New York Journal of Mathematics

New York J. Math. **14** (2008) 733–742.

Note on Frobenius monoidal functors

Brian Day and Craig Pastro

Abstract. It is well known that strong monoidal functors preserve duals. In this short note we show that a weaker version of functor, which we call "Frobenius monoidal", is sufficient. Further properties of Frobenius monoidal functors are developed.

The idea of this note became apparent from Proposition 2.8 in the paper of R. Rosebrugh, N. Sabadini, and R.F.C. Walters [5].

Throughout suppose that $\mathscr{A}$ and $\mathscr{B}$ are strict[1] monoidal categories.

Definition 1. A *Frobenius monoidal functor* is a functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ which is monoidal (F, r, r_0) and comonoidal (F, i, i_0), and satisfies the compatibility conditions

$$ir = (1 \otimes r)(i \otimes 1) : F(A \otimes B) \otimes FC \longrightarrow FA \otimes F(B \otimes C)$$
$$ir = (r \otimes 1)(1 \otimes i) : FA \otimes F(B \otimes C) \longrightarrow F(A \otimes B) \otimes FC,$$

for all $A, B, C \in \mathscr{A}$.

The compact case ($\otimes = \oplus$) of Cockett and Seely's linearly distributive functors [2] are precisely Frobenius monoidal functors, and Frobenius monoidal functors with $ri = 1$ have been called *split monoidal* by Szlachányi in [6].

A *dual situation* in $\mathscr{A}$ is a tuple (A, B, e, n), where A and B are objects of $\mathscr{A}$ and

$$e : A \otimes B \longrightarrow I \qquad\qquad n : I \longrightarrow B \otimes A$$

Received May 16, 2008.

Mathematics Subject Classification. 18A22, 18D10, 18D25.

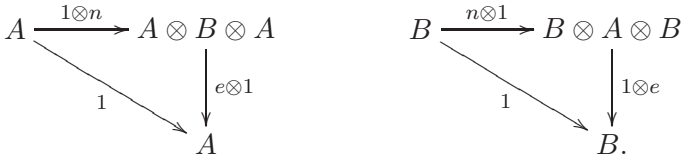
Key words and phrases. Monoidal category, duals, monoidal functor, comonoidal functor, Frobenius monoidal functor, Frobenius algebra.

The first author gratefully acknowledges partial support of an Australian Research Council grant, while the second author is partially supported by GCOE, Kyoto University. This work was completed while the second author was a Ph.D. student at Macquarie University, Australia, and during that time was gratefully supported by an international Macquarie University Research Scholarship and a Scott Russell Johnson Memorial Scholarship.

[1] We have decided to work in the strict setting for simplicity of exposition, however, this is not necessary.

ISSN 1076-9803/08

are morphisms in $\mathscr{A}$, called evaluation and coevaluation respectively, satisfying the "triangle identities":



Theorem 2. *Frobenius monoidal functors preserve dual situations.*

This theorem is actually a special case of the fact that linear functors (between linear bicategories) preserve linear adjoints [1].

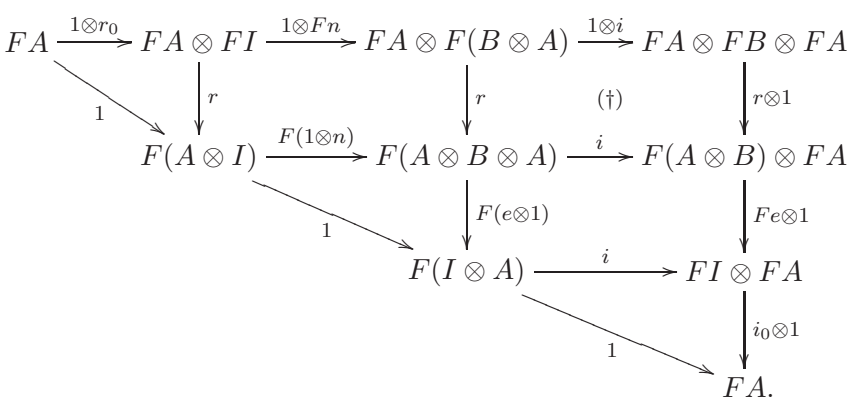
Proof. Suppose that (A, B, e, n) is dual situation in $\mathscr{A}$. We will show that (FA, FB, e, n), where e and n are defined as

