplicial sets*

$$\mathrm{Map}(\varinjlim X_b, Y) \cong \varprojlim \mathrm{Map}(X_b, Y)$$
$$\mathrm{Map}(Y, \varprojlim X_b) \cong \varprojlim \mathrm{Map}(Y, X_b)$$

for any flow Y of **flow**.

Proof. Limits and colimits are calculated pointwise in the category of simplicial sets. Since for every $n \geqslant 0$, the functor $- \otimes \Delta[n]$ is a left adjoint by Proposition 3.3.5, one obtains the following two natural bijections:

$$\begin{aligned} \mathrm{Map}(\varinjlim X_b, Y)_n &\cong \mathbf{flow}((\varinjlim X_b) \otimes \Delta[n], Y) \\ &\cong \mathbf{flow}(\varinjlim (X_b \otimes \Delta[n]), Y) \\ &\cong \varprojlim \mathbf{flow}(X_b \otimes \Delta[n], Y) \\ &\cong \varprojlim \mathrm{Map}(X_b, Y)_n \end{aligned} \tag{1}$$

Here, the first and fourth isomorphisms are by definition of Map, and the second is because $- \otimes \Delta[n]$ is a left adjoint.

$$\begin{aligned} \mathrm{Map}(Y, \varprojlim X_b)_n &\cong \mathbf{flow}(Y \otimes \Delta[n], \varprojlim X_b) \\ &\cong \varprojlim \mathbf{flow}(Y \otimes \Delta[n], X_b) \\ &\cong \varprojlim \mathrm{Map}(Y, X_b)_n. \end{aligned} \tag{2}$$

Here, the first and third isomorphisms are by definition of Map. $\square$

Proposition 3.3.9. *Let K and L be two simplicial sets. Then one has a natural isomorphism of flows $X^{K\times L} = (X^K)^L$ for every flow X of* **flow**.

Proof. If K or L is empty, then $X^{K\times L} = (X^K)^L = \mathbf{1}$ by definition. Suppose now that K and L are both nonempty and connected. The flows $X^{K\times L}$ and $(X^K)^L$ have same set of states X^0. And $\mathbb{P}(X^{K\times L}) \cong \mathbf{TOP}(|\mathrm{K}\times\mathrm{L}|, \mathbb{P}\mathrm{X})$ and $\mathbb{P}((X^K)^L) = \mathbf{TOP}(|\mathrm{L}|, \mathbf{TOP}(|\mathrm{K}|, \mathbb{P}\mathrm{X}))$. Hence the conclusion in this case since **top** is cartesian closed and since there is a homeomorphism $|K \times L| \cong [K| \times |L|$ by [Hov99] Lemma 3.1.8. We treat now the general case where K and L are both nonempty. Let $(K_i)_{i\in I}$ and $(L_j)_{j\in J}$ be the nonempty connected components of K and L respectively. Then $K \times L = \prod_{i\in I}\prod_{j\in J} K_i \times L_j$ and:

$$\begin{aligned} X^{K\times L} &\cong \prod_{i\in I}\prod_{j\in J} X^{K_i\times L_j} \cong \prod_{i\in I}\prod_{j\in J} (X^{K_i})^{L_j} \cong \prod_{j\in J}(\prod_{i\in I} X^{K_i})^{L_j} \cong \prod_{j\in J}(X^K)^{L_j} \\ &\cong (X^K)^L. \end{aligned}$$

Here, the first isomorphism follows since the $K_i \times L_j$'s are nonempty and connected, and the third since the functors $(-)^{L_j}$ are right adjoints. The fourth is by definition of X^K and the fifth by definition of $(-)^L$. □

Proposition 3.3.10. *Let K and L be two simplicial sets. Let X be a flow. Then one has a natural isomorphism of flows $(X \otimes K) \otimes L \cong X \otimes (K \times L)$.*

Proof. Let Y be another flow. Then one has

$$\begin{aligned}\mathbf{flow}((X \otimes K) \otimes L, Y) &\cong \mathbf{flow}(X \otimes K, Y^L) \cong \mathbf{flow}(X, (Y^L)^K) \\ &\cong \mathbf{flow}(X, Y^{L \times K}) \cong \mathbf{flow}(X \otimes (K \times L), Y).\end{aligned}$$

Here, the first, second and fourth isomorphisms are by Proposition 3.3.5 and the third by Proposition 3.3.9. The result now follows by Yoneda's lemma. □

Proposition 3.3.11. *Let K be a simplicial set. Let X and Y be two flows. Then one has a natural isomorphism of simplicial sets*

$$\mathrm{Map}(X \otimes K, Y) \cong \mathrm{Map}(K, \mathrm{Map}(X, Y)).$$

Proof. If $K = \varnothing$, then one has to compare $\mathrm{Map}(X \otimes \varnothing, Y) \cong \mathrm{Map}(\varnothing, Y) \cong \mathbf{1}$ by Proposition 3.3.8 and $\mathrm{Map}(\varnothing, \mathrm{Map}(X, Y)) \cong \mathbf{1}$. So one can suppose the simplicial set K nonempty. By construction of the functor $- \otimes K$ and by Proposition 3.3.8, one can suppose that K is connected as well. Let $n \geqslant 0$. Thus,

$$\begin{aligned}\mathrm{Map}(X \otimes K, Y)_n &\cong \mathbf{flow}(X \otimes (K \times \Delta[n]), Y) \\ &\cong \mathbf{top}(|K \times \Delta[n]|, \mathbf{FLOW}(X, Y)) \\ &\cong \Delta^{\mathrm{op}}\mathbf{Set}(K \times \Delta[n], \mathrm{Map}(X, Y)) \\ &\cong \mathrm{Map}(K, \mathrm{Map}(X, Y))_n.\end{aligned}$$

Here, the first isomorphism is by Proposition 3.3.10, the second by [Gau03] Theorem 7.9, the third by adjunction and by Proposition 3.3.7, and the fourth by definition of Map in $\Delta^{\mathrm{op}}\mathbf{Set}$. □

Proposition 3.3.12. *One has the natural isomorphism of simplicial sets*

$$\mathrm{Map}(X \otimes K, Y) \cong \mathrm{Map}(X, Y^K)$$

for every simplicial set K and every flow X, Y of **flow**.

Proof. One has for any $n \geqslant 0$

$$\begin{aligned}\mathrm{Map}(X \otimes K, Y)_n &\cong \mathbf{flow}(X \otimes K \otimes \Delta[n], Y) \\ &\cong \mathbf{flow}(X \otimes \Delta[n], Y^K) \\ &\cong \mathrm{Map}(X, Y^K)_n.\end{aligned}$$

Here, the first and third isomorphisms are by definition of Map, and the second is by Proposition 3.3.10 and Proposition 3.3.5. □

Lemma 3.3.13. *Let $f : X \longrightarrow Y$ be a morphism of flows. Then the following conditions are equivalent:*

(1) *f is a fibration of flows, that is the continuous map $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a fibration of topological spaces.*
(2) *For any $(\alpha, \beta) \in X^0 \times X^0$, the continuous map $\mathbb{P}f : \mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ is a fibration of topological spaces.*

Proof. A continuous map is a fibration if and only if it satisfies the right lifting property with respect to the inclusion maps $\mathbf{D}^n \longrightarrow \mathbf{D}^n \times [0,1]$. The result comes from the connectedness of both $\mathbf{D}^n$ and $\mathbf{D}^n \times [0,1]$. □

Proposition 3.3.14. *Let $i : A \longrightarrow B$ be a cofibration of flows. Let $p : X \longrightarrow Y$ be a fibration of flows. Then the morphism of simplicial sets*

$$Q(i,p) : \mathrm{Map}(B,X) \longrightarrow \mathrm{Map}(A,X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B,Y)$$

is a fibration of simplicial sets. Moreover if either i or p is trivial, then the fibration $Q(i,p)$ is trivial as well.

Proof. By [GJ99] Proposition II.3.13 p95, it suffices to prove that the morphism of flows

$$X^{\Delta[n]} \longrightarrow X^{\partial\Delta[n]} \times_{Y^{\partial\Delta[n]}} Y^{\Delta[n]}$$

is a fibration (resp. trivial fibration) for any $n \geqslant 0$ as soon as $X \longrightarrow Y$ is a fibration (resp. trivial fibration) of flows. The case of a fibration is the only one treated since the other case is similar.

For $n = 0$, one has to check that

$$X^{\Delta[0]} \longrightarrow \mathbf{1} \times_{\mathbf{1}} Y^{\Delta[0]} \cong Y^{\Delta[0]}$$

is a fibration. Since $\Delta[0]$ is connected, $X^{\Delta[0]} = X$ and $Y^{\Delta[0]} = Y$. So there is nothing to check for $n = 0$.

