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Primitive words and spectral spaces

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Abstract. In Bouacida–Echi–Salhi 1999 and 2000, it was shown that spectral spaces are related to foliation theory. In this paper, we prove that spectral sets and spaces are also related to the relatively new research topic *combinatorics on words*, an area in discrete mathematics motivated in part by computer science.

Let A be a finite alphabet, A^* be the free monoid generated by A (i.e., the set of all finite words over A) and A^+ be the set of nonempty words over A. A nonempty word is called *primitive* if it is not a proper power of another word. Let u be a nonempty word; then there exist a unique primitive word z and a unique integer $k \geq 1$ such that $u = z^k$. The word z is called *the primitive root* of u and is denoted by $z = p_A(u)$.

By a *language* over an alphabet A, we mean any subset of A^*. A language will be called *a primitive language* if it contains the primitive root of all its elements.

The collection $\mathcal{T} := \{O \subseteq A^* \mid p_A^{-1}(O) \subseteq O\}$ defines a topology on A^* (which will be called the *topology of primitive languages*).

Call a $\mathcal{PL}$*-space*, each topological space X which is homeomorphic to A^+ (equipped with the topology of primitive languages) for some finite alphabet A.

The main goal of this paper is to prove that the one-point compactification of a $\mathcal{PL}$-space is a spectral space, providing a new class of spectral spaces in connection with combinatorics on words.

Contents

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1. Introduction and notations

By an *alphabet* we mean a finite nonempty set A. The elements of A are called letters of A. A *finite word* over an alphabet A is a finite sequence of elements of A. The set of all finite words is denoted by A^*. The sequence of zero letters is called the *empty word* and denoted by ε_A. We will denote by A^+ the set of all finite nonempty words. If $u := u_1 \ldots u_n$ is a finite sequence of n letters, then n is called *the length* of the word u and we denote it by $|u|$. Let us denote by A^n the set of all finite words over A of length n. The *concatenation* of two words $u := u_1 \ldots u_n$ and $v := v_1 \ldots v_m$ of lengths respectively n and m is the word $uv := u_1 \ldots u_n v_1 \ldots v_m$ of length $n + m$. The set A^* equipped with the concatenation operation is a monoid with ε_A as a unit element. A *power* of a word u is a word of the form u^k for some $k \in \mathbb{N}$. It is convenient to set $u^0 = \varepsilon_A$, for each word u. When $k \in \mathbb{N}^+ \setminus \{1\}$, we say that u^k is a *proper power* of u (here and throughout the paper $\mathbb{N}^+$ stands for the set of all positive natural numbers).

A word u is said to be a *prefix* (resp. *suffix*, resp. *factor*) of a word v if there exists a word t (resp. t, resp t and s) such that $ut = v$ (resp. $tu = v$, resp. $tus = v$). If $u = vt$, then we set $ut^{-1} := v$ or $v^{-1}u := t$. The prefix of length k of a word u will be denoted by $\mathrm{pref}_k(u)$.

A word is called *primitive* if it is nonempty and not a proper power of another word. The concept of primitive words plays a crucial role in combinatorial theory of words (see [24] and [23]).

Topology has proved to be an essential tool for certain aspects of theoretical computer science. Conversely, the problems that arise in the computational setting have provided new and interesting stimuli for topology. These problems also have increased the interaction between topology and related areas of mathematics such as order theory and topological algebra.

An *Alexandroff space* is a topological space in which any intersection of open sets is open. These spaces were first introduced by Alexandroff in 1937 [1] with the name of *Diskrete Räume*. In [28], Steiner has called Alexandroff spaces *principal spaces*. Topologically, Alexandroff spaces play an important role in the study of the structure of the lattice of topologies on a given set [28]: the lattice of topologies on any set is complemented; moreover, each topology has an Alexandroff topology complement.

It is worth noting that the interest in Alexandroff spaces was a consequence of the very important role of finite spaces in digital topology and the fact that these spaces have all the properties of finite spaces relevant for such theory.

