New York Journal of Mathematics

New York J. Math. **14** (2008) 617–641.

A linearization of Connes' embedding problem

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ABSTRACT. We show that Connes' embedding problem for II_1-factors is equivalent to a statement about distributions of sums of self-adjoint operators with matrix coefficients. This is an application of a linearization result for finite von Neumann algebras, which is proved using asymptotic second-order freeness of Gaussian random matrices.

CONTENTS

1. Introduction

A von Neuman algebra $\mathcal{M}$ is said to be *finite* if it possesses a normal, faithful, tracial state τ. By "finite von Neumann algebra" $\mathcal{M}$, we will always mean such an algebra equipped with a fixed such trace τ. *Connes' embedding problem* asks whether every such $\mathcal{M}$ with a separable predual can be embedded in an ultrapower R^ω of the hyperfinite II_1-factor R in a trace-preserving way. This is well-known to be equivalent to the question of whether a generating set X for $\mathcal{M}$ has microstates, namely, whether there exist matrices over the complex numbers whose mixed moments up to an

Received July 20, 2007 and in revised form October 7, 2008.

Mathematics Subject Classification. 46L10,15A42.

Key words and phrases. Connes Embedding Problem, Horn Problem, random matrices, free probability, sum of matrices.

The first author's research was supported in part by NSERC grant RGPIN/341303-2007.

The second author's research was supported in part by NSF grant DMS-0600814.

ISSN 1076-9803/08

arbitrary given order approximate those of the elements of X with respect to τ, to within an arbitrary given tolerance. (See Section 3 where precise definitions and, for completeness, a proof of this equivalence are given.) We will say that $\mathcal{M}$ posseses *Connes' embedding property* if it embeds in R^ω. (It is known that possession of this property does not depend on the choice of faithful trace τ.)

Seen like this, Connes' embedding probem, which is open, is about a fundamental approximation property for finite von Neumann algebras. There are several important results, due to E. Kirchberg [14], F. Rădulescu [19], [20], [21], [22] and N. Brown [7], that have direct bearing on this problem; see also G. Pisier's paper [18] and N. Ozawa's survey [17].

Recently, H. Bercovici and W. S. Li [6] have proved a property enjoyed by elements in a finite von Neumann algebra that embeds in R^ω. This property is related to a fundamental question about spectra of sums of operators: given Hermitian matrices or, more generally, Hermitian operators A and B with specified spectra, what can the spectrum of $A+B$ be? For $N \times N$ matrices, a description was conjectured by Horn [12] and was eventually proved to be true by work of Klyachko, Totaro, Knutson, Tao and others, if by "spectrum" we mean the *eigenvalue sequence*, namely, the list of eigenvalues repeated according to multiplicity and in nonincreasing order. In this description, the possible spectrum of $A+B$ is a convex subset of $\mathbb{R}^N$ described by certain inequalities, called the *Horn inequalities*. See Fulton's exposition [9] or, for a very abbreviated decription, Section 4 of this paper. We will call this convex set the *Horn body* associated to A and B, and denote it by $S_{\alpha,\beta}$, where α and β are the eigenvalue sequences of A and B, respectively.

Bercovici and Li [5], [6] have studied the analogous question for A and B self-adjoint elements of a finite von Neumann algebra $\mathcal{M}$, namely: if spectral data of A and of B are specified, what are the possible spectral data of $A+B$? Here, by "spectral data" one can take the distribution (i.e., trace of spectral measure) of the operator in question, which is a compactly supported Borel probability measure on $\mathbb{R}$, or, in a description that is equivalent, the *eigenvalue function* of the operator, which is a nonincreasing, right-continuous function on $[0,1)$ that is the nondiscrete version of the eigenvalue sequence.

In [6], for given eigenvalue functions u and v, they construct a convex set, which we will call $F_{u,v}$, of eigenvalue functions. This set can be viewed as a limit (in the appropriate sense) of Horn bodies as $N \to \infty$. They show that the eigenvalue function of $A+B$ must lie in $F_{u,v}$ whenever A and B lie in R^ω and have eigenvalue functions u and, respectively, v.

Bercovici and Li's result provides a concrete method to attempt to show that a finite von Neumann algebra $\mathcal{M}$ does not embed in R^ω: find self-adjoint A and B in $\mathcal{M}$ for which one knows enough about the spectral data of A, B and $A+B$, and find a Horn inequality (or, rather, it's appropriate modification to the setting of eigenvalue functions) that is violated by these.

Their result also inspires two further questions:

Question 1.1. (i) Which Horn inequalitites must be satisfied by the spectral data of self-adjoints A, B and $A+B$ in *arbitrary* finite von Neumann algebras?[1]

(ii) (conversely to Bercovici and Li's result): If we know, for all self-adjoints A and B in an arbitrary finite von Neumann algebra $\mathcal{M}$, calling their eigenvalue functions u and v, respectively, that the eigenvalue function of $A+B$ belongs to $F_{u,v}$, is this equivalent to a positive answer for Connes' embedding problem?

Question (ii) above is easily seen to be equivalent to the same question, but where A and B are assumed to lie in some copies of the matrix algebra $\mathbb{M}_N(\mathbb{C})$ in $\mathcal{M}$, for some $N \in \mathbb{N}$.

Bercovici and Li, in [5], partially answered the first question by showing that all in a subset of the Horn inequalities (namely, the Freede–Thompson inequalities) are always satisfied in arbitrary finite von Neuman algebras.

We attempted to address the second question. We are not able to answer it, but we prove a related result (Theorem 4.6) which answers the analogous question for what we call the *quantum Horn bodies*. These are the like the Horn bodies, but with matrix coefficients. More precisely, if α and β are nonincreasing real sequences of length N and if a_1 and a_2 are self-adjoint $n \times n$ matrices for some n, then the quantum Horn body $K_{\alpha,\beta}^{a_1,a_2}$ is the set of all possible eigenvalue functions of matrices of the form

$$a_1 \otimes U\mathrm{diag}(\alpha)U^* + a_2 \otimes V\mathrm{diag}(\beta)V^* \tag{1}$$

as U and V range over the $N \times N$ unitaries. (In fact, Theorem 4.6 concerns the appropriate union of such bodies over all N — see Section 4 for details.)

Our proof of Theorem 4.6 is an application of a linearization result (Theorem 2.1) in finite von Neumann algebras, which implies that if X_1, X_2, Y_1 and Y_2 are self-adjoint elements of a finite von Neuman algebra and if the distributions (i.e., the moments) of

$$a_1 \otimes X_1 + a_2 \otimes X_2 \tag{2}$$

and

$$a_1 \otimes Y_1 + a_2 \otimes Y_2 \tag{3}$$

agree for all $n \in \mathbb{N}$ and all self-adjoint $a_1, a_2 \in \mathbb{M}_n(\mathbb{C})$, then the mixed moments of the pair (X_1, X_2) agree with the mixed moments of the pair (Y_1, Y_2), i.e., the trace of

$$X_{i_1} X_{i_2} \cdots X_{i_k} \tag{4}$$

[1] It was more recently shown in [4] that all Horn inequalities hold in all finite von Neumann algebras.

agrees with the trace of

$$Y_{i_1}Y_{i_2}\cdots Y_{i_k} \tag{5}$$

for all $k \in \mathbb{N}$ and all $i_1,\ldots,i_k \in \{1,2\}$. This is equivalent to there being a trace-preserving isomorphism from the von Neumann algebra generated by X_1 and X_2 onto the von Neumann algebra generated by Y_1 and Y_2, that sends X_i to Y_i.

This linearization result for von Neumann algebras is quite analogous to one for C*-algebras proved by U. Haagerup and S. Thorbjørnsen [11] (and quoted below as Theorem 2.2). However, our proof of Theorem 2.1 is quite different from that of Haagerup and Thorbjørnsen's result. Our linearization result is not so surprising because, for example, for a proof it would suffice to show that the trace of an arbitrary word of the form (4) is a linear combination of moments of various elements of the form (2). One could imagine that a combinatorial proof by explicit choice of some a_1 and a_2, etc., may be possible. However, our proof does not yield an explicit choice. Rather, it makes a random choice of a_1 and a_2. For this we make use of J. Mingo and R. Speicher's results on second-order freeness of independent GUE random matrices.

Finally, we need more than just the linearization result. We use some ultrapower techniques to reverse quantifiers. In particular, we show that for the von Neumann algebra generated by X_1 and X_2 to be embeddable in R^ω, it suffices that for all self-adjoint matrices a_1 and a_2, there exists Y_1 and Y_2 lying in R^ω such that the distributions of (2) and (3) agree. For this, it is for technical reasons necessary to strengten the linearization result (Theorem 2.1) by restricting the matrices a_1 and a_2 to have spectra in a nontrivial bounded interval $[c,d]$.

To recap: in Section 2 we prove the linearization result, making use of second-order freeness. In Section 3, we review Connes' embedding problem and it's formulation in terms of microstates; then we make an ultrapower argument to prove a result (Theorem 3.4) characterizing embeddability of a von Neumann algebra generated by self-adjoints X_1 and X_2 in terms of distributions of elements of the form (2). In Section 4, we describe the quantum Horn bodies, state some related questions and consider some examples. We finish by rephrasing Connes' embedding problem in terms of the quantum Horn bodies.

2. Linearization

Notation: we let $\mathbb{M}_n(\mathbb{C})$ denote the set of $n\times n$ complex matrices, while $\mathbb{M}_n(\mathbb{C})_{s.a.}$ means the set of self-adjoint elements of $\mathbb{M}_n(\mathbb{C})$. We denote by $\mathrm{Tr} : \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$ the unnormalized trace, and we let $\mathrm{tr} = \frac{1}{n}\mathrm{Tr}$ be the normalized trace (sending the identity element to 1).

