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Local geometry of zero sets of holomorphic functions near the torus

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Abstract. We call a holomorphic function f on a domain in $\mathbb{C}^n$ locally toral at the point P in the n-torus if the intersection of the zero set of f with the n-torus has dimension $n-1$ at P. We study the interplay between the structure of locally toral functions and the geometry of their zero sets.

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0. Introduction

Throughout this paper, we shall let $\mathbb{D}$ denote the unit disk in the complex plane, $\mathbb{T}$ be a unit circle, $\mathbb{E}$ be the complement of the closed unit disk in $\mathbb{C}$ and let $A(\mathbb{D}^n)$ denote the polydisk algebra, the algebra of functions that are continuous on the closure of $\mathbb{D}^n$ and holomorphic on the interior.

When studying function theory on the polydisk $\mathbb{D}^n$, it is often useful to focus on the torus $\mathbb{T}^n$, which is the distinguished boundary of $\mathbb{D}^n$. In several important ways, the behavior of a function in $A(\mathbb{D}^n)$ is controlled by its

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behavior on $\mathbb{T}^n$: not only is $\mathbb{T}^n$ a set of uniqueness, but every function in the algebra attains its maximum modulus on $\mathbb{T}^n$.

We shall say that a variety (by which we always mean an irreducible algebraic set) V is *toral* if its intersection with $\mathbb{T}^n$ is fat enough to be a determining set for holomorphic functions on V (see Section 2 for a precise definition). Otherwise we shall call the variety *atoral*. We shall say that a polynomial in $\mathbb{C}[z_1, \ldots, z_n]$ is toral (respectively, atoral) if the zero set of every irreducible factor is toral (respectively, atoral).

In [2] it was shown that knowing an algebraic set is toral has interesting consequences in function theory and operator theory. In this paper we study the localized versions of this and other related geometric properties of zero sets. In Section 1 we collect standard basic tools for studying the local geometry of an analytic set. In Section 2 we define and analyze the local properties of interest to us. In Section 3 we develop the relationship between the local properties of Section 2 and the global properties studied in [2]. The results in Sections 2 and 3 include constraints, both positive and negative, on which combinations of properties can occur simultaneously. In those sections we also include a number of specific examples showing that various combinations of properties can occur.

Let f be a holomorphic function defined on an open subset G of $\mathbb{C}^n$ and suppose $\tau \in \mathbb{T}^n \cap G$. To localize the notion of torality, we say f is *locally toral at* τ if $f(\tau) = 0$ and, for every neighborhood U of τ in G, there is a neighborhood V of τ in U such that $Z_f \cap V \cap \mathbb{T}^n$ is determining for $Z_f \cap V$. An irreducible holomorphic function may no longer be irreducible if its domain is restricted to a smaller set, however, there is a well-defined notion of locally irreducible at a point. If f is locally irreducible at τ, then we say f is *locally atoral at* τ if $f(\tau) = 0$ and f is not locally toral at τ.

Every toral polynomial is locally toral at some point and vice versa. A polynomial which is toral, locally toral at exactly one point and locally atoral on an arc is presented.

Now let $n = 2$, f be a holomorphic function defined on an open subset G of $\mathbb{C}^2$ and suppose $\tau \in \mathbb{T}^2 \cap G$. A function φ which is holomorphic on an open neighborhood of the closed unit disk is *inner* if $|\varphi(\alpha)| = 1$ for all $\alpha \in \mathbb{D}$ and therefore the zero set of the function $w - \varphi(z)$ lies in $(\mathbb{D} \times \mathbb{D}) \cup (\mathbb{E} \times \mathbb{E}) \cup \mathbb{T}^2$. We define f to be locally inner at τ if $f(\tau) = 0$ and its zero set near τ lies in $(\mathbb{D} \times \mathbb{D}) \cup (\mathbb{E} \times \mathbb{E}) \cup \mathbb{T}^2$ and locally outer at τ if $f(\tau) = 0$ and its zero sets near τ lies in $(\mathbb{D} \times \mathbb{E}) \cup (\mathbb{E} \times \mathbb{D}) \cup \mathbb{T}^2$. Notice that if f is locally inner (or outer) at τ, then the zero set of f does not intersect $\mathbb{S}$ where $\mathbb{S}$ is the set of points $(z, w) \in \mathbb{C}^2$ where either z or w (but not both) is unimodular. With this notation, $\mathbb{C}^2$ is the disjoint union of $\mathbb{T}^2$, $\mathbb{S}$ and the four open "quadrants" in $\mathbb{C}^2$ determined by the sets $\mathbb{D}$ and $\mathbb{E}$, namely, $\mathbb{D} \times \mathbb{D}$, $\mathbb{E} \times \mathbb{D}$, $\mathbb{D} \times \mathbb{E}$, and $\mathbb{E} \times \mathbb{E}$. Any curve is $\mathbb{C}^2$ which intersects two or more of the above quadrants does so by passing through either $\mathbb{S}$ or $\mathbb{T}^2$. Replacing $\mathbb{T}^2$ with $\mathbb{S}$ in the definitions of toral, atoral, locally toral and locally atoral results in our definitions of sidal, asidal, locally sidal and locally asidal, respectively.

If f is locally inner (or locally outer) at τ, then f is locally asidal at τ. Moreover, the product of locally inner and locally outer functions is locally asidal. Every locally asidal function can be decomposed (locally) as a product of a local inner times a local outer.

After defining inner and outer for holomorphic functions of 2 variables (the nonlocalized versions of "locally inner" and "locally outer"), we find that every nonzero atoral polynomial $p(z, w)$ is a product of an inner polynomial and an outer polynomial.

While an inner polynomial is locally inner at each point in its zero set intersect $\mathbb{T}^2$, the converse is not true. The counterexample to the converse is a polynomial p which is locally inner at each point which is both in the zero set of p and in $\mathbb{T}^2$ (and, therefore, locally asidal at each of these points), but is not inner since it is not (globally) sidal.

1. Preliminaries

For the convenience of the reader, in this section we compile a number of elementary results from the literature ([4],[3]) that describe the local properties of 0-sets of holomorphic functions of several variables. In addition, we will indicate, in a general sense, how these results will be used in the subsequent sections of the paper. If $n \geq 1$ and G is an open set in $\mathbb{C}^n$, let $\mathrm{Hol}(G)$ denote the algebra of holomorphic functions on G. If $d \geq 2$ and G_0 is an open set in $\mathbb{C}^{d-1}$, let $\mathrm{Hol}(G_0)[w]$ denote the ring of polynomials in w with coefficients in $\mathrm{Hol}(G_0)$. Let $\mathcal{P}(G_0)$ denote the collection of pseudopolynomials over G_0, i.e., the monic elements of $\mathrm{Hol}(G_0)[w]$. Thus, $P \in \mathcal{P}(G_0)$ if and only if there exists an integer $n \geq 1$ and $s_1, \ldots, s_n \in \mathrm{Hol}(G_0)$ such that

$$P(z, w) = w^n - s_1(z)w^{n-1} + \cdots + (-1)^n s_n(z). \tag{1.1}$$

If G_1 is an open subset of G_0, then we let $P|G_1$ denote the pseudopolynomial over G_1 obtained by restricting the coefficients of P to G_1. Throughout this section, we denote points in $\mathbb{C}^d$ as ordered pairs $\zeta = (z, w)$ with $z \in \mathbb{C}^{d-1}$ and $w \in \mathbb{C}$. If $f \in \mathrm{Hol}(G)$, then we let $Z_f = \{\zeta \mid f(\zeta) = 0 \text{ and } \zeta \in G\}$. If $P \in \mathcal{P}(G_0)$, then we shall view P as an element of $\mathrm{Hol}(G_0 \times \mathbb{C})$. Thus,

$$Z_P = \{(z, w) \in G_0 \times \mathbb{C} \mid P(z, w) = 0\}.$$

If $\zeta \in \mathbb{C}^d$, then we will say that G is a neighborhood of ζ if G is an *open* subset which contains ζ and say that U is a neighborhood of ζ in G if U is an *open* subset of G which contains ζ.

Theorem 1.2 (Weierstrass Preparation Theorem). *If G is an open subset of $\mathbb{C}^d$, $(z_0, w_0) \in G$, $f \in \mathrm{Hol}(G)$ and $f(z_0, w)$, the holomorphic function of the single variable w, has a 0 of order n at $w = w_0$, then there exist*

(i) *a connected neighborhood* G_0 *of* z_0 *in* $\mathbb{C}^{d-1}$,
(ii) *a neighborhood* D_0 *of* w_0 *in* $\mathbb{C}$,
(iii) *an* $h \in \mathrm{Hol}(G_0 \times D_0)$ *such that* $Z_h = \emptyset$, *and*
(iv) *a pseudopolynomial* P *over* G_0

such that $G_0 \times D_0 \subseteq G$,

$$P(z_0, w) = (w - w_0)^n, \tag{1.3}$$

and

$$f(z, w) = P(z, w)h(z, w) \tag{1.4}$$

for all $z \in G_0$ *and all* $w \in D_0$. *Moreover, the representation in* (1.4) *is unique in the following sense: if* Q *is a pseudopolynomial over* G_0, $k \in \mathrm{Hol}(G_0 \times D_0)$, $Z_k = \emptyset$, $Q(z_0, w) = (w - w_0)^m$ *for some* $m \geq 1$, *and* $f = Qk$ *on* $G_0 \times D_0$, *then* $P = Q$ *and* $h = k$.

In light of the uniqueness assertion in Theorem 1.2, the pseudopolynomial P of Theorem 1.2 is referred to as the Weierstrass polynomial of f at (z_0, w_0) over G_0.

Evidently, if $f \in \mathrm{Hol}(G)$, $(z_0, w_0) \in G$, $f(z_0, w_0) = 0$ and the function $f(z_0, w)$ has a zero of finite order at $w = w_0$, then Theorem 1.2 can be used to obtain sets G_0 and D_0 and $P \in \mathcal{P}(G_0)$ such that $Z_f \cap (G_0 \times D_0) = Z_P \cap (G_0 \times D_0)$.

