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Rényi dimension and Gaussian filtering. II

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Abstract. We consider convolving a Gaussian of a varying scale ϵ against a Borel measure μ on Euclidean δ-dimensional space. The L^q norm of the result is differentiable in ϵ. We calculate this derivative and show how the upper order of its growth relates to the lower Rényi dimension of μ. We assume q is strictly between 1 and ∞ and that μ is finite with compact support.

Consider choosing a sequence ϵ_n of scales for the Gaussians

$$g_\epsilon(\mathbf{x}) = \epsilon^{-\delta} e^{-(|\mathbf{x}|/\epsilon)^2}.$$

Let $\|f\|_q$ denote the L^q norm for Lebesgue measure. The differences

$$\left| \|g_{\epsilon_{n+1}} * \mu\|_q - \|g_{\epsilon_n} * \mu\|_q \right|$$

between the norms at adjacent scales ϵ_n and ϵ_{n-1} can be made to grow more slowly than any positive power of n by setting the ϵ_n by a power rule. The correct exponent in the power rule is determined by the lower Rényi dimension.

We calculate and find bounds on the derivative of the Gaussian kernel versions of the correlation integral. We show that a Gaussian kernel version of the Rényi entropy sum is continuous.

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1. Differences of Gaussian filters

Suppose μ is a finite Borel measure on $\mathbb{R}^\delta$ with compact support. If $\delta = 2$ we can think of μ as the abstraction of an image. In the context of image processing, it is common to look at the difference of convolutions of two Gaussians against μ. This is the case in the standard construction of a Laplacian pyramid ([3]). Traditionally, the scales of the kernels vary geometrically. For some purposes, it might be better to set the scales of the convolution kernels in a different pattern.

Given a function g on $\mathbb{R}^\delta$, thought of as acting as a canonical filter kernel on measures, we rescale it as

$$g_\epsilon(\mathbf{x}) = \epsilon^{-\delta} g\left(\epsilon^{-1}\mathbf{x}\right).$$

The most important cases we have in mind are where g is a Gaussian or a function of compact support that approximates a Gaussian. We now consider the problem of selecting some $\epsilon_n \searrow 0$ so that the differences

$$g_{\epsilon_n} * \mu - g_{\epsilon_{n-1}} * \mu$$

behave nicely.

We will use the notation m for Lebesgue measure,

$$\begin{aligned}\|f\|_q &= \left(\int_{\mathbb{R}^\delta} f(\mathbf{x})^q \, dm(\mathbf{x})\right)^{\frac{1}{q}} \\ &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} f(\mathbf{x})^q \, dx_1 dx_2 \cdots dx_\delta\right)^{\frac{1}{q}}.\end{aligned}$$

One criterion for the selection of the scales ϵ_n is to keep

$$\left\|g_{\epsilon_n} * \mu - g_{\epsilon_{n-1}} * \mu\right\|_q$$

approximately constant for some choice of $1 < q < \infty$. We don't see a means of estimating this norm of differences, so we instead look at the difference of norms.

If μ is "fractal," then we expect that setting ϵ_n in a geometric series will lead to exponential growth in

$$\left|\|g_{\epsilon_n} * \mu\|_q - \left\|g_{\epsilon_{n-1}} * \mu\right\|_q\right|.$$

If we set the ϵ_n via a power law, $\epsilon_n = n^{-t}$, we can reasonably hope that this difference is more or less constant.

Recall, say from [1] or [14], that the upper and lower Rényi dimensions of μ for index q are defined by

$$D_q^{\pm}(\mu) = \lim_{\epsilon\to 0}{}^{\sup}_{\inf} \frac{1}{q-1}\frac{\ln\left(S_\mu^q(\epsilon)\right)}{\ln(\epsilon)},$$

where the standard partition function $S_\mu^q(\epsilon)$ is taken to be

$$S_\mu^q(\epsilon) = \sum_{\mathbf{k}\in\mathbb{Z}^\delta} \mu(\epsilon\mathbf{k} + \epsilon\mathbb{I})^q, \tag{1}$$

and where $\mathbb{I}$ is a δ-fold product of the unit interval $[0, 1)$.

The connection with Gaussian convolution is the formula, due to Guérin,

$$\limsup_{x\to\infty} \frac{\ln\left(\|g_{x^{-1}} * \mu\|_q\right)}{\ln(x)} = \frac{q-1}{q}\left(\delta - D_q^-(\mu)\right).$$

See [8] or [12, Lemma 2.3]. For this formula to be valid, we need some restriction on g. For simplicity, we consider the case where g is nonnegative and rapidly decreasing. By rapidly decreasing we mean that any order derivative $g^{(\alpha)}$ of g exists and decays at infinity more rapidly than any negative power of x. Certainly, less is needed. See [8].

Here is the main result.

Theorem 1.1. *Suppose $1 < q < \infty$. Suppose $g : \mathbb{R}^\delta \to \mathbb{R}$ is nonnegative, nontrivial, rapidly decreasing and is radially nonincreasing. Suppose μ is a finite Borel measure on $\mathbb{R}^\delta$ with compact support. Let $I_q(\mu)$ denote the set of positive t for which*

$$\forall\alpha > 0,\ \lim_{n\to\infty} \frac{\left|\|g_{n^{-t}} * \mu\|_q - \left\|g_{(n-1)^{-t}} * \mu\right\|_q\right|}{n^\alpha} = 0.$$

If $D_q^-(\mu) < \delta$ then

$$I_q(\mu) = \left(0, \frac{q}{q-1}\left(\delta - D_q^-(\mu)\right)^{-1}\right],$$

while if $D_q^-(\mu) = \delta$ then $I_q(\mu) = (0, \infty)$.

This is in sharp contrast with what happens if the scales of the kernels are set to geometric growth.

Theorem 1.2. *Suppose q, g and μ are as in Theorem 1.1. If $D_q^-(\mu) < \delta$ then*

$$\limsup_{n\to\infty} \frac{\left|\|g_{2^{-n}} * \mu\|_q - \|g_{2^{-n+1}} * \mu\|_q\right|}{n^\alpha} = \infty \quad (\forall\alpha > 0).$$

The proofs require an estimate on the derivative

$$\frac{d}{d\lambda}\ln\left(\|g_{e^\lambda} * \mu\|_q\right),$$

which we derive in Section 2. Section 3 contains lemmas on the upper order of positive function of a positive variable and completes the proofs of the main theorems.

Let us use the notation

$$\|f\|_{\mu,q} = \left(\int_{\mathbb{R}^\delta} f(\mathbf{x})^q \, d\mu(\mathbf{x}) \right)^{\frac{1}{q}}.$$

The calculations from Section 2 can be adjusted to estimate

$$\frac{d}{d\lambda} \ln \left(\|g_{e^\lambda} * \mu\|_{\mu,q-1} \right).$$

Equivalently, we find bounds on the derivative of

$$\ln \left(\int_{\mathbb{R}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\frac{\mathbf{x}-\mathbf{y}}{e^\lambda} \right) \, d\mu(\mathbf{y}) \right)^q \, d\mu(\mathbf{x}) \right).$$

This is the content of Section 4, which does not depend on Section 3. This should be of interest as it relates to computing the correlation dimension by probabilistic methods.

In Section 5 we consider adjusting the standard partition sum $S_\mu^q(\epsilon)$ by allowing soft cut-offs between the cells (bins). We cannot determine the derivative of the these Gaussian-kernel sums, but do demonstrate they are continuous in ϵ.

2. Differentiating the norm of a filtered measure

Recall that given a function g on $\mathbb{R}^\delta$ we rescale it as

$$g_\epsilon(\mathbf{x}) = \epsilon^{-\delta} g\left(\epsilon^{-1}\mathbf{x}\right).$$

This section's goals are to compute

$$\frac{d}{d\epsilon} \|g_\epsilon * \mu\|_q$$

and to show that

$$\frac{d}{d\lambda} \ln \left(\|g_{e^\lambda} * \mu\|_q \right)$$

is bounded.