It is, also, worth noting that Alexandroff T_0-spaces have been studied as discrete models of continuous spaces in theoretical physics ([15] and [27]).

Recently, Alexandroff topologies proved to be useful for some authors in providing examples and counterexamples in several papers dealing with topology or foliation theory (see for instance [7], [5], [6], and [11]).

Let X be a set. Recall that a map $\mu : 2^X \longrightarrow 2^X$ is said to be a *Kuratowski closure* on X, if it satisfies the following properties:

(1) $\mu(\emptyset) = \emptyset$.
(2) $A \subseteq \mu(A)$, for each $A \in 2^X$.
(3) $\mu(\mu(A)) = \mu(A)$, for each $A \in 2^X$.
(4) $\mu(A \cup B) = \mu(A) \cup \mu(B)$, for each $A, B \in 2^X$.

It is a part of the folklore of general topology that each topology $\mathcal{T}$ on a set X defines a Kuratowski closure ($A \longmapsto \overline{A}$); and conversely, for each Kuratowski closure $\mu : 2^X \longrightarrow 2^X$, there is a unique topology on X such that $\mu(A) = \overline{A}$, for each $A \in 2^X$.

If A is an alphabet, then the map $\mu_A : 2^{A^*} \longrightarrow 2^{A^*}$, defined by $\mu_A(L) = L \cup p_A(L)$ is a Kuratowski closure defining an Alexandroff topology on A^*.

The topology defined previously will be called the *topology of primitive languages* on A^* (*$\mathcal{PL}$-topology*, for short) and we denote it by $\mathcal{PL}(A)$ (in other words, the $\mathcal{PL}$-topology is the topology which has primitive languages as closed sets).

Call a *$\mathcal{PL}$-space*, each topological space X which is homeomorphic to A^+ (equipped with the topology of primitive languages) for some alphabet A.

Let X be a topological space, set $\widetilde{X} = X \cup \{\infty\}$ with the topology whose members are the open subsets of X and all subsets U of $\widetilde{X}$ such that $\widetilde{X} \setminus U$ is a closed compact subset of X. The space $\widetilde{X}$ is called the *Alexandroff extension* of X (or the *one point compactification* of X).

The main goal of this paper is to prove that the one-point compactification of a $\mathcal{PL}$-space is a spectral space, providing a new class of spectral spaces in connection with combinatorics on words (a relatively new research topic in discrete mathematics, mainly, but not only, motivated by computer science).

In order to make this paper as self contained as possible, Section 2 will contain material about words and Alexandroff spaces which is needed throughout.

2. Words and Alexandroff spaces

We begin by recalling some preliminary results.

Lemma 2.1 (Lyndon–Schutzenberger [25]). *Let $u, v \in A^*$ with $uv = vu$. Then there exists a word t such that $u, v \in t^* := \{t^n \mid n \in \mathbb{N}\}$.*

Lemma 2.2 (Lyndon–Schutzenberger [25]). *Let $u \in A^+$. Then there exists a unique primitive word z and a unique integer $k \geq 1$ such that $u = z^k$.*

Notations 2.3. Let $u \in A^+$. By Lemma 2.2, there exists a unique primitive word z and a unique integer $k \geq 1$ such that $u = z^k$.

- The word z is called *the primitive root* of u and is denoted by $z = p_A(u)$.
- The integer k is called *the exponent* of u and is denoted by $k = \mathfrak{e}(u)$.

Let $\mathrm{Pr}(n,k)$ be the number of primitive words of length n over a nonempty alphabet A of size k. In [29], Wang has counted the number $\mathrm{Pr}(n,k)$ by using an inclusion/exclusion argument; this count is usually performed by using Möbius transformations.

Let us recall — see for instance [23] — that we have

$$\mathrm{Pr}(n,k) = \sum_{d|n} k^d \mu\left(\frac{n}{d}\right),$$

where μ is the Möbius function.

It follows that if the alphabet A is not a singleton, then there are infinitely many primitive words over A. This fact will be used extensively in the present paper.