Now, one of the many hurdles to overcome in the quest to understand the 0-sets of holomorphic functions in several variables results from the fact that the ring $\mathrm{Hol}(G_0)$, while an integral domain (when G_0 is connected), fails to be a unique factorization domain. This has the unpleasant consequence that $\mathrm{Hol}(G_0)[w]$ is not a unique factorization domain. It is indeed both a fundamental and fortuitous event that nevertheless $\mathcal{P}(G_0)$, while not a ring, behaves like a unique factorization domain in the following sense.

Theorem 1.5. *If* $d \geq 2$, G_0 *is an open connected set in* $\mathbb{C}^{d-1}$ *and* $P \in \mathcal{P}(G_0)$, *then* P *can be written, uniquely up to order, as a finite product of irreducible elements of* $\mathcal{P}(G_0)$.

In many proofs using Theorem 1.5, it would be desirable to have that if Q is an irreducible factor of P and $G_1 \subset G_0$, then $Q|G_1$ is irreducible. Unfortunately, this is not true. For example, if Q is the pseudopolynomial over $\mathbb{D}$ defined by $Q(z, w) = w^2 - z$, U is an open disk in $\mathbb{D}$, and $0 \notin U$, then Q is irreducible, but $Q|U$ is reducible since $\sqrt{z} \in \mathrm{Hol}(U)$. These considerations motivate the following robust localized notions of irreducibility.

Definition 1.6. Let $d \geq 2$, $z_0 \in \mathbb{C}^{d-1}$, G_0 be a neighborhood of z_0, and $P \in \mathcal{P}(G_0)$. Let us agree to say that P is *locally irreducible at* z_0 if, there exists a neighborhood U of z_0 in G_0 such that for every connected neighborhood V of z_0 in U, $P|V$ is irreducible.

Definition 1.7. If $d \geq 2$, $\zeta \in \mathbb{C}^d$, G is a neighborhood of ζ and $f \in \mathrm{Hol}(G)$, we will say that f is *locally irreducible at* ζ if there exists a neighborhood U of ζ in G such that for every connected neighborhood V of ζ in U, for every $g \in \mathrm{Hol}(V)$ and every $h \in \mathrm{Hol}(V)$, $f|V = gh$ implies either Z_g or Z_h equals the empty set.

Clearly, if G is a connected neighborhood of (z_0, w_0), $f \in \mathrm{Hol}(G)$, and P is a Weierstrass polynomial for f at (z_0, w_0), then f is locally irreducible at (z_0, w_0) if and only if P is locally irreducible at z_0.

An alternate approach to handling considerations pertaining to local irreducibility is to employ the theory of germs of analytic varities. We will not use this theory.

Since the ring of power series which converge on some neighborhood of a given point $z_0 \in \mathbb{C}^{d-1}$ is a unique factorization domain, the following theorems follow from Theorem 1.5.

Theorem 1.8. *If $d \geq 2$, $z_0 \in \mathbb{C}^{d-1}$, G_0 is a neighborhood of z_0, and $P \in \mathcal{P}(G_0)$, then there exists a connected neighborhood G_1 of z_0 in G_0 such that every irreducible factor of $P|G_1$ is locally irreducible at z_0.*

Theorem 1.9. *If $\zeta \in \mathbb{C}^d$, G is a connected neighborhood of ζ, $f \in \mathrm{Hol}(G)$ and f is not identically zero, then there exists a connected neighborhood U of ζ and $f_1, \ldots, f_N \in \mathrm{Hol}(U)$ such that $f|U = f_1 f_2 \cdots f_N$ and each f_j is locally irreducible at ζ.*

The singular points of a pseudopolynomial without multiple factors can be described via a discriminant. Recall that if $p = w^n - s_1 w^{n-1} + \cdots + (-1)^n s_n = (w - r_1) \cdots (w - r_n)$ is a polynomial with complex coefficients, then the coefficients $s_1, \ldots, s_n$ equal symmetric polynomials evaluated at the roots $r_1, \ldots, r_n$ and $\triangle_p$, *the discriminant of* p, is defined by writing the symmetric function $\prod\limits_{i<j} (r_i - r_j)^2$ as a polynomial $\triangle_p$ in the elementary symmetric functions of $r_1, \ldots, r_n$. Thus,

$$\triangle_p = \triangle_p(s_1, \ldots, s_n) = \prod_{i<j} (r_i - r_j)^2 \,.$$

As a consequence, p has a multiple factor if and only if $\triangle_p = 0$. Now, if $P \in \mathcal{P}(G_0)$, then $\triangle_P$ is naturally an element of $\mathrm{Hol}(G_0)$. Furthermore, the following theorem is obtained.

Theorem 1.10. *If G_0 is a connected set in $\mathbb{C}^{d-1}$ and $P \in \mathcal{P}(G_0)$, then P has a multiple factor if and only if $\triangle_P$ is identically 0 on G_0.*

The next theorem shows that if $\triangle_P$ does not vanish at z_0, then, for a sufficiently small neighborhood U of z_0 in G_0, P factors as a product of pseudopolynomials of degree 1.

Theorem 1.11. *Let $z_0 \in \mathbb{C}^{d-1}$, G_0 be a neighborhood of z_0 and P be a pseudopolynomial over G_0 of degree n. If $\triangle_P(z_0) \neq 0$, then there exists a neighborhood U of z_0 in G_0 and n functions $r_1, \dots r_n \in \mathrm{Hol}(U)$ so that*

$$P(z,w) = (w - r_1(z)) \cdots (w - r_n(z))$$

for all $z \in U$ and $w \in \mathbb{C}$.

For $P \in \mathcal{P}(G_0)$ define $S_P(G_0, D_0) = \{(z_0, w_0) \in G_0 \times D_0 \big| \triangle_P(z_0) = 0$ and $w = w_0$ is a repeated root of the one variable polynomial $P(z_0, w)\}$. Thus, if P has no multiple factors, then $S_P(G_0, D_0)$ consists of the singular points of the analytic set $Z_P \cap (G_0 \times D_0)$.

Proposition 1.12. *If G_0 is an open connected set in $\mathbb{C}^{d-1}$, D_0 is an open set in $\mathbb{C}$, $P \in \mathcal{P}(G_0)$, P is irreducible and $Z_P \subseteq G_0 \times D_0$, then $Z_P \backslash S_P(G_0, D_0)$ is connected.*

In applying Proposition 1.12 the following observation is often useful. Notice that if a pseudopolynomial P arises as the Weierstrass polynomial of a function f at (z_0, w_0) over G_0, then condition (1.3) of Theorem 1.2 implies that for a sufficiently small connected neighborhood U of z_0 in G_0,

$$P(z,w) = 0 \text{ and } z \in U \text{ imply } w \in D_0\,. \tag{1.13}$$

If Q is an irreducible factor of $P|U$, then Proposition 1.12 implies that

$$Z_Q \backslash S_Q(U, D_0) \tag{1.14}$$

is connected.

Since every pseudopolynomial is monic, the following theorem holds.

Theorem 1.15. *If G_0 is an open set in $\mathbb{C}^{d-1}$, and both P and Q are pseudopolynomials over G_0, then there exist a quotient $R \in \mathcal{P}(G_0)$ and a remainder $S \in \mathrm{Hol}(G_0)[w]$ such that $P = RQ + S$ and either S is identically zero or the degree of S is less than the degree of Q.*

Combining Theorems 1.5, 1.10 and 1.15, it is easy to see the following Nullstellensatz result for pseudopolynomials: if G_0 is an open connected set in $\mathbb{C}^{d-1}$, $P \in \mathcal{P}(G_0)$, $Q \in \mathcal{P}(G_0)$, and $Z_Q \subseteq Z_P$, then there exists a positive integer n such that Q divides P^n. Therefore, if $(z_0, w_0) \in \mathbb{T}^d$, G is a connected neighborhood of (z_0, w_0), $f \in \mathrm{Hol}(G)$, and $g \in \mathrm{Hol}(G)$, then there exists a neighborhood of the form $G_0 \times D_0$ of (z_0, w_0) in G such that

$$\begin{aligned} &Z_g \subseteq Z_f \text{ implies there exists a positive integer} \\ &\quad n \text{ such that } \ g|(G_0 \times D_0) \text{ divides } (f|(G_0 \times D_0))^n\,. \end{aligned} \tag{1.16}$$

2. The local geometry of analytic sets near the torus

We shall employ standard notations by letting $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$ centered at the origin and $\mathbb{T}$ denote the boundary of $\mathbb{D}$. Thus, for

$d \geq 1$, $\mathbb{D}^d$ is the standard unit polyidisc centered at the origin in $\mathbb{C}^d$ and $\mathbb{T}^d$ is the distinguished boundary of $\mathbb{D}^d$. In addition we set $\mathbb{E} = \mathbb{C} \backslash (\mathbb{D} \cup \mathbb{T})$. For the special case when $d = 2$, note that there are four open "quadrants" in $\mathbb{C}^2$ determined by the sets $\mathbb{D}$ and $\mathbb{E}$, namely, $\mathbb{D} \times \mathbb{D}, \mathbb{E} \times \mathbb{D}, \mathbb{D} \times \mathbb{E}$, and $\mathbb{E} \times \mathbb{E}$. We shall let $\mathbb{S}$ denote the union of the "sides" of these quadrants. Thus,

$$\mathbb{S} = \Big[\big(\mathbb{T} \times \mathbb{C}\big) \cup \big(\mathbb{C} \times \mathbb{T}\big) \Big] \Big\backslash \mathbb{T}^2.$$

Recall from [2] that a polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$ is said to be *toral* if $Z_p \cap \mathbb{T}^n$ is determining for Z_p and p is said to be *atoral* if there does not exist a nonconstant $q \in \mathbb{C}[z_1, \ldots, z_n]$ such that q is toral and q divides p. Here, Z_p denotes the 0-set of p and for a set $X \subseteq \mathbb{C}^n$, we say X is determining for Z_p if $X \cap Z_p$ is a set of uniqueness for $\mathrm{Hol}(Z_p)$, the algebra of holomorphic functions on Z_p.