Lemma 2.1. *Suppose $g : \mathbb{R}^\delta \to \mathbb{R}$ is differentiable, bounded, and has bounded radial derivative. Let $h : \mathbb{R}^\delta \to \mathbb{R}$ be the negative of the radial derivative of g,*

$$h(\mathbf{x}) = -\sum_{j=1}^{\delta} x_j \frac{\partial g}{\partial x_j}.$$

If μ is a finite Borel measure on $\mathbb{R}^\delta$ then

$$\frac{\partial}{\partial \epsilon} \left[(g_\epsilon * \mu)(\mathbf{x}) \right] = \epsilon^{-1} \left((h_\epsilon * \mu)(\mathbf{x}) - \delta (g_\epsilon * \mu)(\mathbf{x}) \right).$$

Proof. For any $\mathbf{w}$,

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\left[g(\epsilon\mathbf{w})\right] &= \sum_{j=1}^{\delta}\left(\left.\frac{\partial g}{\partial x_j}\right|_{\epsilon\mathbf{w}}\right)\frac{\partial}{\partial\epsilon}\left(\epsilon w_j\right)\\ &= \sum_{j=1}^{\delta}\left(\left.\frac{\partial g}{\partial x_j}\right|_{\epsilon\mathbf{w}}\right)w_j\\ &= -\epsilon^{-1}h(\epsilon\mathbf{w}).\end{aligned}$$

Suppose $\mathbf{x}$ is fixed. Assume $0 < a \le \epsilon \le b$ and $\mathbf{y} \in \mathbb{R}^\delta$. Then

$$\begin{aligned}&\frac{\partial}{\partial\epsilon}\left[g_\epsilon(\mathbf{x}-\mathbf{y})\right]\\ &= \frac{\partial}{\partial\epsilon}\left[\epsilon^{-\delta}g(\epsilon^{-1}(\mathbf{x}-\mathbf{y})\right]\\ &= \epsilon^{-\delta}(-\epsilon h(\epsilon^{-1}(\mathbf{x}-\mathbf{y}))(-\epsilon^{-2}) + (-\delta\epsilon^{-\delta-1})g(\epsilon^{-1}(\mathbf{x}-\mathbf{y}))\\ &= \epsilon^{-1}\left(h_\epsilon(\mathbf{x}-\mathbf{y}) - \delta g_\epsilon(\mathbf{x}-\mathbf{y})\right).\end{aligned}$$

and so

$$\left|\frac{\partial}{\partial\epsilon}\left[g_\epsilon(\mathbf{x}-\mathbf{y})\right]\right| \le a^{-\delta-1}H + \delta a^{-\delta-1}G,$$

where G and H are bounds on g and h. Since μ is finite, $g_\epsilon(\mathbf{x}-\mathbf{y})$ is integrable in $\mathbf{y}$. The Dominated Convergence Theorem gives us

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\left[(g_\epsilon * \mu)(\mathbf{x})\right] &= \frac{\partial}{\partial\epsilon}\int_{\mathbb{R}^\delta} g_\epsilon(\mathbf{x}-\mathbf{y})\,d\mu(\mathbf{y})\\ &= \int_{\mathbb{R}^\delta}\epsilon^{-1}\left(h_\epsilon(\mathbf{x}-\mathbf{y}) - \delta g_\epsilon(\mathbf{x}-\mathbf{y})\right)\,d\mu(\mathbf{y})\\ &= \epsilon^{-1}\left((h_\epsilon * \mu)(\mathbf{x}) - \delta(g_\epsilon * \mu)(\mathbf{x})\right).\end{aligned}$$

□

Notation 2.2. We shall use $x \wedge y$ to denote the minimum of two numbers and $x \vee y$ for their maximum.

Theorem 2.3. *Suppose $1 < q < \infty$. Suppose $g : \mathbb{R}^\delta \to \mathbb{R}$ is rapidly decreasing and let $h : \mathbb{R}^\delta \to \mathbb{R}$ denote the negative of the radial derivative of g. Suppose $g \ge 0$ and $h \ge 0$. If μ is a finite Borel measure on $\mathbb{R}^\delta$ with compact support then*

$$\frac{d}{d\epsilon}\left\|g_\epsilon * \mu\right\|_q = \frac{\int_{\mathbb{R}^\delta}(g_\epsilon * \mu)^{q-1}(h_\epsilon * \mu)\,dm}{\epsilon\left\|g_\epsilon * \mu\right\|_q^{q-1}} - \frac{\delta\left\|g_\epsilon * \mu\right\|_q}{\epsilon}$$

and

$$\frac{d}{d\lambda}\ln\left(\left\|g_{e^\lambda} * \mu\right\|_q\right) = \frac{\int_{\mathbb{R}^\delta}(g_{e^\lambda} * \mu)^{q-1}(h_{e^\lambda} * \mu)\,dm}{\int_{\mathbb{R}^\delta}(g_{e^\lambda} * \mu)^q\,dm} - \delta.$$

Proof. Assume $0 < a \leq \epsilon \leq b$.

Pick an integer k with $k > \frac{\delta+1}{q}$. Since g is rapidly decreasing there is a C_1 so that

$$g(\mathbf{x}) \leq C_1(1 \wedge |\mathbf{x}|^{-k}).$$

For all $\mathbf{x}$,

$$\begin{aligned} g_\epsilon(\mathbf{x}) &= \epsilon^{-\delta} g(\epsilon^{-1}\mathbf{x}) \\ &\leq a^{-\delta} C_1(1 \wedge (b^k|\mathbf{x}|^{-k})) \\ &= a^{-\delta} C_1 b^k (b^{-k} \wedge |\mathbf{x}|^{-k}). \end{aligned}$$

If $|\mathbf{y}| \leq \frac{1}{2}|\mathbf{x}|$ and $2b \leq |\mathbf{x}|$ then

$$\begin{aligned} g_\epsilon(\mathbf{x}-\mathbf{y}) &\leq a^{-\delta} C_1 b^k (b^{-k} \wedge |\mathbf{x}-\mathbf{y}|^{-k}) \\ &\leq a^{-\delta} C_1 b^k \left(b^{-k} \wedge \left|\frac{\mathbf{x}}{2}\right|^{-k} \right) \\ &= a^{-\delta} C_1 b^k 2^k |\mathbf{x}|^{-k}. \end{aligned}$$

Suppose

$$\operatorname{supp}(\mu) \subseteq \left\{ \mathbf{y} \in \mathbb{R}^\delta \middle| \, |\mathbf{y}| \leq R \right\}.$$

If $|\mathbf{x}| \geq (2b) \vee (2R)$ then

$$\begin{aligned} (g_\epsilon * \mu)(\mathbf{x}) &= \int_{|\mathbf{y}| \leq R} g_\epsilon(\mathbf{x}-\mathbf{y}) \, d\mu(\mathbf{y}) \\ &\leq \mu\left(\mathbb{R}^\delta\right) a^{-\delta} C_1 b^k 2^k |\mathbf{x}|^{-k}. \end{aligned}$$

If $|\mathbf{x}| \leq (2b) \vee (2R)$ then we have the estimate

$$\begin{aligned} (g_\epsilon * \mu)(\mathbf{x}) &= \int g_\epsilon(\mathbf{x}-\mathbf{y}) \, d\mu(\mathbf{y}) \\ &\leq \int a^{-\delta} C_1 \, d\mu(\mathbf{y}) \\ &= \mu\left(\mathbb{R}^\delta\right) a^{-\delta} C_1. \end{aligned}$$

For some C_2 and C_3,

$$(g_\epsilon * \mu)(\mathbf{x}) \leq C_2(C_3 \wedge |\mathbf{x}|^{-k})$$

for all $\mathbf{x}$.