Now we give some elementary properties of the $\mathcal{PL}$-topology. Note that a systematic study of $\mathcal{PL}$-topologies has been done by Bouallègue–Echi–Naimi in [8].

Let us, first, recall some concepts. Given a poset $(X,\leq)$ and $x \in X$, the *generization* of x in X is $(\downarrow x) = \{y \in X \mid y \leq x\}$, the *specialization* of x in X is $(x \uparrow) = \{y \in X \mid y \geq x\}$. Let X have a topology $\mathcal{T}$ and a partial ordering $\leq$. We say that $\mathcal{T}$ is *compatible* with $\leq$ if $\overline{\{x\}} = (x \uparrow)$ for all $x \in X$.

Let $(X,\mathcal{T})$ be a T_0-space. Then X has a partial ordering $\leq$ *induced by* $\mathcal{T}$ by defining $x \leq y$ if and only if $y \in \overline{\{x\}}$.

Let X be a topological space. Recall that X is said to be *a submaximal space* if each dense subset of X is open [9]. According to Kelley [18], a topological space X is said to be a *door space* if every subset of X is either closed or open. Thus, clearly, every door space is a submaximal space. The converse does not hold.

Let $(X,\mathcal{T})$ be a T_0-space and $\leq$ be the ordering induced by $\mathcal{T}$. A chain $x_0 < x_1 < \cdots < x_n$ of elements of X is said to be of length n; the supremum of the lengths is called the *Krull dimension* of $(X,\mathcal{T})$, which we write as $\dim_K(X,\mathcal{T})$.

Let us recall a characterization of submaximal Alexandroff spaces due to Bezhanishvili et al [4].

Proposition 2.4 ([4, Proposition 4.1])**.** *Let $(X,\mathcal{T})$ be an Alexandroff T_0-space. Then the following statements are equivalent.*

(i) *X is submaximal;*
(ii) $\dim_K(X,\mathcal{T}) \leq 1$.

Recall that, according to [13], a space X is said to *a quasi-Hausdorff space* if for each distinct points $x,y \in X$, either there exists $z \in X$ such that $x,y \in \overline{\{z\}}$, or x and y have disjoint neighborhoods. A space X is said to be *a $T_{\frac{1}{2}}$-space* if each point of X is either closed or open.

The following result sheds light on the $\mathcal{PL}$-topology and provides a class of submaximal spaces which are not door.

Proposition 2.5. *Let A be an alphabet. We equip A^* by its $\mathcal{PL}$-topology. Then the following properties hold.*

(1) *If $u \in A^+$ is a primitive word, then u is a closed point and the smallest open set containing u is $\mathcal{V}_A(u) := \{u^n : n \in \mathbb{N}^+\}$.*
(2) *If $u \in A^+$ is not a primitive word, then u is an open point and*
$$\overline{\{u\}} = \{u, p_A(u)\}.$$
(3) *ε_A is the unique clopen point of A^* (clopen:= closed and open).*
(4) $\dim_K(A^*) = 1$.
(5) *A^* is a $T_{\frac{1}{2}}$-space.*
(6) *A^* is a submaximal space. In addition, A^* is a door space if and only if A is a singleton.*
(7) *A^* is quasi-Hausdorff.*

Proof. Let us, first, note that $\overline{\{\varepsilon_A\}} = \{\varepsilon_A\}$ and $A^* \setminus \{\varepsilon_A\} = A^+ = \mu(A^+)$. Consequently, ε_A is a clopen point of A^*.

(1) Since A^* is an Alexandroff space, the smallest open set containing a word u is $\mathcal{V}_A(u) = \{v \in A^* : u \in \overline{\{v\}}\} = \{v \in A^* : u \in \{v, p_A(v)\}\}$.

If $u \in A^+$ is a primitive word and $v \in \mathcal{V}_A(u)$, then $u = v$ or $u = p_A(v)$. By Lemma 2.2, there exists a unique $n \in \mathbb{N}^+$ such that $v = u^n$.