We wish here to localize the notions of torality and atorality to a point τ on the torus. Accordingly, if G is an open set in $\mathbb{C}^d$ and $f \in \mathrm{Hol}(G)$, let $Z_f = \{\zeta \in G \mid f(\zeta) = 0\}$. Further, if $X \subseteq G$, we say X *is determining for* Z_f if $X \cap Z_f$ is a set of uniqueness for $\mathrm{Hol}(Z_f)$, the algebra of holomorphic functions on Z_f.

Definition 2.1. If G is an open set in $\mathbb{C}^d$, $f \in \mathrm{Hol}(G)$ and $\tau \in \mathbb{T}^d$, then we say f is *locally toral at* τ if $f(\tau) = 0$ and, for every neighborhood U of τ in G, there is a neighborhood V of τ in U such that $Z_f \cap V \cap \mathbb{T}^d$ is determining for $Z_f \cap V$. We say f is *locally atoral at* τ if $f(\tau) = 0$ and there does not exist a neighborhood W of τ in G together with a $g \in \mathrm{Hol}(W)$ such that g is locally toral at τ and $Z_g \subseteq Z_f$.

It easily follows from these definitions that if $f \in \mathrm{Hol}(G)$, $\tau \in \mathbb{T}^d$ and U is a neighborhood of τ in G, then f is locally toral (respectively, atoral) at τ if and only if $f|U$ is locally toral (respectively, atoral) at τ. In addition, if $g \in \mathrm{Hol}(G)$ and $g(\tau) \neq 0$, then f is locally toral (respectively, atoral) at τ if and only if fg is locally toral (respectively, atoral) at τ.

In [2] it was an immediate consequence of the definitions that any irreducible polynomial in $\mathbb{C}[z, w]$ is either toral or atoral. We now prove the localized version of this result. Let $\tau = (\tau_1, \tau_2) \in \mathbb{T}^2$, G be a neighborhood of τ and $f \in \mathrm{Hol}(G)$. We will say that f is *nondegerate* at τ if there exists a neighborhood U of τ in G such that $Z_{z-\tau_1} \cap Z_f \cap U = \{\tau\}$ and say that f is *degenerate* at τ if $f(\tau) = 0$ and f is not nondegenerate at τ. Clearly, for any f, if $f(\tau) = 0$, then f can be written uniquely in the form $f = (z - \tau_1)^m g$ where $m \geq 0$ and either $g(\tau) \neq 0$ or g is nondegenerate. Notice also that if f is nondegenerate at τ, then $f(\tau_1, w)$ has a zero of finite order at $w = \tau_2$.

Proposition 2.2. *Let* $\tau \in \mathbb{T}^2$, *G be a neighborhood of* τ, *and* $f \in \mathrm{Hol}(G)$ *with* $f(\tau) = 0$. *If f is locally irreducible at* τ, *then either f is locally toral at* τ *or f is locally atoral at* τ.

Proof. Suppose f is locally irreducible at τ so that either $Z_{z-\tau_1}\cap U = Z_f\cap U$ for some neighborhood U of τ in G or f is nondegenerate at τ. In the first case, clearly, f is locally toral at τ. In the second case, we let P be a Weierstrass polynomial of f at τ over G_0 so that $Z_f \cap (G_0 \times D_0) = Z_P \cap (G_0 \times D_0)$. Suppose that f is not locally atoral at τ. Thus, there exists a neighborhood W of τ in $G_0 \times D_0$ and $g \in \mathrm{Hol}(W)$ such that g is locally toral at τ and $Z_g \subseteq Z_f$. The local irreducibility of P, the torality of g and (1.16) imply that P is locally toral at τ. Thus, f is locally toral at τ. □

If f is locally atoral at τ, g is a divisor of f and $g(\tau) = 0$, then, by the definition of locally atoral, g is locally atoral at τ. A straightforward modification of the proof of Proposition 1.3 of [2] shows that if f is locally toral at τ, g is a divisor of f and $g(\tau) = 0$, then g is locally toral at τ.

Now, the product of toral (respectively, atoral) polynomials is clearly toral (respectively, atoral). Clearly, the product of functions which are locally atoral is locally atoral. The following proposition shows that the product of functions which are locally toral is locally toral.

Proposition 2.3. *Let $\tau \in \mathbb{T}^2$, G be a neighborhood of τ and $f, g \in \mathrm{Hol}(G)$. If f and g are locally toral at τ, then the product fg is locally toral at τ.*

Proof. Without loss of generality, we may assume that both f and g are nondegenerate at τ. Let $h = fg$. By Theorems 1.2 and 1.8, there exist connected neighborhoods G_0 and G_1 of τ_1, neighborhoods D_0 and D_1 of τ_2, and pseudopolynomials P_f and P_g over G_0 such that $G_0 \times D_0 \subset G$, $G_1 \times D_1 \subset G$,

$$Z_f \cap (G_0 \times D_0) = Z_{P_f} \cap (G_0 \times D_0),$$
$$Z_g \cap (G_1 \times D_1) = Z_{P_g} \cap (G_1 \times D_1),$$

each irreducible factor of P_f is locally irreducible at τ and each irreducible factor of P_g is locally irreducible at τ. Let $R_1, \ldots, R_m, S_1, \ldots, S_n$ be pseudopolynomials over $G_0 \cap G_1$ such that

$$P_f|(G_0 \cap G_1) = R_1 \cdots R_m \quad \text{and} \quad P_g|(G_0 \cap G_1) = S_1 S_2 \cdots S_n \tag{2.4}$$

are the decompositions of P_f and P_g into a product of irreducible pseudopolynomials.

To show h is locally toral at τ, let U be a neighborhood of τ in G. There exists a sufficiently small connected neighborhood V of τ in

$$U \cap ((G_0 \cap G_1) \times (D_0 \cap D_1))$$

such that for $1 \le j \le m$ and $1 \le k \le n$,

$$(Z_{R_j} \cap V)\backslash\{\tau\} \text{ and } (Z_{S_k} \cap V)\backslash\{\tau\} \text{ are connected} \tag{2.5}$$

and

$$(Z_{R_j} \cap V)\backslash\{\tau\} \text{ and } (Z_{S_k} \cap V)\backslash\{\tau\} \text{ do not have any singular points.} \tag{2.6}$$

To show that $Z_h \cap V \cap \mathbb{T}^2$ is determining for $Z_h \cap V$, fix $F \in \mathrm{Hol}(Z_h \cap V)$ with $F|(Z_h \cap V \cap \mathbb{T}^2) = 0$. We shall show that $F = 0$ on $Z_f \cap V$ by showing that $F = 0$ on $Z_{R_j} \cap V$ for each j.

Fix j such that $1 \leq j \leq m$. R_j is locally toral at τ and so there exists a neighborhood W of τ in V such that $Z_{R_j} \cap W \cap \mathbb{T}^2$ is determining for $Z_{R_j} \cap W$. Since $Z_{R_j} \cap W \cap \mathbb{T}^2 \subset Z_f \cap W \cap \mathbb{T}^2 \subset Z_h \cap W \cap \mathbb{T}^2$, $F|(Z_{R_j} \cap W \cap \mathbb{T}^2) = 0$, and, since $Z_{R_j} \cap W \cap \mathbb{T}^2$ is determining for $Z_{R_j} \cap W$, $F|(Z_{R_j} \cap W) = 0$. Now $(Z_{R_j} \cap W) \backslash \{\tau\}$ is a relatively open subset of the connected set $(Z_{R_j} \cap V) \backslash \{\tau\}$ and each point of $(Z_{R_j} \cap V) \backslash \{\tau\}$ is not a singular point of Z_{R_j}. Thus, the use of Theorem 1.11 and analytic continuation yields $F|(Z_{R_j} \cap V) = 0$. Since $F|(Z_{R_j} \cap V) = 0$ for each j, (2.4) shows that $F|(Z_f \cap V) = 0$.

An analogous argument show that $F|(Z_g \cap V) = 0$. Thus, F is identically zero. In summary, therefore, $Z_h \cap V \cap \mathbb{T}^2$ is determining for $Z_h \cap V$ and h is locally toral at τ. $\square$

Corollary 2.2 of [2] shows that every polynomial can be written uniquely, up to multiplicitive constants, as a product of a toral and an atoral polynomial. Proposition 2.2, Proposition 2.3 and Theorem 1.9 imply the following localized version of that result.

Corollary 2.7. *Let $\tau \in \mathbb{T}^2$, G be a neighborhood of τ, and $f \in \mathrm{Hol}(G)$ with $f(\tau) = 0$. If f is neither locally toral nor locally atoral at τ, then there exists a neighborhood U of τ, and an essentially unique factorization $f|U = qr$ for some $q \in \mathrm{Hol}(U)$ which is locally toral at τ and $r \in \mathrm{Hol}(U)$ which is locally atoral at τ.*

Theorem 2.8 below generalizes and proves the following: if $\tau = (\tau_1, \tau_2) \in \mathbb{T}^2$, G_0 is a neighborhood of τ_1, $r \in \mathrm{Hol}(G_0)$ and $r(\tau_1) = \tau_2$, then the pseudopolynomial $w - r(z)$ is locally toral at τ if and only if there exists a connected neighborhood N_1 of τ_1 in G_0 such that, for all $z \in N_1 \cap \mathbb{T}$, $r(z) \in \mathbb{T}$.