We can repeat the previous argument for h_ϵ. Possibly increasing C_2 and C_3 we can have

$$(h_\epsilon * \mu)(\mathbf{x}) \leq C_2(C_3 \wedge |\mathbf{x}|^{-k}).$$

Therefore

$$\begin{aligned}\left|\frac{\partial}{\partial\epsilon}\left[(g_\epsilon * \mu)(\mathbf{x})\right]\right| &= \left|\epsilon^{-1}\left((h_\epsilon * \mu)(\mathbf{x}) - \delta(g_\epsilon * \mu)(\mathbf{x})\right)\right| \\ &\le a^{-1}\left((h_\epsilon * \mu)(\mathbf{x}) + \delta(g_\epsilon * \mu)(\mathbf{x})\right) \\ &\le a^{-1}(\delta+1)C_2(C_3 \wedge |\mathbf{x}|^{-k})\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial}{\partial\epsilon}\left[((g_\epsilon * \mu)(\mathbf{x}))^q\right]\right| &= q\left((g_\epsilon * \mu)(\mathbf{x})\right)^{q-1}\left|\frac{\partial}{\partial\epsilon}\left[(g_\epsilon * \mu)(\mathbf{x})\right]\right| \\ &\le qa^{-1}(\delta+1)(C_2(C_3 \wedge |\mathbf{x}|^{-k}))^q \\ &= qa^{-1}(\delta+1)C_2^q(C_3^q \wedge |\mathbf{x}|^{-qk}),\end{aligned}$$

which is is integrable since k is larger than $\frac{\delta+1}{q}$. We can use dominated convergence again. By Lemma 2.1,

$$\begin{aligned}&\frac{d}{d\epsilon}\left(\|g_\epsilon * \mu\|_q^q\right) \\ &= \int_{\mathbb{R}^\delta} q\left((g_\epsilon * \mu)(\mathbf{x})\right)^{q-1}\epsilon^{-1}\left((h_\epsilon * \mu)(\mathbf{x}) - \delta(g_\epsilon * \mu)(\mathbf{x})\right)\, dm(\mathbf{x}) \\ &= \frac{q}{\epsilon}\left(\int_{\mathbb{R}^\delta}\left((g_\epsilon * \mu)(\mathbf{x})\right)^{q-1}(h_\epsilon * \mu)(\mathbf{x})\, dm(\mathbf{x}) - \delta\int_{\mathbb{R}^\delta}\left((g_\epsilon * \mu)(\mathbf{x})\right)^q\, dm(\mathbf{x})\right).\end{aligned}$$

The two derivative formulas in the statement of the theorem now follow from

$$\frac{d}{d\epsilon}\|g_\epsilon * \mu\|_q = \frac{1}{q}\|g_\epsilon * \mu\|_p^{1-q}\frac{d}{d\epsilon}\left(\|g_\epsilon * \mu\|_q^q\right)$$

and

$$\frac{d}{d\lambda}\ln\left(\|g_{e^\lambda} * \mu\|_q\right) = \frac{\left(\frac{d}{d\epsilon}\big|_{\epsilon=e^\lambda}\|g_\epsilon * \mu\|_q\right)e^\lambda}{\|g_{e^\lambda} * \mu\|_q}.$$ □

Theorem 2.4. *Suppose $1 < q < \infty$. Suppose $g : \mathbb{R}^\delta \to \mathbb{R}$ is rapidly decreasing and let $h : \mathbb{R}^\delta \to \mathbb{R}$ denote the negative of the radial derivative of g. Suppose $g \ge 0$ and $h \ge 0$. If μ is a finite Borel measure on $\mathbb{R}^\delta$ with compact support then*

$$-\delta\epsilon^{-1}\|g_\epsilon * \mu\|_q \le \frac{d}{d\epsilon}\|g_\epsilon * \mu\|_q \le \epsilon^{-1}\|h_\epsilon * \mu\|_q - \delta\epsilon^{-1}\|g_\epsilon * \mu\|_q$$

and

$$-\delta \le \frac{d}{d\lambda}\ln\left(\|g_{e^\lambda} * \mu\|_q\right) = \frac{\|h_{e^\lambda} * \mu\|_q}{\|g_{e^\lambda} * \mu\|_q} - \delta.$$

Proof. The two lower bounds follow trivially from the last theorem and the fact that g and h are nonnegative.

Hölder's inequality gives us

$$\begin{aligned}
&\int_{\mathbb{R}^\delta} ((g_\epsilon * \mu)(\mathbf{x}))^{q-1} \, (h_\epsilon * \mu)(\mathbf{x}) \, dm(\mathbf{x}) \\
&\leq \left(\int_{\mathbb{R}^\delta} ((g_\epsilon * \mu)(\mathbf{x}))^q \, dm(\mathbf{x}) \right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^\delta} ((h_\epsilon * \mu)(\mathbf{x}))^q \, dm(\mathbf{x}) \right)^{\frac{1}{q}} \\
&= \|g_\epsilon * \mu\|_q^{q-1} \|h_\epsilon * \mu\|_q
\end{aligned}$$

and the upper bounds follow. □

Corollary 2.5. *Suppose $1 < q < \infty$. Suppose $g : \mathbb{R}^\delta \to \mathbb{R}$ is nonnegative, nontrivial, rapidly decreasing and is radially nonincreasing. There is a finite constant C so that if μ is a finite Borel measure on $\mathbb{R}^\delta$ with compact support then*

$$-\delta \leq \frac{d}{d\lambda} \ln \left(\|g_{e^\lambda} * \mu\|_q \right) \leq C$$

for all λ. If g is a Gaussian, then we may take $C = 0$.

Proof. We can apply [12, Lemma 2.1] to g. When $S_\mu^q(\epsilon)$ is the partition function (1), this tells us that there is a D so that

$$D^{-1} \leq \frac{\epsilon^{\delta \frac{q-1}{q}} \|g_\epsilon * \mu\|_q}{(S_\mu^q(\epsilon))^{\frac{1}{q}}} \leq D.$$

From the proof of [12, Lemma 2.1] we see that D can be taken to depend only on q and g. This conclusion is valid for the negative radial derivative h as well. This is because $\|h_\epsilon * \mu\|_q$ is invariant under translations of h and clearly h is bounded away from zero on some open set. Therefore

$$\frac{\|h_\epsilon * \mu\|_q}{\|g_\epsilon * \mu\|_q} = \frac{\epsilon^{\delta \frac{q-1}{q}} \|h_\epsilon * \mu\|_q}{(S_\mu^q(\epsilon))^{\frac{1}{q}}} \frac{(S_\mu^q(\epsilon))^{\frac{1}{q}}}{\epsilon^{\delta \frac{q-1}{q}} \|g_\epsilon * \mu\|_q}$$

is bounded above and away from zero.

In the Gaussian case, we know that $\|g_\epsilon * \mu\|_q$ is nonincreasing (cf. [12, Lemma 3.1]) and so the derivative is nonpositive. □

3. Asymptotic indices

Given a function $f > 0$ on the positive reals, the quantities

$$\overline{d}(f) = \limsup_{x \to \infty} \frac{\ln(f(x))}{\ln(x)}$$

and

$$\underline{d}(f) = \liminf_{x \to \infty} \frac{\ln(f(x))}{\ln(x)}$$

are the upper and lower orders of f. These provide a simple way to compare the asymptotic behavior of $f(x)$ to x^c for various powers c. Equivalently, $\overline{d}(f)$ is the smallest extended real number so that

$$c > \overline{d}(f) \implies f(x) \leq x^c \text{ for large } x \tag{2}$$

and $\underline{d}(f)$ is the largest extended real number so that

$$c < \underline{d}(f) \implies f(x) \geq x^c \text{ for large } x. \tag{3}$$

The proof is not complicated. See [10]. For a look at how upper and lower order relate to regular variation, see [2].

In broad terms, if

$$c = \lim_{x\to\infty} \frac{\ln(f(x))}{\ln(x)}$$

exists, then $f(x)$ behaves not so differently from x^c. There is no reason to think that if $f'(x)$ exists it must behave like x^{c-1}. However, if there are some bounds on the derivative of the log-log plot of f then we are able to deduce the upper order of $|f'|$ from the upper order of f.

For the lower order on $|f'|$, we have found no particularly interesting result that can be applied to $\|g_{x^{-1}} * \mu\|_q$. The difficulty is that even if we assume g is a Gaussian we don't know if the derivative of $\|g_{x^{-1}} * \mu\|_q$ is bounded away from zero.

We take the liberty of setting $\ln(0) = -\infty$, and indeed $\frac{\ln(0)}{C} = -\infty$. (This is to accommodate $f'(x) = 0$ at some x and $f(n) = f(n-1)$ at some n.) Both (2) and (3) remain valid.