For $n = 1$, one has to check that

$$X^{\Delta[1]} \longrightarrow X^{\partial\Delta[1]} \times_{Y^{\partial\Delta[1]}} Y^{\Delta[1]}$$

is a fibration. Since $\partial\Delta[1]$ is the discrete two-point simplicial set, then $X^{\partial\Delta[1]} = X \times X$. Since $\Delta[1]$ is connected, one has to check that for any $(\alpha, \beta) \in X^0 \times X^0$, the continuous map

$$\mathbf{TOP}(|\Delta[1]|, \mathbb{P}_{\alpha,\beta}\mathrm{X}) \longrightarrow (\mathbb{P}_{\alpha,\beta}\mathrm{X} \times \mathbb{P}_{\alpha,\beta}\mathrm{X}) \times_{(\mathbb{P}_{\alpha,\beta}\mathrm{Y} \times \mathbb{P}_{\alpha,\beta}\mathrm{Y})} \mathbf{TOP}(|\Delta[1]|, \mathbb{P}_{\alpha,\beta}\mathrm{Y})$$

is a fibration of topological spaces. Using the homeomorphisms

$$\mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\alpha,\beta}X \cong \mathbf{TOP}(\{-1,1\}, \mathbb{P}_{\alpha,\beta}\mathrm{X})$$

and

$$\mathbb{P}_{\alpha,\beta}Y \times \mathbb{P}_{\alpha,\beta}Y \cong \mathbf{TOP}(\{-1,1\}, \mathbb{P}_{\alpha,\beta}\mathrm{Y}),$$

one has to check that the continuous map

$$\mathbf{TOP}(|\Delta[1]|, \mathbb{P}_{\alpha,\beta}\mathrm{X}) \longrightarrow \mathbf{TOP}(\{-1,1\}, \mathbb{P}_{\alpha,\beta}\mathrm{X}) \times_{\mathbf{TOP}(\{-1,1\}, \mathbb{P}_{\alpha,\beta}\mathrm{Y})} \mathbf{TOP}(|\Delta[1]|, \mathbb{P}_{\alpha,\beta}\mathrm{Y})$$

is a fibration of topological spaces. So one has to prove that for any commutative square

$$\begin{tikzcd}
\mathbf{D}^n \times \{0\} \arrow[r] \arrow[d] & \mathbf{TOP}(|\Delta[1]|, \mathbb{P}_{\alpha,\beta}\mathrm{X}) \arrow[d] \\
\mathbf{D}^n \times [0,1] \arrow[r] \arrow[ur, dashed, "k"] & \mathbf{TOP}(\{-1,1\}, \mathbb{P}_{\alpha,\beta}\mathrm{X}) \times_{\mathbf{TOP}(\{-1,1\}, \mathbb{P}_{\alpha,\beta}\mathrm{Y})} \mathbf{TOP}(|\Delta[1]|, \mathbb{P}_{\alpha,\beta}\mathrm{Y}),
\end{tikzcd}$$

the lift k exists. By adjunction, it suffice to prove that the lift k' of the commutative square

$$\begin{tikzcd}
(\mathbf{D}^n \times |\Delta[1]|) \cup (\mathbf{D}^n \times [0,1] \times \{-1,1\}) \arrow[r] \arrow[d] & \mathbb{P}_{\alpha,\beta}X \arrow[d] \\
\mathbf{D}^n \times [0,1] \times |\Delta[1]| \arrow[r] \arrow[ur, dashed, "k'"] & \mathbb{P}_{\alpha,\beta}Y
\end{tikzcd}$$

exists. The inclusion $\{-1,1\} \subset |\Delta[1]|$ is a cofibration. So the left-hand map is the pushout product of a cofibration with a trivial cofibration. By Lemma 3.3.13, the continuous map $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{\alpha,\beta}Y$ is a fibration of topological spaces. The case $n = 1$ is therefore solved.

Consider now the case $n \geqslant 2$. Then both $\Delta[n]$ and $\partial\Delta[n]$ are connected. Therefore one has to check that for any $(\alpha, \beta) \in X^0 \times X^0$, the continuous map

$$\mathbf{TOP}(|\Delta[\mathrm{n}]|, \mathbb{P}_{\alpha,\beta}\mathrm{X}) \longrightarrow \mathbf{TOP}(|\partial\Delta[\mathrm{n}]|, \mathbb{P}_{\alpha,\beta}\mathrm{X}) \times_{\mathbf{TOP}(|\partial\Delta[\mathrm{n}]|, \mathbb{P}_{\alpha,\beta}\mathrm{Y})} \mathbf{TOP}(|\Delta[\mathrm{n}]|, \mathbb{P}_{\alpha,\beta}\mathrm{Y})$$

is a fibration. This holds for the same reason as above because:

(1) The category of topological spaces we are using is cartesian closed.
(2) The inclusion $|\partial\Delta[n]| \longrightarrow |\Delta[n]|$ is a cofibration.
(3) The mapping $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{\alpha,\beta}Y$ is a fibration by Lemma 3.3.13. □

Theorem 3.3.15. *The model category* **flow** *together with the functors* $- \otimes K$, $(-)^K$ *and* $\mathrm{Map}(-,-)$ *assembles to a simplicial model category.*

Proof. By definition, the set $\mathrm{Map}(X,Y)_0$ is the set $\mathbf{flow}(X,Y)$ of morphisms from the flow X to the flow Y. Thus, the identity of Id_X yields for every flow X a simplicial map $\Delta[0] \to \mathrm{Map}(X,X)$. The theorem is then a consequence of Proposition 3.3.11, Proposition 3.3.12 and Proposition 3.3.14. □

Let us conclude this section by an important fact:

Proposition 3.3.16. *Let X and Y be two flows. Let $f, g : X \rightrightarrows Y$ be two morphisms of flows. There exists a simplicial homotopy $H : X \otimes \Delta[1] \to Y$ between f and g if and only if there exists a S-homotopy $\overline{H} : [0,1] \to$* $\mathbf{FLOW}(X,Y)$ *between f and g.*

$$\begin{tikzcd}
(\widehat{0},\widehat{0}) \arrow[r, "(\widehat{0},*)"] \arrow[d, "(*,\widehat{0})"'] \arrow[dr, "(*,*)"] & (\widehat{0},\widehat{1}) \arrow[d, "(*,\widehat{1})"] \\
(\widehat{1},\widehat{0}) \arrow[r, "(\widehat{1},*)"] & (\widehat{1},\widehat{1})
\end{tikzcd}$$

FIGURE 1. The flow $\{\widehat{0} < \widehat{1}\}^2$

Sketch of proof. See Table 2 for the notations. Since $\Delta[1]$ is nonempty and connected, one has the equality $X \otimes \Delta[1] = |\Delta[1]| \boxtimes X$ by definition of the tensor product. And one has the bijection $\mathbf{flow}(|\Delta[1]| \boxtimes X, Y) \cong \mathbf{top}([0,1], \mathbf{FLOW}(X,Y))$ by [Gau03] Theorem 7.9. $\square$

4. Comparing realization functors from precubical sets to flows

4.1. Realizing a precubical set as a flow. A state of the flow associated with the poset $\{\widehat{0} < \widehat{1}\}^n$ (i.e., the product of n copies of $\{\widehat{0} < \widehat{1}\}$) is denoted by a n-uple of elements of $\{\widehat{0}, \widehat{1}\}$. By convention, $\{\widehat{0} < \widehat{1}\}^0 = \{()\}$. The unique morphism/execution path from $(x_1, \dots, x_n)$ to $(y_1, \dots, y_n)$ is denoted by a n-uple $(z_1, \dots, z_n)$ of $\{\widehat{0}, \widehat{1}, *\}$ with $z_i = x_i$ if $x_i = y_i$ and $z_i = *$ if $x_i < y_i$. For example in the flow $\{\widehat{0} < \widehat{1}\}^2$ (cf. Figure 1), one has the algebraic relation $(*,*) = (\widehat{0}, *) * (*, \widehat{1}) = (*, \widehat{0}) * (\widehat{1}, *)$.

Let $\square \to \mathbf{flow}$ be the functor defined on objects by the mapping $[n] \mapsto (\{\widehat{0} < \widehat{1}\}^n)^{\mathrm{cof}}$ and on morphisms by the mapping

$$\delta_i^\alpha \mapsto ((\epsilon_1, \dots, \epsilon_{n-1}) \mapsto (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1}))^{\mathrm{cof}}$$

where the ϵ_i's are elements of $\{\widehat{0}, \widehat{1}, *\}$.

The functor $|-|_{\mathrm{flow}} : \square^{\mathrm{op}}\mathbf{Set} \to \mathbf{flow}$ is then defined by [Gau07c]

$$\boxed{|K|_{\mathrm{flow}} := \varinjlim_{\square[n] \to K} (\{\widehat{0} < \widehat{1}\}^n)^{\mathrm{cof}}.}$$

It is a left adjoint. So it commutes with all small colimits.

The functor $X \mapsto X^0$ from flows to sets is a left adjoint since there is a natural bijection $\mathbf{Set}(X^0, S) \cong \mathbf{flow}(X, \widehat{S})$ where $\widehat{S}$ is the flow defined by $\widehat{S}^0 = S$ and $\mathbb{P}_{\alpha,\beta}\widehat{S} = \{(\alpha,\beta)\}$ with the composition law $(\alpha,\beta) * (\beta,\gamma) = (\alpha,\gamma)$. So one obtains the natural bijections of sets

$$|K|^0_{\mathrm{flow}} \cong \varinjlim_{\square[n] \to K} (\{\widehat{0} < \widehat{1}\}^n)^0 \cong \varinjlim_{\square[n] \to K} \{\widehat{0}, \widehat{1}\}^n \cong \varinjlim_{\square[n] \to K} \square[n]_0 \cong K_0. \tag{3}$$

4.2. Small realization of a precubical set as a flow. The first two propositions will help the reader to understand the differences between the realization functor $|-|_{\mathrm{flow}}$ and the new one $\mathrm{gl}(-)$ which is going to be constructed in this section.

Proposition 4.2.1 (e.g., [Hov99] Lemma 5.2.6)**.** *Let $\mathcal{M}$ be a model category. Consider a pushout diagram of $\mathcal{M}$*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

such that the objects A, B and C are cofibrant and such that the map $A \to C$ is a cofibration. Then D is cofibrant and is the homotopy colimit. In other terms, the commutative diagram above is also a homotopy pushout diagram.