$$e = \big(\ FA \otimes FB \xrightarrow{r} F(A \otimes B) \xrightarrow{Fe} FI \xrightarrow{i_0} I\ \big)$$

$$n = \big(\ I \xrightarrow{r_0} FI \xrightarrow{Fn} F(B \otimes A) \xrightarrow{i} FB \otimes FA\ \big),$$

is a dual situation in $\mathscr{B}$.

The following diagram proves one of the triangle identities.



The square labelled by (†) requires the second Frobenius condition. We remark that to prove the other triangle identity is similar and requires the first Frobenius condition. □

Proposition 3. *Any strong monoidal functor is a Frobenius monoidal functor.*

Proof. Recall that a strong monoidal functor is a monoidal functor and a comonoidal functor for which $r = i^{-1}$ and $r_0 = i_0^{-1}$. The commutativity of

the following diagram proves one of the Frobenius conditions.

$$\begin{array}{ccc} F(A\otimes B)\otimes FC & \xrightarrow{i\otimes 1} & FA\otimes FB\otimes FC \\ {\scriptstyle r}\downarrow\uparrow{\scriptstyle i} & & {\scriptstyle 1\otimes i}\uparrow\downarrow{\scriptstyle 1\otimes r} \\ F(A\otimes B\otimes C) & \xrightarrow{i} & FA\otimes F(B\otimes C). \end{array}$$

The other is similar. □

Proposition 4. *The composite of Frobenius monoidal functors is a Frobenius monoidal functor.*

Proof. Suppose that $F : \mathscr{A} \longrightarrow \mathscr{B}$ and $G : \mathscr{B} \longrightarrow \mathscr{C}$ are Frobenius monoidal functors. It is well known and easy to see that the composite of monoidal (resp. comonoidal) functors is monoidal (resp. comonoidal). We therefore need only prove the Frobenius conditions, one of which follows from the commutativity of the following diagram.

$$\begin{array}{ccccc} GF(A\otimes B)\otimes GFC & \xrightarrow{r} & G(F(A\otimes B)\otimes FC) & \xrightarrow{Gr} & GF(A\otimes B\otimes C) \\ {\scriptstyle Gi\otimes 1}\downarrow & & {\scriptstyle G(i\otimes 1)}\downarrow & (\ddagger) & \downarrow{\scriptstyle Gi} \\ G(FA\otimes FB)\otimes GFC & \xrightarrow{r} & G(FA\otimes FB\otimes FC) & \xrightarrow{G(1\otimes r)} & G(FA\otimes F(B\otimes C)) \\ {\scriptstyle i\otimes 1}\downarrow & (\$) & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ GFA\otimes GFB\otimes GFC & \xrightarrow{1\otimes r} & GFA\otimes G(FB\otimes FC) & \xrightarrow{1\otimes Gr} & GFA\otimes GF(B\otimes C). \end{array}$$

The square labelled by (‡) uses the Frobenius property of F, and the square labelled by ($) uses the Frobenius property of G.

The other Frobenius condition follows from a similar diagram. □

It may be seen that a Frobenius monoidal functor $R : \mathbf{1} \longrightarrow \mathscr{A}$ is a Frobenius algebra in $\mathscr{A}$ [4]. Therefore, we have the following corollary.

