

Refined measurable rigidity and flexibility for conformal iterated function systems

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ABSTRACT. In this paper we investigate aspects of rigidity and flexibility for conformal iterated function systems. For the case in which the systems are not essentially affine we show that two such systems are conformally equivalent if and only if in each of their Lyapunov spectra there exists at least one level set such that the corresponding Gibbs measures coincide. We then proceed by comparing this result with the essentially affine situation. We show that essentially affine systems are far less rigid than nonessentially affine systems, and subsequently we then investigate the extent of their flexibility.

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1. Introduction

In 1982 D. Sullivan published his influential purely measurable form of Mostow's rigidity theorem. It states that if two geometrically finite Kleinian

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groups are conjugate under a Borel map F which is nonsingular with respect to the Patterson measures associated with the two groups, then F agrees almost everywhere with a conformal conjugacy ([Sul82], see also [Sul87], [Sul88] and [Bow79]). Since the appearance of this theorem the concept of measurable rigidity has attracted a great deal of attention, and in the meanwhile numerous generalisations and variations have been obtained. One of these was derived by Hanus and Urbański ([HU99]), who considered nonessentially affine, conformal iterated function systems (see Section 2 for the definitions), and showed that two such systems Φ and Ψ are conformally equivalent if and only if their associated conformal measures μ_Φ and μ_Ψ (each of maximal Hausdorff dimension) coincide up to permutation of the generators. This result can be seen as the starting point for this paper.

Our first goal is to give a multifractal refinement of the result in [HU99]. For ease of exposition, throughout we restrict the discussion to the 1-dimensional, finitely generated case (note that the results in [HU99] include also certain infinitely generated systems; however, in this paper we are mainly concerned with studying conceptual differences between the affine and non-affine settings, and this is done best first for the finitely generated cases). Recall that each system Φ gives rise to its Lyapunov spectrum $u \mapsto \ell_\Phi(u)$, which is given by the multifractal spectrum of the measure of maximal entropy associated with Φ . Moreover, each level set in this spectrum supports a canonical shift-invariant Gibbs measure $\mu_{\Phi,u}$. In a nutshell, our main result for nonessentially affine, conformal iterated function systems is that two such systems Φ and Ψ are conformally equivalent if and only if $\mu_{\Phi,u}$ is equal to $\mu_{\Psi,v}$ up to permutation of the generators, for some $u, v \in \mathbb{R} \setminus \{0\}$ (see Theorem 3.2 for a more complete statement which also involves cohomological equivalence of the associated canonical geometric potential functions, equality of pressure functions as well as equality of Lyapunov spectra).

In the second part of the paper we consider essentially affine, conformal iterated function systems. Note that for nonessentially affine systems a conjugating map between two systems is conformal if and only if it is bi-Lipschitz (see [MU03, Theorem 7.2.4]). Hence, for essentially affine systems bi-Lipschitz conjugacy is the natural substitute for conformal conjugacy. By investigating similar questions as before for the nonessentially affine case, we obtain that from the point of view of multifractal rigidity essentially affine systems behave rather different than nonessentially affine systems. For instance, if for two essentially affine systems Φ and Ψ we have that $\mu_{\Phi,u}$ is equal to $\mu_{\Psi,v}$ up to permutation of the generators, for some $u, v \in \mathbb{R} \setminus \{0\}$, then this does *not necessarily* imply that Φ and Ψ are bi-Lipschitz equivalent. More precisely, we show that equality of $\mu_{\Phi,u}$ and $\mu_{\Psi,u}$ up to permutation of the generators *together* with the equality of the pressure functions P_Φ and P_Ψ at u , for some $u \in \mathbb{R} \setminus \{0\}$, is equivalent to the fact that Φ is bi-Lipschitz equivalent to Ψ , as well as to the facts $P_\Phi = P_\Psi$, $\ell_\Phi = \ell_\Psi$ and

cohomological equivalence of the two canonical geometric potential functions associated with the systems (see Theorem 4.3). These results clearly show that essentially affine systems are less rigid than nonessentially affine systems, and a further investigation of this phenomenon of flexibility is then given in Section 4.3. There, we derive sufficient and necessary conditions for equality of $\mu_{\Phi,u}$ and $\mu_{\Psi,v}$ in terms of the pressure functions and the canonical geometric potential functions (see Theorem 4.5). Also, we show that this situation does in fact occur. Namely, in Proposition 4.8 we obtain that if $\mu_{\Phi,u}$ is given and v fulfils a certain admissibility condition (see Definition 4.7), then there exists an essentially affine system Ψ such that $\mu_{\Phi,u}$ is equal to $\mu_{\Psi,v}$ up to permutation of the generators. Finally, we give a brief discussion of the extent of flexibility of an essentially affine system. The outcome here is that for a nondegenerate Φ the set of systems Ψ for which $\mu_{\Phi,u}$ is equal to $\mu_{\Psi,v}$ up to permutation of the generators, for some $u, v \in \mathbb{R} \setminus \{0\}$, forms a 2-dimensional submanifold of the moduli space of Φ , whereas if Φ is degenerate then this set is a 1-dimensional submanifold (see Proposition 4.10).

2. Preliminaries

2.1. Conformal iterated function systems. Throughout this paper we consider *conformal iterated function systems* (CS) on some connected compact set $X \subset \mathbb{R}$. Recall from [HU99] (see also [MU03]) that these systems are generated by a family Φ of injective contractions $(\varphi_i : X \rightarrow X \mid i \in I)$, for some given finite index set $I := \{1, \dots, d\}$ with at least two elements. Furthermore, Φ satisfies the following conditions, where we use the notation $\varphi_\omega := \varphi_{x_1} \circ \varphi_{x_2} \circ \dots \circ \varphi_{x_n}$ for $\omega = x_1 x_2 \dots x_n \in I^n$.

Strong separation condition:

$$\varphi_i(X) \cap \varphi_j(X) = \emptyset \text{ for all } i, j \in I, i \neq j.$$

Conformality-condition: There exists an open connected set $U \subset \mathbb{R}$ containing X such that φ_i extends to a real-analytic map on U , for each $i \in I$.

Note that the conformality-condition immediately implies the bounded distortion property. That is, we have that the following holds.

Bounded distortion property: There exists $C \geq 1$ such that for all $n \in \mathbb{N}$, $\omega \in I^n$ and $x, y \in U$ we have

$$|\varphi'_\omega(y)| \leq C |\varphi'_\omega(x)|.$$

A central object associated with a CS Φ is its *limit set*

$$\Lambda(\Phi) := \bigcap_{n \in \mathbb{N}} \bigcup_{\omega \in I^n} \varphi_\omega(X).$$

Clearly, $\Lambda(\Phi)$ is the unique nonempty compact subset of \mathbb{R} for which

$$\Lambda(\Phi) = \bigcup_{i \in I} \varphi_i(\Lambda(\Phi)).$$

From a combinatorial point of view Φ is described by the full-shift $\Sigma_d := I^{\mathbb{N}}$. As usual, we assume Σ_d to be equipped with the left-shift map σ . The link between Σ_d and Φ is provided by the canonical bijection $\pi_\Phi : \Sigma_d \rightarrow \Lambda(\Phi)$ which is given by $\pi_\Phi(x_1 x_2 \dots) := \lim_{n \rightarrow \infty} \varphi_{x_1 x_2 \dots x_n}(X)$. Evidently, we can always think of Φ as being a conformal fractal representation of Σ_d .