Proposition 4.2.2. *The functor $|-|_{\mathrm{flow}} : \square^{\mathrm{op}}\mathbf{Set} \to \mathbf{flow}$ is a left adjoint (and therefore is colimit-preserving). Moreover, it satisfies the following properties:*

- *For every $n \geqslant 0$, there is a homotopy pushout diagram of flows*

$$\begin{tikzcd} \mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r] \arrow[d] & {|\partial\square[n+1]|_{\mathrm{flow}}} \arrow[d] \\ \mathrm{Glob}(\mathbf{D}^{n}) \arrow[r, "h"] & {|\square[n+1]|_{\mathrm{flow}}.} \end{tikzcd}$$

- *There exists an objectwise weak S-homotopy equivalence of cocubical flows*
$$|\square[*]|_{\mathrm{flow}} \longrightarrow \{\widehat{0} < \widehat{1}\}^{*}$$
(with always, by convention, $\{\widehat{0} < \widehat{1}\}^{0} = \{()\}$).

Proof. One only has to prove the existence of the homotopy pushout diagram. Equation (3) implies $|\partial\square[n+1]|^{0}_{\mathrm{flow}} = \{\widehat{0},\widehat{1}\}^{n+1}$. By [Gau07c] Theorem 7.8, there is a homotopy equivalence $\mathbf{S}^{n-1} \simeq \mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}|\partial\square[n+1]|_{\mathrm{flow}}$. This yields a morphism of flows

$$t_n : \mathrm{Glob}(\mathbf{S}^{n-1}) \to |\partial\square[n+1]|_{\mathrm{flow}}$$

defined by $t_n(\widehat{0}) = \widehat{0}\dots\widehat{0}$, $t_n(\widehat{1}) = \widehat{1}\dots\widehat{1}$ and

$$\mathbb{P}_{\widehat{0},\widehat{1}}t_n : \mathbb{P}_{\widehat{0},\widehat{1}}\mathrm{Glob}(\mathbf{S}^{n-1}) = \mathbf{S}^{n-1} \to \mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}\,\mathrm{gl}(\partial\square[n+1])$$

is a homotopy equivalence. Then consider the pushout diagram of flows:

$$\begin{tikzcd} \mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r, "t_n"] \arrow[d] & {|\partial\square[n+1]|_{\mathrm{flow}}} \arrow[d] \\ \mathrm{Glob}(\mathbf{D}^{n}) \arrow[r] & Z_{n+1}. \end{tikzcd}$$

By construction, one has the equality $\mathbb{P}_{\alpha,\beta}|\square[n+1]|_{\mathrm{flow}} = \mathbb{P}_{\alpha,\beta}|\partial\square[n+1]|_{\mathrm{flow}}$ for every $(\alpha,\beta) \neq (\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1})$ and there is a pushout diagram of topological spaces

$$\begin{tikzcd} \mathbf{S}^{n-1} \arrow[r, "\mathbb{P}_{\widehat{0},\widehat{1}}t_n"] \arrow[d] & \mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}|\partial\square[n+1]|_{\mathrm{flow}} \arrow[d] \\ \mathbf{D}^n \arrow[r] & \mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}Z_{n+1}. \end{tikzcd}$$

Since the model category **top** is left proper, the map $\mathbf{D}^n \to \mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}Z_{n+1}$ is a weak homotopy equivalence. So the flow Z_{n+1} and $|\square[n+1]|_{\mathrm{flow}}$ are weakly S-homotopy equivalent. Since Z_{n+1} and $|\square[n+1]|_{\mathrm{flow}}$ are both cofibrant and fibrant, there is a S-homotopy equivalence $Z_{n+1} \simeq |\square[n+1]|_{\mathrm{flow}}$. The pushout diagram defining Z_{n+1} is also a homotopy pushout diagram by Proposition 4.2.1. Hence the result. ☐

Proposition 4.2.3. *Loopless flows satisfy the following two facts:*

(1) *If a flow X is loopless, then the reflexive and transitive closure of the set*
$$\{(\alpha,\beta) \in X^0 \times X^0 \text{ such that } \mathbb{P}_{\alpha,\beta}X \neq \varnothing\}$$
induces a partial ordering on X^0.

(2) *The functor $X \mapsto X^0$ from flows to sets induces a functor from the full subcategory of loopless flows to that of partially ordered sets with strictly increasing maps.*

Proof. The first assertion is [Gau06a] Lemma 4.2. The second assertion is then clear. ☐

Theorem 4.2.4. *There exists a colimit-preserving functor* $\mathrm{gl} : \square^{\mathrm{op}}\mathbf{Set} \to$ **flow** *satisfying the following properties:*

(1) *For every $n \geqslant 0$, there is a pushout diagram of flows*
$$\begin{tikzcd} \mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r] \arrow[d] & \mathrm{gl}(\partial\square[n+1]) \arrow[d] \\ \mathrm{Glob}(\mathbf{D}^n) \arrow[r] & \mathrm{gl}(\square[n+1]). \end{tikzcd}$$

(2) *There exists an objectwise weak S-homotopy equivalence of cocubical flows*
$$\mathrm{gl}(\square[*]) \longrightarrow \{\widehat{0} < \widehat{1}\}^*$$
(*with always, by convention,* $\{\widehat{0} < \widehat{1}\}^0 = \{()\}$). *In particular with* $n = 0$, $\mathrm{gl}(\square[0]) = \{()\}$.

Note that the first condition alone does not imply the second condition. Note also that by Proposition 4.2.1, the pushout diagram above is also a homotopy pushout diagram.

Proof. Let us construct the restriction of the functor $\mathrm{gl}(-)$ to the category of n-dimensional precubical sets $\square_n^{op}\mathbf{Set}$ by induction on $n \geqslant 0$. The functor $\mathrm{gl}(-)$ will satisfy the natural isomorphism

$$\mathrm{gl}(K_{\leqslant n}) \cong \varinjlim_{\square[p]\to K_{\leqslant n}} \mathrm{gl}(\square[p])$$

for every precubical set K. One will also prove by induction on $n \geqslant 0$ that:

- For any morphism δ of $\square_n$, the map $\mathrm{gl}(\square[\delta])$ is a relative I^{gl}-cell complex.
- There exists a morphism of cocubical flows of dimension n from $\mathrm{gl}([*])$ to $\{\widehat{0} < \widehat{1}\}^*$ which is an objectwise weak S-homotopy equivalence.
- For all $0 \leqslant p \leqslant n$, the map $\mathrm{gl}(\partial\square[p] \subset \square[p])$ is a cofibration.

For $n = 0$, let $\mathrm{gl}(\square[0]) = \{\widehat{0} < \widehat{1}\}^0$. Note that this defines a functor from $\square_0^{op}\mathbf{Set}$ to $\mathbf{flow}$ and that for any morphism δ of $\square_0$, one has $\mathrm{gl}(\square[\delta]) \in \mathbf{cell}(I^{\mathrm{gl}})$ since $\delta = \mathrm{Id}_{[0]}$ is the only possibility.

Now suppose the construction done for $n \geqslant 0$. Consider the three cocubical flows of dimension n defined by $X([*]) = \mathrm{gl}(\square[*])$, $Y([*]) = |\square[*]|_{\mathrm{flow}}$ and $I([*]) = \{\widehat{0} < \widehat{1}\}^*$ for all $0 \leqslant * \leqslant n$. By induction hypothesis, there exists a morphism of cocubical flows of dimension n from X to I which is an objectwise weak S-homotopy equivalence. And Proposition 4.2.2 provides a morphism of cocubical flows of dimension n from Y to I which is an objectwise weak S-homotopy equivalence. Any map from any cocubical flow to I is an objectwise fibration since for any $n \geqslant 0$, the path space $\mathbb{P}I([n])$ is discrete. Finally, by induction hypothesis, each map $\mathrm{gl}(\partial\square[p] \subset \square[p])$ for $0 \leqslant p \leqslant n$ is a cofibration. And each map $|\partial\square[p] \subset \square[p]|_{\mathrm{flow}}$ for $0 \leqslant p \leqslant n$ is a cofibration by [Gau07c] Proposition 7.6. Theorem 2.3.4 and Proposition 3.3.16 yield a natural S-homotopy equivalence $\mu_{K_{\leqslant n}} : \mathrm{gl}(K_{\leqslant n}) \to |K_{\leqslant n}|_{\mathrm{flow}}$. The precubical set $\partial\square[n+1]$ is of dimension n. So by induction hypothesis, there exists a S-homotopy equivalence

$$\mu_{\partial\square[n+1]} : \mathrm{gl}(\partial\square[n+1]) \xrightarrow{\simeq} |\partial\square[n+1]|_{\mathrm{flow}}.$$

There is also

$$\mathrm{gl}(\partial\square[n+1])^0 \cong |\partial\square[n+1]|_{\mathrm{flow}}^0 \cong \{\widehat{0}, \widehat{1}\}^{n+1}$$

by Equation (3). The continuous map

$$\mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}\mu_{\partial\square[n+1]} : \mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}\,\mathrm{gl}(\partial\square[n+1]) \xrightarrow{\simeq} \mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}|\partial\square[n+1]|_{\mathrm{flow}}$$

is a homotopy equivalence by [Gau03] Corollary 19.8. Using [Gau07c] Theorem 7.8, one deduces that the topological spaces $\mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}\,\mathrm{gl}(\partial\square[n+1])$ and $\mathbf{S}^{n-1}$ are homotopy equivalent. This yields a morphism of flows

$$s_n : \mathrm{Glob}(\mathbf{S}^{n-1}) \to \mathrm{gl}(\partial\square[n+1])$$

defined by:

- $s_n(\widehat{0}) = \widehat{0}\dots\widehat{0}$ with the identifications $\mathrm{gl}(\partial\square[n+1])^0 = \partial\square[n+1]_0 = \{\widehat{0},\widehat{1}\}^{n+1}$.
- $s_n(\widehat{1}) = \widehat{1}\dots\widehat{1}$ with the identifications $\mathrm{gl}(\partial\square[n+1])^0 = \partial\square[n+1]_0 = \{\widehat{0},\widehat{1}\}^{n+1}$.
- $\mathbb{P}_{\widehat{0},\widehat{1}} s_n : \mathbb{P}_{\widehat{0},\widehat{1}}\mathrm{Glob}(\mathbf{S}^{n-1}) = \mathbf{S}^{n-1} \to \mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}\,\mathrm{gl}(\partial\square[n+1])$ is a homotopy equivalence.