Corollary 5. *Frobenius monoidal functors preserve Frobenius algebras. That is, if R is a Frobenius algebra in $\mathscr{A}$ and $F : \mathscr{A} \longrightarrow \mathscr{B}$ is a Frobenius monoidal functor, then FR is a Frobenius algebra in $\mathscr{B}$.*

Example 6. Suppose that $\mathscr{A} = (\mathscr{A}, \otimes, I, c)$ is a braided monoidal category. If $R = (R, \mu, \eta, \delta, \epsilon)$ is a Frobenius algebra in $\mathscr{A}$, then $F = R\otimes - : \mathscr{A} \longrightarrow \mathscr{A}$ is a Frobenius monoidal functor. The monoidal structure (F, r, r_0) is given by

$$r_{A,B} = \left(R\otimes A\otimes R\otimes B \xrightarrow{1\otimes c^{-1}\otimes 1} R\otimes R\otimes A\otimes B \xrightarrow{\mu\otimes 1\otimes 1} R\otimes A\otimes B \right)$$

$$r_0 = \left(I \xrightarrow{\eta} R \right)$$

and the comonoidal structure (F, i, i_0) by

$$i_{A,B} = \left(R \otimes A \otimes B \xrightarrow{\delta \otimes 1 \otimes 1} R \otimes R \otimes A \otimes B \xrightarrow{1 \otimes c \otimes 1} R \otimes A \otimes R \otimes B \right)$$
$$i_0 = \left(R \xrightarrow{\epsilon} I \right).$$

The Frobenius conditions now follow easily from the properties of Frobenius algebras.

This example shows that Frobenius monoidal functors generalize Frobenius algebras much in the same way that monoidal comonads, or comonoidal monads, generalize bialgebras.

The following proposition is a generalization of the fact that morphisms of Frobenius algebras (morphisms which are both algebra and coalgebra morphisms) are isomorphisms. It also generalizes the result that monoidal natural transformations between strong monoidal functors with (left or right) compact domain are invertible.

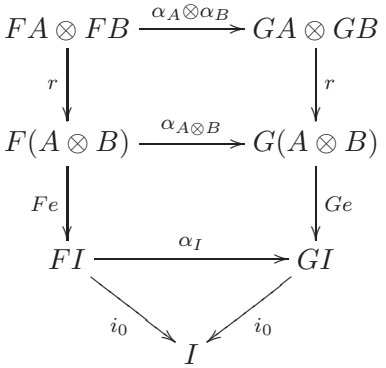
Proposition 7. *Suppose that $F, G : \mathscr{A} \longrightarrow \mathscr{B}$ are Frobenius monoidal functors and that $\alpha : F \longrightarrow G$ is a monoidal and comonoidal natural transformation. If $A \in \mathscr{A}$ is part of a dual situation (i.e., (A, B, e, n) or (B, A, e, n) is a dual situation) then $\alpha_A : FA \longrightarrow GA$ is invertible.*

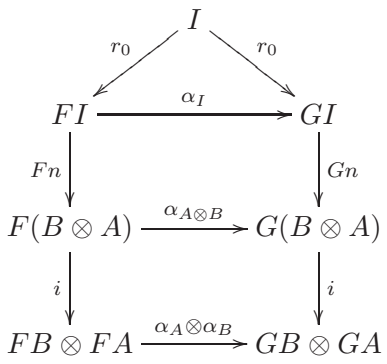
Proof. We shall assume that A is part of the dual situation (A, B, e, n). The case where A is part of a dual situation (B, A, e, n) is treated similarly. The component $\alpha_B : FB \longrightarrow GB$ has mate

$$GA \xrightarrow{1 \otimes n} GA \otimes FB \otimes FA \xrightarrow{1 \otimes \alpha_B \otimes 1} GA \otimes GB \otimes FA \xrightarrow{e \otimes 1} FA$$

which we will show is the inverse to α_A.