Conversely, if $v \in \{u^n : n \in \mathbb{N}^+\}$, then $p_A(v) = u$ (by Lemma 2.2) and thus $u \in \overline{\{v\}}$, proving that $v \in \mathcal{V}_A(u)$. Also, it is clear that u is a closed point of A^*.

(2) Suppose that $u \in A^+$ is not a primitive word. Let $v \in \mathcal{V}_A(u)$. Then $u \in \overline{\{v\}}$. Hence $v \neq \varepsilon_A$, and $p_A(v)$ is a primitive word. So $u \in \{v, p_A(v)\}$ and accordingly $u = v$ (since u is not primitive). It follows that $\mathcal{V}_A(u) = \{u\}$; therefore u is an open point.

(3) Let $u \in A^+$.

- If u is a primitive word, then $\mathcal{V}_A(u) = \{u^n : n \in \mathbb{N}\} \neq \{u\}$. This implies that u is not an open point.
- If u is not primitive, then $\overline{\{u\}} = \{u, p_A(u)\} \neq \{u\}$; and thus u is not a closed point.

(4) and (5) These properties follow immediately from (1), (2).

(6) The fact that A^* is a submaximal space follows immediately from Proposition 2.4.

Now, we will check when A^* is a door space. Two cases are to be considered.

Case 1. Suppose that A is a singleton (say $A = \{a\}$). Then a is the unique primitive word over A. Let S be a nonempty subset of A^*.

- If $a \in S$, then $A^* \setminus S$ contains only nonprimitive words. Hence $A^* \setminus S$ is an open set, by (2). Thus S is a closed set.
- If $a \notin S$, then $a \in A^* \setminus S$ and thus $S = A^* \setminus (A^* \setminus S)$ is an open set.

So in this case A^* is a door space.

Case 2. Suppose that A contains more than one element. Then there are infinitely many primitive words over A. Let $u \in A^+$ be a primitive word and v a nonprimitive word with primitive root distinct from u. Then the subset $S := \{u, v\}$ is neither closed ($p_A(v) \notin S$) nor open ($\mathcal{V}_A(u) = \{u^n : n \in \mathbb{N}^+\} \nsubseteq S$). We conclude that A^* is not a door space.

(7) Let u, v be two distinct words over A. We consider three cases.

Case 1. Suppose that one of the two words is empty. Say, for instance, $u = \varepsilon_A$. Then $\{\varepsilon_A\}$ and $\mathcal{V}_A(p_A(v))$ are disjoint neighborhoods for u, v.

Case 2. Suppose that neither u nor v is primitive; then $\{u\}$ and $\{v\}$ are disjoint neighborhoods for u, v.

Case 3. Suppose that u and v are nonempty words and one of them is primitive (say for example u is a primitive word). Two subcases have, also, to be considered.

- If $p_A(u) \neq p_A(v)$. Then $\mathcal{V}_A(p_A(u))$ and $\mathcal{V}_A(p_A(v))$ are disjoint neighborhoods, respectively, of u and v.
- If $p_A(u) = p_A(v)$, then $u, v \in \overline{\{v\}} = \{u, v\}$.

Therefore, A^* is a quasi-Hausdorff space. □

Let $\leq$ be an ordering on a set X. A subset Y of X is said to be *left-directed* in $(X, \leq)$ if for each $x, y \in Y$, there is some $z \in Y$ such that $z \leq x$ and $z \leq y$.

Recall that a subspace Y of a topological space X is said to be *irreducible* if any two nonempty open sets of Y meet. It is worth noting that Y is irreducible if and only if $\overline{Y}$ is irreducible.

Let $(X, \mathcal{T})$ be a T_0-space and $\leq$ be the ordering induced by the topology $\mathcal{T}$. If Y is a left-directed subset of X, then Y is irreducible in $(X, \mathcal{T})$.

Now, suppose that $(X, \mathcal{T})$ is an Alexandroff space; then a subset $Y \subseteq X$ is irreducible if and only if Y is left-directed (see Hoffman [14, Lemma 1.1]).