The flow $\mathrm{gl}(\square[n+1])$ is then defined by the pushout diagram:

(4)
$$\begin{tikzcd} \mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r, "s_n"] \arrow[d] & \mathrm{gl}(\partial\square[n+1]) \arrow[d] \\ \mathrm{Glob}(\mathbf{D}^n) \arrow[r] & \mathrm{gl}(\square[n+1]). \end{tikzcd}$$

Note that by construction, the map $\mathrm{gl}(\partial\square[n+1] \subset \square[n+1])$ is a cofibration. The $2(n+1)$ inclusions $\square[n] \subset \partial\square[n+1]$ yield the definition of the $\mathrm{gl}(\delta_i^\alpha)$'s for all $\delta_i^\alpha : [n] \to [n+1]$ with $1 \leqslant i \leqslant n+1$ and $\alpha \in \{0,1\}$ as the composites

$$\mathrm{gl}(\delta_i^\alpha) : \mathrm{gl}(\square[n]) \subset \mathrm{gl}(\partial\square[n+1]) \longrightarrow \mathrm{gl}(\square[n+1]).$$

Since the category $\square_{n+1}$ is the quotient of the free category generated by the $\delta_i^\alpha : [p-1] \to [p]$ for $1 \leqslant p \leqslant n+1$ with $1 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$, by the cocubical relations, one has to check the cocubical relation $\mathrm{gl}(\delta_j^\beta) \circ \mathrm{gl}(\delta_i^\alpha) = \mathrm{gl}(\delta_i^\alpha) \circ \mathrm{gl}(\delta_{j-1}^\beta)$ for $i < j$ for every map $\delta_j^\beta \circ \delta_i^\alpha : [n-1] \to [n+1]$ and $\delta_i^\alpha \circ \delta_{j-1}^\beta : [n-1] \to [n+1]$. By induction, each morphism of flows $\mathrm{gl}(\delta_i^\alpha)$ is an inclusion of I^{gl}-cell subcomplexes. The equality $\delta_j^\beta \circ \delta_i^\alpha = \delta_i^\alpha \circ \delta_{j-1}^\beta$ implies that the sources of $\mathrm{gl}(\delta_j^\beta) \circ \mathrm{gl}(\delta_i^\alpha)$ and $\mathrm{gl}(\delta_i^\alpha) \circ \mathrm{gl}(\delta_{j-1}^\beta)$ are the same I^{gl}-cell subcomplex of $\mathrm{gl}(\square[n+1])$.[6] Hence the equality. So the functor from $\square_n$ to **flow** defined by $[p] \mapsto \mathrm{gl}(\square[p])$ for $p \leqslant n$ is extended to a functor from $\square_{n+1}$ to **flow** defined by $[p] \mapsto \mathrm{gl}(\square[p])$ for $p \leqslant n+1$. Let

$$\mathrm{gl}(K_{\leqslant n+1}) = \varinjlim_{\square[p] \to K_{\leqslant n+1}} \mathrm{gl}(\square[p])$$

for all precubical sets K. This construction extends the functor

$$\mathrm{gl} : \square_n^{\mathrm{op}}\mathbf{Set} \to \mathbf{flow}$$

to a functor $\mathrm{gl} : \square_{n+1}^{\mathrm{op}}\mathbf{Set} \to \mathbf{flow}$.

It remains to prove that one has an objectwise weak S-homotopy equivalence of cocubical flows of dimension $n+1$ from $\mathrm{gl}(\square[*])$ to $\{\widehat{0} < \widehat{1}\}^*$ to complete the induction and the proof. The map

$$\mathrm{gl}(\delta_i^\alpha) : \mathrm{gl}(\square[n]) \subset \mathrm{gl}(\partial\square[n+1]) \longrightarrow \mathrm{gl}(\square[n+1])$$

[6]This argument is possible since every element of $\mathbf{cell}(I^{\mathrm{gl}})$ is an (effective) monomorphism of flows by [Gau03] Theorem 10.6. Indeed, a subcomplex of a relative I^{gl}-cell complex is then entirely determined by its set of cells by [Hir03] Proposition 10.6.10 and Proposition 10.6.11.

induces a set map

$$\mathrm{gl}(\delta_i^\alpha)^0 : \mathrm{gl}(\square[n])^0 \subset \mathrm{gl}(\partial\square[n+1])^0 \longrightarrow \mathrm{gl}(\square[n+1])^0.$$

By Equation (3) and Proposition 4.2.3, one obtains a strictly increasing set map

$$\mathrm{gl}(\delta_i^\alpha)^0 : \{\widehat{0} < \widehat{1}\}^n \to \{\widehat{0} < \widehat{1}\}^{n+1}.$$

This yields a morphism of cocubical flows of dimension $n+1$ from $\mathrm{gl}(\square[*])$ to $\{\widehat{0} < \widehat{1}\}^*$. It remains to prove that $\mathrm{gl}(\square[n+1])$ is weakly S-homotopy equivalent to $\{\widehat{0} < \widehat{1}\}^{n+1}$. By construction of $\mathrm{gl}(\square[n+1])$, one has the equality $\mathbb{P}_{\alpha,\beta}\,\mathrm{gl}(\square[n+1]) = \mathbb{P}_{\alpha,\beta}\,\mathrm{gl}(\partial\square[n+1])$ for every $(\alpha,\beta) \neq (\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1})$ and there is a pushout diagram of topological spaces

$$\begin{tikzcd}
\mathbf{S}^{n-1} \arrow[r, "\mathbb{P}_{\widehat{0},\widehat{1}}s_n"] \arrow[d] & \mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}\,\mathrm{gl}(\partial\square[n+1]) \arrow[d] \\
\mathbf{D}^n \arrow[r] & \mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}\,\mathrm{gl}(\square[n+1]).
\end{tikzcd}$$

Since the map $\mathbb{P}_{\widehat{0},\widehat{1}}s_n$ is a weak homotopy equivalence, and since the model category **top** is left proper, the map $\mathbf{D}^n \to \mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}\,\mathrm{gl}(\square[n+1])$ is a weak homotopy equivalence. □

Corollary 4.2.5. *There exist a natural transformation $\mu : \mathrm{gl}(-) \to |-|_{\mathrm{flow}}$ inducing for every precubical set K a natural S-homotopy equivalence $\mu_K : \mathrm{gl}(K) \simeq |K|_{\mathrm{flow}}$ and a natural transformation $\nu : |-|_{\mathrm{flow}} \to \mathrm{gl}(-)$ inducing for every precubical set K a natural S-homotopy equivalence $\nu_K : |K|_{\mathrm{flow}} \simeq \mathrm{gl}(K)$ which is an inverse up to S-homotopy of μ_K.*

Proof. Consider the three cocubical flows $X([*]) = \mathrm{gl}(\square[*])$, $Y([*]) = |\square[*]|_{\mathrm{flow}}$ and $I([*]) = \{\widehat{0} < \widehat{1}\}^*$ for all $* \geqslant 0$. Theorem 4.2.4 and Proposition 4.2.2 yield objectwise weak S-homotopy equivalences $X \to I$ and $Y \to I$. Since the path space $\mathbb{P}I([n])$ is discrete for all $n \geqslant 0$, the two maps $X \to I$ and $Y \to I$ are objectwise trivial fibrations of cocubical flows. Then let us apply Theorem 2.3.3 and let us notice that a simplicial homotopy gives rise to a S-homotopy by Proposition 3.3.16. □

The following theorem gives a sufficient condition for a functor to be a realization functor:

Theorem 4.2.6. *Let $X \to \{\widehat{0} < \widehat{1}\}^*$ be an objectwise weak S-homotopy equivalence of cocubical flows. Assume that for every $n \geqslant 0$, the map $\widehat{X}(\partial\square[n]) \to \widehat{X}(\square[n])$ is a cofibration. Then there exist natural transformations $\mu : \mathrm{gl} \to \widehat{X}$ and $\nu : \widehat{X} \to \mathrm{gl}$ inducing natural S-homotopy equivalences which are inverse to each other up to S-homotopy. In particular, for all $n \geqslant 0$, there is a commutative diagram of flows which is also a homotopy* *pushout diagram*

$$\begin{tikzcd}
\mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r] \arrow[d] & \widehat{X}(\partial\square[n+1]) \arrow[d] \\
\mathrm{Glob}(\mathbf{D}^{n}) \arrow[r, "h"] & \widehat{X}(\square[n+1]) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where the left-hand vertical map is the inclusion of flows $\mathrm{Glob}(\mathbf{S}^{n-1} \subset \mathbf{D}^n)$ *and the right-hand vertical map* $\widehat{X}(\partial\square[n+1] \subset \square[n+1])$.