The main theorem of this section is:

Theorem 2.1. *Let $\mathcal{M}$ be a von Neumann algebra generated by self-adjoint elements $X_1, \ldots, X_k$ and $\mathcal{N}$ be a von Neumann algebra generated by self-adjoint elements $Y_1, \ldots, Y_k$. Let τ be a faithful trace on $\mathcal{M}$ and χ be a faithful trace on $\mathcal{N}$.*

Let $c < d$ be real numbers and suppose that for all $n \in \mathbb{N}$ and all $a_1, \ldots, a_k$ in $\mathbb{M}_n(\mathbb{C})_{s.a.}$ whose spectra are contained in the interval $[c, d]$, the distributions of $\sum_i a_i \otimes X_i$ and $\sum_i a_i \otimes Y_i$ are the same.

Then there exists an isomorphism $\phi : \mathcal{M} \to \mathcal{N}$ such that $\phi(X_i) = Y_i$ and $\chi \circ \phi = \tau$.

The statement of this theorem can be thought of as a version for finite von Neumann algebras of the following C^*-algebra linearization result of Haagerup and Thorbjørnsen.

Theorem 2.2 ([11])**.** *Let A (respectively B) be a unital C^*-algebra generated by self-adjoints $X_1, \ldots, X_k$ (resp. $Y_1, \ldots, Y_k$) such that for all positive integers n and for all $a_0, \ldots, a_k \in \mathbb{M}_n(\mathbb{C})_{s.a.}$,*

$$a_0 \otimes 1 + a_1 \otimes X_1 + \cdots + a_k \otimes X_k \tag{6}$$

and

$$a_0 \otimes 1 + a_1 \otimes Y_1 + \cdots + a_k \otimes Y_k \tag{7}$$

have the same spectrum, then there exists an isomorphism ϕ from A onto B such that $\phi(X_i) = Y_i$.

However, our proof of Theorem 2.1 is quite different from the proof of Theorem 2.2. In addition, there is the notable difference that we do not need to consider matrix coefficients of the identity. In order to simplify our notation, we restrict to proving the $k = 2$ case of Theorem 2.1. We indicate at Remark 2.10 how our proof works in general.

Let $X^\sharp$ be the free monoid generated by free elements x_1, x_2, and

$$\mathbb{C}\langle x_1, x_2 \rangle = \mathbb{C}[X^\sharp]. \tag{8}$$

be the free unital $*$-algebra over self-adjoint elements x_1, x_2.

Let ρ be the rotation action of the integers on the set $X^\sharp$, given by

$$\rho(x_{i_1} \ldots x_{i_n}) = x_{i_2} \ldots x_{i_n} x_{i_1}. \tag{9}$$

Let $X^\sharp/\rho$ denote the set of orbits of this action. Let $\mathcal{I}$ be the vector space spanned by the commutators $[P, Q]$ with $P, Q \in \mathbb{C}\langle x_1, x_2 \rangle$. Recall that an (algebraic) trace is a linear map $\tau : \mathbb{C}\langle x_1, x_2 \rangle \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$. Equivalently, a linear map $\tau : \mathbb{C}\langle x_1, x_2 \rangle \to \mathbb{C}$ is a trace if and only if it vanishes on $\mathcal{I}$.

Lemma 2.3. *For any orbit $O \in X^\sharp/\rho$, let $V_O = \operatorname{span} O \subseteq \mathbb{C}\langle x_1, x_2 \rangle$. Then $\mathbb{C}\langle x_1, x_2 \rangle$ splits as the direct sum*

$$\mathbb{C}\langle x_1, x_2 \rangle = \bigoplus_{O \in X^\sharp/\rho} V_O. \tag{10}$$

Moreover, the commutator subspace $\mathcal{I}$ splits across this direct sum as

$$\mathcal{I} = \bigoplus_{O \in X^\sharp/\rho} V_O \cap \mathcal{I}. \tag{11}$$

Furthermore, $V_O \cap \mathcal{I}$ is of codimension 1 in V_O and we have

$$V_O \cap \mathcal{I} = \left\{ \sum_{x \in O} c_x x \mid c_x \in \mathbb{C}, \sum_{x \in O} c_x = 0 \right\}. \tag{12}$$

Proof. The direct sum decomposition (10) is obvious. From the relation

$$x_{i_1} x_{i_2} \dots x_{i_n} = [x_{i_1} x_{i_2} \dots x_{i_{n-1}}, x_{i_n}] + x_{i_n} x_{i_1} x_{i_2} \dots x_{i_{n-1}}\,, \tag{13}$$

one easily sees

$$\mathcal{I} \subseteq \left\{ \sum_{x \in X^\sharp} c_x x \;\middle|\; c_x \in \mathbb{C}, \sum_{x \in X^\sharp} c_x = 0 \right\} \tag{14}$$

$$\left\{ \sum_{x \in O} c_x x \;\middle|\; c_x \in \mathbb{C}, \sum_{x \in O} c_x = 0 \right\} \subseteq V_O \cap \mathcal{I}, \tag{15}$$

from which the assertions follow. □

An orbit $O \in X^\sharp/\rho$ is a singleton if and only if it is of the form $\{x_i^a\}$ for some $i \in \{1, 2\}$ and some integer $a \geq 0$. For each orbit that is not a singleton, choose a representative of the orbit of the form

$$x = x_1^{a_1} x_2^{b_2} \cdots x_1^{a_n} x_2^{b_n} \tag{16}$$

with $n \geq 1$ and $a_1, \dots, a_n, b_1, \dots, b_n \geq 1$, and collect them together in a set S, of representatives for all the orbits in $X^\sharp/\rho$ that are not singletons.

Let $\widetilde{U}_i$ and $\widetilde{T}_i$ ($i \in \mathbb{N}$) be two families of polynomials, which we will specify later on, such that the degree of each $\widetilde{U}_i$ and $\widetilde{T}_i$ is i. For $x \in S$ written as in (16), we let

$$\widetilde{U}^x = \widetilde{U}_{a_1}(x_1)\widetilde{U}_{b_1}(x_2) \cdots \widetilde{U}_{a_n}(x_1)\widetilde{U}_{b_n}(x_2) \in \mathbb{C}\langle x_1, x_2 \rangle. \tag{17}$$

Lemma 2.4. *The family*

$$\Xi = \{1\} \cup \{\widetilde{T}_a(x_i) \mid a \in \mathbb{N}, i \in \{1, 2\}\} \cup \{\widetilde{U}^x \mid x \in S\} \subseteq \mathbb{C}\langle x_1, x_2 \rangle \tag{18}$$

is linearly independent and spans a space $\mathcal{J}$ such that

$$\mathcal{I} + \mathcal{J} = \mathbb{C}\langle x_1, x_2 \rangle \tag{19}$$

$$\mathcal{I} \cap \mathcal{J} = \{0\}. \tag{20}$$

Proof. For an orbit $O \in X^\sharp/\rho$, the total degree of all $x \in O$ agree; denote this integer by $\deg(O)$. Letting $V_O = \operatorname{span} O$ and using Lemma 2.3, an argument by induction on $\deg(O)$ shows $V_O \subseteq \mathcal{I} + \mathcal{J}$. This implies (19).

To see the linear independece of (18) and to see (20), suppose

$$y = c_0 1 + \sum_{n=1}^{\infty}\Big(c_a^{(1)}\widetilde{T}_a(x_1) + c_a^{(2)}\widetilde{T}_a(x_2)\Big) + \sum_{x\in S} d_x \widetilde{U}^x, \tag{21}$$

for complex numbers c_0, $c_n^{(i)}$ and d_x, not all zero, and let us show $y \notin \mathcal{I}$. We also write

$$y = \sum_{z\in X^\sharp} a_z z \tag{22}$$

for complex numbers a_z.

Suppose $d_x \neq 0$ for some x and let $x \in S$ be of largest degree such that $d_x \neq 0$. Let $O \in X^\sharp/\rho$ be the orbit of x. Then

$$\sum_{z\in O} a_z z = d_x x \notin V_O \cap \mathcal{I}. \tag{23}$$

By the direct sum decomposition (11), we get $y \notin \mathcal{I}$.

On the other hand, if $c_n^{(i)} \neq 0$ for some $i \in \{1,2\}$ and some $n \geq 1$. Suppose n is the largest such that $c_n^{(i)} \neq 0$. Then $a_{x_i^n} = c_n^{(i)} \neq 0$, and $y \notin \mathcal{I}$.

Finally, if $d_x = 0$ for all $x \in S$ and if $c_n^{(i)} = 0$ for some $i \in \{1,2\}$ and some $n \geq 1$, then we are left with $c_0 \neq 1$ and $y = c_0 1 \notin \mathcal{I}$. □

We recall that a Gaussian unitary ensemble (also denoted by *GUE*) is the probability distribution of the random matrix $Z_N + Z_N^*$ on $\mathbb{M}_N(\mathbb{C})$, where Z_N has independent complex gaussian entries of variance $1/2N$. This distribution has a density proportional to $e^{-N\mathrm{Tr}X^2}$ with respect to the Lebesgue measure on the self-adjoint real matrices. A classical result of Wigner [26] states that the empirical eigenvalue distribution of a *GUE* converges as $N \to \infty$ in moments to Wigner's semicircle distribution

$$\frac{1}{2\pi} 1_{[-2,2]}(x)\sqrt{4-x^2}dx. \tag{24}$$

If we view the X_N for various N as matrix-valued random variables over a commone probability space, then almost surely, the largest and smallest eigenvalues of X_N converge as $N \to \infty$ to ± 2, respectively. This was proved by Bai and Yin [3] (see also [2]). See [10] for further discussion and an alternative proof.