We also consider the special situation in which all the φ_i are in particular affine transformations. In this case the system is called *affine iterated function system* (AS), and occasionally it will also be referred to as an *affine fractal representation* of Σ_d . One of the major issues of this paper is to study certain deformations of a given CS $\Phi = (\varphi_i : X \rightarrow X \mid i \in I)$. More precisely, let $\Psi := (\psi_i : Y \rightarrow Y \mid i \in I)$ be some other CS defined on some connected compact set $Y \subset \mathbb{R}$. Then Ψ is called a *deformation* of Φ if there exists a bi-Lipschitz map $h : \Lambda(\Phi) \rightarrow \Lambda(\Psi)$ such that after some permutation of the generators,

$$\psi_i = h \circ \varphi_i \circ h^{-1}, \text{ for each } i \in I.$$

A map h of this type will be called a *fractal correspondence*. In particular, if Φ is an AS, that is if Ψ is a deformation of an affine iterated function system, then Ψ will be referred to as *essentially affine iterated function system* (EAS). On the other hand, if Φ is a CS which is not an EAS then Φ will be called *nonessentially affine iterated function system* (NAS).

Let us also introduce the *deformation space* $\mathcal{T}(\Sigma_d)$ associated with Σ_d . This is given by

$$\mathcal{T}(\Sigma_d) := \{\Psi : \Psi \text{ is a CS on } \Sigma_d\}.$$

Clearly, $\mathcal{T}(\Sigma_d)$ relates to Σ_d similarly to the relationship between the Teichmüller space for a Riemann surface and the associated fundamental group. We then decompose the space $\mathcal{T}(\Sigma_d)$ into the two disjoint deformation spaces

$$\begin{aligned} \mathcal{T}_E(\Sigma_d) &:= \{\Psi : \Psi \text{ is an EAS on } \Sigma_d\}, \\ \mathcal{T}_N(\Sigma_d) &:= \{\Psi : \Psi \text{ is an NAS on } \Sigma_d\}. \end{aligned}$$

Also, we introduce an equivalence relation on $\mathcal{T}(\Sigma_d)$ as follows. Two systems $\Phi, \Psi \in \mathcal{T}(\Sigma_d)$ are said to be equivalent ($\Phi \sim \Psi$) if and only if there exists a fractal correspondence $h : \Lambda(\Phi) \rightarrow \Lambda(\Psi)$ between them. Finally, recall that a CS is called *degenerate* if it is equivalent to an AS $\Psi = (\psi_i : X \rightarrow X \mid i \in I)$ for which $\psi'_i = \psi'_j$, for all $i, j \in I$. It is easy to see that for a degenerate EAS the multifractal analysis in this paper is trivial.

2.2. Thermodynamic and multifractal formalism for CS. Let $\Phi = (\varphi_i : X \rightarrow X \mid i \in I) \in \mathcal{T}(\Sigma_d)$ be given, and let δ_Φ refer to the Hausdorff dimension of $\Lambda(\Phi)$. Throughout, we require the following standard concepts

from thermodynamic formalism, and we assume that the reader is familiar with the basics of this formalism (see e.g., [Bow75], [MU03], [Pes97], [Rue78]). Here we use the common notations

$$[x_1 \dots x_n] := \{y = (y_1 y_2 \dots) \in \Sigma_d : y_i = x_i \text{ for } i = 1, \dots, n\},$$

$$S_n f := \sum_{k=0}^{n-1} f \circ \sigma^k.$$

- The *canonical geometric potential* $I_\Phi : \Sigma_d \rightarrow \mathbb{R}$ associated with Φ is given by $I_\Phi(x) := \log |\varphi'_{x_1}(\varphi_{x_1}^{-1}(\pi_\Phi(x)))|$ for all $x = (x_1 x_2 \dots) \in \Sigma_d$. This function is Hölder continuous.
- μ_Φ refers to a Gibbs measure on Σ_d for the potential $\delta_\Phi I_\Phi$.
- ℓ_Φ refers to the *Lyapunov spectrum* of Φ , given for $\alpha \in \mathbb{R}$ by

$$\ell_\Phi(\alpha) := \dim_H \left(\pi_\Phi \left(\left\{ x \in \Sigma_d : \lim_{n \rightarrow \infty} \frac{S_n I_\Phi(x)}{-n} = \alpha \right\} \right) \right).$$

- $\mathcal{P}_{\Sigma_d} : \mathcal{C}(\Sigma_d) \rightarrow \mathbb{R}$ denotes the *pressure function*, given for a potential function $f : \Sigma_d \rightarrow \mathbb{R}$ by

$$\mathcal{P}_{\Sigma_d}(f) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \exp(\sup_{x \in [\omega]} S_n f(x)).$$

Also, for $u \in \mathbb{R}$ we define

$$P_\Phi(u) := \mathcal{P}_{\Sigma_d}(u I_\Phi)$$

and

$$\alpha_\Phi(u) := -P'_\Phi(u) = - \int I_\Phi d\mu_{\Phi,u}.$$

Here, $\mu_{\Phi,u}$ refers to the σ -invariant *Gibbs measure* on Σ_d for the potential function $u I_\Phi - P_\Phi(u)$, where ‘Gibbs’ means as usual that for all $n \in \mathbb{N}$, $(x_1, \dots, x_n) \in I^n$ and $x \in [x_1 \dots x_n]$,

$$\mu_{\Phi,u}([x_1, \dots, x_n]) \asymp e^{u S_n I_\Phi(x) - n P_\Phi(u)}.$$

Furthermore, we let $m_{\Phi,u}$ denote the $(u I_\Phi - P_\Phi(u))$ -conformal measure within the measure class of $\mu_{\Phi,u}$, given by

$$\frac{dm_{\Phi,u} \circ \varphi_i}{dm_{\Phi,u}} = |\varphi'_i|^u e^{-P_\Phi(u)}, \text{ for all } i \in I.$$

Finally, throughout we require the following notions of equivalence in connection with two given $\Phi, \Psi \in \mathcal{T}(\Sigma_d)$.

- The Gibbs measures $\mu_{\Phi,u}$ and $\mu_{\Psi,v}$ are *equal up to permutation* ($\mu_{\Phi,u} \cong \mu_{\Psi,v}$) if and only if $\mu_{\Phi,u} = \mu_{\Psi_0,v}$ for some system Ψ_0 obtained from Ψ by a permutation of the generators of Ψ .

- The potentials I_Φ and I_Ψ are *cohomologically equivalent* ($I_\Phi \simeq I_\Psi$) if I_Φ is cohomologous to I_{Ψ_0} , for some system Ψ_0 obtained from Ψ by a permutation of the generators of Ψ . (Recall that two functions $f, g : \Sigma_d \rightarrow \mathbb{R}$ are *cohomologous* if there exists a continuous function $e : \Sigma_d \rightarrow \mathbb{R}$ such that $f - g = e - e \circ \sigma$.)

For the type of iterated function systems which we consider in this paper the calculation of the Lyapunov spectrum is basically an application of the multifractal analysis of the measure of maximal entropy for cookie-cutter Cantor sets. The following proposition summarises the outcome of this analysis. Subsequently, we will outline the proof employing the down-to-earth approach given in [Fal97]. Note that the proposition also immediately follows from the multifractal formalism for growth rates developed in [KS04]. Also, note that in here the function β_Φ is the inverse of the function α_Φ , that is $\beta_\Phi \circ \alpha_\Phi = \text{id}$.