Proof. Since the path space $\mathbb{P}\{\widehat{0} < \widehat{1}\}^n$ of the flow $\{\widehat{0} < \widehat{1}\}^n$ is discrete for all $n \geqslant 0$, the map $X \to \{\widehat{0} < \widehat{1}\}^*$ is an objectwise trivial fibration of flows. Then apply Theorem 2.3.3 and Theorem 4.2.4 to obtain the natural transformations μ and ν. One obtains the commutative diagram of flows

$$\begin{tikzcd}
\mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r] \arrow[d] & \mathrm{gl}(\partial\square[n+1]) \arrow[r, "\mu_{\partial\square[n+1]}"] \arrow[d] & \widehat{X}(\partial\square[n+1]) \arrow[d] \\
\mathrm{Glob}(\mathbf{D}^{n}) \arrow[r] & \mathrm{gl}(\square[n+1]) \arrow[r, "\mu_{\square[n+1]}"] \arrow[ul, phantom, "\ulcorner", very near start] & \widehat{X}(\square[n+1]),
\end{tikzcd}$$

and therefore the commutative diagram of flows

$$\begin{tikzcd}
\mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r] \arrow[d] & \mathrm{gl}(\partial\square[n+1]) \arrow[r, "\mu_{\partial\square[n+1]}"] \arrow[d] & \widehat{X}(\partial\square[n+1]) \arrow[r, equal] \arrow[d] & \widehat{X}(\partial\square[n+1]) \arrow[d] \\
\mathrm{Glob}(\mathbf{D}^{n}) \arrow[r] & \mathrm{gl}(\square[n+1]) \arrow[r, "\phi_n"] \arrow[ul, phantom, "\ulcorner", very near start] \arrow[rr, bend right, "\mu_{\square[n+1]}"'] & T_n \arrow[r, "\psi_n"] \arrow[ul, phantom, "\ulcorner", very near start] & \widehat{X}(\square[n+1]).
\end{tikzcd}$$

The map ϕ_n is a weak S-homotopy equivalence since **flow** is left proper by [Gau07b] Theorem 7.4. So by the two-out-of-three property, the map ψ_n is a weak S-homotopy equivalence as well. Hence the homotopy pushout of flows. □

5. Realizing a precubical set as a small globular complex

5.1. Globular complex. A globular complex is, like a *d*-space, a local pospace and a stream, a topological space with an additional structure modeling time irreversibility. We refer to [Gau05a] for further explanations about the following list of definitions.[7] The original definition of a globular complex can be found in [GG03] but this old definition is slightly different and less tractable than the one of [Gau05a]. So it will not be used.

[7] The paper [Gau07a] gives an interpretation of the category of globular complexes as the full subcategory of cellular objects of a combinatorial model category. This point of view will not be used in this work.

A *multipointed topological space* (X, X^0) is a pair such that X^0 is a subset of the topological space X. A morphism of multipointed topological spaces $f : (X, X^0) \longrightarrow (Y, Y^0)$ is a continuous map $f : X \longrightarrow Y$ such that $f(X^0) \subset Y^0$. The corresponding category is denoted by $\mathbf{MTop}$. The category of multipointed spaces is cocomplete. Let Z be a topological space. The *(topological) globe of* Z, which is denoted by $\mathrm{Glob}^{\mathrm{top}}(Z)$, is the multipointed space $(|\mathrm{Glob}^{\mathrm{top}}(Z)|, \{\widehat{0}, \widehat{1}\})$ where the topological space $|\mathrm{Glob}^{\mathrm{top}}(Z)|$ is the quotient of $\{\widehat{0}, \widehat{1}\} \sqcup (Z \times [0,1])$ by the relations $(z,0) = (z',0) = \widehat{0}$ and $(z,1) = (z',1) = \widehat{1}$ for any $z, z' \in Z$ (cf. Figure 2). In particular, $\mathrm{Glob}^{\mathrm{top}}(\varnothing)$ is the multipointed space $(\{\widehat{0}, \widehat{1}\}, \{\widehat{0}, \widehat{1}\})$. If Z is a singleton, then the globe of Z is denoted by $\overrightarrow{I}^{\mathrm{top}}$. Let

$$I^{\mathrm{gl,top}} := \{\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^n), n \geqslant 0\}.$$

A *globular precomplex* is a λ-sequence for some ordinal λ of multipointed topological spaces $X : \lambda \longrightarrow \mathbf{MTop}$ such that $X \in \mathbf{cell}(I^{\mathrm{gl,top}})$ and such that $X_0 = (X^0, X^0)$ with X^0 a discrete space. This λ-sequence is characterized by a presentation ordinal λ, and for any $\beta < \lambda$ by an integer $n_\beta \geqslant 0$ and an attaching map $\phi_\beta : \mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n_\beta - 1}) \longrightarrow X_\beta$. The family $(n_\beta, \phi_\beta)_{\beta<\lambda}$ is called the *globular decomposition* of X. A morphism of globular precomplexes $f : X \longrightarrow Y$ is a morphism of multipointed spaces still denoted by f from $\varinjlim X$ to $\varinjlim Y$. If X is a globular precomplex, then the *underlying topological space* of the multipointed space $\varinjlim X$ is denoted by $|X|$. Let X be a globular precomplex. A morphism of globular precomplexes $\tau : \overrightarrow{I}^{\mathrm{top}} \longrightarrow X$ is *nondecreasing* if there exist $t_0 = 0 < t_1 < \cdots < t_n = 1$ such that:

(1) $\tau(t_i) \in X^0$ for all $0 \leqslant i \leqslant n$.
(2) $\tau(]t_i, t_{i+1}[) \subset \mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^{n_{\beta_i}} \backslash \mathbf{S}^{n_{\beta_i}-1})$ for some $(n_{\beta_i}, \phi_{\beta_i})$ of the globular decomposition of X.
(3) For $0 \leqslant i < n$, there exists $z_\tau^i \in \mathbf{D}^{n_{\beta_i}} \backslash \mathbf{S}^{n_{\beta_i}-1}$ and a strictly increasing continuous map $\psi_\tau^i : [t_i, t_{i+1}] \longrightarrow [0,1]$ such that $\psi_\tau^i(t_i) = \widehat{0}$ and $\psi_\tau^i(t_{i+1}) = \widehat{1}$ and for any $t \in [t_i, t_{i+1}]$, $\tau(t) = (z_\tau^i, \psi_\tau^i(t))$.

In particular, the restriction $\tau \upharpoonright_{]t_i, t_{i+1}[}$ of τ to $]t_i, t_{i+1}[$ is one-to-one. The set of nondecreasing morphisms from $\overrightarrow{I}^{\mathrm{top}}$ to X is denoted by $\mathbb{P}^{\mathrm{top}}(X)$. A morphism of globular precomplexes $f : X \longrightarrow Y$ is *nondecreasing* if the canonical set map $\mathbf{top}([0,1], |X|) \longrightarrow \mathbf{top}([0,1], |Y|)$ induced by composition by f yields a set map $\mathbb{P}^{\mathrm{top}}(X) \longrightarrow \mathbb{P}^{\mathrm{top}}(Y)$. In other terms, one has the commutative diagram of sets

$$\begin{tikzcd}
\mathbb{P}^{\mathrm{top}}(X) \arrow[r] \arrow[d, "\subset"'] & \mathbb{P}^{\mathrm{top}}(Y) \arrow[d, "\subset"'] \\
\mathbf{top}([0,1], |X|) \arrow[r] & \mathbf{top}([0,1], |Y|).
\end{tikzcd}$$

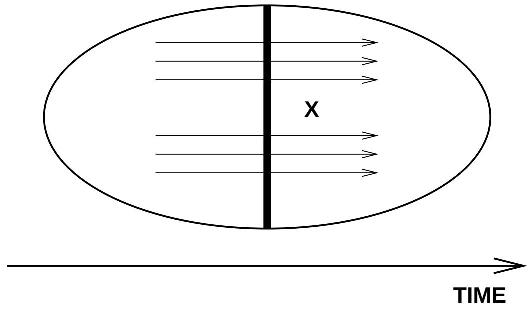

FIGURE 2. Symbolic representation of $\mathrm{Glob}^{\mathrm{top}}(X)$ for some topological space X.

A *globular complex* X is a globular precomplex such that the attaching maps ϕ_β are nondecreasing. A morphism of globular complexes is a morphism of globular precomplexes which is nondecreasing. The category of globular complexes together with the morphisms of globular complexes as defined above is denoted by **glTop**.

5.2. S-homotopy equivalence of globular complex. Let X and U be two globular complexes. Let $\mathbf{glTOP}(X,U)$ be the set $\mathbf{glTop}(X,U)$ equipped with the Kelleyfication of the compact-open topology. Let $f,g : X \rightrightarrows U$ be two morphisms of globular complexes. Then a *S-homotopy* is a continuous map $H : [0,1] \to \mathbf{glTOP}(X,U)$ with $H_0 = f$ and $H_1 = g$. This situation is denoted by $f \sim_S g$. The S-homotopy relation defines a congruence on the category **glTop**. If there exists a map $f' : U \to X$ with $f \circ f' \sim_S \mathrm{Id}_U$ and $f' \circ f \sim_S \mathrm{Id}_X$, then f is called a *S-homotopy equivalence*. The class of S-homotopy equivalences of globular complexes is denoted by $\mathcal{SH}$.

Since the S-homotopy relation of globular complexes is associated with a cylinder functor ([Gau05a] Corollary II.4.9), there is an isomorphism of categories

$$\mathbf{glTop}[\mathcal{SH}^{-1}] \cong \mathbf{glTop}/{\sim_S}$$

between the localization of the category of globular complexes by the S-homotopy equivalences and the quotient of the category of globular complexes by S-homotopy (see the proof of [Gau05a] Theorem V.4.1 and also [Gau03] Theorem 4.7).

5.3. Realizing a globular complex as a flow. By Theorem III.3.1 of [Gau05a], there exists a unique functor $\mathrm{cat} : \mathbf{glTop} \longrightarrow \mathbf{cell}(I^{\mathrm{gl}}_+) \subset \mathbf{flow}$ such that:

(1) If $X = X^0$ is a discrete globular complex, then $\mathrm{cat}(X)$ is the flow X^0.
(2) If $Z = \mathbf{S}^{n-1}$ or $Z = \mathbf{D}^n$ for some integer $n \geqslant 0$, then

$$\mathrm{cat}(\mathrm{Glob}^{\mathrm{top}}(Z)) = \mathrm{Glob}(Z).$$

(3) For any globular complex X with globular decomposition $(n_\beta, \phi_\beta)_{\beta<\lambda}$, for any limit ordinal $\beta \leqslant \lambda$, the canonical morphism of flows

$$\varinjlim_{\alpha<\beta} \mathrm{cat}(X_\alpha) \longrightarrow \mathrm{cat}(X_\beta)$$

is an isomorphism of flows.