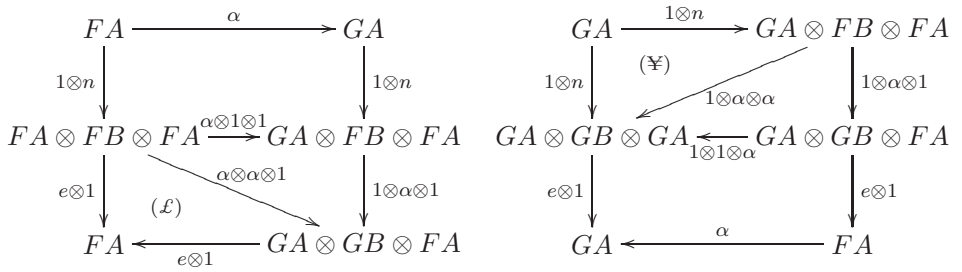
If α is both monoidal and comonoidal then the diagrams





commute. The following diagrams prove that α_A is invertible.The first diagram above says exactly that the triangle labelled by $(\pounds)$ below commutes.

The second diagram above that the triangle labelled by (¥) below commutes.



□

An obvious corollary to this proposition is:

Corollary 8. *Let $\mathscr{A}$ be a category in which every object is part of a dual situation (e.g., a left or right compact category), $\mathscr{B}$ a monoidal category, and $F, G : \mathscr{A} \longrightarrow \mathscr{B}$ Frobenius monoidal functors. Any monoidal and comonoidal natural transformation $\alpha : F \longrightarrow G$ is a natural isomorphism.*

Denote by $\mathrm{Frob}(\mathscr{A}, \mathscr{B})$ the category of Frobenius monoidal functors from $\mathscr{A}$ to $\mathscr{B}$ and *all* natural transformations between them.

Proposition 9 (cf. [5] Prop. 2.10)**.** *If $\mathscr{B}$ is a braided monoidal category, then* $\mathrm{Frob}(\mathscr{A}, \mathscr{B})$ *is a braided monoidal category with the pointwise tensor product of functors.*

Proof. Consider the pointwise tensor product of Frobenius monoidal functors $F, G : \mathscr{A} \longrightarrow \mathscr{B}$. That is,

$$(F \otimes G)A = FA \otimes GA.$$

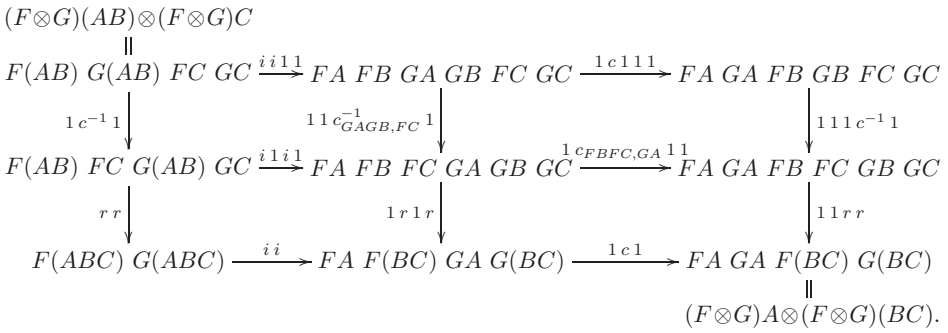
It is obviously an associative and unital tensor product with unit $I(A) = I$ for all $A \in \mathscr{A}$.

We may define morphisms as follows:

$$\begin{aligned} r &= (r \otimes r)(1 \otimes c^{-1} \otimes 1) : (F \otimes G)A \otimes (F \otimes G)B \longrightarrow (F \otimes G)(A \otimes B) \\ r_0 &= r_0 \otimes r_0 : I \longrightarrow (F \otimes G)I \\ i &= (1 \otimes c \otimes 1)(i \otimes i) : (F \otimes G)(A \otimes B) \longrightarrow (F \otimes G)A \otimes (F \otimes G)B \\ i_0 &= i_0 \otimes i_0 : (F \otimes G)I \longrightarrow I. \end{aligned}$$

That these morphisms provide a monoidal and a comonoidal structure on $F \otimes G$ is not too difficult to show, and is omitted here. The following diagram proves the first Frobenius condition, where the "$\otimes$" symbol has

been removed as a space spacing mechanism.