A T_0-space X is said to be *sober* if each nonempty irreducible closed subset C of X has a generic point (that is, there exists $a \in C$ such that $C = \overline{\{a\}}$).

The following result gives a complete characterization of Alexandroff sober spaces.

Theorem 2.6. *Let X be an Alexandroff T_0-space and $\leq$ be the ordering induced by the topology of X. Then the following statements are equivalent:*

(1) *X is a sober space.*
(2) *Each decreasing sequence of $(X, \leq)$ is stationary.*

Proof. (1) $\Longrightarrow$ (2) First, let us show that if Y is an irreducible subset of X, then Y has an infimum.

Indeed, since $\overline{Y}$ is an irreducible closed set of X, it has a generic point a; $\overline{\{a\}} = \overline{Y}$. Thus $a \leq y$, for each $y \in \overline{Y}$. On the other hand, if $z \in X$ is such that $z \leq y$, for each $y \in Y$, then $Y \subseteq \overline{\{z\}}$. Therefore, $\overline{Y} \subseteq \overline{\{z\}}$; and consequently, $z \leq a$; proving that a is an infimum of Y.

Let $(x_n)_{n\in\mathbb{N}}$ be a decreasing sequence of $(X, \leq)$. Since $Y = \{x_n \mid n \in \mathbb{N}\}$ is left-directed, then Y is an irreducible subset of X and consequently, it has an infimum a in $\overline{Y}$, by the previous comment; so that $\overline{\{a\}} = \overline{Y}$.

But $(\downarrow a)$ is an open subset of X containing a, this yields $(\downarrow a) \cap Y \neq \emptyset$. Hence there is $p \in \mathbb{N}$ such that $x_p \in (\downarrow a)$. Therefore,

$$x_n \leq x_p \leq a \leq x_n \leq x_p$$

for each $n \geq p$, proving that the sequence $(x_n)_{n\in\mathbb{N}}$ is stationary.

(2)$\Longrightarrow$(1) Let C be an irreducible closed subset of X. Suppose that C has no generic point. Pick $x_0 \in C$. Then $\overline{\{x_0\}} \subseteq C$; so that there is $y_0 \in C$ such that $y_0 \notin \overline{\{x_0\}}$; consequently, $\overline{\{x_0\}} \cup \overline{\{y_0\}} \subseteq C$. Since C is left-directed [14, Lemma 1.1], there is $x_1 \in C$ such that $x_1 \leq x_0$ and $x_1 \leq y_0$.

Necessarily, $x_0 \not\leq x_1$; if not $\overline{\{x_0\}} = \overline{\{x_1\}}$ and thus $y_0 \in \overline{\{x_1\}} = \overline{\{x_0\}}$, a contradiction. One may do the same thing for $x_1 \in C$ in order to get $x_2 \in C$ such that $x_2 \leq x_1$ and $x_1 \not\leq x_2$ etc …This procedure provides a decreasing sequence $(x_n)_{n\in\mathbb{N}}$ of elements of C which is not stationary, a contradiction. It follows that C has a generic point. □

Remark 2.7. The assertion that if Y is an irreducible subset of a sober space X then Y has an infimum is proved in the Johnstone's book *Stone spaces* [16] (it is a part of the proof of [16, Lemma I.1.9]).

3. Spectral sets and spaces

A topology $\mathcal{T}$ on a set X is said to be *spectral* [13] if and only if the following axioms hold:

(1) $\mathcal{T}$ is sober (that is, every nonempty irreducible closed subset of X is the closure of a unique point).
(2) $(X, \mathcal{T})$ is compact.
(3) The compact open subsets of X form a basis of $\mathcal{T}$.
(4) The family of compact open subsets of X is closed under finite intersections.

In lattice theory, a spectral space is showed to be homeomorphic to the prime spectrum of a bounded (with 0 and 1) distributive lattice.