(4) For any globular complex X with globular decomposition $(n_\beta, \phi_\beta)_{\beta<\lambda}$, for any $\beta < \lambda$, one has the pushout of flows

$$\begin{tikzcd} \mathrm{Glob}(\mathbf{S}^{n_\beta-1}) \arrow[r, "\mathrm{cat}(\phi_\beta)"] \arrow[d] & \mathrm{cat}(X_\beta) \arrow[d] \\ \mathrm{Glob}(\mathbf{D}^{n_\beta}) \arrow[r] & \mathrm{cat}(X_{\beta+1}). \end{tikzcd}$$

The properties of the functor cat used in this paper are summarized in the statement below:

Theorem 5.3.1. *One has:*

- *The functor* cat *induces for each pair* (X, U) *of globular complexes a surjective set map* $\mathbf{glTop}(X, U) \to \mathbf{flow}(\mathrm{cat}(X), \mathrm{cat}(U))$ ([Gau05a, Corollary IV.3.15]).
- *For each flow* $X \in \mathbf{cell}(I^{\mathrm{gl}}_+)$*, there exists a globular complex* X^{top} *with*

$$\mathrm{cat}(X^{\mathrm{top}}) = X$$

  ([Gau05a, Theorem V.4.1] *and* [Gau06b, Theorem 6.1]).
- *The functor* $\mathrm{cat} : \mathbf{glTop} \to \mathbf{flow}$ *induces a category equivalence*

$$\mathbf{glTop}[\mathcal{SH}^{-1}] \simeq \mathbf{Ho}(\mathbf{flow})$$

  between the localization of **glTop** *by the S-homotopy equivalences and the homotopy category of flows* ([Gau05a, Theorem V.4.2]).
- *There exists a unique functor* $|-| : \mathbf{glTop}[\mathcal{SH}^{-1}] \to \mathbf{Ho}(\mathbf{top})$ *such that the following diagram of categories is commutative:*

$$\begin{tikzcd} \mathbf{glTop} \arrow[r, "|-|"] \arrow[d, "\gamma_{\mathbf{glTop}}"'] & \mathbf{top} \arrow[d, "\gamma_{\mathbf{top}}"] \\ \mathbf{glTop}[\mathcal{SH}^{-1}] \arrow[r, "|-|"] & \mathbf{Ho}(\mathbf{top}) \end{tikzcd}$$

  where $\gamma_{\mathbf{glTop}} : \mathbf{glTop} \to \mathbf{glTop}[\mathcal{SH}^{-1}]$ *is the canonical functor from the category of globular complexes to its localization by the S-homotopy equivalences* ([Gau05a, Corollary VII.2.3]).

5.4. Realizing a precubical set as a small globular complex.

Proposition 5.4.1 (Dual of [ML98] Exercise III.4.8b)**.** *Let $\mathcal{C}$ be a category. Consider a commutative diagram of $\mathcal{C}$:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & F \end{tikzcd}$$

such the square $ABDE$ is a pushout diagram. Then the square $ACDF$ is a pushout diagram if and only if the square $BCEF$ is a pushout diagram.

Theorem 5.4.2. *There exists a functor* $\mathrm{gl}^{\mathrm{top}} : \square^{\mathrm{op}}\mathbf{Set} \to \mathbf{glTop}$ *such that for every precubical set K, there is a natural isomorphism of flows* $\mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(K)) \cong \mathrm{gl}(K)$.

The functor $\mathrm{gl}^{\mathrm{top}}(-)$ which is going to be constructed *essentially* coincides with the functor from precubical sets to globular complexes constructed in [GG03]. Essentially means not exactly. Indeed, the old definition of globular complex given in [GG03] and the new one given in [Gau05a] are not exactly the same. For example, with the new definition, an execution path is locally strictly increasing: see the remark in [Gau05a] between Definition II.2.14 and Definition II.2.15. Another difference: Hausdorff spaces are used in [GG03]. Weak Hausdorff spaces are used here and in [Gau05a]. Moreover, the construction given in the following proof is more tractable than the construction given in [GG03] thanks to the use of the cocomplete category of multipointed topological spaces. Note that the functor cat is not colimit-preserving. It only preserves globular decompositions of globular complexes. So the proof is a little bit more complicated than expected. Intuitively, the construction of $\mathrm{gl}^{\mathrm{top}}(K)$ consists of replacing each globe $\mathrm{Glob}(\mathbf{D}^n)$ of $\mathrm{gl}(K)$ by a topological globe $\mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^n)$.

Proof of Theorem 5.4.2. First of all, let us construct the restriction of the functor $\mathrm{gl}^{\mathrm{top}}(-)$ to $\square_n^{\mathrm{op}}\mathbf{Set}$ and let us prove the existence of a natural isomorphism $\mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n})) \cong \mathrm{gl}(K_{\leqslant n})$ by induction on $n \geqslant 0$. The functor $\mathrm{gl}^{\mathrm{top}}(-)$ will satisfy the natural isomorphism

$$\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n}) \cong \varinjlim_{\square[p]\to K_{\leqslant n}} \mathrm{gl}^{\mathrm{top}}(\square[p])$$

for every precubical set K, where the colimit is taken in the category of multipointed topological spaces $\mathbf{MTop}$. So viewed as a functor from $\square_n^{\mathrm{op}}\mathbf{Set}$ to $\mathbf{MTop}$, the functor $\mathrm{gl}^{\mathrm{top}}(-)$ is a left adjoint. We will also prove by induction on n that for any morphism δ of $\square_n$, the morphism of globular complexes $\mathrm{gl}^{\mathrm{top}}(\square(\delta))$ is an element of $\mathbf{cell}(I^{\mathrm{gl,top}})$.

For $n = 0$, let $\mathrm{gl}^{\mathrm{top}}(K_{\leqslant 0}) = K_0$. We have done since $\mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant 0})) = K_0$. Note this defines a functor from $\square_0^{\mathrm{op}}\mathbf{Set}$ to $\mathbf{glTop}$ which is colimit-preserving. Note also that for any morphism δ of $\square_0$, one has $\mathrm{gl}^{\mathrm{top}}(\square[\delta]) \in \mathbf{cell}(I^{\mathrm{gl,top}})$, $\delta = \mathrm{Id}_{[0]}$ being the only possibility.

Now suppose the construction done for $n \geqslant 0$. The precubical set $\partial\square[n+1]$ is of dimension n. So the globular complex $\mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])$ is already defined by induction hypothesis and one has the isomorphism of flows

$$\mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])) \cong \mathrm{gl}(\partial\square[n+1]).$$

Since the set map

$$\mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}), \mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])) \to \mathbf{flow}(\mathrm{Glob}(\mathbf{S}^{n-1}), \mathrm{gl}(\partial\square[n+1]))$$

is onto by Theorem 5.3.1, there exists a morphism of globular complexes

$$s_n^{\mathrm{top}} : \mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}) \to \mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])$$

with $\mathrm{cat}(s_n^{\mathrm{top}}) = s_n$, s_n being the map defined in the proof of Theorem 4.2.4. Let $\mathrm{gl}^{\mathrm{top}}(\square[n+1])$ be the multipointed topological space defined by the pushout diagram of multipointed topological spaces

$$\begin{tikzcd} \mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}) \arrow[r, "s_n^{\mathrm{top}}"] \arrow[d] & \mathrm{gl}^{\mathrm{top}}(\partial\square[n+1]) \arrow[d] \\ \mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^{n}) \arrow[r] & \mathrm{gl}^{\mathrm{top}}(\square[n+1]). \end{tikzcd} \tag{5}$$

The globular decomposition of the multipointed space $\mathrm{gl}^{\mathrm{top}}(\square[n+1])$ is obtained by considering the globular decomposition of the globular complex $\mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])$ and by adding the globular cell

$$\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^{n})$$

with the attaching map s_n^{top}. So the multipointed space $\mathrm{gl}^{\mathrm{top}}(\square[n+1])$ is a globular complex.

The $2(n+1)$ inclusions $\square[n] \subset \partial\square[n+1]$ yield the definition of the $\mathrm{gl}^{\mathrm{top}}(\delta_i^\alpha)$'s for all $\delta_i^\alpha : [n] \to [n+1]$ with $1 \leqslant i \leqslant n+1$ and $\alpha \in \{0,1\}$ as the composites:

$$\mathrm{gl}^{\mathrm{top}}(\delta_i^\alpha) : \mathrm{gl}^{\mathrm{top}}(\square[n]) \subset \mathrm{gl}^{\mathrm{top}}(\partial\square[n+1]) \longrightarrow \mathrm{gl}^{\mathrm{top}}(\square[n+1]).$$

Since the category $\square_{n+1}$ is the quotient of the free category generated by the $\delta_i^\alpha : [p-1] \to [p]$ for $1 \leqslant p \leqslant n+1$ with $1 \leqslant i \leqslant p$ and $\alpha \in \{0,1\}$, by the cocubical relations, one has to check the cocubical relation $\mathrm{gl}^{\mathrm{top}}(\delta_j^\beta) \circ \mathrm{gl}^{\mathrm{top}}(\delta_i^\alpha) = \mathrm{gl}^{\mathrm{top}}(\delta_i^\alpha) \circ \mathrm{gl}^{\mathrm{top}}(\delta_{j-1}^\beta)$ for $i < j$ for every map $\delta_j^\beta \circ \delta_i^\alpha : [n-1] \to [n+1]$ and $\delta_i^\alpha \circ \delta_{j-1}^\beta : [n-1] \to [n+1]$. By induction, each morphism of globular complexes $\mathrm{gl}^{\mathrm{top}}(\delta_i^\alpha)$ is an inclusion of $I^{\mathrm{gl,top}}$-cell subcomplexes. The equality $\delta_j^\beta \circ \delta_i^\alpha = \delta_i^\alpha \circ \delta_{j-1}^\beta$ implies that the sources of $\mathrm{gl}^{\mathrm{top}}(\delta_j^\beta) \circ \mathrm{gl}^{\mathrm{top}}(\delta_i^\alpha)$ and

$\mathrm{gl}^{\mathrm{top}}(\delta_i^\alpha)\circ\mathrm{gl}^{\mathrm{top}}(\delta_{j-1}^\beta)$ are the same $I^{\mathrm{gl,top}}$-cell subcomplex of $\mathrm{gl}^{\mathrm{top}}(\square[n+1])$.[8] Hence the equality.