The bottom left square commutes by the Frobenius condition, and the others by properties of the braiding. The second Frobenius condition follows from a similar diagram. So, $F \otimes G$ is a Frobenius monoidal functor.

The braiding $c_{F,G} : F \otimes G \longrightarrow G \otimes F$ is given on components by

$$(c_{F,G})_A = c_{FA,GA} : FA \otimes GA \longrightarrow GA \otimes FA. \qquad \square$$

Corollary 10. *If $\mathscr{B}$ is a braided monoidal category and $\mathscr{A}$ is a self-dual compact category, meaning that for any object $A \in \mathscr{A}$, (A, A, e, n) is a dual situation in $\mathscr{A}$, then* $\mathrm{Frob}(\mathscr{A}, \mathscr{B})$ *is a self-dual braided compact category.*

Proof. By Theorem 2 Frobenius monoidal functors preserve duals, and therefore, for any $A \in \mathscr{A}$, (FA, FA, e, n) is a dual situation in $\mathscr{B}$. $\square$

Recall that, if $\mathscr{A}$ is a small monoidal category, and if small colimits exist and commute with the tensor product in $\mathscr{B}$, then the equations

$$F * G = \int^{A,B} \mathscr{A}(A \otimes B, -) \cdot FA \otimes FB$$
$$J = \mathscr{A}(I, -) \cdot I,$$

where $\cdot$ denotes copower, describe the *convolution monoidal structure* on the functor category $[\mathscr{A}, \mathscr{B}]$ (cf. [3]).

Monoidal functors are monoids in $[\mathscr{A}, \mathscr{B}]$ with the convolution tensor product. The following theorem attempts a similar description for Frobenius monoidal functors as Frobenius algebras in $[\mathscr{A}, \mathscr{B}]$. Unfortunately, it is not entirely successful as there is no natural counit. This is also the reason that comonoidal functors are not comonoids in $[\mathscr{A}, \mathscr{B}]$. However, what we have is:

Theorem 11. *If $\mathscr{A}$ is a small monoidal category and $\mathscr{B}$ is a monoidal category having all small colimits commuting with tensor (so that we may form the functor category $[\mathscr{A}, \mathscr{B}]$ with the convolution tensor product), then any Frobenius monoidal functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ for which the canonical evaluation*

morphism

$$(\flat) \qquad \int^{A,B,C} \mathscr{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C) \longrightarrow F$$

is an isomorphism, becomes an algebra with a comultiplication which satisfies the Frobenius identities in the convolution functor category $[\mathscr{A}, \mathscr{B}]$.

Note that, by the Yoneda lemma, the equation $(\flat)$ *is satisfied by all the functors* $F : \mathscr{A} \longrightarrow \mathscr{B}$ *if* $\mathscr{A}$ *is a closed monoidal category and the canonical evaluation morphism*

$$(\sharp) \qquad \int^{B,C} \mathscr{A}(A, B \otimes C \otimes [B \otimes C, -]) \longrightarrow \mathscr{A}(A, -)$$

is an isomorphism for all $A \in \mathscr{A}$.

Before we prove Theorem 11 we will need the following lemma.

Lemma 12. *Assuming equation* $(\flat)$ *in Theorem* 11*, we may also derive the two variable version, that is, that the canonical evaluation morphism*

$$\int^{A,B} \mathscr{A}(A \otimes B, -) \cdot F(A \otimes B) \longrightarrow F$$

is an isomorphism.