Lemma 3.1. *Suppose $f : [1,\infty) \to (0,\infty)$ is differentiable and that for some finite constant C,*

$$\left|\frac{d}{dx}\ln\left(f\left(e^x\right)\right)\right| \leq C. \tag{4}$$

Given any nondecreasing sequence x_n with limit ∞, if

$$\lim_{n\to\infty} \frac{\ln(x_{n+1})}{\ln(x_n)} = 1 \tag{5}$$

then

$$\limsup_{n\to\infty} \frac{\ln(f(x_n))}{\ln(x_n)} = \limsup_{x\to\infty} \frac{\ln(f(x))}{\ln(x)}.$$

Proof. The bound (4) implies

$$\left|\ln\left(f\left(e^x\right)\right) - \ln\left(f\left(e^y\right)\right)\right| \leq C\left|x-y\right|$$

or

$$\left|\ln\left(f\left(x\right)\right) - \ln\left(f\left(y\right)\right)\right| \leq C\left|\ln(x) - \ln(y)\right|.$$

The rest of the proof mimics that of [12, Lemma 4.1], and is omitted. □

Lemma 3.2. *Suppose $f : [1, \infty) \to (0, \infty)$ is differentiable. If, for some finite constant C,*

$$\left| \frac{d}{dx} \ln \left(f \left(e^x \right) \right) \right| \leq C$$

for all x, then

$$\begin{aligned} \limsup_{n\to\infty} \frac{\ln |f(n) - f(n-1)|}{\ln(n)} &= \limsup_{x\to\infty} \frac{\ln |f'(x)|}{\ln(x)} \\ &= \limsup_{x\to\infty} \frac{\ln \left(f(x) \right)}{\ln(x)} - 1. \end{aligned}$$

Proof. Suppose f is a function with the bounds $\pm C$ on the slope of its log-log plot. For each n, the Mean Value Theorem gives us a number x_n in the range

$$n - 1 \leq x_n \leq n$$

for which

$$f(n) - f(n-1) = f'(x_n).$$

From basic facts about nets we obtain

$$\begin{aligned} \limsup_{n\to\infty} \frac{\ln |f(n) - f(n-1)|}{\ln(n)} &= \limsup_{n\to\infty} \frac{\ln |f'(x_n)|}{\ln(n)} \\ &\leq \limsup_{n\to\infty} \frac{\ln |f'(x_n)|}{\ln(x_n)} \\ &\leq \limsup_{x\to\infty} \frac{\ln |f'(x)|}{\ln(x)}. \end{aligned}$$

Let $g(x) = \ln \left(f \left(e^x \right) \right)$ so that $|g'(x)| \leq C$ and

$$g'(x) = \frac{e^x f' \left(e^x \right)}{f \left(e^x \right)}.$$

This can be rewritten as

$$f'(x) = g'(\ln(x)) x^{-1} f(x)$$

and so we have

$$\left| f'(x) \right| \leq C x^{-1} f(x).$$

Therefore

$$\begin{aligned} \limsup_{x\to\infty} \frac{\ln |f'(x)|}{\ln(x)} &\leq \limsup_{x\to\infty} \frac{\ln C - \ln(x) + \ln(f(x))}{\ln(x)} \\ &= \limsup_{x\to\infty} \frac{\ln(f(x))}{\ln(x)} - 1. \end{aligned}$$

To finish, we must show

$$\limsup_{x\to\infty} \frac{\ln(f(x))}{\ln(x)} - 1 \leq \limsup_{n\to\infty} \frac{\ln |f(n) - f(n-1)|}{\ln(n)}.$$

We can apply Lemma 3.1, because

$$1 \leq \frac{\ln(x_{n+1})}{\ln(x_n)} \leq \frac{\ln(n+1)}{\ln(n-1)} \to 1,$$

and this tells us that it will suffice to show

$$\limsup_{n\to\infty} \frac{\ln(f(n))}{\ln(n)} - 1 \leq \limsup_{n\to\infty} \frac{\ln|f(n) - f(n-1)|}{\ln(n)}.$$

Let

$$m = \limsup_{n\to\infty} \frac{\ln|f(n) - f(n-1)|}{\ln(n)}.$$

Suppose we are given $\delta > 0$. Then pick $c \neq -1$ with

$$m < c < m + \delta.$$

(If $m = \infty$ we have nothing to prove. If $m = -\infty$ then modify this to picking $c \neq -1$ less then any given finite number C.) There is a natural number n_0 so that

$$n \geq n_0 \implies |f(n) - f(n-1)| \leq n^c.$$

For large n,

$$\begin{aligned} f(n) &= f(n_0) + \sum_{k=n_0+1}^{n} |f(k) - f(k-1)| \\ &\leq f(n_0) + \sum_{k=n_0+1}^{n} k^c \\ &\leq f(n_0) + \int_{n_0}^{n+1} y^c \, dy \\ &\leq f(n_0) + \frac{1}{c+1}(n+1)^{c+1}. \end{aligned}$$

For large n,

$$f(n) \leq n^{m+\delta+1}.$$

Therefore

$$\limsup_{n\to\infty} \frac{\ln(f(n))}{\ln(n)} \leq m + \delta + 1.$$

Since this is true for all $\delta > 0$, we are done. (If $m = -\infty$ then we obtain this lim sup is less than C, for all finite C, and so is also $-\infty$.) □

Lemma 3.3. *Suppose $f : [1, \infty) \to (0, \infty)$ is differentiable and that there is a finite constant C so that*

$$\left| \frac{d}{dx} \ln\left(f\left(e^x\right)\right) \right| \leq C$$

for all x. If

$$\overline{d}(f) = \limsup_{x\to\infty} \frac{\ln(f(x))}{\ln(x)} > 0$$

then

$$\left\{ t > 0 \,\middle|\, \forall \alpha > 0, \lim_{n\to\infty} \frac{\left|f(n^t) - f((n-1)^t)\right|}{n^\alpha} = 0 \right\} = \left(0, \overline{d}(f)^{-1}\right].$$

If $\overline{d}(f) = 0$ *then*

$$\left\{ t > 0 \,\middle|\, \forall \alpha > 0, \lim_{n\to\infty} \frac{\left|f(n^t) - f((n-1)^t)\right|}{n^\alpha} = 0 \right\} = (0, \infty).$$

Proof. For a sequence $a_n > 0$, it is routine to show that

$$\limsup_n \frac{\ln(a_n)}{\ln(n)} \leq 0 \iff \forall \alpha > 0, \lim_{n\to\infty} \frac{a_n}{n^\alpha} = 0. \tag{6}$$

Let $h(x) = \ln(f(e^x))$ so that $|h'(x)| \leq C$. With $t > 0$ to be specified below, define $g(x) = f(x^t)$. Then

$$\left| \frac{d}{dx} \ln\left(g\left(e^x\right)\right) \right| \leq tC$$

and we may apply Lemma 3.2 to g.

As to the upper order:

$$\begin{aligned} \limsup_{x\to\infty} \frac{\ln(g(x))}{\ln(x)} &= \limsup_{x\to\infty} \frac{\ln(f(x^t))}{\ln(x)} \\ &= \limsup_{x\to\infty} \frac{\ln(f(x))}{\ln(x^{\frac{1}{t}})} \\ &= t\overline{d}(f). \end{aligned}$$

By Lemma 3.2,

$$\begin{aligned} \limsup_{n\to\infty} \frac{\ln\left(\left|f(n^t) - f((n-1)^t)\right|\right)}{\ln(n)} &= \limsup_{n\to\infty} \frac{\ln\left(|g(n) - g(n-1)|\right)}{\ln(n)} \\ &= \limsup_{x\to\infty} \frac{\ln\left(g(x)\right)}{\ln(x)} - 1 \\ &= t\overline{d}(f) - 1. \end{aligned}$$

If $t > \overline{d}(f)$ then by (6) there exists $\alpha > 0$ so that

$$\frac{f(n^t) - f((n-1)^t)}{n^\alpha} \not\to 0.$$

If $t \leq \overline{d}(f)$ then

$$\frac{f(n^t) - f((n-1)^t)}{n^\alpha} \to 0$$

for all positive α. □

Theorem 1.1 now follows, since if

$$f(x) = \|g_{x^{-1}} * \mu\|_q$$

then by [8], or [12, Lemma 2.3],

$$\overline{d}(f) = \frac{q-1}{q}\left(\delta - D_q^-(\mu)\right).$$

To prove Theorem 1.2 requires only the following lemma.