We recall that the Chebyshev polynomials of the first kind T_k are the monic polynomials orthogonal with respect to the weight

$$1_{(-1,1)}(x)(1-x^2)^{-1/2}dx$$

and are also given by $T_k(\cos\theta) = \cos(k\theta)$. Alternatively, they are determined by their generating series

$$\sum_{k\geq 0} T_k(x)t^k = \frac{1-tx}{1-2tx+t^2}. \tag{25}$$

The following result is random matrix folklore, but it is implied by more general results of Johansson ([13], Cor 2.8):

Proposition 2.5. *Let X_N be the* GUE *of dimension N and T_n the Chebyshev polynomial of first kind. Let*

$$\alpha_n = \frac{1}{2\pi}\int_{-2}^{2} T_n(t)\sqrt{4-t^2}\,dt. \tag{26}$$

Then for every $m \in \mathbb{N}$, the real random vector

$$2\left(\frac{\mathrm{Tr}(T_n(X_N)) - N\alpha_n}{\sqrt{n}}\right)_{n=1}^{m} \tag{27}$$

tends in distribution as $N \to \infty$ toward a vector of independent standard real Gaussian variables.

Consider two GUE random matrix ensembles $(X_N)_{N\in\mathbb{N}}$ and $(Y_N)_{N\in\mathbb{N}}$, that are independent from each other (for each N). Voiculescu proved [24] that these converge in moments to free semicircular elements s_1 and s_2 having first moment zero and second moment 1, meaning that we have

$$\lim_{N\to\infty} E(\mathrm{tr}(X_N^{k_1}Y_N^{\ell_1}\cdots X_N^{k_m}Y_N^{\ell_m})) = \tau(s_1^{k_1}s_2^{\ell_2}\cdots s_1^{k_m}s_2^{\ell_m}) \tag{28}$$

for all $m \geq 1$ and $k_i, \ell_i \geq 0$, (where τ is a trace with respect to which s_1 and s_2 are semicircular and free). Of course, by freeness, this implies that if p_i and q_i are polynomials such that $\tau(p_i(s_1)) = 0 = \tau(q_i(s_2))$ for all $i \in \{1,\dots,m\}$, then

$$\lim_{N\to\infty} E(\mathrm{tr}(p_1(X_N)q_1(Y_N)\cdots p_m(X_N)q_m(Y_N))) = 0. \tag{29}$$

Mingo and Speicher [16] have proved some remarkable results about the related fluctuations, namely, the (magnified) random variables (30) below. These are asymptotically Gaussian and provide examples of the phenomenon of second-order freeness, which has been treated in a recent series of papers [16], [15], [8]. In particular, the following theorem is a straightforward consequence of some of the results in [16].

Theorem 2.6. *Let X_N and Y_N be independent GUE random matrix ensembles. Let s be a $(0,1)$-semicircular element with respect to a trace τ. Let $m \geq 1$ and let $p_1,\dots,p_m,q_1,\dots,q_m$ be polynomials with real coefficients such that $\tau(p_i(s)) = \tau(q_i(s)) = 0$ for each i. Then the random variable*

$$\mathrm{Tr}(p_1(X_N)q_1(Y_N)\cdots p_m(X_N)q_m(Y_N)) \tag{30}$$

converges in moments as $N \to \infty$ to a Gaussian random variable. Moreover, if $\widetilde{m} \geq 1$ and if $\widetilde{p}_1,\dots,\widetilde{p}_{\widetilde{m}},\widetilde{q}_1,\dots,\widetilde{q}_{\widetilde{m}}$ are real polynomials such that

$\tau(\widetilde{p}_i(s)) = \tau(\widetilde{q}_i(s)) = 0$ for each i, then

$$\lim_{N\to\infty} E\big(\mathrm{Tr}(p_1(X_N)q_1(Y_N)\cdots p_m(X_N)q_m(Y_N)) \tag{31}$$

$$\cdot\, \overline{\mathrm{Tr}(\widetilde{p}_1(X_N)\widetilde{q}_1(Y_N)\cdots \widetilde{p}_{\widetilde{m}}(X_N)\widetilde{q}_{\widetilde{m}}(Y_N))}\,\big) \tag{32}$$

$$= \begin{cases} \displaystyle\sum_{\ell=0}^{m-1}\prod_{j=1}^{m} \tau(p_j(s)\widetilde{p}_{j+\ell}(s))\tau(q_j(s)\widetilde{q}_{j+\ell}(s)), & m = \widetilde{m}, \\ 0, & m \neq \widetilde{m}, \end{cases} \tag{33}$$

where the subscripts of p and q are taken modulo m. Furthermore, for any polynomial r, we have

$$\lim_{N\to\infty} E\big(\mathrm{Tr}(p_1(X_N)q_1(Y_N)\cdots p_m(X_N)q_m(Y_N))\mathrm{Tr}(r(X_N))\big) = 0 \tag{34}$$

$$\lim_{N\to\infty} E\big(\mathrm{Tr}(p_1(X_N)q_1(Y_N)\cdots p_m(X_N)q_m(Y_N))\mathrm{Tr}(r(Y_N))\big) = 0. \tag{35}$$

If $\mathfrak{A}$ is any unital algebra and if $a_1, a_2 \in \mathfrak{A}$, we let

$$\mathrm{ev}_{a_1,a_2} : \mathbb{C}\langle x_1, x_2\rangle \to \mathfrak{A} \tag{36}$$

be the algebra homomorphism given by

$$\mathrm{ev}_{a_1,a_2}(P) = P(a_1, a_2). \tag{37}$$

In the corollary below, which follows directly from Theorem 2.6 and Proposition 2.5, we take as $\mathfrak{A}$ the algebra of random matrices (over a fixed probability space) whose entries have moments of all orders.

Corollary 2.7. *Let u and v be real numbers with $u < v$. Let A_N, B_N be independent copies of*

$$\frac{u+v}{2}\mathrm{Id} + \frac{v-u}{2}X \tag{38}$$

where X is distributed as the GUE *of dimension N. Let*

$$\widetilde{T}_i(x) := T_i\left(\frac{2}{v-u}x - \frac{u+v}{v-u}\right) \tag{39}$$

and

$$\widetilde{U}_i(x) := U_i\left(\frac{2}{v-u}x - \frac{u+v}{v-u}\right). \tag{40}$$

If $y \in S$, then we have

$$\lim_{N\to\infty} E(\mathrm{tr}\circ \mathrm{ev}_{A_N,B_N}(y)) = 0, \tag{41}$$

and we let $\beta(y) = 0$. If $y = x_i^n$ for $i \in \{1,2\}$ and $n \in \mathbb{N}$, then we have

$$\lim_{N\to\infty} E(\mathrm{tr}\circ \mathrm{ev}_{A_N,B_N}(y)) = \alpha_n\,, \tag{42}$$

where α_n is as in (26), *and we set $\beta(y) = \alpha_n$.*

Then the random variables

$$(\,(\mathrm{Tr}\circ \mathrm{ev}_{A_N,B_N})(y) - N\beta(y)\,)_{y\in\Xi\setminus\{1\}}\,, \tag{43}$$

where Ξ is as in Lemma 2.4, converge in moments as $N \to \infty$ to independent, nontrivial, centered, Gaussian variables.

The following lemma is elementary and we will only use it in the especially simple case of $\delta = 0$. We will use it to see that for a sequence z_N of random variables converging in moments to a nonzero random variable, we have that $\mathrm{Prob}(z_N \neq 0)$ is bounded away from zero as $N \to \infty$. This is all unsurprising and well-known, but we include proofs for completeness.

Lemma 2.8. *Let y be a random variable with finite first and second moments, denoted m_1 and m_2. Suppose $y \geq 0$ and $m_1 > 0$. Then for every $\delta > 0$ satisfying*

$$0 \leq \delta < \min\left(\frac{m_2}{2m_1}, m_1\right), \tag{44}$$

there is w, a continuous function of m_1, m_2 and δ, such that $0 \leq w < 1$ and

$$\mathrm{Prob}(y \leq \delta) \leq w. \tag{45}$$

More precisely, we may choose

$$w = \begin{cases} \dfrac{-m_2 + 2\delta m_1 + \sqrt{m_2^2 - 4\delta m_2(m_1 - \delta)}}{2\delta^2}, & \delta > 0, \\ 1 - \dfrac{m_1^2}{m_2}, & \delta = 0. \end{cases} \tag{46}$$

Proof. Say that y is a random variable on a probability space (Ω, μ) and let $V \subseteq \Omega$ be the set where y takes values $\leq \delta$. Using the Cauchy–Schwarz inequality, we get

$$m_1 \leq \delta\mu(V) + \int_{V^c} y \, d\mu \leq \delta\mu(V) + m_2^{1/2}(1 - \mu(V))^{1/2}, \tag{47}$$

which yields

$$\delta^2\mu(V)^2 + (m_2 - 2\delta m_1)\mu(V) + m_1^2 - m_2 \leq 0. \tag{48}$$

If $\delta = 0$, then this gives $\mu(V) \leq 1 - \frac{m_1^2}{m_2} =: w$. When $\delta > 0$, consider the polynomial

$$p(x) = \delta^2 x^2 + (m_2 - 2\delta m_1)x + m_1^2 - m_2. \tag{49}$$

Its minimum value occurs at $x = \frac{2\delta m_1 - m_2}{2\delta^2} < 0$ and we have

$$p(0) = m_1^2 - m_2 \leq 0$$

(by the Cauchy–Schwarz inequality) and $p(1) = (\delta - m_1)^2 > 0$. Therefore, letting r_2 denote the larger of the roots of p, we have $0 \leq r_2 < 1$. Moreover, if $x \geq 0$ and $p(x) \leq 0$, then $x \leq r_2$. Taking $w = r_2$, we conclude that $\mu(V) \leq w$, and we have the formula (46). It is easy to see that w is a continuous function of m_1, m_2 and δ. $\square$

Lemma 2.9. *Let $c < d$ be real numbers. For matrices a_1 and a_2, consider the maps* $\mathrm{Tr} \circ \mathrm{ev}_{a_1,a_2} : \mathbb{C}\langle x_1, x_2\rangle \to \mathbb{C}$. *Then we have*

$$\bigcap_{\substack{N\in\mathbb{N} \\ a_1,a_2\in\mathbb{M}_N(\mathbb{C})_{s.a.} \\ c\cdot 1\le a_i\le d\cdot 1,\,(i=1,2)}} \ker(\mathrm{Tr} \circ \mathrm{ev}_{a_1,a_2}) = \mathcal{I}. \tag{50}$$

Proof. The inclusion $\supseteq$ in (50) follows from the trace property.