Let Spec(R) denote the set of prime ideals of a commutative ring R with identity, ordered by inclusion, and call a partial ordered set *spectral* if it is order isomorphic to Spec(R) for some ring R [22]. Such spectral sets are of interest not only in (topological) ring and lattice theory, but also in computer science, in particular, in domain theory (introduced in 1970 by Dana Scott as a mathematical theory of computation for the semantics of programming languages [26]).

Although many results about spectral sets have been obtained by Dobbs, Hochster, Fontana, Lewis, Ohm and Kaplansky (see for instance, [10], [12], [13], [17], [21] and [22]), a complete algebraic characterization of spectral sets still seems very far off.

Recall that if R is a ring, then the *Zariski topology* (the hull kernel topology) for $\operatorname{Spec}(R)$ is defined by letting $C \subseteq \operatorname{Spec}(R)$ be closed if and only if there exists an ideal $\mathcal{A}$ of R such that $C = \{\mathcal{P} \in \operatorname{Spec}(R) \mid \mathcal{P} \supseteq \mathcal{A}\}$.

In the remarkable paper [13], Hochster has proved that spectral spaces are exactly topological spaces homeomorphic to the prime spectrum of a ring equipped with the Zariski topology.

One can, obviously, see that $(X, \leq)$ is spectral if and only if there exists an order compatible spectral topology on X.

By an *Alexandroff-spectral space* (*A-spectral space*, for short), we mean a topological space such that its one-point compactification (Alexandroff compactification) is a spectral space [3].

Before recalling the main result of [3], let us rewrite [3, Definition 1.5]; but with a slight change as done in [11].

Definitions 3.1. Let X be a topological space and U a subset of X.

(1) U is said to be *intersection compact open*, or ICO, if for each compact open subset O of X, $U \cap O$ is compact.
(2) U is said to be *intersection compact closed*, or ICC, if for each compact closed subset O of X, $U \cap O$ is compact.
(3) U is said to be *intersection compact open closed*, or ICOC, if it is ICO and ICC.
(4) Let $\mathcal{P}$ be a property. U is said to be *co-$\mathcal{P}$* if $X \setminus U$ satisfies $\mathcal{P}$.

A complete characterization of A-spectral spaces has been given by Belaid–Echi–Gargouri.

Theorem 3.2 ([3])**.** *A space X is A-spectral if and only if the following axioms hold.*

(1) *$(X, \mathcal{T})$ is a sober space.*
(2) *X has a basis of compact open sets which is closed under finite intersections.*
(3) *For each compact closed subset C of X, there exists a co-compact* ICOC *open subset O of X such that $O \subseteq X \setminus C$.*

An *up-spectral space* is defined to be a topological space satisfying the axioms of a spectral space with the exception that X is not necessarily compact [2]. Up-spectral spaces have been introduced and studied by Belaid and Echi [2], in order to give some substantial information on a conjecture about spectral sets raised by Lewis and Ohm in 1976 [22].

Using [3, Theorem 2.2], it is clear that an A-spectral space is up-spectral. A natural question is whether an up-spectral space is necessarily A-spectral?

The above question has been answered by Echi and Gargouri in [11].

Next, we recall a concept introduced in [11].

Definition 3.3 ([11])**.** Let X be a topological space and U a subset of X. Call U a T-subset of X, if U is a closed compact co-ICO subset of X.

Recall that a link between up-spectral spaces and A-spectral space has been given by Echi and Gargouri as follows.

Theorem 3.4 ([11, Theorem 1.8]). *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is A-spectral;*
(2) *X satisfies the following properties:*
  (*i*) *X is up-spectral.*
  (*ii*) *For each compact closed subset C of X, there exists a T-subset D of X such that $C \subseteq D$.*

Now we are in a position to state the main result of this paper.