Proposition 2.1. *Let $\Phi \in \mathcal{T}(\Sigma_d)$ nondegenerate be given. Then there exists a real-analytic function $\beta_\Phi : (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$, $\alpha_-, \alpha_+ > 0$, such that $\ell_\Phi(\alpha) = 0$ for all $\alpha \notin [\alpha_-, \alpha_+]$ and such that for all $\alpha \in [\alpha_-, \alpha_+]$,*

$$\ell_\Phi(\alpha) = \beta_\Phi(\alpha) + \frac{P_\Phi(\beta_\Phi(\alpha))}{\alpha}.$$

Proof. (Sketch) Let ν refer to the measure of maximal entropy for (Σ_d, σ) . Then ν is a Gibbs measure for the potential function φ constant equal to the negative of the topological entropy $h_{\text{top}} := \log d$. Hence, we in particular have $\nu([\omega]) \asymp \exp(S_n \varphi(x)) = d^{-n}$ for all $n \in \mathbb{N}, \omega \in I^n$ and $x \in [\omega]$. Trivially, we have $\varphi < 0$ and $\mathcal{P}_{\Sigma_d}(\varphi) = 0$, which shows that ν can be analysed by standard multifractal analysis (see e.g., [Fal97]). This gives that there exists a well-defined, strictly decreasing, real-analytic function $\gamma_\Phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathcal{P}_{\Sigma_d}(\gamma_\Phi(t) I_\Phi + t\varphi) = 0$, for all $t \in \mathbb{R}$. In order to determine the Hausdorff dimension spectrum

$$\{\dim_H(E_\tau) : \tau \in \mathbb{R}\},$$

where

$$E_\tau := \left\{ x \in \Lambda(\Phi) : \lim_{n \rightarrow \infty} \frac{\log \nu(\pi_\Phi^{-1}(B(x, r)))}{\log r} = \tau \right\},$$

one considers the Legendre transform of γ_Φ , given by $f(\tau) = \inf\{\gamma_\Phi(t) + t\tau : t \in \mathbb{R}\}$, or what is equivalent $f(\tau) = \gamma_\Phi(t_\tau) + t_\tau \tau$ where t_τ is determined by $\gamma'_\Phi(t_\tau) = -\tau$. In particular, there exists a maximal interval (τ_-, τ_+) on which f is continuous, concave and strictly positive; outside this interval f vanishes. Now, the key observation is that there exists a Gibbs measure ν_τ for the potential function $\gamma_\Phi(t_\tau) I_\Phi + t_\tau \varphi$ which is concentrated on $\pi_\Phi^{-1}(E_\tau)$. (Note that the measure ν_τ coincides with the measure $\mu_{\Phi, \gamma_\Phi(t_\tau)}$ which we already introduced above). Hence, we have for all $n \in \mathbb{N}, \omega \in I^n$ and $x \in [\omega]$,

$$\nu_\tau([\omega]) \asymp \exp(\gamma_\Phi(t_\tau) S_n I_\Phi(x) + t_\tau S_n \varphi(x)).$$

Since by the bounded distortion property $\exp(S_n I_\Phi(x)) \asymp |\pi_\Phi([\omega])|$, the mass distribution principle immediately gives $\dim_H(E_\tau) = f(\tau)$. To finish the proof, note that

$$\begin{aligned} \pi_\Phi^{-1}(E_\tau) &= \left\{ x = (x_1 x_2 \dots) \in \Sigma_d : \lim_{n \rightarrow \infty} \frac{-n h_{\text{top}}}{\log |\pi_\Phi([x_1 \dots x_n])|} = \tau \right\} \\ &= \left\{ x \in \Sigma_d : \lim_{n \rightarrow \infty} \frac{S_n I_\Phi(x)}{-n} = \frac{h_{\text{top}}}{\tau} \right\}. \end{aligned}$$

This shows that $f(\tau) = \ell_\Phi(\alpha)$, for $\alpha := h_{\text{top}}/\tau$. Finally, define $\beta_\Phi(\alpha) := \gamma_\Phi(t_\tau)$ and note that $\mathcal{P}_{\Sigma_d}(\gamma_\Phi(t_\tau) I_\Phi + t_\tau \varphi) = 0$ immediately implies that $P_\Phi(\beta_\Phi(\alpha)) = t_\tau h_{\text{top}}$. Using this and rewriting the above in terms of α , the result follows. \square

3. Multifractal rigidity for NAS

For the proof of the main result of this section (Theorem 3.2) we require the following proposition. Note that for $u = \delta_\Phi$ this result was obtained by Mauldin and Urbański [MU03, Theorem 6.1.3]. Since it is straightforward to adapt the arguments in [MU03] to our multifractal situation here, we will only give an outline of the proof emphasising the major changes which have to be made.

Proposition 3.1. *Let $\Phi \in \mathcal{T}_N(\Sigma_d)$ and $u \in \mathbb{R} \setminus \{0\}$ be given, and let $m_{\Phi,u}$ refer to the $(uI_\Phi - P_\Phi(u))$ -conformal measure in the measure class of $\mu_{\Phi,u}$. Then there exists an open connected set $W \supset X$ such that $d\mu_{\Phi,u}/dm_{\Phi,u}$ has a positive real-analytic extension to W .*

Proof. (Sketch) The first step consists of applying Arzelà–Ascoli to obtain that

$$F : C(X) \rightarrow C(X), F(g) := e^{-P_\Phi(u)} \sum_{i \in I} |\varphi'_i|^u g \circ \varphi_i$$

is an almost periodic operator, that is $\{F^n(g) : n \in \mathbb{N}\}$ is relatively compact with respect to the sup-norm for every $g \in C(X)$ (see [MU03, Lemma 6.1.1]). Also, the Gibbs-property of $\mu_{\Phi,u}$ immediately implies that $F^n(\mathbf{1})$ is uniformly bounded away from zero and infinity, for each $n \in \mathbb{N}$.

The second step is to use the above results to show that there exists a unique positive continuous function $\rho : X \rightarrow \mathbb{R}^+$ such that (see [MU03, Theorem 6.1.2])

$$F(\rho) = \rho, \quad \int \rho dm_{\Phi,u} = 1 \quad \text{and} \quad \rho|_{\Lambda(\Phi)} = \frac{d\mu_{\Phi,u}}{dm_{\Phi,u}}.$$

The final step is to consider the sequence of functions $(b_n)_{n \in \mathbb{N}}$, given by

$$b_n(z) := \sum_{|\omega|=n} |\varphi'_\omega(z)|^u e^{-nP_\Phi(u)}.$$

One verifies that each b_n is defined locally on a sufficiently large neighbourhood of each $w \in X$, where it is analytic, uniformly bounded and equicontinuous (see [MU03], proof of Theorem 6.1.3). It then follows that (b_n) has a subsequence converging to an analytic function which locally extends ρ . Since X is compact and simply connected, this provides us with a globally defined analytic extension of ρ , which is uniformly bounded from above and below. \square

The following theorem gives the main results of this section. Here, the main outcome is that if we have equality up to permutation of two Gibbs measures associated with two points in the Lyapunov spectra of two NAS, then the two systems are already bi-Lipschitz equivalent. Therefore, the theorem represents a refinement of the Hanus–Urbański rigidity theorem mentioned in the introduction (see also Corollary 3.3).

Theorem 3.2 (Multifractal rigidity for NAS). *Let $\Phi, \Psi \in \mathcal{T}_N(\Sigma_d)$ and $u, v \in \mathbb{R} \setminus \{0\}$ be given. Then the following three statements are equivalent:*

- (i) $\mu_{\Phi, u} \cong \mu_{\Psi, v}$.
- (ii) $\Phi \sim \Psi$ and $u = v$.
- (iii) $I_\Phi \simeq I_\Psi$ and $u = v$.

Also, the following two statements are equivalent:

- (iv) $P_\Phi = P_\Psi$.
- (v) $\ell_\Phi = \ell_\Psi$.

Furthermore, each of the statements in (i)–(iii) implies the statements in (iv) and (v).

Proof. The implications (ii) \implies (i), (iii) \implies (i) and (iii) \implies (iv), as well as the equivalence of (iv) and (v) follow exactly as in the case $\Phi \in \mathcal{T}_E(\Sigma_d)$, and for this we refer to Theorem 4.3 in Section 4.2.