Proof. The canonical evaluation morphism

$$\int^{A,B,C} \mathscr{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C) \xrightarrow{h} \int^{A,B} \mathscr{A}(A \otimes B, -) \cdot F(A \otimes B)$$

is a retraction of either of the canonical morphisms in the opposite direction, let us say, k. We may compose the canonical morphism

$$\int^{A,B} \mathscr{A}(A \otimes B, -) \cdot F(A \otimes B) \longrightarrow F$$

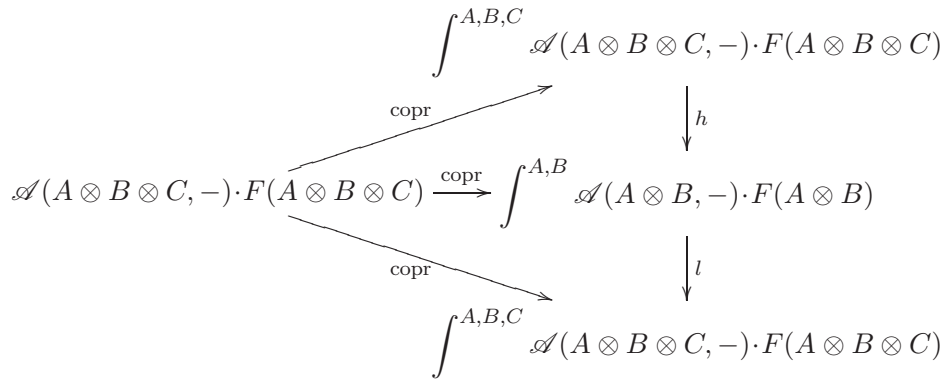
with the isomorphism of our assumption

$$F \xrightarrow{\cong} \int^{A,B,C} \mathscr{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C)$$

to get a morphism

$$\int^{A,B} \mathscr{A}(A \otimes B, -) \cdot F(A \otimes B) \xrightarrow{l} \int^{A,B,C} \mathscr{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C),$$

which makes the diagram



commute. Therefore $lh = 1$. We also have

$$hl = hlhk = hk = 1,$$

where the last step holds as k is a retraction of h, so l is an isomorphism, and hence the canonical evaluation morphism

$$\int^{A,B} \mathscr{A}(A\otimes B,-)\cdot F(A\otimes B) \longrightarrow F$$

is an isomorphism. □

A consequence of Lemma 12 is that we may write

$$\begin{aligned}
F * F &= \int^{X,C} \mathscr{A}(X\otimes C,-)\cdot FX\otimes FC \\
&\cong \int^{X,C} \mathscr{A}(X\otimes C,-)\cdot \Big(\int^{A,B} \mathscr{A}(A\otimes B,X)\cdot F(A\otimes B)\Big)\otimes FC \\
&\cong \int^{X,A,B,C} (\mathscr{A}(X\otimes C,-)\times \mathscr{A}(A\otimes B,X))\cdot (F(A\otimes B)\otimes FC) \\
&\cong \int^{A,B,C} \Big(\int^{X} \mathscr{A}(X\otimes C,-)\times \mathscr{A}(A\otimes B,X)\Big)\cdot (F(A\otimes B)\otimes FC) \\
&\cong \int^{A,B,C} \mathscr{A}(A\otimes B\otimes C,-)\cdot F(A\otimes B)\otimes FC, \qquad \text{(Yoneda)}
\end{aligned}$$

and similarly,

$$F * F \cong \int^{A,B,C} \mathscr{A}(A\otimes B\otimes C,-)\cdot FA\otimes F(B\otimes C).$$