Lemma 3.4. *Suppose $f : [1,\infty) \to (0,\infty)$ is differentiable. If, for some finite constant C,*

$$\left|\frac{d}{dx}\ln\left(f\left(e^x\right)\right)\right| \le C$$

for all x, and if

$$\overline{d}(f) = \limsup_{x\to\infty}\frac{\ln(f(x))}{\ln(x)} > 0,$$

then

$$\limsup_{n\to\infty}\frac{\ln\left|f(2^n) - f(2^{n-1})\right|}{\ln(n)} = \infty.$$

Proof. Suppose for some $\alpha > 0$ there is an n_0 so that

$$n \ge n_0 \implies \left|f(2^n) - f(2^{n-1})\right| \le n^\alpha.$$

Then

$$n \ge n_0 \implies f(2^n) \le f\left(2^{n_0}\right) + n^{\alpha+1}.$$

Suppose $\beta > 0$. Then for some $n_1 \ge n_0$,

$$n \ge n_1 \implies f\left(2^{n_0}\right) + n^{\alpha+1} \le 2^{n\beta}.$$

Therefore

$$\limsup_{n\to\infty}\frac{\ln\left(f\left(2^n\right)\right)}{\ln\left(2^n\right)} \le \beta.$$

Lemma 3.1 tells us

$$\limsup_{n\to\infty}\frac{\ln\left(f\left(x\right)\right)}{\ln\left(x\right)} = \limsup_{n\to\infty}\frac{\ln\left(f\left(2^n\right)\right)}{\ln\left(2^n\right)} = 0. \qquad \square$$

4. Gaussian kernel correlation integrals

The probabilistic interpretations of the correlation integral

$$\int \mu(\mathbf{x} + \epsilon\mathbb{B})^{q-1}\, d\mu(\mathbf{x})$$

make it a common tool for determining the Rényi dimensions of μ. Here

$$\mathbb{B} = \left\{\mathbf{x} \in \mathbb{R}^\delta \,\middle|\, |\mathbf{x}| \le 1\right\}$$

and μ is a Borel probability measure on $\mathbb{R}^\delta$. When $q = 2$ the correlation integral is

$$\int \mu(B_\epsilon(\mathbf{x}))\, d\mu(\mathbf{x}) = \Pr\left\{|X_1 - X_2| \leq \epsilon\right\},$$

where X_1 and X_2 are random locations in the probability space $\left(\mathbb{R}^\delta, \mu\right)$. Using a sharp cut-off for the allowed distance seems unwise in a numerical situation, as is discussed in [4, 5, 6, 7, 8, 11, 13, 16, 17].

Consider the expectation of the scalar-valued random variable

$$G\left(\epsilon^{-1}\,|X_1 - X_2|\right)$$

for a function such as a Gaussian $G(x) = e^{-x^2}$, or any $G \geq 0$ that is positive at 0 and rapidly decreasing. This expectation can be rewritten as follows. Let $g(\mathbf{x}) = G(|\mathbf{x}|)$ and

$$g_\epsilon(\mathbf{x}) = \epsilon^{-\delta} G(\epsilon^{-1}|\mathbf{x}|).$$

Then

$$\begin{aligned} E\left[G\left(\epsilon^{-1}\,\|X_1 - X_2\|\right)\right] &= \epsilon^\delta \int\int g_\epsilon\left(\mathbf{x} - \mathbf{y}\right)\, d\mu(\mathbf{y})\, d\mu(\mathbf{x}) \\ &= \epsilon^\delta \left\|g_\epsilon * \mu\right\|_{\mu,1}. \end{aligned}$$

As is shown in [1],

$$\begin{aligned} D_q^{\pm}(\mu) &= \lim_{\epsilon\to 0}{}^{\sup}_{\inf} \frac{1}{q-1} \frac{\ln\left(\int_{\mathbb{R}^\delta} \left(\epsilon^\delta g_\epsilon * \mu\right)^{q-1}\, d\mu\right)}{\ln(\epsilon)} \\ &= \delta + \lim_{\epsilon\to 0}{}^{\sup}_{\inf} \frac{\ln\left(\left\|g_\epsilon * \mu\right\|_{\mu,q-1}\right)}{\ln(\epsilon)} \end{aligned}$$

and more specifically there is a constant $C \neq 0$ so that

$$C^{-1} \leq \frac{\int_{\mathbb{R}^\delta} \epsilon^\delta \left(g_\epsilon * \mu\right)^{q-1}\, d\mu}{S_\mu^p(\epsilon)} \leq C$$

for all ϵ.

The Rényi dimensions of μ can be computed as

$$\lim_{\lambda\to-\infty}{}^{\sup}_{\inf} \frac{\ln(P_\mu(e^\lambda))}{\lambda}$$

for at least the following six choices of partition function. (See [1, 8, 9, 15] and Section 5.)

$$P_\mu^q(\epsilon) = \left(S_\mu^q(\epsilon)\right)^{\frac{1}{q-1}} = \left(\sum_{\mathbf{j}\in\mathbb{Z}^\delta} \mu(\epsilon\mathbf{j} + \epsilon\mathbb{I})^q\right)^{\frac{1}{q-1}}; \tag{7}$$

$$P_\mu^q(\epsilon) = \left(\int_{\mathbb{R}^\delta} \mu(\mathbf{x} + \epsilon\mathbb{B})^{q-1}\, d\mu(\mathbf{x})\right)^{\frac{1}{q-1}}; \tag{8}$$

$$P_\mu^q(\epsilon) = \left(\int_{\mathbb{R}^\delta} \mu(\mathbf{x} + \epsilon \mathbb{B})^q \frac{1}{\epsilon^\delta} \, dm(\mathbf{x}) \right)^{\frac{1}{q-1}}; \tag{9}$$

$$P_\mu^q(\epsilon) = \left(\sum_{\mathbf{j} \in \mathbb{Z}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon} \right) \, d\mu(\mathbf{y}) \right)^{q-1} \right)^{\frac{1}{q-1}}; \tag{10}$$

$$P_\mu^q(\epsilon) = \left(\int_{\mathbb{R}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\frac{\mathbf{x} - \mathbf{y}}{\epsilon} \right) \, d\mu(\mathbf{y}) \right)^{q-1} \, d\mu(\mathbf{x}) \right)^{\frac{1}{q-1}}; \tag{11}$$

$$P_\mu^q(\epsilon) = \left(\int_{\mathbb{R}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\frac{\mathbf{x} - \mathbf{y}}{\epsilon} \right) \, d\mu(\mathbf{y}) \right)^{q} \frac{1}{\epsilon^\delta} \, dm(\mathbf{x}) \right)^{\frac{1}{q-1}}. \tag{12}$$

The functions (7) and (8) can be discontinuous in ϵ. (A sum of two point masses shows this.) Assuming μ has compact support, we find that (10) is continuous for $1 < q < \infty$ (Theorem 5.2), that (11) is continuous for $1 < q < \infty$ and differentiable for $2 < q < \infty$, (Theorems 4.5 and 4.2), and that (12) is differentiable for $1 < q < \infty$ (Theorem 2.3). Added smoothness should be an advantage in computational situations, as was pointed out in [17].

It is not clear if the function in (9) is continuous whenever μ is finite with compact support.

The bound on the last partition function that we found in Section 2 used a different normalizing constant. Recall there is a C so that

$$-\delta \leq \frac{d}{d\lambda} \ln \left(\|g_{e^\lambda} * \mu\|_q \right) \leq C.$$

In the Gaussian case we had $C = 0$. Since

$$\begin{aligned} &\ln \left(\left(\int_{\mathbb{R}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\frac{\mathbf{x} - \mathbf{y}}{e^\lambda} \right) \, d\mu(\mathbf{y}) \right)^q \frac{1}{e^{\delta\lambda}} \, dm(\mathbf{x}) \right)^{\frac{1}{q-1}} \right) \\ &= \ln \left(e^{\delta\lambda} \, \|g_{e^\lambda} * \mu\|_q^{\frac{q}{q-1}} \right) \\ &= \delta\lambda + \frac{q}{q-1} \ln \left(\|g_{e^\lambda} * \mu\|_q \right) \end{aligned}$$

we have

$$\frac{\delta}{1-q} \leq \frac{d}{d\lambda} \ln \left(\left(\int_{\mathbb{R}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\frac{\mathbf{x} - \mathbf{y}}{e^\lambda} \right) \, d\mu(\mathbf{y}) \right)^q \frac{1}{e^{\delta\lambda}} \, dm(\mathbf{x}) \right)^{\frac{1}{q-1}} \right) \leq C_1.$$

In the Gaussian case, we may take $C_1 = \delta$.