Theorem 2.8. *Let $\zeta = (z_0, w_0) \in \mathbb{T}^2$, G be a neighborhood of ζ and $f \in \mathrm{Hol}(G)$ be such that ζ is not a singular point of Z_f, $f(\zeta) = 0$, and f is nondegenerate at ζ. f is locally toral at ζ if and only if there exist a neighborhood N_1 of z_0, a neighborhood N_2 of w_0 and a function $r \in \mathrm{Hol}(N_1)$ such that*

$$Z_f \cap (N_1 \times N_2) = \{(z, r(z)) : z \in N_1\} \tag{2.9}$$

and

$$r(z) \in \mathbb{T} \text{ whenever } z \in N_1 \cap \mathbb{T}\,. \tag{2.10}$$

Proof. Suppose that f is locally toral at ζ. By shrinking G if necessary, we may assume that Z_f does not have any singular points. By Theorem 1.2, there exists a neighborhood $G_0 \times D_0$ of ζ in G and P a Weierstruass polynomial of f at ζ over G_0 such that $Z_f \cap (G_0 \times D_0) = Z_P \cap (G_0 \times D_0)$. Without loss of generality, we may assume that $0 \notin G_0$. By the discussion preceeding (1.13), we may assume $Z_P \cap (G_0 \times \mathbb{C}) = Z_P \cap (G_0 \times D_0)$. Let Q be a pseudopolynomial over G_0 without multiple factors such that $Z_P = Z_Q$. Since $Z_f \cap (G_0 \times D_0) = Z_Q \cap (G_0 \times D_0)$ and Z_f does not have any singular points, $S_Q(G_0, D_0) = \emptyset$ and $\triangle_Q(z) \neq 0$ for all $z \in G_0$. Since P is a Weierstrass polynomial at ζ and $Z_Q = Z_P$, $Q(z_0, w) = (w - w_0)^m$ where m is the degree of Q. To show that $m = 1$, note that since ζ is not a singular point of Z_f and $Z_f \cap (G_0 \times D_0) = Z_Q \cap (G_0 \times D_0)$, ζ is not a singular point of Z_Q and $\triangle_Q(z_0) \neq 0$. Thus, by Theorem 1.11, $m = 1$ and there exists $r \in \mathrm{Hol}(G_0)$ such that $Q(z, w) = w - r(z)$. For a sufficiently small neighborhood U of z_0 in G_0, $\overline{r(1/\bar{z})}$ is well-defined and nonzero and since Q is locally toral at ζ by construction, $Z_Q \cap U \cap \mathbb{T}^2$ is determining for $Z_Q \cap U$. Therefore, since $w - 1/\overline{r(1/\bar{z})}$ vanishes on $Z_Q \cap U \cap \mathbb{T}^2$, $w = 1/\overline{r(1/\bar{z})}$ for $z \in Z_Q \cap U$ and so $r(z) \in \mathbb{T}$ whenever $z \in U \cap \mathbb{T}$. Thus, (2.9) and (2.10) hold with $N_1 = U$ and $N_2 = D_0$.

Now suppose that $N_1 \times N_2$ is a neighborhood of ζ and (2.9) and (2.10) hold. Let Q be the pseudopolynomial over N_1 defined by $Q(z, w) = w - r(z)$. To show Q is locally toral at (z_0, w_0), let U be a neighborhood of (z_0, w_0) in G. By the discussion preceeding (1.13), there exist a connected neighborhood V_1 of z_0 and a neighborhood V_2 of w_0 such that $V_1 \times V_2 \subseteq U, V_1 \times V_2 \subseteq G_0 \times D_0$ and $Z_Q \cap (V_1 \times \mathbb{C}) = Z_Q \cap (V_1 \times V_2)$. With this choice of V_1 and V_2,

$$Z_Q \cap (V_1 \times V_2) = \{(z, r(z)) : z \in V_1\}. \tag{2.11}$$

To show $Z_Q \cap (V_1 \times V_2) \cap \mathbb{T}^2$ is determining for $Z_Q \cap (V_1 \times V_2)$, fix $F \in \mathrm{Hol}(Z_Q \cap (V_1 \times V_2))$ with $F|(Z_Q \cap (V_1 \times V_2) \cap \mathbb{T}^2) = 0$. By (2.9), $(z, r(z)) \in \mathbb{T}^2$ whenever $z \in V_1 \cap \mathbb{T}$ and so $F(z, r(z)) = 0$ whenever $z \in V_1 \cap \mathbb{T}$. Since $V_1 \cap \mathbb{T}$ is determining for V_1, (2.11) implies that F vanishes on $Z_Q \cap (V_1 \times V_2)$. Hence, $Z_Q \cap (V_1 \times V_2) \cap \mathbb{T}^2$ is determining for $Z_Q \cap (V_1 \times V_2)$, Q is locally toral at τ and, consequently, f is locally toral at τ. $\square$

Recall from [2] that $p \in \mathbb{C}[z_1, \ldots, z_n]$ is atoral if and only if there exists an algebraic set A such that dim $A \leq n - 2$ and $Z_p \cap \mathbb{T}^2 \subseteq A$. When $n = 2$, this result simply says that p is atoral *if and only if* $Z_p \cap \mathbb{T}^2$ is finite. Thus, Proposition 2.12 below, which gives a characterization of local atorality, does not come as a great surprise.

Proposition 2.12. *Let $\tau \in \mathbb{T}^2, G$ be a neighborhood of τ and $f \in \mathrm{Hol}(G)$ with $f(\tau) = 0$. f is locally atoral at τ if and only if τ is an isolated point of $Z_f \cap \mathbb{T}^2$.*

Proof. First assume that τ is an isolated point of $Z_f \cap \mathbb{T}^2$. We argue by contradiction. If f is not locally atoral at τ, then there exist a neighborhood W of τ and $g \in \mathrm{Hol}(W)$ such that g is locally toral at τ and $Z_g \subseteq Z_f$. Since τ is isolated, there exists an open set $U \subseteq \mathbb{C}^2$ such that $Z_g \cap U \cap \mathbb{T}^2 = \{\tau\}$. Since g is locally toral at τ, there exists a neighborhood V of τ in U such that

$Z_g \cap V \cap \mathbb{T}^2$ is determining for $Z_f \cap V$. Since $Z_g \cap V \cap \mathbb{T}^2 \subseteq Z_g \cap U \cap \mathbb{T}^2 = \{\tau\}$, $\{\tau\}$ is determining for $Z_g \cap V$, an impossibility.

Now assume that $\tau = (\tau_1, \tau_2)$ is not an isolated point of $Z_f \cap \mathbb{T}^2$. If $f(\tau_1, w)$ is identically 0, then f is degenerate at τ and hence is not locally atoral at τ. Therefore, we may assume that $f(\tau_1, w)$ has a zero of finite order n at $w = \tau_2$. Let G_0, D_0, P and h be the sets and functions guaranteed by Theorem 1.2 with P the Weierstrass polynomial for f at τ over G_0. Without loss of generality we may assume that $0 \notin D_0$ and D_0 is symmetric, i.e., if $z \in D_0$, then $1/\bar{z} \in D_0$. Similarly, we may assume that $0 \notin G_0$ and G_0 is symmetric. Since Theorem 1.2 asserts that $P(\tau_1, 0) = (-\tau_2)^n, P(z, 0) \neq 0$ on a neighborhood of τ_1. As $s_n(z) = (-1)^n P(z, 0)$, we may choose G_0 to be sufficiently small so that

$$s_n(z) \neq 0 \quad \text{for} \quad z \in G_0. \tag{2.13}$$

Now since τ is not isolated, P has an irreducible factor $Q \in \mathcal{P}(G_0)$ such that there is a sequence of distinct points $\tau^\ell \in \mathbb{T}^2 \cap (G_0 \times \mathbb{C})$ such that $\tau^\ell \to \tau$ and

$$Q(\tau^\ell) = 0 \quad \text{for all} \quad \ell. \tag{2.14}$$

Furthermore, since $f(\tau_1, w)$ is not identically 0, we may assume that $\tau_1^\ell \neq \tau_1^j$ if $\ell \neq j$. Let m be the degree of Q and $t_1, \ldots, t_m \in \mathrm{Hol}(G_0)$ be such that

$$Q(z, w) = w^m - t_1(z) w^{m-1} + \cdots + (-1)^m \ t_m(z)\,.$$

Now, recalling that $0 \notin G_0$, $0 \notin D_0$, and that both G_0 and D_0 are symmetric, we define $R \in \mathcal{P}(G_0)$ by

$$R(z, w) = \frac{(-1)^m}{\overline{t_m(1/\bar{z})}} w^m \overline{Q\left(\frac{1}{\bar{z}}, \frac{1}{\bar{w}}\right)}. \tag{2.15}$$

Here, $t_m \neq 0$ on G_0 by (2.13).

Now consider $QR \in \mathcal{P}(G_0)$. By (2.14), $Q(\tau^\ell) = 0$ for $\ell \geq 1$ and by (2.15) $R(\tau^\ell) = 0$ for $\ell \geq 1$. Hence $\triangle_{QR}$ vanishes on the sequence $\{\tau^\ell\}$, a set of uniqueness for G_0, and we see that $\triangle_{QR}$ is identically 0 on G_0. Since Q and R are irreducible pseudopolynomials and $\triangle_{QR}$ is identically zero, $Q = R$ by Theorem 1.10.

Now notice that by construction $Z_Q \cap (G_0 \times D_0) \subseteq Z_f \cap (G_0 \times D_0)$. Thus, as we need to show that f is not locally atoral, the proof of the proposition will be complete if we can show Q is locally toral at (z_0, w_0). Accordingly, assume that U is a neighborhood of τ in $G_0 \times D_0$. By the remark following Proposition 1.12, there exists a neighborhood V_0 of z_0 in G_0 and a neighborhood D_1 of w_0 in D_0 such that $V_0 \times D_1 \subseteq U, S_Q(V_0, D_1) = \{\tau\}$, and

$$(Z_Q \cap (V_0 \times D_1)) \backslash \{\tau\}$$

is connected. We claim that $Z_Q \cap (V_0 \times D_1) \cap \mathbb{T}^2$ is determining for $Z_Q \cap (V_0 \times D_1)$. Since $\tau^\ell \to \tau$, there exists $\ell \geq 1$ such that $\tau^\ell \in V_0 \times D_1$.