Proof of Theorem 11. Using the isomorphisms of equation (♭) and Lemma 12 one of the Frobenius equations may be written as

$$\begin{array}{ccc} \int^{A,B,C} \mathscr{A}(A\otimes B\otimes C,-)\cdot F(A\otimes B)\otimes FC & \xrightarrow{\int 1\otimes i\otimes 1} & \int^{A,B,C} \mathscr{A}(A\otimes B\otimes C,-)\cdot FA\otimes FB\otimes FC \\ \Big\downarrow{\scriptstyle \int 1\otimes r} & & \Big\downarrow{\scriptstyle \int 1\otimes 1\otimes r} \\ \int^{A,B,C} \mathscr{A}(A\otimes B\otimes C,-)\cdot F(A\otimes B\otimes C) & \xrightarrow{\int 1\otimes i} & \int^{A,B,C} \mathscr{A}(A\otimes B\otimes C,-)\cdot FA\otimes F(B\otimes C). \end{array}$$

This diagram is seen to commute as F is a Frobenius monoidal functor. The other Frobenius equation follows from a similar diagram.

To prove the second part of the theorem, assume that $\mathscr{A}$ is a closed monoidal category and that equation (♯) holds. The following calculation verifies the claim.

$$\begin{aligned} &\int^{A,B,C} \mathscr{A}(A\otimes B\otimes C,-)\cdot F(A\otimes B\otimes C) \\ &\qquad \cong \int^{A,B,C} \mathscr{A}(C,[A\otimes B,-])\cdot F(A\otimes B\otimes C) && (\mathscr{A}\text{ closed}) \\ &\qquad \cong \int^{A,B} F(A\otimes B\otimes [A\otimes B,-]) && (\text{Yoneda}) \\ &\qquad \cong \int^{X,A,B} \mathscr{A}(X, A\otimes B\otimes [A\otimes B,-])\cdot FX && (\text{Yoneda}) \\ &\qquad \cong \int^{X} \mathscr{A}(X,-)\otimes FX && (\sharp) \\ &\qquad \cong F && (\text{Yoneda}) \end{aligned}$$

□

Acknowledgements. The authors would like to thank Ross Street for several helpful comments.

References

[1] COCKETT, J. R. B.; KOSLOWSKI, J.; SEELY, R. A. G. Introduction to linear bicategories. The Lambek Festschrift: mathematical structures in computer science (Montreal, QC, 1997). *Math. Struct. Comp. Science* **10** (2000) 165–203. MR1770230 (2001e:03104), Zbl 0991.18007.

[2] COCKETT, J. R. B.; SEELY, R. A. G. Linearly distributive functors. Special volume on the occasion of the 60th birthday of Professor Michael Barr (Montreal, QC, 1997). *J. Pure Appl. Algebra* **143** (1999) 155–203. MR1731041 (2000m:18012), Zbl 0946.18005.

[3] DAY, BRIAN. On closed categories of functors. 1970 *Reports of the Midwest Category Seminar* IV, 1–38. Lecture Notes in Mathematics, **137**. *Springer, Berlin*, 1970. MR0272852 (42 #7733), Zbl 0203.31402.

[4] LAWVERE, F. WILLIAM. Ordinal sums and equational doctrines. *Seminar on Triples and Categorical Homology Theory* (ETH, Zrich, 1966/67), 141–155. Lecture Notes in

Mathematics 80. *Springer, Berlin*, 1969. Also Reprints in Theory and Applications of Categories **18** (2008) 113–121. MR0240158 (39 #1512), Zbl 0165.03204.

[5] ROSEBRUGH, R.; SABADINI, N.; WALTERS, R. F. C. Generic commutative separable algebras and cospans of graphs. *Theory Appl. Categories* **15** (2005) 164–177. MR2210579 (2007b:68036), Zbl 1087.18003.

[6] SZLACHÁNYI, KORNÉL. Finite quantum groupoids and inclusions of finite type. *Mathematical physics in mathematics and physics* (Siena, 2000), 393–407. Fields Inst. Comm., 30. *Amer. Math. Soc., Providence, RI*, 2001. MR1867570 (2002j:18007), Zbl 1022.18007.

DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NEW SOUTH WALES 2109 AUSTRALIA

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502 JAPAN
craig@kurims.kyoto-u.ac.jp

This paper is available via http://nyjm.albany.edu/j/2008/14-31.html.