In this section we prove that for $2 \leq q < \infty$, there is a constant C depending on g and q so that for any finite Borel measure μ of compact

support,

$$0 \leq \frac{d}{d\lambda} \ln \left(\left(\int_{\mathbb{R}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\frac{\mathbf{x}-\mathbf{y}}{e^\lambda} \right) \, d\mu(\mathbf{y}) \right)^{q-1} \, d\mu(\mathbf{x}) \right)^{\frac{1}{q-1}} \right) \leq C.$$

We will need a lower bound on

$$\int_{\mathbb{R}^\delta} \epsilon^\delta \, (h_\epsilon * \mu)^{q-1} \, d\mu,$$

where h is the negative of the radial derivative of g. Since $h(\mathbf{0}) = 0$ we need a small modification of a result in [1]. We are restricting our attention to the case $1 < q < \infty$, which allows us to avoid the technicalities encountered in [1].

Here we use the notation from [12], so $\mu^{(\epsilon)}$ is the sequence over $\mathbb{Z}^\delta$ given by

$$\mu^{(\epsilon)}_{\mathbf{n}} = \mu(\epsilon\mathbf{n} + \epsilon\mathbb{I}).$$

Lemma 4.1. *Assume that $g \geq 0$ is rapidly decreasing and that $1 < q < \infty$. There is a finite constant C so that for any finite Borel measure μ on $\mathbb{R}^\delta$,*

$$\frac{\epsilon^\delta \, \|g_\epsilon * \mu\|_{\mu,q-1}}{(S^q_\mu(\epsilon))^{\frac{1}{q-1}}} \leq C$$

for all $\epsilon > 0$. If also $g(\mathbf{0}) > 0$ then there is a $c > 0$ so that

$$c \leq \frac{\epsilon^\delta \, \|g_\epsilon * \mu\|_{\mu,q-1}}{(S^q_\mu(\epsilon))^{\frac{1}{q-1}}}.$$

Proof. Define Γ over $\mathbb{Z}^\delta$ by

$$\Gamma_{\mathbf{n}} = \sup\{g(\mathbf{x}) \mid \mathbf{x} \in \mathbf{n} + \mathbb{D}\},$$

where

$$\mathbb{D} = (-1,1) \times (-1,1) \times \cdots \times (-1,1).$$

Repeating an argument from [1], we find

$$\begin{aligned}
&\|g_\epsilon * \mu\|^{q-1}_{\mu,q-1} \\
&= \epsilon^{-(q-1)\delta} \int \left(\int g(\epsilon^{-1}(\mathbf{x}-\mathbf{y})) \, d\mu(\mathbf{y}) \right)^{q-1} \, d\mu(\mathbf{x}) \\
&= \epsilon^{-(q-1)\delta} \sum_{\mathbf{j}\in\mathbb{Z}^\delta} \int_{\epsilon\mathbf{j}+\epsilon\mathbb{I}} \left(\sum_{\mathbf{k}\in\mathbb{Z}^\delta} \int_{\epsilon\mathbf{k}+\epsilon\mathbb{I}} g(\epsilon^{-1}(\mathbf{x}-\mathbf{y})) \, d\mu(\mathbf{y}) \right)^{q-1} \, d\mu(\mathbf{x}) \\
&\leq \epsilon^{-(q-1)\delta} \sum_{\mathbf{j}\in\mathbb{Z}^\delta} \left(\sum_{\mathbf{k}\in\mathbb{Z}^\delta} \Gamma_{\mathbf{j}-\mathbf{k}} \mu^{(\epsilon)}_{\mathbf{k}} \right)^{q-1} \mu^{(\epsilon)}_{\mathbf{j}}.
\end{aligned}$$

Hölder's inequality and Young's convolution inequality now tell us

$$\begin{aligned}\|g_\epsilon * \mu\|_{\mu,q-1}^{q-1} &\le \epsilon^{-(q-1)\delta}\left(\left\|\Gamma * \mu^{(\epsilon)}\right\|_q\right)^{q-1}\left\|\mu^{(\epsilon)}\right\|_q \\ &\le \epsilon^{-(q-1)\delta}\|\Gamma\|_1^{q-1}\left\|\mu^{(\epsilon)}\right\|_q^q,\end{aligned}$$

i.e.,

$$\frac{\epsilon^\delta \|g_\epsilon * \mu\|_{\mu,q-1}}{(S_\mu^p(\epsilon))^{\frac{1}{q-1}}} \le \|\Gamma\|_1.$$

If g is positive at the origin, then since it is continuous we can rescale g using $\widetilde{g} = g_\eta$ with the same properties as g, but with

$$\inf\{\widetilde{g}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{D}\} > 0$$

and $g_\epsilon = \widetilde{g}_{\eta\epsilon}$. We can compare $\|g_\epsilon * \mu\|_{\mu,q-1}$ and $\|\widetilde{g}_{\eta\epsilon} * \mu\|_{\mu,q-1}$ as follows:

$$\begin{aligned}\frac{\epsilon^\delta \|g_\epsilon * \mu\|_{\mu,q-1}}{\left(S_\mu^\delta(\epsilon)\right)^{\frac{1}{q-1}}} &= \frac{\epsilon^\delta \|\widetilde{g}_{\eta\epsilon} * \mu\|_{\mu,q-1}}{\left(S_\mu^\delta(\epsilon)\right)^{\frac{1}{q-1}}} \\ &= \left(\eta^{-\delta}\right)\frac{\eta^\delta\epsilon^\delta \|\widetilde{g}_{\eta\epsilon} * \mu\|_{\mu,q-1}}{(S_\mu^q(\eta\epsilon))^{\frac{1}{q-1}}}\frac{(S_\mu^q(\eta\epsilon))^{\frac{1}{q-1}}}{(S_\mu^q(\epsilon))^{\frac{1}{q-1}}}.\end{aligned}$$

By [12, Theorem 3.4], there are constants A and B so that

$$e^{-A-B|\ln(\eta)|} \le \frac{S_\mu^q(\eta\epsilon)}{S_\mu^q(\epsilon)} \le e^{A+B|\ln(\eta)|}$$

for all ϵ. Therefore

$$\frac{\epsilon^\delta \|g_\epsilon * \mu\|_{\mu,q-1}}{(S_\mu^q(\epsilon))^{\frac{1}{q-1}}} \le \left(\eta^{-\delta}e^{\frac{A+B|\ln(\eta)|}{q-1}}\right)\frac{(\eta\epsilon)^\delta \|\widetilde{g}_{\eta\epsilon} * \mu\|_{\mu,q-1}}{(S_\mu^q(\eta\epsilon))^{\frac{1}{q-1}}},$$

and it suffices to prove the result in the case where

$$\inf\{g(\mathbf{x}) \mid \mathbf{x} \in \mathbb{D}\} > 0.$$

Let

$$\gamma_{\mathbf{n}} = \inf\{g(\mathbf{x}) \mid \mathbf{x} \in \mathbf{n} + \mathbb{D}\}.$$

As above, we find

$$\|g_\epsilon * \mu\|_{\mu,q-1}^{q-1} \ge \epsilon^{-(q-1)\delta}\sum_{\mathbf{j}\in\mathbb{Z}^\delta}\left(\sum_{\mathbf{k}\in\mathbb{Z}^\delta}\gamma_{\mathbf{j}-\mathbf{k}}\mu_{\mathbf{k}}^{(\epsilon)}\right)^{q-1}\mu_{\mathbf{j}}^{(\epsilon)}$$

and so

$$\begin{aligned}\|g_\epsilon * \mu\|_{\mu,q-1}^{q-1} &\geq \epsilon^{-(q-1)\delta} \sum_{\mathbf{j}\in\mathbb{Z}^\delta} \left(\gamma_{\mathbf{0}}\mu_{\mathbf{j}}^{(\epsilon)}\right)^{q-1} \mu_{\mathbf{j}}^{(\epsilon)} \\ &= \gamma_{\mathbf{0}}\epsilon^{-(q-1)\delta} \sum_{\mathbf{j}\in\mathbb{Z}^\delta} \left(\mu_{\mathbf{j}}^{(\epsilon)}\right)^{q}\end{aligned}$$

and

$$\epsilon^\delta \|g_\epsilon * \mu\|_{\mu,q-1} \geq \gamma_{\mathbf{0}}^{\frac{1}{q-1}} \left(S_\mu^q(\epsilon)\right)^{\frac{1}{q-1}}. \qquad \square$$