Let $c < u < v < d$ and make the choice of polynomials $\widetilde{T}_i$ and $\widetilde{U}_i$ described in Corollary 2.7. Letting Ξ and $\mathcal{J}$ be as in Lemma 2.4, for each $y \in \mathcal{J}\backslash\{0\}$, we will find matrices a_1 and a_2 such that

$$\mathrm{Tr}(\mathrm{ev}_{a_1,a_2}(y)) \neq 0. \tag{51}$$

By (19) and (20) of Lemma 2.4, this will suffice to show $\subseteq$ in (50). Rather than find a_1 and a_2 explicitly, we make use of random matrices.

We may write

$$y = c_0 1 + \sum_{n=1}^{\infty}\Big(c_a^{(1)}\widetilde{T}_a(x_1) + c_a^{(2)}\widetilde{T}_a(x_2)\Big) + \sum_{x\in S} d_x \widetilde{U}^x, \tag{52}$$

with c_0, $c_n^{(i)}$ and d_x, not all zero. If c_0 is the only nonzero coefficient, then y is a nonzero constant multiple of the identity and any choice of a_1 and a_2 gives (51). So assume some $c_n^{(i)} \neq 0$ or $d_x \neq 0$. Let A_N and B_N be the independent $N \times N$ random matrices as described in Corollary 2.7. Extend the function $\beta : \Xi\backslash\{1\} \to \mathbb{R}$ that was defined in Corollary 2.7 to a function $\beta : \mathcal{J} \to \mathbb{R}$ by linearity and by setting $\beta(1) = 1$. By that corollary, the random variable

$$z_N := \mathrm{Tr} \circ \mathrm{ev}_{A_N,B_N}(y) - N\beta(y) \tag{53}$$

converges as $N \to \infty$ in moments to a Gausian random variable with some nonzero variance σ^2. It is now straightforward to see that

$$\mathrm{Prob}\big(\mathrm{Tr} \circ \mathrm{ev}_{A_N,B_N}(y) \neq 0\big) \tag{54}$$

is bounded away from zero as $N \to \infty$. Indeed, If $\beta(y) \neq 0$, then since $N\beta(y) \to \pm\infty$ and since the second moment of z_N stays bounded as $N \to \infty$, the quantity (54) stays bounded away from zero as $N \to \infty$. On the other hand, if $\beta(y) = 0$, then considering the second and fourth moments of z_N and applying Lemma 2.8, we find $w < 1$ such that for all N sufficiently large, we have $\mathrm{Prob}(z_N \neq 0) \geq 1 - w$. Thus, also in this case, the quantity (54) is bounded away from zero as $N \to \infty$.

By work of Haagerup and Thorbjørnsen (see Equation (3.7) and the next displayed equation of [10]), we have

$$\lim_{N\to\infty} \mathrm{Prob}(c \cdot 1 \leq A_N \leq d \cdot 1) = 1, \tag{55}$$

and also for B_N. Combining boundedness away from zero of (54) with (55), for some N sufficiently large, we can evaluate A_N and B_N on a set of nonzero measure to obtain $a_1, a_2 \in \mathbb{M}_N(\mathbb{C})$ so that $\mathrm{Tr} \circ \mathrm{ev}_{a_1,a_2}(y) \neq 0$ and $c \cdot 1 \leq a_i \leq d \cdot 1$ for $i = 1, 2$. $\square$

Proof of Theorem 2.1. As mentioned before, we concentrate on the case $k = 2$, and the other cases follow similarly. By the Gelfand–Naimark–Segal representation theorem, it is enough to prove that for all monomials P in k noncommuting variables, we have

$$\tau(P(X_i)) = \chi(P(Y_i)). \tag{56}$$

Rephrased, this amounts to showing that we have

$$\tau \circ \mathrm{ev}_{X_1,X_2}(x) = \chi \circ \mathrm{ev}_{Y_1,Y_2}(x) \tag{57}$$

for all $x \in X^\sharp$. By hypothesis, for all $p \geq 0$, all $N \in \mathbb{N}$ and all $a_1, a_2 \in \mathbb{M}_N(\mathbb{C})$ we have

$$\mathrm{tr} \otimes \tau((a_1 \otimes X_1 + a_2 \otimes X_2)^p) = \mathrm{tr} \otimes \chi((a_1 \otimes Y_1 + a_2 \otimes Y_2)^p). \tag{58}$$

Developing the right-hand side minus the left-hand side of (58) gives that the equality

$$\sum_{i_1,\dots i_p \in \{1,2\}} \mathrm{tr}(a_{i_1} \dots a_{i_p})(\tau(X_{i_1} \dots X_{i_p}) - \chi(Y_{i_1} \dots Y_{i_p})) = 0 \tag{59}$$

holds true for any choice $a_1, a_2 \in \mathbb{M}_N(\mathbb{C})_{s.a.}$. This equation can be rewritten as

$$\sum_{x \in S_p} c_x \big((\mathrm{tr} \circ \mathrm{ev}_{a_1,a_2})(x)\big)\big(\tau \circ \mathrm{ev}_{X_1,X_2}(x) - \chi \circ \mathrm{ev}_{Y_1,Y_2}(x)\big) = 0, \tag{60}$$

where $S_p \subset X^\sharp$ is a set representatives, one from each orbit in $X^\sharp/\rho$, of the monomials of degree p, and where c_x is the cardinality of each class.

Suppose, for contradiction, that (57) fails for some $x \in S_p$. Let

$$y = \sum_{x \in S_p} c_x \big(\tau \circ \mathrm{ev}_{X_1,X_2}(x) - \chi \circ \mathrm{ev}_{Y_1,Y_2}(x)\big) x \in \mathbb{C}\langle x_1, x_2 \rangle. \tag{61}$$

By Lemma 2.3, $y \notin \mathcal{I}$. By Lemma 2.9, there are $N \in \mathbb{N}$ and $a_1, a_2 \in \mathbb{M}_N(\mathbb{C})$ such that $c1 \leq a_i \leq d1$ for $i = 1, 2$ and $\mathrm{tr} \circ \mathrm{ev}_{a_1,a_2}(y) \neq 0$. But $\mathrm{tr} \circ \mathrm{ev}_{a_1,a_2}(y)$ is the left-hand side of (60), and we have a contradiction. $\square$

Remark 2.10. We only proved the result for $k = 2$. The proof for arbitrary k is actually exactly the same. The only difference is that the notations in the definition of second-order freeness is more cumbersome, but Theorem 2.6 as well as the other lemmas are unchanged.

Remark 2.11. The main ingredient in the proof of Theorem 2.1 is to provide a method of constructing $a_1, a_2 \in \mathbb{M}_N(\mathbb{C})_{s.a.}$ such that

$$(\mathrm{Tr} \circ \mathrm{ev}_{a_1,a_2})(y) \neq 0, \tag{62}$$

whenever this is not ruled out by reasons of symmetry. Our approach is probabilistic, and makes unexpected use of second-order freeness. In particular, our approach is nonconstructive. It would be interesting to find a direct approach. For an alternative approach, see [23].

It is natural to wonder how much one can shrink the choice of matrices from which a_1 and a_2 in Remark 2.11 are drawn. We would like to point out here that in Lemma 2.9 we need at least infinitely many values of N. More precisely, we can prove the following:

Proposition 2.12. *For each $N_0 \in \mathbb{N}$, we have*

$$\bigcap_{\substack{N \le N_0 \\ a_1,a_2 \in \mathbb{M}_N(\mathbb{C})_{s.a.}}} \ker(\mathrm{Tr} \circ \mathrm{ev}_{a_1,a_2}) \supsetneqq \mathcal{I}. \tag{63}$$

Proof. Without loss of generality (for example, by taking $N_0!$), it will be enough to prove

$$\bigcap_{a_1,a_2 \in \mathbb{M}_N(\mathbb{C})_{s.a.}} \ker(\mathrm{Tr} \circ \mathrm{ev}_{a_1,a_2}) \supsetneqq \mathcal{I} \tag{64}$$

for each $N \in \mathbb{N}$.

Following the proof of Theorem 2.1, let $W_p = \mathrm{span}\,\{x + \mathcal{I} \mid x \in S_p\}$ be the degree p vector subspace of the quotient of vector spaces $\mathbb{C}\langle x_1, x_2\rangle/\mathcal{I}$. The dimension of W_p is at least $2^p/p$.