So the functor from $\square_n$ to $\mathbf{glTop}$ defined by $[p]\mapsto \mathrm{gl}^{\mathrm{top}}(\square[p])$ for $p\leqslant n$ is extended to a functor from $\square_{n+1}$ to $\mathbf{MTop}$ defined by $[p]\mapsto \mathrm{gl}^{\mathrm{top}}(\square[p])$ for $p\leqslant n+1$. Let

$$\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n+1}) = \varinjlim_{\square[p]\to K_{\leqslant n+1}} \mathrm{gl}^{\mathrm{top}}(\square[p])$$

for all precubical sets K. This construction extends the functor $\mathrm{gl}^{\mathrm{top}}: \square_n^{\mathrm{op}}\mathbf{Set}\to\mathbf{glTop}$ to a functor $\mathrm{gl}^{\mathrm{top}}:\square_{n+1}^{\mathrm{op}}\mathbf{Set}\to\mathbf{MTop}$ which is still colimit-preserving since it is still a left adjoint. So one obtains the commutative diagram of multipointed spaces

$$\begin{tikzcd}
\bigsqcup_{x\in K_{n+1}}\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}) \arrow[r, "\bigsqcup s_n^{\mathrm{top}}"] \arrow[d] & \bigsqcup_{x\in K_{n+1}}\mathrm{gl}^{\mathrm{top}}(\partial\square[n+1]) \arrow[r] \arrow[d] & \mathrm{gl}^{\mathrm{top}}(K_{\leqslant n}) \arrow[d] \\
\bigsqcup_{x\in K_{n+1}}\mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^{n}) \arrow[r] & \bigsqcup_{x\in K_{n+1}}\mathrm{gl}^{\mathrm{top}}(\square[n+1]) \arrow[r] & \mathrm{gl}^{\mathrm{top}}(K_{\leqslant n+1}).
\end{tikzcd}$$

The left-hand square is a pushout by definition of $\mathrm{gl}^{\mathrm{top}}(\square[n+1])$. The right-hand square is a pushout since $\mathrm{gl}^{\mathrm{top}}:\square^{\mathrm{op}}\mathbf{Set}\to\mathbf{MTop}$ is colimit-preserving and since for every precubical set K, there is a pushout diagram of sets

$$\begin{tikzcd}
\bigsqcup_{x\in K_{n+1}}\partial\square[n+1] \arrow[r] \arrow[d] & K_{\leqslant n} \arrow[d] \\
\bigsqcup_{x\in K_{n+1}}\square[n+1] \arrow[r] & K_{\leqslant n+1}
\end{tikzcd}$$

where the sum is over $x\in K_{n+1}=\square^{\mathrm{op}}\mathbf{Set}(\square[n+1],K)$ and where the corresponding map $\partial\square[n+1]\to K_{\leqslant n}$ is the composite $\partial\square[n+1]\subset\square[n+1]\xrightarrow{x} K_{\leqslant n}$. Since pushout diagrams compose by Proposition 5.4.1, one obtains the pushout diagram of multipointed spaces

(6)
$$\begin{tikzcd}
\bigsqcup_{x\in K_{n+1}}\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}) \arrow[r] \arrow[d] & \mathrm{gl}^{\mathrm{top}}(K_{\leqslant n}) \arrow[d] \\
\bigsqcup_{x\in K_{n+1}}\mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^{n}) \arrow[r] & \mathrm{gl}^{\mathrm{top}}(K_{\leqslant n+1}),
\end{tikzcd}$$

[8] This argument is possible since every element of $\mathbf{cell}(I^{\mathrm{gl,top}})$ is an (effective) monomorphism of multipointed topological spaces by [Gau06b] Theorem 8.2. Indeed, since $\mathbf{MTop}$ is cocomplete, a subcomplex of a relative $I^{\mathrm{gl,top}}$-cell complex is then entirely determined by its set of cells by [Hir03] Proposition 10.6.10 and Proposition 10.6.11.

and then, by construction of the functor cat : **glTop** → **flow**, the pushout diagram of flows

$$\begin{tikzcd}
\bigsqcup_{x\in K_{n+1}} \mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r] \arrow[d] & \mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n})) \arrow[d] \\
\bigsqcup_{x\in K_{n+1}} \mathrm{Glob}(\mathbf{D}^{n}) \arrow[r] & \mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n+1})).
\end{tikzcd}$$

Diagram (6) above yields a globular decomposition for the multipointed space $\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n+1})$, proving that the functor $\mathrm{gl}^{\mathrm{top}}(-)$ is actually a functor from $\square_{n+1}^{\mathrm{op}}\mathbf{Set}$ to **glTop**. By construction of the functor cat : **glTop** → **flow**, there is a pushout diagram of flows

$$\begin{tikzcd}
\bigsqcup\limits_{x\in K_{n+1}} \mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r, "\sqcup s_n"] \arrow[d] & \bigsqcup\limits_{x\in K_{n+1}} \mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])) \arrow[d] \\
\bigsqcup\limits_{x\in K_{n+1}} \mathrm{Glob}(\mathbf{D}^{n}) \arrow[r] & \bigsqcup\limits_{x\in K_{n+1}} \mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(\square[n+1])).
\end{tikzcd}$$

So by Proposition 5.4.1, one obtains the pushout diagram of flows:[9]

(7)
$$\begin{tikzcd}
\bigsqcup_{x\in K_{n+1}} \mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])) \arrow[r] \arrow[d] & \mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n})) \arrow[d] \\
\bigsqcup_{x\in K_{n+1}} \mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(\square[n+1])) \arrow[r] & \mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n+1})).
\end{tikzcd}$$

The diagram of solid arrows of Figure 3 is commutative for the following reasons:

- The back face is commutative and is a pushout diagram of flows by Diagram (7).
- The front face is commutative and is a pushout diagram of flows since the functor gl : $\square^{\mathrm{op}}\mathbf{Set} \to \mathbf{flow}$ is colimit-preserving.
- Apply the functor cat to Diagram (5). One obtains the pushout diagram of flows

$$\begin{tikzcd}
\mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r, "\mathrm{cat}(s_n^{\mathrm{top}})"] \arrow[d] & \mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])) \arrow[d] \\
\mathrm{Glob}(\mathbf{D}^{n}) \arrow[r] & \mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(\square[n+1])).
\end{tikzcd}$$

  Diagram (4) and the equality $\mathrm{cat}(s_n^{\mathrm{top}}) = s_n$ imply the commutativity of the left-hand face.

[9]Let us repeat that the functor cat is not colimit-preserving. So the use of Proposition 5.4.1 seems to be necessary to obtain Diagram (7).

$$\begin{tikzcd}
\bigsqcup\limits_{x\in K_{n+1}} \operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])) \arrow[rr] \arrow[dr, "\cong"] \arrow[dd] & & \operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n})) \arrow[dr, "\cong"] \arrow[dd] & \\
& \bigsqcup\limits_{x\in K_{n+1}} \mathrm{gl}(\partial\square[n+1]) \arrow[rr] \arrow[dd] & & \mathrm{gl}(K_{\leqslant n}) \arrow[dd] \\
\bigsqcup\limits_{x\in K_{n+1}} \operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(\square[n+1])) \arrow[rr] \arrow[dr, "\cong"] & & \operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n+1})) \arrow[dr, dashed] & \\
& \bigsqcup\limits_{x\in K_{n+1}} \mathrm{gl}(\square[n+1]) \arrow[rr] & & \mathrm{gl}(K_{\leqslant n+1}).
\end{tikzcd}$$

Figure 3. Isomorphism $\operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n+1})) \cong \mathrm{gl}(K_{\leqslant n+1})$.

- Finally, the top face is commutative since there is a natural isomorphism
$$\operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(K)) \cong \mathrm{gl}(K)$$
for all precubical sets K of dimension n by induction hypothesis.

Hence the existence of an isomorphism of flows
$$\operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n+1})) \cong \mathrm{gl}(K_{\leqslant n+1})$$
for every precubical set K. The isomorphism is natural for the following reasons:

- The map $\bigsqcup_{x\in K_{n+1}} \operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])) \to \operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n}))$ is natural with respect to K since it is the image by the functor $\operatorname{cat} \circ \mathrm{gl}^{\mathrm{top}}(-)$ of the natural map of precubical sets $i(K,n) : \bigsqcup_{x\in K_{n+1}} \partial\square[n+1] \to K_{\leqslant n}$.
- The map $\bigsqcup_{x\in K_{n+1}} \mathrm{gl}(\partial\square[n+1]) \to \mathrm{gl}(K_{\leqslant n})$ is natural with respect to K since it is the image by the functor $\mathrm{gl}(-)$ of the natural map of precubical sets $i(K,n)$.
- There is a natural isomorphism $\operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(L)) \cong \mathrm{gl}(L)$ with respect to L for every n-dimensional precubical set L by induction hypothesis. Apply this fact for $L = K_{\leqslant n}$ and $L = \bigsqcup_{x\in K_{n+1}} \partial\square[n+1]$.
- Morphisms of precubical sets of the form $\bigsqcup_{x\in K_{n+1}} A \to \bigsqcup_{x\in K_{n+1}} B$ are natural with respect to K for every morphism of precubical sets $A \to B$.