Theorem 4.2. *Suppose $2 \leq q < \infty$. Suppose $g : \mathbb{R}^\delta \to \mathbb{R}$ is rapidly decreasing and let $h : \mathbb{R}^\delta \to \mathbb{R}$ denote the negative of the radial derivative of g. Suppose $g \geq 0$ and $h \geq 0$. If μ is a finite Borel measure on $\mathbb{R}^\delta$ then*

$$\frac{d}{d\epsilon}\|g_\epsilon * \mu\|_{\mu,q-1} = \frac{\int_{\mathbb{R}^\delta} (g_\epsilon * \mu)^{q-2}\, h_\epsilon * \mu\, d\mu}{\epsilon \|g_\epsilon * \mu\|_{\mu,q-1}^{q-2}} - \frac{\delta \|g_\epsilon * \mu\|_{\mu,q-1}}{\epsilon}$$

and

$$\frac{d}{d\lambda} \ln\left(\|g_{e^\lambda} * \mu\|_{\mu,q-1}\right) = \frac{\int_{\mathbb{R}^\delta} (g_{e^\lambda} * \mu)^{q-2}\, h_{e^\lambda} * \mu\, d\mu}{\int_{\mathbb{R}^\delta} (g_{e^\lambda} * \mu)^{q-1}\, d\mu} - \delta.$$

Proof. For ϵ restricted to some interval $[a, b]$, it follows from Lemma 2.1 that

$$\left|\frac{\partial}{\partial\epsilon}\left((g_\epsilon * \mu)(\mathbf{x})\right)^{q-1}\right| \leq (q-1)a^{-(q-1)(\delta+1)} \|\mu\|_1^{q-1} \|g\|_\infty^{q-2} \left(\|h\|_\infty + \delta \|g\|_\infty\right).$$

Dominated convergence yields

$$\begin{aligned}&\frac{d}{d\epsilon}\left(\int_{\mathbb{R}^\delta} (g_\epsilon * \mu)^{q-1}\, d\mu\right) \\ &\qquad = \frac{q-1}{\epsilon}\left(\int_{\mathbb{R}^\delta} (g_\epsilon * \mu)^{q-2}\, h_\epsilon * \mu\, d\mu - \delta \|g_\epsilon * \mu\|_{\mu,q-1}^{q-1}\right)\end{aligned}$$

and so

$$\begin{aligned}&\frac{d}{d\epsilon}\|g_\epsilon * \mu\|_{\mu,q-1} \\ &= \frac{1}{\epsilon}\|g_\epsilon * \mu\|_{\mu,q-1}^{2-q}\left(\int_{\mathbb{R}^d} (g_\epsilon * \mu)^{q-2}\, h_\epsilon * \mu\, d\mu - \delta \|g_\epsilon * \mu\|_{\mu,q-1}^{q-1}\right) \\ &= \frac{\int_{\mathbb{R}^\delta} (g_\epsilon * \mu)^{q-2}\, h_\epsilon * \mu\, d\mu}{\epsilon \|g_\epsilon * \mu\|_{\mu,q-1}^{q-2}} - \frac{\delta \|g_\epsilon * \mu\|_{\mu,q-1}}{\epsilon}.\end{aligned}$$

We use

$$\frac{d}{d\lambda} \ln\left(\|g_{e^\lambda} * \mu\|_{\mu,q-1}\right) = \frac{\left(\frac{d}{d\epsilon}\big|_{\epsilon=e^\lambda} \|g_\epsilon * \mu\|_{\mu,q-1}\right) e^\lambda}{\|g_{e^\lambda} * \mu\|_{\mu,q-1}}$$

and find

$$\frac{d}{d\lambda}\ln\left(\|g_{e^\lambda}*\mu\|_{\mu,q-1}\right)=\frac{\int_{\mathbb{R}^\delta}(g_{e^\lambda}*\mu)^{q-2}\,h_{e^\lambda}*\mu\,d\mu}{\|g_{e^\lambda}*\mu\|_{\mu,q-1}^{q-1}}-\delta. \qquad \square$$

Theorem 4.3. *Suppose $2\le q<\infty$. Suppose $g:\mathbb{R}^\delta\to\mathbb{R}$ is rapidly decreasing and let $h:\mathbb{R}^\delta\to\mathbb{R}$ denote the negative of the radial derivative of g. Suppose $g\ge 0$ and $h\ge 0$. If μ is a finite Borel measure of compact support on $\mathbb{R}^d$ then*

$$\begin{aligned}-\delta\epsilon^{-1}\|g_\epsilon*\mu\|_{\mu,q-1}&\le\frac{d}{d\epsilon}\left(\|g_\epsilon*\mu\|_{\mu,q-1}\right)\\&\le\epsilon^{-1}\|h_\epsilon*\mu\|_{\mu,q-1}-\delta\epsilon^{-1}\|g_\epsilon*\mu\|_{\mu,q-1}\end{aligned}$$

and

$$-\delta\le\frac{d}{d\lambda}\ln\left(\|g_{e^\lambda}*\mu\|_{\mu,q-1}\right)\le\frac{\|h_{e^\lambda}*\mu\|_{\mu,q-1}}{\|g_{e^\lambda}*\mu\|_{\mu,q-1}}-\delta.$$

Proof. If $2<q<\infty$, we can apply Hölder's inequality and we find

$$\begin{aligned}&\int_{\mathbb{R}^\delta}(g_\epsilon*\mu)^{q-2}\,(h_\epsilon*\mu)\,d\mu\\&\qquad\le\left(\int_{\mathbb{R}^\delta}(g_\epsilon*\mu)^{q-1}\,d\mu\right)^{\frac{q-2}{q-1}}\left(\int_{\mathbb{R}^\delta}(h_\epsilon*\mu)^{q-1}\,d\mu\right)^{\frac{1}{q-1}}.\end{aligned}$$

This is trivially true as well when $q=2$. We can rewrite this as

$$\int_{\mathbb{R}^\delta}(g_\epsilon*\mu)^{q-2}\,(h_\epsilon*\mu)\,d\mu\le\|g_\epsilon*\mu\|_{\mu,q-1}^{q-2}\,\|h_\epsilon*\mu\|_{\mu,q-1}$$

and the inequalities follow from the last result and the fact that $g_\epsilon*\mu$ and $h_\epsilon*\mu$ are nonnegative. $\square$

Corollary 4.4. *Suppose $2\le q<\infty$. Suppose $g:\mathbb{R}^\delta\to\mathbb{R}$ is nonnegative, nontrivial, rapidly decreasing and is radially nonincreasing. There is a finite constant C so that if μ is a finite Borel measure of compact support on $\mathbb{R}^\delta$ then*

$$0\le\frac{d}{d\lambda}\ln\left(\left(\int_{\mathbb{R}^\delta}\left(\int_{\mathbb{R}^\delta}g\left(\frac{\mathbf{x}-\mathbf{y}}{e^\lambda}\right)\,d\mu(\mathbf{y})\right)^{q-1}\,d\mu(\mathbf{x})\right)^{\frac{1}{q-1}}\right)\le C.$$