Consider the commutative polynomial algebra

$$\mathbb{C}[x_{11}, \ldots, x_{NN}, y_{11}, \ldots, y_{NN}]$$

in the $2N^2$ variables $\{x_{ij}, y_{ij} \mid 1 \le i, j \le N\}$. Consider matrices

$$X = (x_{ij}),\ Y = (y_{ij}) \in \mathbb{M}_N(\mathbb{C}) \otimes \mathbb{C}[x_{11}, \ldots, x_{NN}, y_{11}, \ldots, y_{NN}] \tag{65}$$

over this ring. In this setting,

$$\phi := (\mathrm{Tr} \otimes \mathrm{id}_{\mathbb{C}[x_{11},\ldots,x_{NN},y_{11},\ldots,y_{NN}]}) \circ \mathrm{ev}_{X,Y} \tag{66}$$

is a $\mathbb{C}$-linear map from $\mathbb{C}\langle x_1, x_2\rangle$ to $\mathbb{C}[x_{11}, \ldots, x_{NN}, y_{11}, \ldots, y_{NN}]$ that vanishes on $\mathcal{I}$ and every map of the form $\mathrm{Tr} \circ \mathrm{ev}_{a_1,a_2}$ for $a_1, a_2 \in \mathbb{M}_N(\mathbb{C})$ is ϕ composed with some evaluation map on the polynomial ring

$$\mathbb{C}[x_{11}, \ldots, x_{NN}, y_{11}, \ldots, y_{NN}].$$

Therefore, we have

$$\ker \phi \subseteq \bigcap_{a_1,a_2 \in \mathbb{M}_N(\mathbb{C})_{s.a.}} \ker(\mathrm{Tr} \circ \mathrm{ev}_{a_1,a_2}). \tag{67}$$

We denote also by ϕ the induced map

$$\mathbb{C}\langle x_1, x_2\rangle/\mathcal{I} \to \mathbb{C}[x_{11}, \ldots, x_{NN}, y_{11}, \ldots, y_{NN}]. \tag{68}$$

Clearly, ϕ maps W_p into the vector space of homogeneous polynomials in $\mathbb{C}[x_{11}, \ldots, x_{NN}, y_{11}, \ldots, y_{NN}]$ of degree p. The space of homogenous polynomials of degree p in M variables has dimension equal to the binomial coefficient $\binom{p+M-1}{M-1}$. Therefore, there exists a constant $C > 0$, depending on N, such that ϕ maps into a subspace of complex dimension $\leq Cp^{N^2-1}$. For fixed N, there is p large enough so that one has $2^p/p > Cp^{N^2-1}$. Therefore, by the rank theorem, the kernel of ϕ restricted to W_p must be nonempty. Combined with (67), this proves (64). □

3. Application to embeddability

We begin by recalling the ultrapower construction. Let R denote the hyperfinite II_1-factor and τ_R its normalized trace. Let ω be a free ultrafilter on $\mathbb{N}$ and let I_ω denote the ideal of $\ell^\infty(\mathbb{N}, R)$ consisting of those sequences $(x_n)_{n=1}^\infty$ such that $\lim_{n\to\omega} \tau_R((x_n)^* x_n) = 0$. Then R^ω is the quotient $\ell^\infty(\mathbb{N}, R)/I_\omega$, which is actually a von Neumann algebra.

Let $\mathcal{M}$ be a von Neumann algebra with normal, faithful, tracial state τ.

Definition 3.1. The von Neumann algebra $\mathcal{M}$ is said to have *Connes' embedding property* if $\mathcal{M}$ can be embedded into an ultra power R^ω of the hyperfinite von Neumann algebra R in a trace-preserving way.

Definition 3.2. If $X = (x_1, \ldots, x_n)$ is a finite subset of

$$\mathcal{M}_{s.a.} := \{x \in \mathcal{M} \mid x^* = x\},$$

we say that X *has matricial microstates* if for every $m \in \mathbb{N}$ and every $\epsilon > 0$, there is $k \in \mathbb{N}$ and there are self-adjoint $k \times k$ matrices $A_1, \ldots, A_n$ such that whenever $1 \leq p \leq m$ and $i_1, \ldots, i_p \in \{1, \ldots, n\}$, we have

$$|\mathrm{tr}_k(A_{i_1} A_{i_2} \cdots A_{i_p}) - \tau(x_{i_1} x_{i_2} \cdots x_{i_p})| < \epsilon, \tag{69}$$

where tr_k is the normalized trace on $\mathbb{M}_k(\mathbb{C})$.

It is not difficult to see that if X has matricial microstates, then for every $m \in \mathbb{N}$ and $\epsilon > 0$, there is $K \in \mathbb{N}$ such that for every $k \geq K$ there are matrices $A_1, \ldots, A_n \in \mathbb{M}_k(\mathbb{C})$ whose mixed moments approximate those of X in the sense specified above. Also, as proved by an argument of Voiculescu [25], if X has matricial microstates, then each approximating matrix A_j above can be chosen to have norm no greater than $\|x_j\|$.

The following result is well-known. For future reference, we briefly describe a proof.

Proposition 3.3. *Let $\mathcal{M}$ be a von Neumann algebra with seperable predual and τ a normal, faithful, tracial state on $\mathcal{M}$. Then the following are equivalent:*

(i) *$\mathcal{M}$ has Connes' embedding property.*
(ii) *Every finite subset $X \subseteq \mathcal{M}_{s.a.}$ has matricial microstates.*
(iii) *If $Y \subseteq M_{s.a.}$ is a generating set for $\mathcal{M}$, then every finite subset X of Y has matricial microstates.*

In particular, if Y is a finite generating set of $\mathcal{M}$ then the above conditions are equivalent to Y having matricial microstates.

Proof. The implication (i) $\Longrightarrow$ (ii) follows because if

$$X = (x_1, \dots, x_n) \subseteq (R^\omega)_{s.a.},$$

then choosing any representatives of the x_j in $\ell^\infty(\mathbb{N}, R)$, we find elements $a_1, \dots, a_n$ of R whose mixed moments up to order m approximate those of the x_j as closely as desired. Now we use that any finite subset of R is approximately (in $\| \, \|_2$-norm) contained in some copy $\mathbb{M}_k(\mathbb{C}) \subseteq R$, for some k sufficiently large.

The implication (ii) $\Longrightarrow$ (iii) is evident.

For (iii) $\Longrightarrow$ (i), we may without loss of generality suppose that

$$Y = \{x_1, x_2, \dots\}$$

for some sequence $(x_j)_1^\infty$ possibly with repetitions. Fix $m \in \mathbb{N}$, let $k \in \mathbb{N}$ and let $A_1^{(m)}, \dots, A_m^{(m)} \in \mathbb{M}_k(\mathbb{C})$ be matricial microstates for $x_1, \dots, x_m$ so that (69) holds for all $p \leq m$ and for $\epsilon = 1/m$, and assume $\|A_i^{(m)}\| \leq \|x_i\|$ for all i. Choose a unital $*$-homomorphism $\pi_k : \mathbb{M}_k(\mathbb{C}) \hookrightarrow R$, and let $a_i^m = \pi_k(A_i^{(m)})$. Let $b_i = (a_i^m)_{m=1}^\infty \in \ell^\infty(\mathbb{N}, R)$, where we set $a_i^m = 0$ if $i > m$. Let z_i be the image of b_i in R^ω. Then $z_1, z_2, \dots$ has the same joint distribution as $x_1, x_2, \dots$, and this yields an embedding $M \hookrightarrow R^\omega$ sending x_i to z_i. $\square$

A direct consequence of Theorem 2.1 is:

Theorem 3.4. *Suppose that a von Neumann algebra $\mathcal{M}$ with trace τ is generated by self-adjoint elements x_1 and x_2. Let $c < d$ be real numbers. Then $\mathcal{M}$ has Connes' embedding property if and only if there exists*

$$y_1, y_2 \in (R^\omega)_{s.a.}$$

such that for all $a_1, a_2 \in \mathbb{M}_n(\mathbb{C})_{s.a.}$ whose spectra are contained in $[c, d]$,

$$\operatorname{distr}(a_1 \otimes x_1 + a_2 \otimes x_2) = \operatorname{distr}(a_1 \otimes y_1 + a_2 \otimes y_2). \tag{70}$$

In this section we will prove that Connes' embedding property is equivalent to a weaker condition.

Lemma 3.5. *Suppose that a von Neumann algebra $\mathcal{M}$ with trace τ is generated by self-adjoint elements x_1 and x_2. Let $c < d$ be real numbers and for every $n \in \mathbb{N}$, let E_n be a dense subset of the set of all elements of $\mathbb{M}_n(\mathbb{C})$ whose spectra are contained in the interval $[c, d]$. Then $\mathcal{M}$ has Connes' embedding property if and only if for all finite sets I and all choices of $n(i) \in I$ and $a_1^i, a_2^i \in E_{n(i)}$, $(i \in I)$, there exists $y_1, y_2 \in R^\omega_{s.a.}$ such that*

$$\operatorname{distr}(x_1) = \operatorname{distr}(y_1) \tag{71}$$

$$\operatorname{distr}(x_2) = \operatorname{distr}(y_2) \tag{72}$$

$$\operatorname{distr}(a_1^i \otimes x_1 + a_2^i \otimes x_2) = \operatorname{distr}(a_1^i \otimes y_1 + a_2^i \otimes y_2), \quad (i \in I). \tag{73}$$

Proof. Necessity is clear.