Since $\triangle_Q(\tau_1^\ell) \neq 0$, Theorem 1.11 implies that there exists a symmetric neighborhood N_1 of τ_1^ℓ, a symmetric neighborhood N_2 of τ_2^ℓ and a function $r \in \mathrm{Hol}(N_1)$ such that $r(\tau_1^\ell) = \tau_2^\ell$ and

$$Z_Q \cap (N_1 \times N_2) = \{(z, r(z)) | z \in N_1\}\,. \tag{2.16}$$

Recalling (2.15) and that $Q = R$, we see immediately, that for $z \in N_1$, $r(z) = \frac{1}{r\left(\frac{1}{\bar{z}}\right)}$, i.e.,

$$|r(z)| = 1 \quad \text{whenever} \quad z \in N_1 \cap \mathbb{T}. \tag{2.17}$$

To show $Z_Q \cap (V_0 \times D_1) \cap \mathbb{T}^2$ is determining for $Z_Q \cap (V_0 \times D_1)$, fix $F \in \mathrm{Hol}(Z_Q \cap (V_0 \times D_1))$ with $F|(Z_Q \cap (V_0 \times D_1) \cap \mathbb{T}^2) = 0$. Since

$$F|(Z_Q \cap (N_1 \times N_2) \cap \mathbb{T}^2)$$

is identically zero, (2.15) and (2.17) and Theorem 2.8 imply

$$F|(Z_Q \cap (N_1 \times N_2))$$

is identically zero. Now $(Z_Q \cap (N_1 \times N_2)) \backslash \{\tau\}$ is a relatively open subset of the connected set $(Z_Q \cap (V_0 \times D_1)) \backslash \{\tau\}$ and each point of $(Z_Q \cap (V_0 \times D_1)) \backslash \{\tau\}$ is not a singular point of Z_Q by (2.16). Thus the use of Theorem 1.11 and analytic continuation yields $F|(Z_Q \cap (V_1 \times D_0)) = 0$. Thus, $Z_Q \cap (V_1 \times D_0) \cap \mathbb{T}^2$ is determining for $Z_Q \cap (V_1 \times D_0)$, Q is locally toral at τ and, consequently, f is locally toral at τ. Since $Z_Q \subset Z_f$ and Q is locally toral at τ, f is not locally atoral at τ. $\square$

Before continuing to the geometry of toral points we wish to formalize an additional fact about the geometry of atoral points (Proposition 2.19 below). Note the following definition is exactly the same as Definition 2.1 with $\mathbb{T}^2$ replaced by $\mathbb{S}$.

Definition 2.18. Let $\tau \in \mathbb{T}^2, G$ be a neighborhood of τ, and $f \in \mathrm{Hol}(G)$. Let us agree to say f is *locally sidal at* τ if $f(\tau) = 0$ and for every neighborhood U of τ in G there exists a neighborhood V of τ in U such that $Z_f \cap V \cap \mathbb{S}$ is determining for $Z_f \cap V$. We say f is *locally asidal at* τ if $f(\tau) = 0$ and there does not exist a neighborhood W of τ in G together with a $g \in \mathrm{Hol}(W)$ such that g is locally sidal at τ and $Z_g \subseteq Z_f$.

The following facts follow in a direct way using the ideas in the proof of Proposition 2.12.

Proposition 2.19. *Let $\tau \in \mathbb{T}^2, G$ be a neighborhood of τ, and $f \in \mathrm{Hol}(G)$. f is locally asidal at τ if and only if $f(\tau) = 0$ and there exists a neighborhood U of τ in G such that $Z_f \cap U \cap \mathbb{S} = \emptyset$.*

Corollary 2.20. *Let $\tau \in \mathbb{T}^2, G$ be a neighborhood of τ and $f \in \mathrm{Hol}(G)$. If $f(\tau) = 0$ and f is neither locally sidal nor locally asidal at τ, then there exists a neighborhood U of τ in G and functions $f_1, f_2 \in \mathrm{Hol}(U)$ such that f_1 is locally sidal at τ, f_2 is locally asidal at τ and $f|U = f_1 f_2$. Furthermore,*

f_1 and f_2 are essentially unique on U (i.e., if $f|U = g_1 g_2$ is any other factorization of f with g_1 locally sidal at τ and g_2 locally asidal at τ, then there exists a nonvanishing $u \in \mathrm{Hol}(U)$ such that $g_1 = uf_1$).

Proposition 2.21. *Let $\tau \in \mathbb{T}^2$, G a neighborhood of τ, and $f \in \mathrm{Hol}(G)$. If f is locally atoral at τ, then f is locally sidal at τ.*

While the converse of Proposition 2.21 is false, nevertheless, it is true that in a certain generic sense, locally sidal points are locally atoral. We make this assertion precise later in this section.

Before continuing, note that Propositions 2.21, 2.2 and 2.3 imply that if $f(\tau) = 0$ and f is locally asidal at τ, then f is locally toral at τ. If τ is a nonsingular point of Z_f, then the following local conditions which are weaker than local asidality guarantee local torality. In the following corollary, the first conclusion follows from Theorem 2.8 and the second follows from the first by considering $g(z,w) = f(w,z)$.

Corollary 2.22. *Let $\tau \in \mathbb{T}^2$, G be a neighborhood of τ and $f \in \mathrm{Hol}(G)$ with $f(\tau) = 0$. Suppose τ is not a singular point of Z_f. If W is a neighborhood of τ in G and $Z_f \cap W \cap (\mathbb{T} \times \mathbb{C}) \subseteq \mathbb{T}^2$, then f is locally toral at τ. If W is a neighborhood of τ in G and $Z_f \cap W \cap (\mathbb{C} \times \mathbb{T}) \subseteq \mathbb{T}^2$, then f is locally toral at τ.*

We now turn to the geometry of toral points. Recall from [2] that if p is a toral polynomial of degree (m,n), then each of the following equivalent statements is true.

$$Z_p \text{ is symmetric, } \left(\text{i.e., } (z,w) \in Z_p \cap (\mathbb{C}^*)^2 \Rightarrow (1/\bar{z}, 1/\overline{w}) \in Z_p\right). \tag{2.23}$$

$$\begin{gathered} p \text{ is essentially symmetric,} \\ \left(\text{i.e., } z^m w^n \overline{p\left(\frac{1}{\bar{z}}, \frac{1}{\overline{w}}\right)} = \sigma p(z,w) \text{ for some nonzero constant } \sigma\right). \end{gathered} \tag{2.24}$$

We localize the notions in (2.23) and (2.24) to a point on the torus in the following definition.

Definition 2.25. Let us agree to say a set $S \subseteq \mathbb{C}^2$ is *symmetric* if $S \subseteq (\mathbb{C}\backslash\{0\})^2$ and $1/\bar{\zeta} \in S$ whenever $\zeta \in S$. Here, if $\zeta = (z,w)$, then $1/\bar{\zeta} = (1/\bar{z}, 1/\overline{w})$. If $U \subseteq \mathbb{C}^2$ is an open symmetric set and $f \in \mathrm{Hol}(U)$, we say f is *essentially symmetric* if there exists a nonvanishing $\sigma \in \mathrm{Hol}(U)$ such that $\overline{f\left(1/\bar{\zeta}\right)} = \sigma(\zeta) f(\zeta)$ for all $\zeta \in U$. If $\tau \in \mathbb{T}^2$, G is a neighborhood of τ and $f \in \mathrm{Hol}(G)$, we say *Z_f is locally symmetric at τ* if there exists a neighborhood U of τ in G such that $Z_f \cap U$ is symmetric. In addition, we say *f is locally essentially symmetric at τ* if there exists a symmetric neighborhood U of τ in G such that $f|U$ is essentially symmetric.

Proposition 2.26. *Let $\tau \in \mathbb{T}^2$, G a neighborhood of τ, and $f \in \mathrm{Hol}(G)$. If f is locally toral at τ, then f is locally essentially symmetric at τ.*

Proof. By Theorem 1.9, it suffices to show that if f is locally toral at τ and f is locally irreducible at τ, then f is locally essentially symmetric at τ. Suppose f is locally toral at τ and f is locally irreducible at τ. By Proposition 2.12, τ is not an isolated point of $Z_f \cap \mathbb{T}^2$. Thus, the proof of Proposition 2.12 shows that for sufficiently small neighborhood G_0 and D_0,

$$P(z,w) = \sigma(z,w)\overline{P(1/\overline{z}, 1/\overline{w})} \text{ for } (z,w) \in G_0 \times D_0$$

for a nonvanishing $\sigma \in \mathrm{Hol}(G_0 \times D_0)$. Thus, P, and therefore f, is locally essentially symmetric at τ. □

Now recall that atoral points necessarily are sidal. This is not the case for toral points. Indeed, toral points come in 3 types: sidal, asidal, and neither sidal nor asidal. However, toral points that are neither sidal nor asidal arise from sidal toral points and asidal toral points in a particularly simple manner, as the following proposition asserts.

Proposition 2.27. *Let $\tau \in \mathbb{T}^2$, G a neighborhood of τ, and $f \in \mathrm{Hol}(G)$. If f is locally toral at τ, then one of the following holds:*

(i) *f is locally sidal at τ.*
(ii) *f is locally asidal at τ.*
(iii) *There exists a neighborhood U of τ in G and essentially unique f_1 and $f_2 \in \mathrm{Hol}(U)$ such that f_1 is locally toral and sidal at τ, f_2 is locally toral and asidal at τ, and $f|U = f_1 f_2$.*

As the following examples show, all three cases mentioned in Proposition 2.27 can occur.

Example 2.28. If $\beta \in \mathbb{T}$, $p(z,w) = w - \beta z$ or $p(z,w) = zw - \beta$, then p is both locally toral and locally asidal at each point in $Z_p \cap \mathbb{T}^2$.