On the basis of the assumption that (i) \implies (ii) holds, the implication (i) \implies (iii) can be obtained as follows. Assume that $\mu_{\Phi, u} \cong \mu_{\Psi, v}$. We then have $uI_\Phi \simeq vI_\Psi + c$, for some constant c . Also, since (i) \implies (ii) holds, we have that $u = v$ and $\Phi \sim \Psi$. It hence follows that $I_\Phi - I_\Psi \simeq c$ and $\delta_\Phi = \delta_\Psi$. Consequently, $0 = P_\Phi(\delta_\Phi) - P_\Psi(\delta_\Psi) = c\delta_\Phi$. Since $\delta_\Phi \neq 0$, this implies that $c = 0$, and hence (iii) follows.

It remains to show (i) \implies (ii). For this note that by applying a suitable permutation if necessary, we can assume without loss of generality that $\mu_{\Phi, u} = \mu_{\Psi, v}$. Let $h := \pi_\Psi \circ \pi_\Phi^{-1} : \Lambda(\Phi) \rightarrow \Lambda(\Psi)$ refer to the associated measurable map. Clearly, $\psi_i \circ h = h \circ \varphi_i$ for all $i \in I$. The aim is to show that there exists an open neighbourhood of $\Lambda(\varphi)$ such that h extends to a real-analytic map on this neighbourhood. For ease of notation we will not distinguish between measures on Σ_d and measures on the corresponding limit sets arising from representations of Σ_d . For $\omega \in I^n$, define $J_{\varphi_\omega, u} := d\mu_{\Phi, u} \circ \varphi_\omega / d\mu_{\Phi, u}$, and similar $J_{\psi_\omega, v}$ for the system Ψ . We then have for each

$i \in I$,

$$J_{\varphi_i, u} = \frac{d\mu_{\Phi, u} \circ \varphi_i}{d\mu_{\Phi, u}} = \frac{d\mu_{\Psi, v} \circ \psi_i \circ h}{d\mu_{\Psi, v} \circ h} = \frac{d\mu_{\Psi, v} \circ \psi_i}{d\mu_{\Psi, v}} \circ h = J_{\psi_i, v} \circ h.$$

On the other hand, we have

$$\begin{aligned} J_{\psi_i, v} &= \frac{d\mu_{\Psi, v} \circ \psi_i}{dm_{\Psi, v} \circ \psi_i} \frac{dm_{\Psi, v} \circ \psi_i}{dm_{\Psi, v}} \frac{dm_{\Psi, v}}{d\mu_{\Psi, v}} \\ &= \frac{d\mu_{\Psi, v}}{dm_{\Psi, v}} \circ \psi_i \frac{dm_{\Psi, v} \circ \psi_i}{dm_{\Psi, v}} \left(\frac{d\mu_{\Psi, v}}{dm_{\Psi, v}} \right)^{-1}. \end{aligned}$$

Also, since $m_{\Psi, v}$ is the $(vI_{\Psi} - P_{\Psi}(v))$ -conformal measure in the measure class of $\mu_{\Psi, v}$, we have

$$\frac{dm_{\Psi, v} \circ \psi_i}{dm_{\Psi, v}} = |\psi'_i|^v e^{-P_{\Psi}(v)}, \text{ for all } i \in I.$$

Now, the conformality condition in the definition of a CS guarantees that $|\psi'_i|^v$ has a real-analytic extension to an open neighbourhood of X . Hence, by combining these observations with Proposition 3.1, it follows that there exists $W \supset X$ such that $J_{\psi_i, v}$ has a real-analytic extension $\tilde{J}_{\psi_i, v}$ to W . In the same way we obtain a real-analytic extension $\tilde{J}_{\varphi_i, u}$ for the system Φ . Next, note that since $\Psi \in \mathcal{T}_N(\Sigma_d)$, there exists $j \in I$ such that $\tilde{J}_{\psi_j, v}$ is not equal to a constant. Since $\tilde{J}_{\varphi_j, u} = \tilde{J}_{\psi_j, v} \circ h$, the same holds for $\tilde{J}_{\varphi_j, u}$ (note, h is defined on the perfect set $\Lambda(\Phi)$). In particular, the set of zeros of $\tilde{J}'_{\varphi_j, u}$, and $\tilde{J}'_{\psi_j, v}$ respectively, can not have points of accumulation in X , and Y respectively. Therefore, there exists $x \in \Lambda(\Phi)$ such that $\tilde{J}'_{\varphi_j, u}(x) \neq 0$ and $\tilde{J}'_{\psi_j, v}(h(x)) \neq 0$. This implies that there exists an inverse branch $\tilde{J}_{\psi_j, v}^{-1}$ which is analytic in a neighbourhood of $\tilde{J}_{\varphi_j, u}(x)$ such that $\tilde{J}_{\psi_j, v}^{-1}(\tilde{J}_{\varphi_j, u}(x)) = h(x)$. By choosing a neighbourhood $W' \subset X$ of x sufficiently small, we obtain that $\tilde{J}_{\psi_j, v}^{-1} \circ \tilde{J}_{\varphi_j, u}$ is well-defined and bijective on W' , and

$$\tilde{J}_{\psi_j, v}^{-1} \circ \tilde{J}_{\varphi_j, u}(y) = h(y), \text{ for all } y \in W' \cap \Lambda(\Phi).$$

It now follows that there exists $\omega \in I^n$, for some $n \in \mathbb{N}$, such that $\varphi_{\omega}(X) \subset W'$. Hence, there exists $W'' \supset X$ on which $\psi_{\omega}^{-1} \circ \tilde{J}_{\psi_j, v}^{-1} \circ \tilde{J}_{\varphi_j, u} \circ \varphi_{\omega}$ is real-analytic and such that $\psi_{\omega}^{-1} \circ \tilde{J}_{\psi_j, v}^{-1} \circ \tilde{J}_{\varphi_j, u} \circ \varphi_{\omega}$ coincides with h on $W'' \cap \Lambda(\Phi)$. \square

The following corollary is an immediate consequence of the previous theorem. We remark that the fact that $\mu_{\Phi} \cong \mu_{\Psi}$ implies that the two Lyapunov spectra coincide is somehow characteristic for nonessentially affine systems. Namely, as we will see in Section 4.1, in this respect essentially affine systems behave rather different. Also, note that the equivalence of (i) and (ii) is precisely the content of the Hanus–Urbański rigidity theorem.

Corollary 3.3. *For $\Phi, \Psi \in \mathcal{T}_N(\Sigma_d)$, the following statements are equivalent:*

- (i) $\mu_\Phi \cong \mu_\Psi$.
- (ii) $\Phi \sim \Psi$.

In particular, we also have

$$\mu_\Phi \cong \mu_\Psi \implies \ell_\Phi = \ell_\Psi.$$

Remark. Recently, it has been shown in [PW05] that for cocompact Fuchsian groups the pressure function is *not* a complete invariant of isometry, that is equality of the pressure functions of two isomorphic cocompact Fuchsian groups does not necessarily imply that the two associated Riemann surfaces are isometric. This result suggests that one might expect that for two systems $\Phi, \Psi \in \mathcal{T}_N(\Sigma_d)$ we have that $P_\Phi = P_\Psi$ does not necessarily imply $\Phi \sim \Psi$. However, the argument in [PW05] relies on Buser's constructive example of isospectral but nonisometric, compact Riemann surfaces (see [Bus92]), and it is currently not clear (at least to the authors) how to adapt this construction to the situation of an NAS.

4. Multifractal rigidity and flexibility for EAS

4.1. Deformation spaces for EAS. We require some elementary observations concerning how to switch forward and backward between two given essentially affine iterated function systems. The content of the following lemma is quite certainly well-known to experts in this area. However, for the reader's convenience we decided to include its proof.