For sufficiency, we'll use an ultraproduct argument. Let $(a_1^i, a_2^i)_{i\in\mathbb{N}}$ be an enumeration of a countable, dense subset of the disjoint union $\sqcup_{n\geq 1} E_n \times E_n$. We let $n(i)$ be such that $a_1^i, a_2^i \in \mathbb{M}_{n(i)}(\mathbb{C})$. For each $m \in \mathbb{N}$, let y_1^m, y_2^m be elements of R^ω satisfying $\mathrm{distr}(y_j^m) = \mathrm{distr}(x_j)$ and

$$\mathrm{distr}(a_1^i \otimes x_1 + a_2^i \otimes x_2) = \mathrm{distr}(a_1^i \otimes y_1^m + a_2^i \otimes y_2^m) \tag{74}$$

for all $i \in \{1, \ldots, m\}$. In particular, $\|y_j^m\| = \|x_j\|$ for $j = 1, 2$ and all m. Let

$$b_j^m = (b_{j,n}^m)_{n=1}^\infty \in \ell^\infty(\mathbb{N}, R) \tag{75}$$

be such that $\|b_j^m\| \leq \|x_j\| + 1$ and the image of b_j^m in R^ω is y_j^m $(j = 1, 2)$. This implies that for all $p \in \mathbb{N}$ and all $i \in \{1, \ldots, m\}$, we have

$$\lim_{k\to\omega} \mathrm{tr}_{n(i)} \otimes \tau_R\big((a_1^i \otimes b_{1,k}^m + a_2^i \otimes b_{2,k}^m)^p\big) = \mathrm{tr}_{n(i)} \otimes \tau\big((a_1^i \otimes x_1 + a_2^i \otimes x_2)^p\big), \tag{76}$$

which in turn implies that there is a set F_m belonging to the ultrafilter ω such that for all $p, i \in \{1, \ldots, m\}$ and all $k \in F_m$, we have

$$\big|\mathrm{tr}_{n(i)} \otimes \tau_R\big((a_1^i \otimes b_{1,k}^m + a_2^i \otimes b_{2,k}^m)^p\big) - \mathrm{tr}_{n(i)} \otimes \tau\big((a_1^i \otimes x_1 + a_2^i \otimes x_2)^p\big)\big| < \frac{1}{m}. \tag{77}$$

For $q \in \mathbb{N}$, let $k(q) \in \cap_{m=1}^q F_m$ and for $j = 1, 2$, let

$$b_j = (b_{j,k(q)}^q)_{q=1}^\infty \in \ell^\infty(\mathbb{N}, R). \tag{78}$$

Then for all $i, p \in \mathbb{N}$, we have

$$\lim_{q\to\infty} \mathrm{tr}_{n(i)} \otimes \tau_R\big((a_1^i \otimes b_{1,k(q)}^q + a_2^i \otimes b_{2,k(q)}^q)^p\big) = \mathrm{tr}_{n(i)} \otimes \tau\big((a_1^i \otimes x_1 + a_2^i \otimes x_2)^p\big), \tag{79}$$

Let y_j be the image in R^ω of b_j. Then we have

$$\mathrm{distr}(a_1^i \otimes x_1 + a_2^i \otimes x_2) = \mathrm{distr}(a_1^i \otimes y_1 + a_2^i \otimes y_2) \tag{80}$$

for all $i \in \mathbb{N}$. By density, we have that (70) holds for all $n \in \mathbb{N}$ and all $a_1, a_2 \in \mathbb{M}_n(\mathbb{C})_{s.a.}$ having spectra in $[c, d]$. Therefore, by Theorem 3.4, $\mathcal{M}$ is embeddable in R^ω. □

Theorem 3.6. *Suppose that a von Neumann algebra $\mathcal{M}$ with trace τ is generated by self-adjoint elements x_1 and x_2 and suppose that both x_1 and x_2 are positive and invertible. Then $\mathcal{M}$ has Connes' embedding property if and only if for all $n \in \mathbb{N}$ and all $a_1, a_2 \in \mathbb{M}_n(\mathbb{C})_+$ there exists $y_1, y_2 \in R^\omega_{s.a.}$ such that*

$$\mathrm{distr}(x_1) = \mathrm{distr}(y_1) \tag{81}$$

$$\mathrm{distr}(x_2) = \mathrm{distr}(y_2) \tag{82}$$

$$\mathrm{distr}(a_1 \otimes x_1 + a_2 \otimes x_2) = \mathrm{distr}(a_1 \otimes y_1 + a_2 \otimes y_2) \tag{83}$$

hold.

Proof. Again, necessity is clear.

For the reverse implication, we will show that the conditions of Lemma 3.5 are satisfied. Suppose that for all $n \in \mathbb{N}$ and all $a_1, a_2 \in \mathbb{M}_n(\mathbb{C})_+$, there exist y_1 and y_2 such that (81)–(83) hold. Let $K > 1$ be such that

$$\|x_j\| \leq K, \qquad \|x_j^{-1}\| \leq K. \tag{84}$$

Let E_n be the set of all elements of $\mathbb{M}_n(\mathbb{C})_{s.a.}$ having spectra in the interval $[K, K^2]$. We will show that the condition appearing in Lemma 3.5 is satisfied for these sets. Let $I = \{1, 2, \ldots, m\}$ and for every $i \in I$ let $n(i) \in \mathbb{N}$, and $a_1^i, a_2^i \in E_{n(i)}$. We will find $y_1, y_2 \in R^\omega$ such that (71)–(73) hold. For any $j \in \{1, 2\}$ and $i \in I$, the spectrum of $a_j^i \otimes x_j$ lies in the interval

$$[1, K^3]. \tag{85}$$

Let $N = \sum_{i=1}^m n(i)$ and let $a_1, a_2 \in \mathbb{M}_N(\mathbb{C})$ be the block diagonal matrices

$$a_j = \oplus_{i=1}^m K^{4i} a_j^i, \quad (j = 1, 2). \tag{86}$$

By hypothesis, there exists $y_1, y_2 \in R^\omega$ such that (81)–(83) hold. We have

$$a_1 \otimes x_1 + a_2 \otimes x_2 = \oplus_{i=1}^m K^{4i}(a_1^i \otimes x_1 + a_2^i \otimes x_2) \tag{87}$$

and similarly for $a_1 \otimes y_1 + a_2 \otimes y_2$. Since the spectrum of $a_j^i \otimes x_j$ lies in $[1, K^3]$ for all j and i, the spectrum of $a_1^i \otimes x_1 + a_2^i \otimes x_2$ lies in $[2, 2K^3]$ as does the spectrum of $a_1^i \otimes y_1 + a_2^i \otimes y_2$. Since the intervals in the family $([2K^{4i}, 2K^{4i+3}])_{i=1}^m$ are pairwise disjoint, it follows that for every $i \in \{1, \ldots, m\}$, the projections

$$(0_{n(1)} \oplus \cdots 0_{n(i-1)} \oplus I_{n(i)} \oplus 0_{n(i+1)} \oplus \cdots 0_{n(m)}) \otimes 1_{\mathcal{M}}$$
$$(0_{n(1)} \oplus \cdots 0_{n(i-1)} \oplus I_{n(i)} \oplus 0_{n(i+1)} \oplus \cdots 0_{n(m)}) \otimes 1_{R^\omega}$$

arise as the spectral projections of

$$a_1 \otimes x_1 + a_2 \otimes x_2 \quad \text{and} \quad a_1 \otimes y_1 + a_2 \otimes y_2,$$

respectively, for the inverval $[2K^{4i}, 2K^{4i+3}]$. Cutting by these spectral projections, we thus obtain that the distributions of $a_1^i \otimes x_1 + a_2^i \otimes x_2$ and $a_1^i \otimes y_1 + a_2^i \otimes y_2$ are the same, as required. □

4. Quantum Horn bodies

Let $\mathbb{R}_{\geq}^N$ denote the set of N-tuples of real numbers listed in nonincreasing order. The *eigenvalue sequence* of an $N \times N$ self-adjoint matrix is its sequence of eigenvalues repeated according to multiplicity and in nonincreasing order, so as to lie in $\mathbb{R}_{\geq}^N$. Consider $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $\beta = (\beta_1, \ldots, \beta_N)$ in $\mathbb{R}_{\geq}^N$. Let $S_{\alpha,\beta}$ be the set of all possible eigenvalue sequences $\gamma = (\gamma_1, \ldots, \gamma_N)$ of $A + B$, where A and B are self-adjoint $N \times N$ matrices with eigenvalue sequences α and β, respectively. Thus, $S_{\alpha,\beta}$ is the set of all eigenvalue sequences of $N \times N$-matrices of the form

$$U \mathrm{diag}(\alpha) U^* + V \mathrm{diag}(\beta) V^*, \qquad (U, V \in \mathbb{U}_N), \tag{88}$$

where $\mathbb{U}_N$ is the group of $N \times N$-unitary matrices. Klyatchko, Totaro, Knutson and Tao described the set $S_{\alpha,\beta}$ in terms first conjectured by Horn. See Fulton's exposition [9]. Taking traces, clearly every $\gamma \in S_{\alpha,\beta}$ must satisfy

$$\sum_{k=1}^{N} \gamma_k = \sum_{i=1}^{N} \alpha_i + \sum_{j=1}^{N} \beta_j. \tag{89}$$

Consider the inequality

$$\sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j \geq \sum_{k\in K} \gamma_k. \tag{90}$$

for a triple (I, J, K) of subsets of $\{1, \dots, N\}$. Horn defined sets T_r^N of triples (I, J, K) of subsets of $\{1, \dots, N\}$ of the same cardinality r, by the following recursive procedure. Set

$$U_r^N = \left\{ (I, J, K) \;\middle|\; \sum_{i\in I} i + \sum_{j\in J} j = \sum_{k\in K} k + \frac{r(r+1)}{2} \right\}. \tag{91}$$

When $r = 1$, set $T_1^N = U_1^N$. Otherwise, let

$$T_r^N = \left\{ (I, J, K) \in U_r^N \;\middle|\; \sum_{f\in F} i_f + \sum_{g\in G} j_g \leq \sum_{h\in H} k_h + \frac{p(p+1)}{2}, \right. \\ \left. \text{for all } p < r \text{ and } (F, G, H) \in T_p^r \right\}. \tag{92}$$

The result of Klyatchko, Totaro, Knutson and Tao is that $S_{\alpha,\beta}$ consists of those elements $\gamma \in \mathbb{R}^N_{\geq}$ such that the equality (89) holds and the inequality (90) holds for every triple $(I, J, K) \in \bigcup_{r=1}^{N-1} T_r^N$. We will refer to $S_{\alpha,\beta}$ as the *Horn body* of α and β. It is, thus, a closed, convex subset of $\mathbb{R}^N_{\geq}$.