Example 2.29. Let $\alpha \in \mathbb{D}\backslash\{0\}$, $q_\alpha(z,w) = (1 - \overline{\alpha}z)zw - (z - \alpha)$ and $r_\alpha(z) = \frac{z-\alpha}{z(1-\overline{\alpha}z)}$. Clearly, q_α is irreducible,

$$Z_{q_\alpha} = \{(z, r_\alpha(z)) : z \in \mathbb{C}\backslash\{0, 1/\overline{\alpha}\}\},$$

and Z_{q_α} does not have any singular points. Since $r_\alpha(z) \in \mathbb{T}$ whenever $z \in \mathbb{T}$, Theorem 2.8 implies q_α is locally toral at each point in $Z_{q_\alpha} \cap \mathbb{T}^2$. Now if $a_\alpha = \frac{1}{\overline{\alpha}}\left[|\alpha|^2 + i|\alpha|\sqrt{1-|\alpha^2|}\right]$ and $b_\alpha = \frac{1}{\overline{\alpha}}\left[|\alpha|^2 - i|\alpha|\sqrt{1-|\alpha^2|}\right]$, then $a_\alpha, b_\alpha \in \mathbb{T}$ and for $\tau_1 \in \mathbb{T}$, $r'_\alpha(\tau_1) = 0$ if and only if $\tau_1 = a_\alpha$ or $\tau_1 = b_\alpha$. In these cases, r_α is not one-to-one on any neighborhood of τ_1 and so p is locally toral and locally sidal at $(a_\alpha, r_\alpha(a_\alpha))$ and at $(b_\alpha, r_\alpha(b_\alpha))$.

Example 2.30. Let $\alpha \in \mathbb{D}\backslash\{0\}$ and q_α, r_α and a_α be as in Example 2.29. For an appropriate choice at $\beta \in \mathbb{T}$, $(a_\alpha, r_\alpha(a_\alpha))$ is a zero of $w - \beta z$. It is easy to see that if $\zeta = (a_\alpha, r_\alpha(a_\alpha))$, then $w - \tau_1 z$ is locally asidal at ζ and q_α is locally sidal at ζ and $(w - \beta z)q_\alpha(z,w)$ is neither locally sidal nor locally asidal at ζ.

It would appear that the local geometry of sidal toral points can be quite complex. However, much can be said about the geometry of asidal points. If f is locally asidal at τ, then Proposition 2.19 implies that there exists a neighborhood U of τ in G such that $Z_f \cap U \cap \mathbb{S} = \emptyset$, i.e.,

$$Z_f \cap U \subseteq (\mathbb{D}\times\mathbb{D}) \cup (\mathbb{D}\times\mathbb{E}) \cup (\mathbb{E}\times\mathbb{D}) \cup (\mathbb{E}\times\mathbb{E}) \cup \mathbb{T}^2. \tag{2.31}$$

Two special cases of (2.31) would be:

$$Z_f \cap U \subseteq (\mathbb{D}\times\mathbb{D}) \cup (\mathbb{E}\times\mathbb{E}) \cup \mathbb{T}^2 \tag{2.32}$$

and

$$Z_f \cap U \subseteq (\mathbb{D}\times\mathbb{E}) \cup (\mathbb{E}\times\mathbb{D}) \cup \mathbb{T}^2. \tag{2.33}$$

In light of these two special possibilities we make the following definition.

Definition 2.34. Let $\tau \in \mathbb{T}^2$, G a neighborhood of τ and $f \in \mathrm{Hol}(G)$ with $f(\tau) = 0$. We say f is *locally inner* (respectively, *locally outer*) *at* τ if there exists a neighborhood U of τ in G such that (2.32) (respectively, (2.33)) holds.

The following theorem together with Theorems 1.2 and 1.11 show that if f is locally toral at τ, f is nondegenerate at τ and τ is not a singular point of Z_f, then either f is locally inner at τ, f is locally outer at τ or f is locally sidal at τ. Moreover, we can determine which it is using the angular derivative, the definition of which we now recall. Let $\tau = (\tau_1, \tau_2) \in \mathbb{T}^2$, G_0 a neighborhood of τ_1 and $r \in \mathrm{Hol}(G_0)$ be such that $r(z) \in \mathbb{T}$ whenever $z \in G_0 \cap \mathbb{T}$. Since

$$i\frac{d}{d\theta}\mathrm{Arg}(r(e^{i\theta})) = \frac{d}{d\theta}\log(r(e^{i\theta})) = \frac{1}{r(e^{i\theta})}\frac{d}{d\theta}r(e^{i\theta}) = \frac{r'(e^{i\theta})e^{i\theta}i}{r(e^{i\theta})},$$

the angular derivative of r at τ is

$$A_r(\tau) = \frac{\tau_1 r'(\tau_1)}{r(\tau_1)}.$$

Now, if f is a holomorphic function, f is locally toral at τ, and τ is not a singular point of Z_f, then using Theorems 1.2 and 1.11 and the following lemma can be used to determine if f is locally inner at τ, is locally outer at τ or is locally sidal at τ.

Lemma 2.35. *Let $\tau_1 \in \mathbb{T}$, G_0 a neighborhood of τ_1, and $r \in \mathrm{Hol}(G_0)$ be such that $r(z) \in \mathbb{T}$ whenever $z \in \mathbb{T} \cap G_0$ and $P(z,w) = (z,w) = w - r(z)$. Then the following hold.*

(i) *If $A_r(\tau_1) > 0$, then P is locally inner at $(\tau_1, r(\tau_1))$.*
(ii) *If $A_r(\tau_1) < 0$, then P is locally outer at $(\tau_1, r(\tau_1))$.*
(iii) *If $A_r(\tau_1) = 0$, then r is not one-to-one on any neighborhood of τ_1 and so P is locally sidal at $(\tau_1, r(\tau_1))$.*

Example 2.36. If $p(z,w) = w - z$, then p is locally inner at each point of $Z_p \cap \mathbb{T}^2$. If $q(z,w) = zw - 1$, then q is locally outer at each ponit of $Z_q \cap \mathbb{T}^2$.

Example 2.37. Let $\alpha \in \mathbb{D}\backslash\{0\}$ and q_α, r_α, a_α and b_α be as in Example 2.29. If $z_0 \in \mathbb{C}\backslash\{0, 1/\overline{\alpha}\}$ and G_0 is a connected neighborhood of z_0 in $\mathbb{C}\backslash\{0, 1/\overline{\alpha}\}$, then $Q(z,w) = w - r_\alpha(z)$ is the Weierstrass polynomial of q_α at $(z_0, r_\alpha(z_0))$ over G_0. Since $A_{r_\alpha}\left(\frac{|\alpha|}{\alpha}\right) > 0$, $A_{r_\alpha}\left(-\frac{|\alpha|}{\alpha}\right) < 0$, $A_{r_\alpha}(a_\alpha) = A_{r_\alpha}(b_\alpha) = 0$ and $A_{r_\alpha}(\tau_1) \neq 0$ whenever $\tau \in \mathbb{T}\backslash\{a_\alpha, b_\alpha\}$, Lemma 2.35 implies q_α is locally inner at each point in the arc of $\mathbb{T}\backslash\{a_\alpha, b_\alpha\}$ which contains $\frac{|\alpha|}{\alpha}$ and q_α is locally outer at each point in the arc of $\mathbb{T}\backslash\{a_\alpha, b_\alpha\}$ which contains $\frac{|\alpha|}{\alpha}$.

Proposition 2.38. *Let $\tau \in \mathbb{T}^2$, G a neighborhood of τ and $f \in \mathrm{Hol}(G)$ with $f(\tau) = 0$. If f is locally inner at τ or f is locally outer at τ, then f is locally toral at τ.*

Proof. If f is either locally inner or locally outer at τ, then f is locally asidal at τ and, therefore, f is locally toral at τ. □

Theorem 2.39. *Let $\tau \in \mathbb{T}^2$, G a neighborhood of τ and $f \in \mathrm{Hol}(G)$. If f is locally asidal at τ, then there exist a neighborhood U of τ in G and $f_1, f_2 \in \mathrm{Hol}(U)$ such that $f|U = f_1 f_2$, f_1 is locally inner at τ and f_2 is locally outer at τ. The factorization is unique in the following sense: if $g_1, g_2 \in \mathrm{Hol}(U)$, $f|U = g_1 g_2, g_1$ is locally inner at τ and g_2 is locally outer at τ, then there exists $u \in \mathrm{Hol}(U)$ such that $g_1 = uf$ and $Z_u = \emptyset$.*

Proof. The uniqueness assertion follows from Definition 2.34 and Theorem 1.5.

It suffices to show that if f is locally asidal at τ and locally irreducible at τ, then f is either locally inner at τ or locally outer at τ. Suppose f is locally asidal at τ and locally irreducible at τ. By Proposition 2.2, the definition of locally irreducible, the comments preceding (1.13) and the fact that each singular point of Z_f is isolated in Z_f, there exists a connected neighborhood G_0 of τ_1, a neighborhood D_0 of τ_2, and $P \in \mathcal{P}(G_0)$ such that $G_0 \times D_0 \subseteq G$ and P is the Weierstrass polynomial of f at τ over G_0, $Z_P \cap (G_0 \times D_0) = Z_P \cap (G_0 \times \mathbb{C})$, $S_P(G_0, D_0) \subset \{\tau\}$, and $P|U$ is irreducible whenever U is a connected neighborhood of τ in G_0. Thus, by Proposition 1.12, $Z_P\backslash\{\tau\}$ is connected.

Since $S_P(G_0, D_0) \subset \{\tau\}$, $\triangle_P(z) \neq 0$ whenever $z \in G_0\backslash\{\tau_1\}$. Since $\triangle_P \in \mathrm{Hol}(G_0)$ and $G_0 \subseteq \mathbb{C}$, $Z_{\triangle_P}$ is finite and there exists a neighborhood G_1 of τ_1 in G_0 such that $Z_{\triangle_P|G_1} \subseteq \{\tau_1\}$.