Lemma 4.1. *Let $\Phi = (\varphi_i : X \rightarrow X \mid i \in I), \Psi = (\psi_i : Y \rightarrow Y \mid i \in I) \in \mathcal{T}_E(\Sigma_d)$ be given. Then there exists a Hölder continuous homeomorphism $h : \Lambda(\Phi) \rightarrow \Lambda(\Psi)$ such that*

$$\varphi_i \circ h = h \circ \psi_i, \text{ for all } i \in I.$$

Moreover, if $\varphi'_i = \psi'_i$ for all $i \in I$, then h is bi-Lipschitz.

Proof. Let Φ, Ψ be given as stated in the lemma. Without loss of generality we can assume that $X = Y = [0, 1]$ and that both systems are affine. For each $n \in \mathbb{N}$, we define a piecewise linear map h_n by induction as follows. For $i \in I$ let $I_i := \varphi_i(\Lambda(\Phi))$ and $J_i := \psi_i(\Lambda(\Psi))$, and define

$$h_{0,i} : \text{Conv}(I_i) \rightarrow \text{Conv}(J_i)$$

to be the uniquely determined linear surjection from I_i onto J_i , where Conv refers to the convex hull. The map $h_0 := \sum_{i=1}^n h_{0,i}$ is piecewise linear and maps $\bigcup_{i \in I} \text{Conv}(I_i)$ onto $\bigcup_{i \in I} \text{Conv}(J_i)$. Similarly, for each $\omega \in I^n, i \in I$ and $n \in \mathbb{N}$, let $h_{\omega,i}$ be the uniquely determined linear surjection which maps $\text{Conv}(\varphi_{\omega i}(\Lambda(\Phi)))$ onto the set $\text{Conv}(\psi_{\omega i}(\Lambda(\Psi)))$. Hence,

$$h_n := \sum_{\omega \in I^n} \sum_{i \in I} h_{\omega,i}$$

is a piecewise linear surjection with

$$h_n \left(\bigcup_{\omega \in I^n} \bigcup_{i \in I} \text{Conv}(\varphi_\omega(I_i)) \right) = \bigcup_{\omega \in I^n} \bigcup_{i \in I} \text{Conv}(\psi_\omega(J_i)).$$

Also, one readily verifies that h_n converges uniformly to a continuous function $h := \lim_{n \rightarrow \infty} h_n$. The fact that h is Hölder continuous with Hölder exponent

$$s := \min \{ \log(\psi'_i) / \log(\varphi'_i) : i \in I \}$$

can be seen as follows. Let $x = \pi_\Phi(x_1 x_2 \dots)$ and $y = \pi_\Phi(y_1 y_2 \dots)$ be two distinct elements of $\Lambda(\Phi)$. If $x_1 \neq y_1$ then the assertion follows immediately, and hence we can assume without loss of generality that $x_1 = y_1$. Then there exists a smallest $n \in \mathbb{N}$ such that $x_{n+1} \neq y_{n+1}$ and $x_i = y_i$, for all $1 \leq i \leq n$. The open set condition gives that there exists $c > 0$ such that $|x - y| \geq c \prod_{i=1}^n \varphi'_{x_i}$. Using this, we obtain

$$|h(x) - h(y)| \leq \prod_{i=1}^n \psi'_{x_i} \leq \prod_{i=1}^n \varphi'^s_{x_i} = \frac{1}{c^s} \left(c \prod_{i=1}^n \varphi'_{x_i} \right)^s \leq \frac{1}{c^s} |x - y|^s.$$

The remainder of the proposition is now straightforward. \square

Note that we necessarily have that each equivalence class in $\mathcal{T}_E(\Sigma_d) / \sim$ contains an affine fractal representation. Also, note that each affine fractal representation $\Phi = (\varphi_i : X \rightarrow X \mid i \in I)$ can be parameterized by its *contraction rate vector* $(\varphi'_1, \dots, \varphi'_d)$, and the previous lemma shows that this vector has to be unique up to permutations of its entries. Therefore, as an immediate consequence of the previous lemma we obtain the following.

Proposition 4.2. *There exists a canonical bijection from $\mathcal{T}_E(\Sigma_d) / \sim$ onto*

$$\left\{ (\lambda_1, \dots, \lambda_d) \in (\mathbb{R}^+)^d : \sum_{i=1}^d \lambda_i < 1 \right\} / \Pi_d.$$

Here, Π_d refers to the group of permutations of the elements in I .

4.2. Multifractal rigidity for EAS. The goal of this section is to study rigidity for essentially affine iterated function systems. We show that for these systems one can only obtain a multifractal version of Sullivan's purely measurable rigidity theorem which is significantly weaker than the one for the nonessentially affine situation which we obtained in the previous section.

The following theorem states the main result of this section. In there it is shown that in the EAS setting there is a 1-1 correspondence between the space of pressure functions and the moduli space $\mathcal{T}_E(\Sigma_d) / \sim$. Also, the theorem in particular gives that for essentially affine systems equivalence of μ_Φ and μ_Ψ alone does in general not imply that the pressure functions of the systems coincide. In fact, as we will see in Section 4.3, this will only be the case if the two systems are equivalent. Clearly, this can be seen as

a first instance exhibiting the difference between the essentially affine and the nonessentially affine settings. Also note that unlike in the nonessentially affine situation (see the remark at the end of Section 3), the following theorem gives that for the essentially affine cases it is clear that $P_\Phi = P_\Psi$ is equivalent to $\Phi \sim \Psi$.

Theorem 4.3 (Multifractal rigidity for EAS). *For $\Phi, \Psi \in \mathcal{T}_E(\Sigma_d)$ nondegenerate, the following statements are equivalent:*

- (i) $\mu_{\Phi,u} \cong \mu_{\Psi,u}$ and $P_\Phi(u) = P_\Psi(u)$, for some $u \in \mathbb{R} \setminus \{0\}$.
- (ii) $\Phi \sim \Psi$.
- (iii) $I_\Phi \simeq I_\Psi$.
- (iv) $P_\Phi = P_\Psi$.
- (v) $\ell_\Phi = \ell_\Psi$.

Proof. Let $\Phi = (\varphi_i : X \rightarrow X \mid i \in I)$, $\Psi = (\psi_i : Y \rightarrow Y \mid i \in I) \in \mathcal{T}_E(\Sigma_d)$ be two given nondegenerate systems.

(i) \implies (ii): Suppose that $\mu_{\Phi,u} = \mu_{\Psi,u}$, for some $u \in \mathbb{R} \setminus \{0\}$. We then have for each $n \in \mathbb{N}$ and $\omega \in I^n$,

$$\begin{aligned} |\varphi_\omega(\Lambda(\Phi))| &\asymp (\mu_{\Phi,u} \circ \pi_\Phi^{-1}(\varphi_\omega(\Lambda(\Phi))))^{1/u} e^{nP_\Phi(u)/u} \\ &= (\mu_{\Psi,u} \circ \pi_\Psi^{-1}(\psi_\omega(\Lambda(\Psi))))^{1/u} e^{nP_\Psi(u)/u} \asymp |\psi_\omega(\Lambda(\Psi))|. \end{aligned}$$

We can now proceed similar as in Proposition 4.1 to build up a bi-Lipschitz map $h : \Lambda(\Phi) \rightarrow \Lambda(\Psi)$ as the limit of piecewise linear surjections. (Note that the existence of h can be obtained alternatively by applying Theorem 2.2 in [HU99]).

(ii) \implies (i): Suppose that $\Phi \sim \Psi$, and note that a bi-Lipschitz conjugacy does not alter the pressure function. Hence, similar as in the previous case, we obtain for each $n \in \mathbb{N}$, $\omega \in I^n$ and $u \in \mathbb{R}$,

$$\begin{aligned} \mu_{\Phi,u} \circ \pi_\Phi^{-1}(\varphi_\omega(\Lambda(\Phi))) &\asymp |\varphi_\omega(\Lambda(\Phi))|^u e^{-nP_\Phi(u)} \asymp |h(\varphi_\omega(\Lambda(\Phi)))|^u e^{-nP_\Phi(u)} \\ &\asymp |\psi_\omega(\Lambda(\Psi))|^u e^{-nP_\Psi(u)} \asymp \mu_{\Psi,u} \circ \pi_\Psi^{-1}(\psi_\omega(\Lambda(\Psi))). \end{aligned}$$

Therefore, using the ergodicity of $\mu_{\Phi,u}$ and $\mu_{\Psi,u}$, it follows that $\mu_{\Phi,u} = \mu_{\Psi,u}$.