Theorem 3.5. *Let $(X, \mathcal{T})$ be a $\mathcal{PL}$-space and $\leq$ be the ordering defined on X by: $u \leq v$ if and only if $v \in \overline{\{u\}}$. Then the following properties hold.*

(1) *$(X, \leq)$ is a spectral set.*
(2) *$(X, \mathcal{T})$ is an up-spectral space.*
(3) *$(X, \mathcal{T})$ is spectral if and only if X has a unique closed point (in other words, any alphabet associated to X is a singleton).*
(4) *$(X, \mathcal{T})$ is an A-spectral space.*

Proof. One may suppose without loss of generality that $X = A^+$ (equipped with the topology of primitive languages), for some finite alphabet A.

(1) Let $\mathrm{Prim}(A)$ be the set of all primitive words over A. For $u \in \mathrm{Prim}(A)$, we let X_u be the set $\{u^n \mid n \in \mathbb{N}^+\}$. Then

$$X = \bigcup_{u \in \mathrm{Prim}(A)} X_u$$

is an ordered disjoint union, where the ordering on X_u is defined by $u^n \leq u^m$ if and only if $n = m$ or $m = 1$.

By [22, Teorem 4.1], in order to prove that $(X, \leq)$ is spectral, it is enough to show that $(X_u, \leq)$ is spectral, for each $u \in \mathrm{Prim}(A)$. But this is an immediate consequence of the characterization of L(eft)-spectral spaces done by Dobbs–Fontana–Papick in [10]. Therefore, $(X, \leq)$ is a spectral set.

(2)

– *$(X, \mathcal{T})$ is a sober space.*
  This follows immediately from Proposition 2.5(4) and Theorem 2.6.
– *$(X, \mathcal{T})$ has a basis of compact open sets which is closed under finite intersections.*
  It is easily seen that $\mathcal{B} := \{\mathcal{V}_A(u) \mid u \in A^*\} \cup \{\emptyset\}$ is a basis of compact open sets; and for each two words $u, v \in A^*$, the two sets $\mathcal{V}_A(u)$ and $\mathcal{V}_A(v)$ are comparable or do not meet.

(3) According to (2), X is spectral if and only if it is compact.

– If A is not a singleton, then there are infinitely many primitive words. Hence

$$X := \bigcup_{u\in \mathrm{Prim}(A)} \mathcal{V}_{_A}(u)$$

is an open covering of X which has no finite subcover. Thus X is not compact.

– Now, if A is a singleton (say for example $A = \{a\}$), then the unique closed point (the unique primitive word) of A^* is the word a. In this case, $X = \mathcal{V}_{_A}(a)$; and thus X is compact.

(4) To prove that X is A-spectral, let us use Theorem 3.2. Since X is up-spectral, it is enough to show that $(X, \mathcal{T})$ satisfies Condition (3) of Theorem 3.2.

Let C be a compact closed set of X. Then there exist finitely many primitive words $p_1, p_2, \dots, p_n$ over A such that

$$C \subseteq \mathcal{V}_{_A}(p_1) \cup \mathcal{V}_{_A}(p_2) \cup \dots \mathcal{V}_{_A}(p_n).$$

Set

$$O := \bigcup_{u\in \mathrm{Prim}(A)\setminus\{p_1,p_2,\dots,p_n\}} \mathcal{V}_{_A}(u);$$

then, clearly, O is a clopen set of X. It is easy to check that each closed set of a topological space is ICOC. Thus O is a co-compact ICOC open subset of X such that $O \subseteq X \setminus C$.

Therefore, $(X, \mathcal{T})$ is an A-spectral space. □

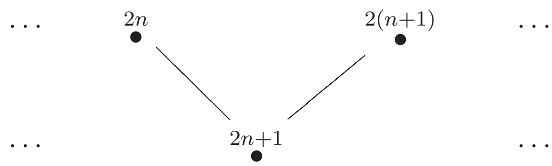
Next, we give an example a topological space satisfying Properties (1), (2) of Theorem 3.5 and Conditions (4), (5), (6) of Proposition 2.5 which is not a $\mathcal{PL}$-space.