To show that P is either locally inner or locally outer at τ, fix $z_0 \in G_1\backslash\{\tau_1\}$ and let w_0 be such that $P(z_0, w_0) = 0$. By Theorem 1.11, there exists a neighborhood V of τ_1 in G_1 and $r \in \mathrm{Hol}(V)$ such that $r(z_0) = w_0$ and $P(z, r(z)) = 0$ for $z \in V$. Since f is locally asidal at τ, $r(z) \in \mathbb{T}$ for $z \in V \cap \mathbb{T}$. Furthermore, if $r'(z_1) = 0$ for some $z_1 \in V \cap \mathbb{T}$, then there would necessarily exist a point z_2 near z_1 in $V\backslash\mathbb{T}$ such that $|r(z_2)| = 1$. Since f is locally asidal at τ, no such z_1 exists and we see that $r' \neq 0$ on $V \cap \mathbb{T}$.

Accordingly, either

$$r(V \cap \mathbb{D}) \subseteq \mathbb{D} \quad \text{and} \quad r(V \cap \mathbb{E}) \subseteq \mathbb{E} \tag{2.40}$$

or

$$r(V \cap \mathbb{D}) \subseteq \mathbb{E} \quad \text{and} \quad r(V \cap \mathbb{E}) \subseteq \mathbb{D}. \tag{2.41}$$

Noting that (2.40) implies that

$$\{(z, r(z)) : z \in V\} \subseteq (\mathbb{D} \times \mathbb{D}) \cup (\mathbb{E} \times \mathbb{E}) \cup \mathbb{T}^2$$

and that (2.41) implies that

$$\{(z, r(z)) : z \in V\} \subseteq (\mathbb{D} \times \mathbb{E}) \cup (\mathbb{E} \times \mathbb{D}) \cup \mathbb{T}^2$$

we have shown that if $(z_0, w_0) \in (Z_P \cap \mathbb{T}^2)\backslash\{\tau\}$, then there exists a relatively open neighborhood $U_{(z_0,w_0)}$ of (z_0, w_0) in $Z_P\backslash\{\tau\}$ such that exactly one of the following inclusions is obtained:

$$U_{(z_0,w_0)} \subseteq (\mathbb{D} \times \mathbb{D}) \cup (\mathbb{E} \times \mathbb{E}) \cup \mathbb{T}^2, \tag{2.42}$$

$$U_{(z_0,w_0)} \subseteq (\mathbb{D} \times \mathbb{E}) \cup (\mathbb{E} \times \mathbb{D}) \cup \mathbb{T}^2. \tag{2.43}$$

Now, since the asidality of P at τ implies that if $(z_0, w_0) \in Z_P\backslash\mathbb{T}^2$, then (z_0, w_0) is an element of one of the open sets $(\mathbb{D} \times \mathbb{D}) \cup (\mathbb{E} \times \mathbb{E})$ or $(\mathbb{D} \times \mathbb{E}) \cup (\mathbb{E} \times \mathbb{D})$, we see that in fact for *every* point (z_0, w_0) in $Z_P\backslash\{\tau\}$, there exists a relatively open neighborhood of (z_0, w_0) in $Z_P\backslash\{\tau\}$ such that either (2.42) or (2.43) holds, but not both. Consequently, since $Z_P\backslash\{\tau\}$ is connected, either $Z_P\backslash\{\tau\} \subseteq (\mathbb{D} \times \mathbb{D}) \cup (\mathbb{E} \times \mathbb{E}) \cup \mathbb{T}^2$ and P is locally inner at τ or $Z_P\backslash\{\tau\} \subseteq (\mathbb{D} \times \mathbb{E}) \cup (\mathbb{E} \times \mathbb{D}) \cup \mathbb{T}^2$ and P is locally outer at τ. $\square$

Before we consider the global properties of zero sets of globally defined polynomials, we summarize our classification of points τ in the 0-sets of locally irreducible holomorphic functions. Indeed, if $\tau \in \mathbb{T}^2$, f is holomorphic on a neighborhood G of τ, and f is locally irreducible at τ, then one of the following occurs.

(i) f could be locally toral and locally sidal at τ.
(ii) f could be locally inner at τ.
(iii) f could be locally outer at τ.
(iv) f could be locally atoral and locally sidal at τ.

If τ is not a singular point of Z_f and f is locally toral at τ, then we can determine whether (i), (ii) or (iii) above occur via Lemma 2.35.

Example 2.36 gives examples of (ii) and (iii) above.

Example 2.29 gives an example of (i) above.

Proposition 2.21 states that (iv) occurs if f is locally atoral at τ. The polynomial $2 - \tau_1 z - \tau_2 w$ is locally atoral at τ and therefore is an example of (iv) above.

For $\alpha \in \mathbb{D}$, let $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$. If $\tau_1 \in \mathbb{T}$, then $B_\alpha(\tau_1) \in \mathbb{T}$ and

$$A_{B_\alpha}(\tau_1) = \frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2}. \tag{2.44}$$

Example 2.45. Let $r(z) = \dfrac{zB_{1/3}(z)}{B_{1/2}(z)}$,

$$U = \{z \in \mathbb{C} : |z| < 3/2 \text{ and } z \neq 1/2\},$$

and $f(z,w) = w - r(z)$ for $(z,w) \in U \times \mathbb{C}$. For $\tau_1 = \cos(t) + i\sin(t)$, $t \in \mathbb{R}$,

$$A_r(\tau_1) = \frac{12(5-2\cos(t))\sin^2\left(\frac{t}{2}\right)}{(5-3\cos(t))(5-4\cos(t))}.$$

Thus, $A_r(1) = 0$ and $A_r(\tau_1) > 0$ whenever $\tau_1 \in \mathbb{T}\backslash\{1\}$. Thus, f is locally sidal at $(1,1)$ and f is locally inner at each point of $(Z_f \cap \mathbb{T}^2)\backslash\{(1,1)\}$.

Example 2.46. Let $r(z) = \dfrac{zB_{1/4}(z)}{B_{1/2}(z)}$,

$$U = \{z \in \mathbb{C} : |z| < 3/2 \text{ and } z \neq 1/2\},$$

and $f(z,w) = w - r(z)$ for $(z,w) \in U \times \mathbb{C}$. For $\tau_1 = \cos(t) + i\sin(t)$, $t \in \mathbb{R}$,

$$A_r(\tau_1) = \frac{32\cos^2(t) - 144\cos(t) + 109}{(5-4\cos(t))(17-8\cos(t))}.$$

Thus, if we set $a = \arccos\left(\frac{18-\sqrt{106}}{8}\right)$, then f is locally outer at each point in the set $\{(e^{it}, r(e^{it})) : -a < t < a\}$, f is locally inner at each point in the set $\{(e^{it}, r(e^{it})) : a < t < 2\pi - a\}$, and f is locally sidal at $(e^{ia}, r(e^{ia}))$ and $(e^{-ia}, r(e^{-ia}))$.

3. The global geometry of algebraic sets

In this section we shall discuss the various connections between the concepts of torality and atorality for polynomials in $\mathbb{C}^2$ introduced in [2] and the concepts of local torality and local atorality for holomorphic functions introduced in the previous section. We then shall extend the local concepts of sidality, asidality, inner and outer from the previous section to the context of globally defined algebraic sets in $\mathbb{C}^2$.

Theorem 3.1. *Let p be an irreducible polynomial. The following are equivalent.*

(i) *p is toral.*
(ii) *There exists $\tau \in \mathbb{T}^2$ such that p is locally toral at τ.*
(iii) *There exists $\tau \in Z_p \cap \mathbb{T}^2$ such that $\nabla p(\tau) \neq 0$ and Z_p is locally symmetric at τ.*

Proof. We shall show that (i)→(iii)→(ii)→(i). First assume that (i) holds. Since p is toral, Theorem 2.4 of [2] implies that $Z_p \cap \mathbb{T}^2$ is infinite. Since p is irreducible, $Z_{\nabla p}$ is finite. Hence there exists $\tau \in Z_p \cap \mathbb{T}^2$ such that $\nabla p(\tau) \neq 0$. Now, by Proposition 1.5 of [2], Z_p is symmetric. Hence Z_p is locally symmetric at τ and (iii) holds.

Now assume that (iii) holds. Since $\nabla p(\tau) \neq 0$, τ is not a singular point of Z_p and, by Theorems 1.11 and 1.2, there exists a neighborhood G_0 of τ_1, a neighborhood D_0 of τ_2 and $r \in \mathrm{Hol}(G_0)$ such that

$$Z_f \cap (G_0 \times D_0) = \{(z, r(z)) : z \in G_0\}.$$

Thus, since Z_f is locally symmetric, $r(z) \in \mathbb{T}$ whenever $z \in \mathbb{T}$. By Theorem 2.8, f is locally toral at τ and (ii) holds.

Finally assume that (ii) holds. By Proposition 2.12, $Z_p \cap \mathbb{T}^2$ is an infinite set. By Theorem 2.4 of [2], p is not atoral and therefore, since p is irreducible, p is toral. □

Theorem 3.1 cannot be much improved since the following examples illustrate the facts that the zeros of toral polynomials on the torus need not be locally toral (i.e., (i) does not imply (ii) for *every* τ) and that local symmetry at a singular point need not imply local torality (i.e., (iii) does not imply (ii) without the assumption that $\nabla p(\tau) \neq 0$). To work efficiently with pseudopolynomials of degree 2 in w, we state the following lemma.