Proof. By Lemma 4.1, we have an upper bound on $\|h_{e^\lambda}*\mu\|_{\mu,q-1}$ and a lower bound on $\|g_{e^\lambda}*\mu\|_{\mu,q-1}$ that depends only on q and g. Therefore Theorem 4.3 gives us a C_1 so that

$$-\delta\le\frac{d}{d\lambda}\ln\left(\|g_{e^\lambda}*\mu\|_{\mu,q-1}\right)\le C_1$$

for all μ and all λ. As to the partition function,

$$\ln\left(\left(\int_{\mathbb{R}^\delta}\left(\int_{\mathbb{R}^\delta} g\left(\frac{\mathbf{x}-\mathbf{y}}{e^\lambda}\right)\,d\mu(\mathbf{y})\right)^{q-1}\,d\mu(\mathbf{x})\right)^{\frac{1}{q-1}}\right) = \delta\lambda + \ln\left(\|g_{e^\lambda} * \mu\|_{\mu,q-1}\right)$$

and so

$$0 \leq \frac{d}{d\lambda}\ln\left(\left(\int_{\mathbb{R}^\delta}\left(\int_{\mathbb{R}^\delta} g\left(\frac{\mathbf{x}-\mathbf{y}}{e^\lambda}\right)\,d\mu(\mathbf{y})\right)^{q-1}\,d\mu(\mathbf{x})\right)^{\frac{1}{q-1}}\right) \leq C_1 + \delta. \qquad \square$$

Theorem 4.5. *Assume that $g \geq 0$ is rapidly decreasing and that $1 < q < \infty$. For any finite Borel measure μ on $\mathbb{R}^\delta$ with compact support,*

$$\int_{\mathbb{R}^\delta} (g_\epsilon * \mu)^{q-1}\,d\mu(\mathbf{x})$$

varies continuously in ϵ.

Proof. Assume $0 < a \leq \epsilon \leq b$. If G is a bound on g, then $a^{-\delta}G$ is a bound on g_ϵ and so

$$((g_\epsilon * \mu)(\mathbf{y})) \leq a^{-\delta(q-1)}G^{q-1}\|\mu\|_1^{q-1}.$$

Since μ is a finite measure, we can apply the dominated convergence theorem and have

$$\lim_{\epsilon\to\eta}\int_{\mathbb{R}^\delta}(g_\epsilon * \mu)^{q-1}\,d\mu(\mathbf{x}) = \int_{\mathbb{R}^\delta}(g_\eta * \mu)^{q-1}\,d\mu(\mathbf{x}). \qquad \square$$

5. Gaussian kernel Rényi entropy sums

There is a smooth version of the partition function

$$\sum_{\mathbf{j}\in\mathbb{Z}^\delta}\mu(\epsilon\mathbf{j} + \epsilon\mathbb{I})^q$$

that eliminates the sharp cut-off at the boundary of the cells in the grid,

$$\sum_{\mathbf{j}\in\mathbb{Z}^\delta}\left(\int_{\mathbb{R}^\delta} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon}\right)\,d\mu(\mathbf{y})\right)^q.$$

Although we have not determined if this creates a partition function that is differentiable, it does at least give continuity.

What we have in mind for g is either a Gaussian, or a smooth function between 0 and 1 that equals the characteristic function for $\mathbb{I}$ except close to the boundary of $\mathbb{I}$.

Recall from [12] that we say a finite Borel measure μ on $\mathbb{R}^d$ is *q-finite* if $S_\mu^q(1) < \infty$. This is automatic if $1 < q < \infty$.

First we show that this modified Rényi entropy sum still leads to $D_q^\pm(\mu)$.

Theorem 5.1. *Assume that $g \geq 0$ is rapidly decreasing, with $g(\mathbf{0}) > 0$, and that $0 < q < \infty$, $q \neq 1$. There is a constant C so that, for any Borel measure μ on $\mathbb{R}^\delta$,*

$$C^{-1} \leq \frac{\sum_{\mathbf{j}\in\mathbb{Z}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon}\right)\, d\mu(\mathbf{y})\right)^q}{\sum_{\mathbf{j}\in\mathbb{Z}^\delta} \mu(\epsilon\mathbf{j} + \epsilon\mathbb{I})^q} \leq C$$

for all $\epsilon > 0$.

Proof. The proof is almost identical to that of [12, Lemma 2.3], and we again use the notation used there.

Notice that

$$\sum_{\mathbf{j}\in\mathbb{Z}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon}\right)\, d\mu(\mathbf{y})\right)^q = \sum_{\mathbf{j}\in\mathbb{Z}^\delta} \left(\sum_{\mathbf{k}\in\mathbb{Z}^\delta} \int_{\epsilon\mathbf{k}+\epsilon\mathbb{I}} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon}\right)\, d\mu(\mathbf{y})\right)^q.$$

If

$$\mathbf{y} \in \epsilon\mathbf{k} + \epsilon\mathbb{I}$$

then

$$\mathbf{j} - \frac{\mathbf{y}}{\epsilon} \in (\mathbf{j} - \mathbf{k}) - \mathbb{I} \subseteq (\mathbf{j} - \mathbf{k}) - \mathbb{D}.$$

Therefore

$$\sum_{\mathbf{j}\in\mathbb{Z}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon}\right)\, d\mu(\mathbf{y})\right)^q \leq \sum_{\mathbf{j}\in\mathbb{Z}^d} \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} \Gamma_{\mathbf{j}-\mathbf{k}}\mu(\epsilon\mathbf{k} + \epsilon\mathbb{I})\right)^q$$

and

$$\sum_{\mathbf{j}\in\mathbb{Z}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon}\right)\, d\mu(\mathbf{y})\right)^q \geq \sum_{\mathbf{j}\in\mathbb{Z}^d} \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} \gamma_{\mathbf{j}-\mathbf{k}}\mu(\epsilon\mathbf{k} + \epsilon\mathbb{I})\right)^q.$$

Therefore

$$\left\|\gamma * \mu^{(\epsilon)}\right\|_q^q \leq \sum_{\mathbf{j}\in\mathbb{Z}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon}\right)\, d\mu(\mathbf{y})\right)^q \leq \left\|\Gamma * \mu^{(\epsilon)}\right\|_q^q$$

and the rest of the proof follows that of [12, Lemma 2.3]. □

Theorem 5.2. *Assume that $g \geq 0$ is rapidly decreasing and that $0 < q < \infty$, $q \neq 1$. For any finite Borel measure μ on $\mathbb{R}^\delta$ with compact support, the sum*

$$\sum_{\mathbf{j}\in\mathbb{Z}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon}\right)\, d\mu(\mathbf{y})\right)^q$$

varies continuously in ϵ.

Proof. Since

$$\sum_{\mathbf{j}\in\mathbb{Z}^\delta} \left(\int_{\mathbb{R}^\delta} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon}\right)\, d\mu(\mathbf{y})\right)^q = \epsilon^{\delta q} \sum_{\mathbf{j}\in\mathbb{Z}^\delta} (g_\epsilon * \mu(\epsilon\mathbf{j}))^q$$

it will suffice to prove the continuity of

$$\sum_{\mathbf{j}\in\mathbb{Z}^\delta} (g_\epsilon * \mu(\epsilon\mathbf{j}))^q \,.$$

Again we restrict ϵ to some the interval $[a, b]$.

We know that g is bounded by some $G < \infty$. We have the bound

$$\epsilon^{-\delta} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon}\right) \leq a^{-\delta} G,$$

and since μ is a finite measure, we can apply dominated convergence and conclude that

$$g_\epsilon * \mu(\epsilon\mathbf{j}) = \int_{\mathbb{R}^\delta} \epsilon^{-\delta} g\left(\mathbf{j} - \frac{\mathbf{y}}{\epsilon}\right) \, d\mu(\mathbf{y})$$

varies continuously in ϵ.

We saw in the proof of Theorem 2.3 that there is a bound

$$(g_\epsilon * \mu)(\mathbf{x}) \leq C_2(C_3 \wedge |\mathbf{x}|^{-k}),$$

where k is taken to be an integer larger than $\frac{\delta+1}{q}$. Therefore

$$((g_\epsilon * \mu)(\epsilon\mathbf{j}))^q \leq a^{-qk} C_2(b^{qk} C_3^q \wedge |\mathbf{j}|^{-qk}).$$

We can apply dominated convergence, this time with the measure being counting measure on $\mathbb{Z}^\delta$, and conclude that

$$\sum_{\mathbf{j}\in\mathbb{Z}^\delta} (g_\epsilon * \mu(\epsilon\mathbf{j}))^q$$

varies continuously in ϵ. □

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