First, let us recall that the study of the geometric and topological properties of digital images is the goal of digital topology. In the process of digitizing a movie, some situations are often represented by subspaces and quotients of locally finite topological spaces, so the study of these topological spaces is important. The digital line, also known as the *Khalimsky line*, is the major building bloc of the digital n-space. The Khalimsky line is the set of the integers $\mathbb{Z}$, equipped with the topology $\mathcal{K}$, generated by the subbase $\mathcal{G}_{\mathcal{K}} = \{\{2n-1, 2n, 2n+1\} \mid n \in \mathbb{Z}\}$ [19], [20]. Hence a set $U \subseteq \mathbb{Z}$ is open in $\mathcal{K}$ if and only if whenever $x \in U$ is an even integer, then $x-1, x+1 \in U$.

Example 3.6. The Khalimsky line $(\mathbb{Z}, \mathcal{K})$ satisfies the following properties:

(1) $\mathcal{K}$ is an Alexandroff topology.

(2) $\overline{\{2n\}} = \{2n\}$ and $\overline{\{2n+1\}} = \{2n, 2n+1, 2n+2\}$, for each $n \in \mathbb{Z}$. Let $\leq_{\mathcal{K}}$ be the ordering induced by the Khalimsky topology. Then the ordered set $(\mathbb{Z}, \leq_{\mathcal{K}})$ is a 1-dimensional poset which looks like



(3) For each $n \in \mathbb{Z}$, $\{2n\}$ is closed and $\{2n+1\}$ is open in $(\mathbb{Z}, \mathcal{K})$. Thus $(\mathbb{Z}, \mathcal{K})$ is a $T_{\frac{1}{2}}$-space.
(4) $(\mathbb{Z}, \mathcal{K})$ is a submaximal space.
(5) $\mathcal{K}$ is a quasi-Hausdorff topology.
(6) $(\mathbb{Z}, \leq_{\mathcal{K}})$ is a spectral set.
(7) $(\mathbb{Z}, \mathcal{K})$ is a noncompact A-spectral space.
(8) $(\mathbb{Z}, \mathcal{K})$ is not a $\mathcal{PL}$-space.

Proof. Clearly, Properties (1), (2), (3) and (5) are straightforward.

(4) This property follows immediately from Proposition 2.4 and the fact that the digital line is 1-dimensional.

(6) Let X_1 be the set of elements of height 1 and X_0 the set of elements of height 0. Since for each $x \in X_0$, $(x \uparrow) \cap (y \uparrow) = \emptyset$, for all but finitely many $y \in X_0$, and also for each $x \in X_1$, $(\downarrow x) \cap (\downarrow y) = \emptyset$, for all but finitely many $y \in X_1$, the poset $(\mathbb{Z}, \leq_{\mathcal{K}})$ is spectral, by [22, Corollary 5.10].

(7) Of course, $((\downarrow 2n), n \in \mathbb{Z})$ is an open covering of $\mathbb{Z}$ which has no finite subcovering. Thus $(\mathbb{Z}, \mathcal{K})$ is not compact.

- The fact that the digital line is 1-dimensional implies that it is a sober space (by Theorem 2.6).
- The collection $\mathcal{B} := \{(\downarrow x) \mid x \in \mathbb{Z}\} \cup \{\emptyset\}$ is a basis of compact open sets of the digital line, which is closed under finite intersections.
- It is easily seen that compact subsets of the digital line are exactly finite sets (since $(\downarrow x)$ is finite, for each $x \in \mathbb{Z}$). Thus each subset of $\mathbb{Z}$ is ICOC. It follows that Condition (3) of Theorem 3.2 is satisfied.

Therefore, $(\mathbb{Z}, \mathcal{K})$ is A-spectral.

(8) If $X = A^+$ for some alphabet A, then for each primitive word u over A, $\mathcal{V}_A(u) = (\downarrow u)$ is an infinitely countable set. But, for the digital line, the set $(\downarrow x)$ has cardinality 1 or 3, for each $x \in \mathbb{Z}$. It follows that the digital line is not a $\mathcal{PL}$-space. □

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