Lemma 3.2. *Let $s, p, r_1, r_2 \in \mathbb{C}$. If $P(w) = w^2 - sw + p = (w - r_1)(w - r_2)$, then the following hold.*

(i) *P is a symmetric polynomial if and only if $|p| = 1$ and $\bar{s}p = s$.*
(ii) *If $|p| = 1$ and $|s| < 2$, then P has two distinct unimodular roots.*
(iii) *If $|p| = 1$ and $|s| = 2$, then P has a double unimodular root.*
(iv) *If $|p| = 1$ and $|s| > 2$, then P has two roots r_1 and r_2 neither of which is unimodular and $|r_1||r_2| = 1$.*

Example 3.3. Let $h(x) = \sum_{n=0}^{2} a_n e^{inx}$ be a trigonometric polynomial of degree 2 such that $h(x) > 0$ for all real numbers x and there exist numbers a, b, c and d such that $0 < a < b < c < d$, $h > 4$ on $[0, a)$, $h < 4$ on (b, c), $h > 4$ on (c, d), $h(d) = 4$ and $h > 4$ on $(d, 2\pi]$. Let $\tau_1 = e^{id}$. By Fejer's Theorem, there exists a polynomial $s(z)$ of degree two such that $s(e^{ix}) = h(x)$. Let $p(z) = s(z)/\overline{s(1/\bar{z})}$ and $P(z, w) = w^2 - s(z)w + p(z)$. Now $|s(\tau_1)| = 2$ and $|s(z)| > 2$ whenever $z \in \mathbb{T}$, z is near τ_1 and $z \neq \tau_1$. Thus, by Lemma 3.2, there exists $\tau_2 \in \mathbb{T}$ such that $P(\tau_1, \tau_2) = 0$ and for a sufficiently small neighborhood G_0 of τ_1, $Z_P \cap \mathbb{T}^2 \cap (G_0 \times \mathbb{C}) = \{\tau\}$. Thus, P is locally atoral at τ. By clearing denominators, we obtain a polynomial of degree (4,2) which is irreducible, toral and is locally atoral at a single point.

Example 3.4. Let $q(z, w) = (3z + 1)w^2 - (z + 3)(3z + 1)w + z(z + 3)$, $s(z) = z + 3$ and $p(z) = \frac{z(z+3)}{3z+1}$. It is easy to show that q is irreducible, q is essentially symmetric (see (2.24)) and

$$Z_q = \{(z, w) \in \mathbb{C}^2 \mid z \neq -1/3 \text{ and } w^2 - s(z)w + p(z) = 0\}.$$

If $z \in \mathbb{T}\backslash\{-1\}$, then $|s(z)| > 2$. Thus, by Lemma 3.2, $Z_q \cap \mathbb{T}^2 = \{(-1, 1)\}$. In summary, q is irreducible, q is locally atoral at $(-1, 1)$ and Z_q is locally symmetric at $(-1, 1)$. Also $\nabla q(-1, 1) = 0$. Thus, condition (iii) of Theorem 3.1. without the '$\nabla p(\tau) \neq 0$" does not imply condition (i) of Theorem 3.1.

Corollary 3.5. *If $p \in \mathbb{C}[z, w]$, then p is atoral if and only if p is locally atoral at each point in $Z_p \cap \mathbb{T}^2$.*

We now globalize Definition 2.18.

Definition 3.6. Let us agree to say that a polynomial $p = p(z, w) \in \mathbb{C}[z, w]$ is *sidal* if $\mathbb{S}$ is determining for Z_p and that p is *asidal* if no nonconstant divisor of p is sidal.

Noting that the logic of Definition 3.6 is parallel to that of Definition 1.2 of [2] (with $\mathbb{T}^2$ replaced by $\mathbb{S}$, we see that the following analogs of Corollaries 2.1 and 2.2 from [2] are obtained.

Proposition 3.7. *Let p be a nonzero polynomial in $\mathbb{C}[z, w]$. The following are equivalent.*

(i) *p is sidal (respectively, asidal).*
(ii) *Each irreducible factor of p is sidal (respectively, asidal).*
(iii) *Every divisor of p is sidal (respectively, asidal).*

Proposition 3.8. *Let p be a nonzero polynomial in $\mathbb{C}[z, w]$. There exist an essentially unique factorization $p = qr$ with q sidal and r asidal.*

Notice that Proposition 3.8 is a global analog of Corollary 2.20 from the previous section.

Now, while it is true that one can prove Propositions 3.7 and 3.8 above by following the arguments from [2], it is also true that the propositions can be deduced from the following simple geometric characterizations on sidal and asidal polynomials.

Theorem 3.9 (cf. Proposition 2.19)**.** *If $p \in \mathbb{C}[z, w]$, then p is asidal if and only if $Z_p \cap \mathbb{S} = \emptyset$.*

Theorem 3.10. *If $p \in \mathbb{C}[z, w]$ and p is irreducible, then p is sidal if and only if $Z_p \cap \mathbb{S} \neq \emptyset$.*

The reason that Theorems 3.9 and 3.10 are so much simpler than Theorems 2.4 and 2.7 from [2] is due to the fact that if $p \in \mathbb{C}[z, w]$, then $Z_p \cap \mathbb{S}$ does not have isolated points. This fact also yields immediate proofs of the theorems.

Now recall Theorem 2.39 which asserted that if f is locally asidal at $\tau \in \mathbb{T}^2$, then f has an essentially unique factorization $f = f_1 f_2$ with f_1 locally inner at τ and f_2 locally outer at τ. Following Definition 2.34, we introduce the following definition.

Definition 3.11. If $p \in \mathbb{C}[z,w]$, we say p is *inner* if $Z_p \subset (\mathbb{D}\times\mathbb{D})\cap(\mathbb{E}\times\mathbb{E})\cap\mathbb{T}^2$ and we say p is *outer* if $Z_p \subset (\mathbb{D}\times\mathbb{E})\cap(\mathbb{E}\times\mathbb{D})\cap\mathbb{T}^2$.

Every polynomial $p \in \mathbb{C}[z,w]$ which is inner or outer is toral as can be seen by the following proposition.

Proposition 3.12. *If $p \in \mathbb{C}[z,w]$ and p is asidal, then p is toral.*

Notice that the product of an inner polynomial and outer polynomial is asidal. The converse is true and is shown in the following theorem.

Theorem 3.13. *Let $p(z,w) \in \mathbb{C}[z,w]$. If $Z_p \cap \mathbb{S} = \emptyset$, then there exists $p_1, p_2 \in \mathbb{C}[z,w]$ such that $p = p_1p_2$, p_1 is inner and p_2 is outer.*

Proof. We need only show that if p is irreducible and $Z_p \cap \mathbb{S} = \emptyset$, then p is either inner or outer. By Corollary 2.1 in [2], it suffices to show that if p is irreducible and $Z_p \cap \mathbb{S} = \emptyset$, then p is toral. Suppose p is irreducible and $Z_p \cap \mathbb{S} = \emptyset$. Since p is irreducible, $\triangle_p$ is not identically zero. Since $\triangle_p \in \mathbb{C}[z]$, $Z_{\triangle_p}$ is finite and $Z_p \backslash S_p(\mathbb{C},\mathbb{C})$ is connected. Let U_1 and U_2 be the sets

$$U_1 = \{(z,w) \in Z_p : \text{there exists a neighborhood } V \text{ of } (z,w) \text{ such that } Z_p \cap V \subseteq (\mathbb{D}\times\mathbb{D}) \cup (\mathbb{E}\times\mathbb{E}) \cup \mathbb{T}^2\}$$

and

$$U_2 = \{(z,w) \in Z_p : \text{there exists a neighborhood } V \text{ of } (z,w) \text{ such that } Z_p \cap V \subseteq (\mathbb{D}\times\mathbb{E}) \cup (\mathbb{E}\times\mathbb{D}) \cup \mathbb{T}^2\}.$$

By Theorem 2.39, $Z_p \cap \mathbb{T}^2 \subseteq U_1 \cup U_2$. Clearly,

$$Z_p \cap (\mathbb{D}\times\mathbb{D}) \subseteq U_1, \qquad Z_p \cap (\mathbb{D}\times\mathbb{E}) \subseteq U_2,$$
$$Z_p \cap (\mathbb{E}\times\mathbb{D}) \subseteq U_2 \qquad Z_p \cap (\mathbb{E}\times\mathbb{E}) \subseteq U_1.$$

Since $Z_p \cap \mathbb{S} = \emptyset$,

$$Z_p = (Z_p\cap(\mathbb{D}\times\mathbb{D}))\cup(Z_p\cap(\mathbb{D}\times\mathbb{E}))\cup(Z_p\cap(\mathbb{E}\times\mathbb{D}))\cup(Z_p\cap(\mathbb{E}\times\mathbb{E}))\cup(Z_p\cap\mathbb{T}^2).$$

Thus $Z_p = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$ and both U_1 and U_2 are open. Thus $U_1 = \emptyset$ and p is outer or $U_2 = \emptyset$ and p is inner. $\square$

We close this section by remarking that every polynomial in $\mathbb{C}[z,w]$ is a product of irreducible polynomials in $\mathbb{C}[z,w]$. Here is a summary of the types of irreducible polynomials which we have encountered.

Example 3.14. Example 3.3 gives an example of a toral polynomial which is locally atoral at some point.

Example 3.15. Let $q(z,w) = (az+1)w^2 - (z+a)(az+1)w + z(z+a)$. In the case that $a = 3$, Example 3.4 shows that $Z_q \cap \mathbb{T}^2$ is a singleton set, q is atoral and q is symmetric.

Example 3.16. If $a > 3$ and $q(z,w) = (az+1)w^2-(z+a)(az+1)w+z(z+a)$, then $Z_q \cap \mathbb{T}^2 = \emptyset$, q is atoral and q is symmetric.

Example 3.17. Recall that for $\alpha \in \mathbb{D}$, we set $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$. Let

$$r(z) = \frac{zB_{2/5}(z)}{B_{1/2}(z)},$$

$U = \{z \in \mathbb{C} : |z| < 3/2 \text{ and } z \neq 1/2\}$, and $f(z,w) = w - r(z)$ for $(z,w) \in U \times \mathbb{C}$. For $t \in \mathbb{R}$ and $\tau_1 = \cos(t) + i\sin(t)$,

$$A_r(\tau_1) = \frac{80\cos^2(t) - 240\cos(t) + 163}{(5 - 4\cos(t))(29 - 20\cos(t))}.$$

Thus, $A_r(\tau_1) > 0$ for all $\tau_1 \in \mathbb{T}$ and, by Lemma 2.35, f is locally inner and locally asidal at each point in $Z_f \cap \mathbb{T}^2$. For z near $1/2$, $r(z) > 1$ and therefore, f is not inner since f is sidal. Hence, locally inner does not imply inner and locally asidal does not imply asidal.

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