(i) \iff (iii): This is an immediate consequence of the fact that $\mu_{\Phi,u}$ and $\mu_{\Psi,u}$ are Gibbs measures for the potential $uI_\Phi - P_\Phi(u)$, and $uI_\Psi - P_\Psi(u)$ respectively.

(iii) \implies (iv): This follows from the definition of the pressure function.

(iv) \iff (v): This follows since P_Φ and ℓ_Φ are a Legendre transform pair.

(iv) \implies (iii): Suppose that $P_\Phi = P_\Psi$, and let Φ_a and Ψ_a be the affine fractal representations within the equivalence classes $[\Phi], [\Psi] \in \mathcal{T}_E(\Sigma_d)/\sim$. Also, let $(\lambda_1, \dots, \lambda_d)$ and (ρ_1, \dots, ρ_d) refer to the contraction rate vectors associated with Φ_a , and Ψ_a respectively. Using the fact that (i) implies (iv), we obtain

$$P_{\Phi_a} = P_\Phi = P_\Psi = P_{\Psi_a}.$$

Since for affine systems the pressure function at u is equal to the logarithm of the sum of the contraction rates raised to the power u , it follows that

$$\log \sum_{i=1}^d \lambda_i^u = P_{\Phi_a}(u) = P_{\Psi_a}(u) = \log \sum_{i=1}^d \rho_i^u, \text{ for all } u \in \mathbb{R}.$$

We can now employ a finite inductive argument as follows. Let the λ_i and ρ_i be ordered by their sizes such that $\lambda_{i_1} \geq \lambda_{i_2} \geq \dots \geq \lambda_{i_d}$ and $\rho_{j_1} \geq \rho_{j_2} \geq \dots \geq \rho_{j_d}$. Since $\sum_{i=1}^d \lambda_i^u = \sum_{i=1}^d \rho_i^u$, it follows

$$\left(\frac{\lambda_{i_1}}{\rho_{j_1}} \right)^u = \frac{1 + \sum_{m=2}^d (\rho_{j_m} / \rho_{j_1})^u}{1 + \sum_{m=2}^d (\lambda_{i_m} / \lambda_{i_1})^u}.$$

Since for each $u \geq 0$ the right-hand side in the latter equality lies between $1/d$ and d , we deduce, by letting u tend to infinity, that the assumption $\lambda_{i_1} \neq \rho_{j_1}$ gives rise to an immediate contradiction. Hence, we have that $\lambda_{i_1} = \rho_{j_1}$. For the inductive step assume that for some $k \in I$ we have $\lambda_{i_n} = \rho_{j_n}$, for all $n \in \{1, \dots, k\}$. We then have $\sum_{m=k+1}^d \lambda_{i_m}^u = \sum_{m=k+1}^d \rho_{j_m}^u$, and hence

$$\left(\frac{\lambda_{i_{k+1}}}{\rho_{j_{k+1}}} \right)^u = \frac{1 + \sum_{m=k+2}^d (\rho_{j_m} / \rho_{j_{k+1}})^u}{1 + \sum_{m=k+2}^d (\lambda_{i_m} / \lambda_{i_{k+1}})^u}.$$

As above, the right-hand side in the latter equality lies between $1/d$ and d , and hence, by letting u tend to infinity, we get an immediate contradiction to the assumption $\lambda_{i_{k+1}} \neq \rho_{j_{k+1}}$. This shows that the contraction rate vectors $(\lambda_1, \dots, \lambda_d)$ and (ρ_1, \dots, ρ_d) coincide up to a permutation. Combining this observation with the fact that (i) implies (iii), it follows that

$$I_{\Phi} \simeq I_{\Phi_a} = I_{\Psi_a} \simeq I_{\Psi}.$$

This completes the proof of the theorem. \square

The following corollary is an immediate consequence of the previous theorem. Note that a comparison of the statement in here with Corollary 3.3 (see also Theorem 3.2) clearly shows in which respect essentially affine systems have to be considered as being less rigid than nonessentially affine systems. Also, we remark that it is straightforward to incorporate the degenerate cases.

Corollary 4.4. *For $\Phi, \Psi \in \mathcal{T}_E(\Sigma_d)$, the following statements are equivalent:*

- (i) $\mu_{\Phi} \cong \mu_{\Psi}$ and $\delta_{\Phi} = \delta_{\Psi}$.
- (ii) $\Phi \sim \Psi$.

Moreover, we have

$$\mu_{\Phi} \cong \mu_{\Psi} \text{ and } \delta_{\Phi} = \delta_{\Psi} \implies \ell_{\Phi} = \ell_{\Psi}.$$

4.3. Multifractal flexibility for EAS and applications to Lyapunov spectra. As shown in Theorem 4.3, if for two essentially affine systems Φ and Ψ we have that $\mu_\Phi \cong \mu_\Psi$, then this does not necessarily imply that the two systems are equivalent, nor that their pressure functions coincide. This naturally raises the question of what can be said about the pressure functions in case $\mu_\Phi \cong \mu_\Psi$ and $\delta_\Phi \neq \delta_\Psi$. The following theorem gives a complete answer to this question.

Theorem 4.5 (Multifractal flexibility for EAS). *For $\Phi, \Psi \in \mathcal{T}_E(\Sigma_d)$ and $u, v \in \mathbb{R} \setminus \{0\}$, the following three statements are equivalent:*

- (i) $\mu_{\Phi, u} \cong \mu_{\Psi, v}$.
- (ii) $I_\Phi \simeq \frac{v}{u} I_\Psi + \frac{P_\Phi(u) - P_\Psi(v)}{u}$.
- (iii) $P_\Phi(s) = P_\Psi\left(s \cdot \frac{v}{u}\right) + s \cdot \frac{P_\Phi(u) - P_\Psi(v)}{u}$, for all $s \in \mathbb{R}$.

Furthermore, each of the statements in (i)–(iii) implies

- (iv) $\alpha_\Phi(u) \ell_\Phi(\alpha_\Phi(u)) = \alpha_\Psi(v) \ell_\Psi(\alpha_\Psi(v))$.

Proof. The equivalence (ii) \iff (iii) can be obtained by exactly the same means as the equivalence (iii) \iff (iv) in Theorem 4.3. Hence, it is sufficient to show that (i) \iff (ii).

(i) \iff (ii): By using a permutation of the generators if necessary, we can assume without loss of generality that $\mu_{\Phi, u} = \mu_{\Psi, v}$. It is then a standard result for Gibbs measure that this is equivalent to $uI_\Phi \simeq vI_\Psi + P_\Phi(u) - P_\Psi(v)$, giving that all three statements are equivalent.

To finish the proof, it remains to show that (i) and (ii) imply (iv). For this, we have by Proposition 2.1,

$$\begin{aligned} \alpha_\Phi(u) \ell_\Phi(\alpha_\Phi(u)) &= u\alpha_\Phi(u) + P_\Phi(u) = - \int (uI_\Phi - P_\Phi(u)) d\mu_{\Phi, u} \\ &= - \int (vI_\Psi - P_\Psi(v)) d\mu_{\Psi, v} = v\alpha_\Psi(v) + P_\Psi(v) \\ &= \alpha_\Psi(v) \ell_\Psi(\alpha_\Psi(v)). \end{aligned} \quad \square$$

For the special case in which $u = \delta_\Phi$ and $v = \delta_\Psi$, the previous theorem has the following immediate implication.