The analogue of this situation occuring in finite von Neumann algebras has been considered by Bercovici and Li [5], [6]; let us summarize part of what they have done. We denote by $\mathcal{F}$ the set of all right-continuous, nonincreasing, bounded functions $\lambda : [0, 1) \to \mathbb{R}$. Let $\mathcal{M}$ be a von Neumann algebra with normal, faithful, tracial state τ and let $a = a^* \in \mathcal{M}$. The *distribution* of a is the Borel measure μ_a, supported on the spectrum of a, such that

$$\tau(a^n) = \int_{\mathbb{R}} t^n \, d\mu_a(t) \qquad (n \geq 1). \tag{93}$$

The *eigenvalue function* of a is $\lambda_a \in \mathcal{F}$ defined by

$$\lambda_a(t) = \sup\{x \in \mathbb{R} \mid \mu_a((x, \infty)) > t\}. \tag{94}$$

We call $\mathcal{F}$ the set of all eigenvalue functions. It is an affine space, where we take scalar multiples and sums of functions in the usual way. Identifying $\mathcal{F}$ with the set of all compactly supported Borel measures on the real line, it is a subspace of the dual of $C(\mathbb{R})$. We endow $\mathcal{F}$ with the weak*-topology inherited from this pairing.

It is clear that for every $\lambda \in \mathcal{F}$ and every II_1-factor $\mathcal{M}$, there is $a = a^* \in \mathcal{M}$ such that $\lambda_a = \lambda$. Note that if $\mathcal{M} = M_N(\mathbb{C})$ and if $a = a^* \in M_N(\mathbb{C})$ has eigenvalue sequence $\alpha = (\alpha_1, \ldots, \alpha_N)$, then its eigenvalue function is given by

$$\lambda_a(t) = \alpha_j, \quad \frac{j-1}{N} \le t < \frac{j}{N}, \qquad (1 \le j \le N). \tag{95}$$

In this way, $\mathbb{R}^N_{\geq}$ is embedded as a subset $\mathcal{F}^{(N)}$ of $\mathcal{F}$, and the affine structure on $\mathcal{F}^{(N)}$ inherited from $\mathcal{F}$ corresponds to the usual one on $\mathbb{R}^N_{\geq}$ coming from the vector space structure of $\mathbb{R}^N$.

For a set $(I, J, K) \in T^N_r$, consider the triple $(\sigma^N_I, \sigma^N_J, \sigma^N_K)$, where for $F \subseteq \{1, 2, \ldots, N\}$, we set

$$\sigma^N_F = \bigcup_{i \in F} \left[\frac{i-1}{N}, \frac{i}{N}\right). \tag{96}$$

Let

$$\mathcal{T} = \bigcup_{N=1}^{\infty} \bigcup_{r=1}^{N-1} \left\{(\sigma^N_I, \sigma^N_J, \sigma^N_K) \mid (I, J, K) \in T^N_r\right\}. \tag{97}$$

Theorem 4.1 ([6], Theorem 3.2)**.** *For any $u, v, w \in \mathcal{F}$, there exists self-adjoint elements a and b in the ultrapower R^ω of the hyperfinite II_1-factor with $u = \lambda_a$, $v = \lambda_b$ and $w = \lambda_{a+b}$ if and only if*

$$\int_0^1 u(t)\,dt + \int_0^1 v(t)\,dt = \int_0^1 w(t)\,dt \tag{98}$$

and, for every $(\omega_1, \omega_2, \omega_3) \in \mathcal{T}$, we have

$$\int_{\omega_1} u(t)\,dt + \int_{\omega_2} v(t)\,dt \ge \int_{\omega_3} w(t)\,dt. \tag{99}$$

Given eigenvalue functions $u, v \in \mathcal{F}$, let $F_{u,v}$ be the set of all $w \in \mathcal{F}$ such that (98) holds and (99) holds for every $(\omega_1, \omega_2, \omega_3) \in \mathcal{T}$. Since the functions in $F_{u,v}$ are uniformly bounded, we see that $F_{u,v}$ is a compact, convex subset of $\mathcal{F}$.

Now we consider an alternative formulation of a special case of Theorem 4.1. Let $N \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}^N_{\geq}$. For $d \in \mathbb{N}$, let

$$K_{\alpha,\beta,d} = \{\lambda_C \mid C = \mathrm{diag}(\alpha) \otimes 1_d + U(\mathrm{diag}(\beta) \otimes 1_d)U^*,\ U \in \mathbb{U}_{Nd}\}. \tag{100}$$

For $d = 1$, this is just the set of eigenvalue functions corresponding to the Horn body $S_{\alpha,\beta}$. Let

$$K_{\alpha,\beta,\infty} = \overline{\bigcup_{d \ge 1} K_{\alpha,\beta,d}}\,. \tag{101}$$

As a consequence of Bercovici and Li's results we have the following.

Proposition 4.2. *Let $\alpha, \beta \in \mathbb{R}^N_{\geq}$ and let $u = \lambda_{\mathrm{diag}(\alpha)}$ and $v = \lambda_{\mathrm{diag}(\beta)}$ be the correspoding eigenvalue functions. Then*

$$K_{\alpha,\beta,\infty} = F_{u,v} \tag{102}$$

is a compact, convex subset of $\mathcal{F}$.

If Connes' embedding problem has a positive solution, then for every II_1-factor $\mathcal{M}$ and every $a, b \in \mathcal{M}_{s.a.}$ whose eigenvalue functions are u and v, respectively, we have $\lambda_{a+b} \in K_{\alpha,\beta,\infty}$.

Proof. The inclusion $\subseteq$ in (102) is clear. For the reverse inclusion, let $w \in F_{u,v}$. Then (98) holds and (99) holds for every $(\omega_1, \omega_2, \omega_3) \in \mathcal{T}$. For $n \in \mathbb{N}$, let $w^{(n)} \in \mathcal{F}$ be obtained by averaging over the intervals of length $1/n$, namely,

$$w^{(n)}(t) = \int_{(i-1)/n}^{i/n} f(s)\,ds, \qquad \left(\frac{i-1}{n} \leq t < \frac{i}{n}, \quad i \in \{1, 2, \ldots, n\}\right). \tag{103}$$

Then $w^{(n)}$ corresponds to an eigenvalue sequence $\gamma \in \mathbb{R}^n_{\geq}$. We have

$$\int_0^1 u(t)\,dt + \int_0^1 v(t)\,dt = \int_0^1 w^{(n)}(t)\,dt \tag{104}$$

and, for every $(\omega_1, \omega_2, \omega_3) = (\sigma^n_I, \sigma^n_J, \sigma^n_K) \in \mathcal{T}$ for $(I, J, K) \in T^n_r$, we have

$$\int_{\omega_1} u(t)\,dt + \int_{\omega_2} v(t)\,dt \geq \int_{\omega_3} w^{(n)}(t)\,dt. \tag{105}$$

Therefore, taking $n = Nd$ to be a multiple of N, by the theorem formerly known as Horn's conjecture, we have $\gamma \in S_{\alpha\otimes 1_d, \beta\otimes 1_d}$ and, consequently, $w^{(Nd)} \in K_{\alpha,\beta,d}$. Since $w^{(Nd)}$ converges as $d \to \infty$ to w, we have $w \in K_{\alpha,\beta,\infty}$. This proves the equality (102).

The final statement is a consequence of Bercovici and Li's result, Theorem 4.1. $\square$

Bercovici and Li's results provide a means of trying to find a II_1-factor $\mathcal{M}$ that lack's Connes' embedding property: namely, by finding self-adjoint elements $a, b \in \mathcal{M}$ such that $\lambda_{a+b} \notin F_{\lambda_a, \lambda_b}$; this amounts to finding some $(I, J, K) \in T^N_r$ such that

$$\int_{\sigma^N_I} \lambda_a(t)\,dt + \int_{\sigma^N_J} \lambda_b(t)\,dt < \int_{\sigma^N_K} \lambda_{a+b}(t)\,dt. \tag{106}$$

On the other hand we will use Theorem 3.6 to see that Connes' embedding problem is equivalent to an anlogous question about versions of the Horn body with "matrix coefficients."

Let $a_1, a_2 \in \mathbb{M}_n(\mathbb{C})_{s.a.}$, and $\alpha, \beta \in \mathbb{R}^N_{\geq}$. We introduce the set $K^{a_1,a_2}_{\alpha,\beta}$ of the eigenvalue functions of all matrices of the form

$$a_1 \otimes U\mathrm{diag}(\alpha)U^* + a_2 \otimes V\mathrm{diag}(\beta)V^*, \qquad (U, V \in \mathbb{U}_N). \tag{107}$$

Although, for reasons that will be immediately apparent, we choose to view $K^{a_1,a_2}_{\alpha,\beta}$ as a subset of $\mathcal{F}$, we may equally well consider the corresponding eigenvalue sequences and view $K^{a_1,a_2}_{\alpha,\beta}$ as a subset of $\mathbb{R}^{nN}_{\geq}$. Comparing to (88), the set $K^{a_1,a_2}_{\alpha,\beta}$ is seen to be the analogue of the Horn body $S_{\alpha,\beta}$, but with "coefficients" a_1 and a_2. We will refer to these sets as *quantum Horn bodies*.

The example below shows that $K^{a_1,a_2}_{\alpha,\beta}$ need not be convex, even in the case where a_1, a_2 commute.