The induction is now complete. One has
$$\operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(K)) \cong \varinjlim \operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(K_{\leqslant n}))$$
by definition of the functor cat. And one has $\mathrm{gl}(K) \cong \varinjlim \mathrm{gl}(K_{\leqslant n})$ since the functor $\mathrm{gl}(-)$ is colimit-preserving. Hence a natural isomorphism of flows $\operatorname{cat}(\mathrm{gl}^{\mathrm{top}}(K)) \cong \mathrm{gl}(K)$. $\square$

Note that the functor $\mathrm{gl}^{\mathrm{top}} : \square^{\mathrm{op}}\mathbf{Set} \to \mathbf{glTop}$ is not unique. It is entirely characterized up to isomorphism of functors by the noncanonical choice of the maps $s_n : \mathrm{Glob}(\mathbf{S}^{n-1}) \to \mathrm{gl}(\partial\square[n+1])$ and of the maps $s_n^{\mathrm{top}} : \mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}) \to \mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])$ for all $n \geqslant 0$. Let $\gamma_{\mathbf{flow}} : \mathbf{flow} \to \mathbf{Ho}(\mathbf{flow})$ be the canonical functor from the category of flows to its homotopy category. Let us denote by

$$\overline{\mathrm{cat}} : \mathbf{glTop}[\mathcal{SH}^{-1}] \simeq \mathbf{Ho}(\mathbf{top}) : \overline{\mathrm{cat}}^{-1}$$

the equivalence of categories between the globular complexes up to S-homotopy and the homotopy category of flows (see Theorem 5.3.1).

Theorem 5.4.3. *The functor* $\gamma_{\mathbf{flow}} \circ |-|_{\mathrm{flow}} : \square^{\mathrm{op}}\mathbf{Set} \to \mathbf{Ho}(\mathbf{flow})$ *factors up to an isomorphism of functors as a composite*

$$\square^{\mathrm{op}}\mathbf{Set} \xrightarrow{\mathrm{hogl}^{\mathrm{top}}} \mathbf{glTop}/\!\sim_S \longrightarrow \mathbf{Ho}(\mathbf{flow}).$$

The functor $\mathrm{hogl}^{\mathrm{top}} : \square^{\mathrm{op}}\mathbf{Set} \to \mathbf{glTop}/\!\sim_S$ *is unique up to isomorphism of functors.*

In other terms, the functor $\mathrm{gl}^{\mathrm{top}}(-)$ constructed in Theorem 5.4.2 is unique up to a natural S-homotopy of globular complexes.

Proof. Since there is an isomorphism of categories

$$\mathbf{glTop}[\mathcal{SH}^{-1}] \cong \mathbf{glTop}/\!\sim_S,$$

let us identify the two categories. Let

$$\boxed{\mathrm{hogl}^{\mathrm{top}} = \gamma_{\mathbf{glTop}} \circ \mathrm{gl}^{\mathrm{top}}}.$$

Then one obtains the isomorphisms of functors

$$\overline{\mathrm{cat}} \circ \mathrm{hogl}^{\mathrm{top}} = \gamma_{\mathbf{flow}} \circ \mathrm{cat} \circ \mathrm{gl}^{\mathrm{top}} \cong \gamma_{\mathbf{flow}} \circ |-|_{\mathrm{flow}}$$

by Theorem 5.4.2 and Theorem 4.2.4. Hence the existence. Take two functors $\mathrm{hogl}_1^{\mathrm{top}} : \square^{\mathrm{op}}\mathbf{Set} \to \mathbf{glTop}/\!\sim_S$ and $\mathrm{hogl}_2^{\mathrm{top}} : \square^{\mathrm{op}}\mathbf{Set} \to \mathbf{glTop}/\!\sim_S$ satisfying the condition of the theorem. Then

$$\overline{\mathrm{cat}} \circ \mathrm{hogl}_1^{\mathrm{top}} = \overline{\mathrm{cat}} \circ \mathrm{hogl}_2^{\mathrm{top}} = \gamma_{\mathbf{flow}} \circ |-|_{\mathrm{flow}}.$$

So one has the isomorphisms of functors

$$\mathrm{hogl}_1^{\mathrm{top}} \cong \overline{\mathrm{cat}}^{-1} \circ (\overline{\mathrm{cat}} \circ \mathrm{hogl}_1^{\mathrm{top}}) \cong \overline{\mathrm{cat}}^{-1} \circ (\overline{\mathrm{cat}} \circ \mathrm{hogl}_2^{\mathrm{top}}) \cong \mathrm{hogl}_2^{\mathrm{top}}. \qquad \square$$

6. Globular and cubical underlying homotopy type

6.1. Definition of the globular and cubical underlying homotopy type. Let $\square \to \mathbf{top}$ be the functor defined on objects by the mapping $[n] \mapsto [0,1]^n$ and on morphisms by the mapping

$$\delta_i^\alpha \mapsto ((\epsilon_1, \ldots, \epsilon_{n-1}) \mapsto (\epsilon_1, \ldots, \epsilon_{i-1}, \alpha, \epsilon_i, \ldots, \epsilon_{n-1}))\,.$$

The functor $|-|_{\mathrm{space}} : \square^{op}\mathbf{Set} \to \mathbf{top}$ is then defined by

$$\boxed{|K|_{\mathrm{space}} := \varinjlim_{\square[n]\to K} [0,1]^n.}$$

It is a left adjoint. So it commutes with all small colimits. The purpose of this section is the comparison of this functor with the *underlying homotopy type functor* defined by the composite [Gau05a]:

$$\boxed{\Omega : \mathbf{flow} \xrightarrow{\gamma_{\mathbf{flow}}} \mathbf{Ho}(\mathbf{flow}) \xrightarrow{\overline{\mathrm{cat}}^{-1}} \mathbf{glTop}[\mathcal{SH}^{-1}] \xrightarrow{|-|} \mathbf{Ho}(\mathbf{top}).}$$

6.2. Comparison of the two functors.

Theorem 6.2.1. *For every precubical set K, there is a natural isomorphism of homotopy types $\gamma_{\mathbf{top}}(|K|_{\mathrm{space}}) \cong \Omega(|K|_{\mathrm{flow}})$.*

Proof. Consider the three cocubical topological spaces:

- $X([*]) = [0,1]^*$ for all $* \geqslant 0$,
- $Y([*]) = |\,\mathrm{gl}^{\mathrm{top}}(\square[*])|$ for all $* \geqslant 0$,
- $I([*]) = \{0\}$ for all $* \geqslant 0$.

There exist a unique map $X \to I$ and a unique map $Y \to I$ which are both objectwise weak homotopy equivalences and objectwise fibrations. The cocubical object X satisfies the hypotheses of Theorem 2.3.3 in an obvious way. The proof of Theorem 5.4.2 implies the pushout diagram of spaces

$$\begin{tikzcd}
|\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1})| \arrow[r, "s_n^{\mathrm{top}}"] \arrow[d] & |\,\mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])| \arrow[d] \\
|\mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^{n})| \arrow[r] & |\,\mathrm{gl}^{\mathrm{top}}(\square[n+1])|
\end{tikzcd}$$

for every $n \geqslant 0$. Since the continuous map $|\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1})| \to |\mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^{n})|$ is a cofibration of topological spaces (see the proof of [Gau06b] Theorem 8.2), the map $|\,\mathrm{gl}^{\mathrm{top}}(\partial\square[n+1])| \to |\,\mathrm{gl}^{\mathrm{top}}(\square[n+1])|$ is a cofibration as well. So the cocubical space Y satisfies the hypotheses of Theorem 2.3.3 as well. Hence there exists a natural homotopy equivalence $|K|_{\mathrm{space}} \simeq |\,\mathrm{gl}^{\mathrm{top}}(K)|$ with respect to K. So there exists a natural isomorphism of homotopy types $\gamma_{\mathbf{top}}(|K|_{\mathrm{space}}) \cong \gamma_{\mathbf{top}}(|\,\mathrm{gl}^{\mathrm{top}}(K)|)$. The proof is complete after the following sequence of natural isomorphisms:

$$\begin{aligned}
\gamma_{\mathbf{top}}(|K|_{\mathrm{space}}) &\cong \gamma_{\mathbf{top}}(|\,\mathrm{gl}^{\mathrm{top}}(K)|) \cong |\gamma_{\mathbf{glTop}}(\mathrm{gl}^{\mathrm{top}}(K))| \\
&\cong |\overline{\mathrm{cat}}^{-1} \circ \overline{\mathrm{cat}} \circ \gamma_{\mathbf{glTop}}(\mathrm{gl}^{\mathrm{top}}(K))| \\
&\cong |\overline{\mathrm{cat}}^{-1} \circ \gamma_{\mathbf{flow}}(\mathrm{cat}(\mathrm{gl}^{\mathrm{top}}(K)))| \\
&\cong |\overline{\mathrm{cat}}^{-1} \circ \gamma_{\mathbf{flow}}(|K|_{\mathrm{flow}})| \\
&= \Omega(|K|_{\mathrm{flow}}).
\end{aligned}$$

Here, the first isomorphism is by the results above, the second and fourth are by Theorem 5.3.1, the fifth by Theorem 5.4.2, and the last equality by definition of Ω. $\square$

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