Corollary 4.6. *For $\Phi, \Psi \in \mathcal{T}_E(\Sigma_d)$, the following statements are equivalent:*

- (i) $\mu_\Phi \cong \mu_\Psi$.
- (ii) $I_\Phi \simeq \delta_\Psi/\delta_\Phi \cdot I_\Psi$.
- (iii) $P_\Phi(s) = P_\Psi(\delta_\Psi/\delta_\Phi \cdot s)$, for all $s \in \mathbb{R}$.

Our next aim is to show that there exist systems which are not bi-Lipschitz equivalent but which nevertheless admit multifractal measures which coincide up to permutation of the generators. For this note that using Theorem 4.5, we have

$$\frac{u}{v}I_\Phi = I_\psi + \frac{P_\Phi(u) - P_\Psi(v)}{v} < \frac{P_\Phi(u) - P_\Psi(v)}{v}.$$

By monotonicity of the pressure function, it therefore follows

$$P_\Phi\left(\frac{u}{v}\right) < \frac{P_\Phi(u) - P_\Psi(v)}{v}.$$

This observation motivates the following notion of admissibility.

Definition 4.7. Let $\Phi \in \mathcal{T}_E(\Sigma_d)$ and $u, v, p \in \mathbb{R}$ such that $v \neq 0$ be given. The triple (u, v, p) is called Φ -admissible if and only if

$$P_\Phi\left(\frac{u}{v}\right) < \frac{P_\Phi(u) - p}{v}.$$

Proposition 4.8 (Flexibility of Lyapunov spectra for EAS (I)). *Let a non-degenerate $\Phi \in \mathcal{T}_E(\Sigma_d)$ be given, and let (u, v, p) be a Φ -admissible triple. Then there exists $[\Psi] \in \mathcal{T}_E(\Sigma_d)/\sim$ (which is unique up to permutations of the generators of Ψ) such that*

$$\mu_{\Phi, u} \cong \mu_{\Psi, v} \quad \text{and} \quad p = P_\Psi(v).$$

Proof. Without loss of generality we can assume that Φ is an AS. Let $(\lambda_1, \dots, \lambda_d) \in (\mathbb{R}^+)^d$ be the contraction rate vector associated with Φ , and let (u, v, p) be a given Φ -admissible triple. Then define

$$\rho_n := \left(e^{p - P_\Phi(u)} \lambda_n^u \right)^{1/v}, \quad \text{for each } n \in \{1, \dots, d\}.$$

An elementary calculation immediately shows that the Φ -admissibility of (u, v, p) is equivalent to $\sum_{n=1}^d \rho_n < 1$. Hence, by Corollary 4.2 there exists an affine fractal representation $\Psi = (\psi_i : [0, 1] \rightarrow (0, 1) \mid i \in I) \in \mathcal{T}_E(\Sigma_d)$ whose contraction rate vector is (ρ_1, \dots, ρ_d) . Next, observe that

$$\begin{aligned} uI_\Phi - P_\Phi(u) - (vI_\Psi - P_\Psi(v)) &= -p + P_\Psi(v) = -p + \lim_{k \rightarrow \infty} \frac{1}{k} \log \left(\sum_{n=1}^d \rho_n^v \right)^k \\ &= -p + \log \left(e^{p - P_\Phi(u)} \sum_{n=1}^d \lambda_n^u \right) \\ &= -P_\Phi(u) + \log \sum_{n=1}^d \lambda_n^u = 0. \end{aligned}$$

This shows that the potentials $uI_\Phi - P_\Phi(u)$ and $vI_\Psi - P_\Psi(v)$ coincide, and also that $p = P_\Psi(v)$. It follows that the Gibbs measures corresponding to these potentials have to be equal up to permutation, that is $\mu_{\Phi, u} \cong \mu_{\Psi, v}$. \square

We end this section by giving a brief discussion of the extent of flexibility of an EAS. For this it is more convenient to work with the *moduli space of* Σ_d

$$\mathcal{M}_E(\Sigma_d) := \mathcal{T}_E(\Sigma_d) / \sim,$$

where without loss of generality we always assume that an equivalence class in $\mathcal{M}_E(\Sigma_d)$ is represented by the unique affine system contained in it. Now, first note that there clearly always is a trivial measure-wise overlap between the Lyapunov spectra of two EAS, namely $\mu_{\Phi,0} \cong \mu_{\Psi,0}$ for all $\Phi, \Psi \in \mathcal{T}_E(\Sigma_d)$. As we have seen above, for EAS there also is the possibility of nontrivial overlaps, and we will now see that these are generically represented by 2-dimensional submanifolds of $\mathcal{M}_E(\Sigma_d)$.

Definition 4.9. Two systems $\Phi, \Psi \in \mathcal{M}_E(\Sigma_d)$ are called *Lyapunov-related* if and only if there exist $u, v \in \mathbb{R} \setminus \{0\}$ such that $\mu_{\Phi,u} \cong \mu_{\Psi,v}$.

Using Theorem 4.5 (ii), we immediately see that if Φ and Ψ are Lyapunov-related, that is $\mu_{\Phi,u} \cong \mu_{\Psi,v}$ for some $u, v \in \mathbb{R} \setminus \{0\}$, then for each $s \in \mathbb{R} \setminus \{0\}$ there exists $t \in \mathbb{R} \setminus \{0\}$ such that $\mu_{\Phi,s} \cong \mu_{\Psi,t}$ (simply choose $t = s \cdot v/u$). More precisely, we have the following proposition which shows that for a nondegenerate Φ the set of systems which are Lyapunov-related to Φ forms a 2-dimensional submanifold of $\mathcal{M}_E(\Sigma_d)$, whereas if Φ is degenerate then this set is a 1-dimensional submanifold. Note that here, the case $d = 2$ appears to be special since it permits only exactly two equivalence classes modulo Lyapunov-relatedness, namely the diagonal in $\mathcal{T}_E(\Sigma_d)$ and the complement of it in $\mathcal{M}_E(\Sigma_d)$ (see Figure 1). In all other cases there is a continuum of such equivalence classes.

Proposition 4.10 (Flexibility of Lyapunov spectra for EAS (II)).

- (i) *The ‘Lyapunov relation’ is an equivalence relation on $\mathcal{M}_E(\Sigma_d)$.*
- (ii) *Let $\Phi \in \mathcal{M}_E(\Sigma_d)$ be given, and let $(\lambda_1, \dots, \lambda_d)$ be the contraction rate vector of Φ . Then the following holds for the equivalence class $[[\Phi]]$ of Φ modulo the Lyapunov relation. If Φ is degenerate, then $[[\Phi]]$ is equal to*

$$\{\Psi \in \mathcal{M}_E(\Sigma_d) : \rho_i = t, \text{ for all } i \in I, \text{ for some } t \in (0, 1/d)\}.$$

If Φ is nondegenerate, then $[[\Phi]]$ is equal to

$$\{\Psi \in \mathcal{M}_E(\Sigma_d) : \rho_i = t \cdot \lambda_i^s, \text{ for all } i \in I, \text{ for some } s, t \in \mathbb{R} \setminus \{0\}\}.$$

Here, (ρ_1, \dots, ρ_d) refers to the contraction rate vector of the system Ψ .