Example 4.3. Let

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{108}$$

and let $\alpha = \beta = (2, 1)$. Then the 4×4 matrices of the form (107) are all unitary conjugates of the matrices

$$R_t = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1+t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 2-t \end{pmatrix}, \tag{109}$$

for $0 \leq t \leq 1$. One easily finds the eigenvalues $\lambda_1(t) \geq \lambda_2(t) \geq \lambda_3(t) \geq \lambda_4(t)$ of R_t to be

$$\lambda_1(t) = \frac{15}{2} + \frac{1}{2}\sqrt{25 - 16t} \tag{110}$$

$$\lambda_2(t) = \begin{cases} \frac{9}{2} + \frac{1}{2}\sqrt{9 - 8t}, & 0 \leq t \leq t_1 \\ \frac{15}{2} - \frac{1}{2}\sqrt{25 - 16t}, & t_1 \leq t \leq 1 \end{cases} \tag{111}$$

$$\lambda_3(t) = \begin{cases} \frac{15}{2} - \frac{1}{2}\sqrt{25 - 16t}, & 0 \leq t \leq t_1 \\ \frac{9}{2} + \frac{1}{2}\sqrt{9 - 8t}, & t_1 \leq t \leq 1 \end{cases} \tag{112}$$

$$\lambda_4(t) = \frac{9}{2} - \frac{1}{2}\sqrt{9 - 8t}, \tag{113}$$

where $t_1 = \frac{3}{2}\sqrt{65} - \frac{23}{2} \approx 0.593$. Then the set $\{(\lambda_1(t), \ldots, \lambda_4(t)) \mid 0 \leq t \leq 1\}$ is a 1-dimensional subset of 4-space that is far from being convex. For example, a plot of the projection of this set onto the last two coordinates is the curve in Figure 1. The upper part of this curve is a line segment, while the lower part is not.

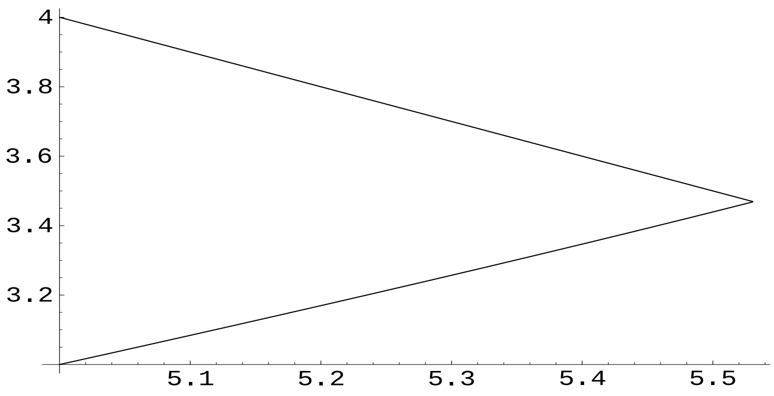
Extending the notions introduced above, for integers $d \geq 1$, let $K^{a_1,a_2}_{\alpha,\beta,d}$ be the set of the eigenvalue functions of all matrices of the form

$$a_1 \otimes U(\mathrm{diag}(\alpha) \otimes 1_d)U^* + a_2 \otimes V(\mathrm{diag}(\beta) \otimes 1_d)V^*, \qquad (U, V \in \mathbb{U}_{Nd}). \tag{114}$$

If d' divides d, then we have

$$K^{a_1,a_2}_{\alpha,\beta,d'} \subseteq K^{a_1,a_2}_{\alpha,\beta,d}\,. \tag{115}$$

FIGURE 1. A parametric plot of λ_4 (vertical axis) and λ_3 (horizontal axis).



Let us define

$$K^{a_1,a_2}_{\alpha,\beta,\infty} = \overline{\bigcup_{d\in\mathbb{N}} K^{a_1,a_2}_{\alpha,\beta,d}}\,, \tag{116}$$

where the closure is in the weak*-topology for $\mathcal{F}$ described earlier in this section. Note that the set $K^{a_1,a_2}_{\alpha,\beta,\infty}$ is compact.

Question 4.4. Though Example 4.3 shows that $K^{a_1,a_2}_{\alpha,\beta}$ need not be convex, is it true that $K^{a_1,a_2}_{\alpha,\beta,\infty}$ must be convex, or even that $K^{a_1,a_2}_{\alpha,\beta,d}$ must be convex for all d sufficiently large? Note that it is clear that $K^{a_1,a_2}_{\alpha,\beta,\infty}$ is convex with respect to the affine structure on $\mathcal{F}$ that arises from taking convex combinations of measures, under the correspondence between $\mathcal{F}$ and the set of Borel probability measures on $\mathbb{R}$. However, we are interested in the other affine structure of $\mathcal{F}$, resulting from addition of functions on $[0,1)$.

For $a_1,a_2\in\mathbb{M}_n(\mathbb{C})_{s.a.}$ with eigenvalue sequences $\gamma_1,\gamma_2\in\mathbb{R}^n_{\geq}$, we obviously have

$$K^{a_1,a_2}_{\alpha,\beta} \subseteq K_{\gamma_1\otimes\alpha,\gamma_2\otimes\beta} \tag{117}$$

and

$$K^{a_1,a_2}_{\alpha,\beta,\infty} \subseteq K_{\gamma_1\otimes\alpha,\gamma_2\otimes\beta,\infty}. \tag{118}$$

The following example shows that these inclusions can be strict.

Example 4.5. Let

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{119}$$

One directly sees that for any eigenvalue sequences α and β of length N and any $U,V\in\mathbb{U}_N$, the eigenvalue sequence of

$$a_1\otimes U(\mathrm{diag}(\alpha)\otimes 1_d)U^* + a_2\otimes V(\mathrm{diag}(\beta)\otimes 1_d)V^* \tag{120}$$

is the re-ordering of the concatenation of α and β. Thus, $K_{\alpha,\beta}^{a_1,a_2}$ has only one element. Moreover, dilating α to $\alpha \otimes 1_d$ does not change the corresponding eigenvalue functions of

$$a_1 \otimes U\mathrm{diag}(\alpha \otimes 1_d)U^* + a_2 \otimes V\mathrm{diag}(\alpha \otimes 1_d)V^*. \tag{121}$$

This shows that $K_{\alpha,\alpha,\infty}^{a_1,a_2}$ has only one element. Now we easily get

$$K_{\alpha,\beta,\infty}^{a_1,a_2} \neq K_{\alpha\oplus 0_N,\beta\oplus 0_N}, \tag{122}$$

where $\alpha \oplus 0_N$ means the eigenvalue sequence of $a_1 \otimes \mathrm{diag}(\alpha)$, etc.

For $\mathcal{M}$ a II$_1$-factor, we define $L_{\alpha,\beta,\mathcal{M}}^{a_1,a_2}$ to be the set of all eigenvalue functions of all operators of the form

$$a_1 \otimes x_1 + a_2 \otimes x_2 \in \mathbb{M}_n(\mathbb{C}) \otimes \mathcal{M}, \tag{123}$$

where x_1 and x_2 are self-adjoint elements of $\mathcal{M}$ whose eigenvalue functions agree with those of the matrices $\mathrm{diag}(\alpha)$ and $\mathrm{diag}(\beta)$, respectively (see (95) for an explicit description of the latter). It is easliy seen that we have

$$K_{\alpha,\beta,\infty}^{a_1,a_2} = L_{\alpha,\beta,R^\omega}^{a_1,a_2}. \tag{124}$$

Let

$$L_{\alpha,\beta}^{a_1,a_2} = \bigcup_{\mathcal{M}} L_{\alpha,\beta,\mathcal{M}}^{a_1,a_2}, \tag{125}$$

where the union is over all II$_1$-factors $\mathcal{M}$ with separable predual (acting on a specific separable Hilbert space, say). Using an ultraproduct argument, one can show that $L_{\alpha,\beta}^{a_1,a_2}$ is closed in $\mathcal{F}$ and compact. Also, one obviously has

$$K_{\alpha,\beta,\infty}^{a_1,a_2} \subseteq L_{\alpha,\beta}^{a_1,a_2}. \tag{126}$$

Theorem 3.6 gives us the following equivalent formulation of the embedding question.

Theorem 4.6. *The following are equivalent:*

(i) *Every II$_1$-factor $\mathcal{M}$ with separable predual has Connes' embedding property.*

(ii) *For all integers $n, N \geq 1$ and all $a_1, a_2 \in \mathbb{M}_n(\mathbb{C})_{s.a.}$, and $\alpha, \beta \in \mathbb{R}_{\geq}^N$, we have*

$$K_{\alpha,\beta,\infty}^{a_1,a_2} = L_{\alpha,\beta}^{a_1,a_2}. \tag{127}$$

Proof. Clearly, (i) implies $L_{\alpha,\beta}^{a_1,a_2} = L_{\alpha,\beta,R^\omega}^{a_1,a_2}$. We then get (127) from (124).

Suppose (ii) holds. It is well-known that to solve Connes' embedding problem in the affirmative, it will suffice to show that every tracial von Neuman algebra $\mathcal{M}$ that is generated by two self-adjoints x_1 and x_2 is embeddable in R^ω.

So suppose $\mathcal{M}$ is generated by self-adjoints x_1 and x_2. By Proposition 3.3, it will suffice to show that x_1 and x_2 have matricial microstates. Approximating x_1 and x_2, if necessary, we may without loss of generality assume that the eigenvalue functions of both belong to $\mathcal{F}^{(N)}$ for some $N \in \mathbb{N}$, namely, that they correspond to sequences α and, respectively, β in $\mathbb{R}^N_{\geq}$. By adding constants, if necessary, we may without loss of generality assume that x_1 and x_2 are positive and invertible. Let $n \in \mathbb{N}$ and let $a_1, a_2 \in \mathbb{M}_n(\mathbb{C})$. Using (124) and (127), there are $y_1, y_2 \in R^\omega$ such that (81)–(83) of Theorem 3.6 hold. So by that theorem, the pair x_1, x_2 has matricial microstates. $\square$

Note. We recently learned of a result of Mikaël de la Salle [23] that gives more direct proofs of our Lemma 2.9 and Theorem 2.1. We also learned about the work [1] of Akemann, Anderson and Weaver, that seems related to Theorem 2.1.

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