Proof. The assertion in (i) is an immediate consequence of the definition of the relation \cong . Furthermore, the first part in (ii) follows since for degenerate systems the Lyapunov spectrum is trivial. For the second part of (ii) we proceed as follows. Let $\Phi, \Psi \in \mathcal{M}_E(\Sigma_d)$ be two nondegenerate systems with contraction rate vector $(\lambda_1, \dots, \lambda_d)$, and (ρ_1, \dots, ρ_d) respectively. First, if

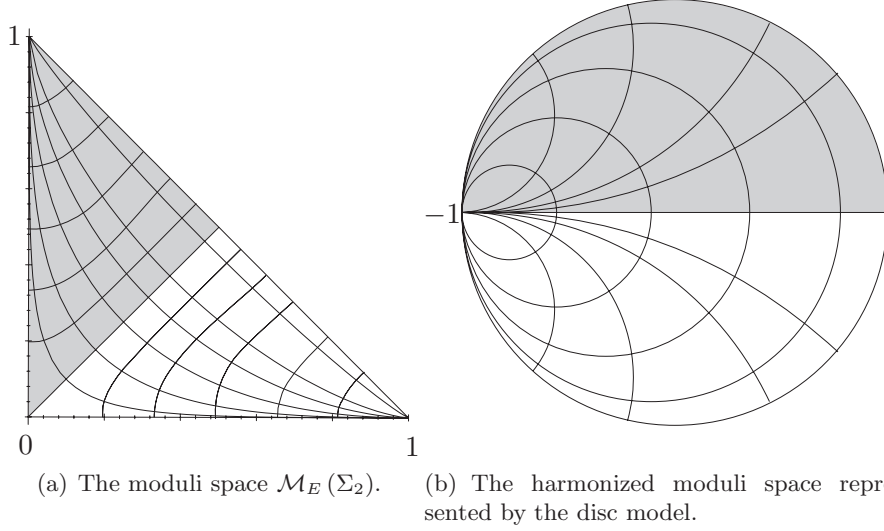


Figure 1: (a) The shaded (or alternatively, the nonshaded) region of the simplex parameterizes the moduli space $\mathcal{M}_E(\Sigma_2) := \mathcal{T}_E(\Sigma_2)/\sim$. The major axes (where at least one generator disappeared) are not included, whereas the anti-diagonal opposite to the origin (where the limit set is the whole space X) is included. The degenerate cases are found on the diagonal. The lines with endpoints in $(0,1)$ and $(1,0)$ represent ‘iso-dimensionals’ (i.e., the Hausdorff-dimension is constant on each of these lines), whereas the ‘ortho-dimensionals’ (lines orthogonal to the iso-dimensionals) are the lines of maximal decent of the Hausdorff dimension.

(b) The ‘harmonized model’ of the moduli space $\mathcal{M}_E(\Sigma_2)$, where the unit intervals on the major axis are compressed to the singleton $\{-1\} \in \mathbb{S}^1$. Here, the iso-dimensionals give rise to the horocyclic foliation centred at $\{-1\}$, whereas the ortho-dimensionals are hyperbolic geodesics with one endpoint at $\{-1\}$.

Φ and Ψ are Lyapunov-related, then Theorem 4.5 implies that there exist $u, v \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{pmatrix} \log \lambda_1 & 1 \\ \vdots & \vdots \\ \log \lambda_d & 1 \end{pmatrix} \begin{pmatrix} u \\ -P_\Phi(u) \end{pmatrix} = \begin{pmatrix} \log \rho_1 & 1 \\ \vdots & \vdots \\ \log \rho_d & 1 \end{pmatrix} \begin{pmatrix} v \\ -P_\Psi(v) \end{pmatrix}.$$

This implies that for some $a, b \in \mathbb{R}, a \neq 0$, we have

$$\begin{pmatrix} \log \lambda_1 \\ \vdots \\ \log \lambda_d \end{pmatrix} = \begin{pmatrix} \log \rho_1 & 1 \\ \vdots & \vdots \\ \log \rho_d & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

This settles one direction of the equality. For the reverse direction, assume that

$$\begin{pmatrix} \log \lambda_1 \\ \vdots \\ \log \lambda_d \end{pmatrix} \in \text{span} \left(\begin{pmatrix} \log \rho_1 \\ \vdots \\ \log \rho_d \end{pmatrix}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right).$$

We then have that $I_\Phi = vI_\Psi + u$, for uniquely determined $u, v \in \mathbb{R}$, $v \neq 0$, giving that $u = P_\Phi(1) - P_\Psi(v)$. Hence, it follows that $I_\Phi = vI_\Psi + P_\Phi(1) - P_\Psi(v)$, which gives $\mu_{\Phi,1} = \mu_{\Psi,v}$. This shows that Φ and Ψ are Lyapunov-related. \square

References

- [Bow75] BOWEN, R. Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Mathematics, 470. *Springer-Verlag, Berlin*, 1975. [MR0442989](#) (56 #1364), [Zbl 0308.28010](#).
- [Bow79] BOWEN, R. Hausdorff dimension of quasicircles. *Inst. Hautes Études Sci. Publ. Math.* (1979), no. 50, 11–25. [MR556580](#) (81g:57023), [Zbl 0439.30032](#).
- [Bus92] BUSER, P. Geometry and spectra of compact Riemann surfaces. Progress in Mathematics, 106, *Birkhäuser Boston Inc., Boston, MA*, 1992. [MR1183224](#) (93g:58149), [Zbl 0770.53001](#).
- [Den05] DENKER, M. Einführung in die Analysis dynamischer Systeme. Springer-Lehrbuch. *Springer-Verlag, Berlin*, 2005. [MR2281846](#), [Zbl 1057.37001](#).
- [Fal97] FALCONER, K. Techniques in fractal geometry. *John Wiley & Sons Ltd., Chichester*, 1997. [MR1449135](#) (99f:28013), [Zbl 0869.28003](#).
- [HU99] HANUS, P.; URBAŃSKI, M. Rigidity of infinite one-dimensional iterated function systems. *Real Anal. Exchange* **24** (1998/99), no. 1, 275–287. [MR1691751](#) (2001c:37036), [Zbl 0945.28005](#).
- [KS04] KESSEBÖHMER, M.; STRATMANN, B.O. A multifractal formalism for growth rates and applications to geometrically finite Kleinian groups. *Ergodic Theory Dynam. Systems* **24** (2004), no. 1, 141–170. [MR2041265](#) (2005e:37061), [Zbl 1115.37024](#).
- [MU03] MAULDIN, R.D.; URBAŃSKI, M. Graph directed Markov systems. Geometry and dynamics of limit sets. Cambridge Tracts in Mathematics, 148, *Cambridge University Press, Cambridge*, 2003. [MR2003772](#) (2006e:37036), [Zbl 1033.37025](#).
- [Pes97] PESIN, Y.B. Dimension theory in dynamical systems. Contemporary views and applications. Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1997. [MR1489237](#) (99b:58003), [Zbl 0895.58033](#).
- [PW05] POLLICOTT, M.; WEISS, H. Free energy as a geometric invariant. *Comm. Math. Phys.* **260** (2005), no. 2, 445–454. [MR2177326](#) (2006i:37070), [Zbl 1092.53031](#).
- [Rue78] RUELLE, D. Thermodynamic formalism. The mathematical structures of classical equilibrium statistical mechanics, With a foreword by Giovanni Gallavotti and Gian-Carlo Rota. Encyclopedia of Mathematics and its Applications, vol. 5, *Addison-Wesley Publishing Co., Reading, Mass.*, 1978. [MR511655](#) (80g:82017), [Zbl 0401.28016](#).
- [Sul82] SULLIVAN, D. Discrete conformal groups and measurable dynamics. *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), no. 1, 57–73. [MR634434](#) (83c:58066), [Zbl 0489.58027](#).
- [Sul87] SULLIVAN, D. Quasiconformal homeomorphisms in dynamics, topology, and geometry. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986) (Providence, RI), *Amer. Math. Soc.*, 1987, pp. 1216–1228. [MR934326](#) (90a:58160), [Zbl 0698.58030](#).

- [Sul88] SULLIVAN, D. Differentiable structures on fractal like sets, determined by intrinsic scaling functions on dual Cantor sets. Nonlinear evolution and chaotic phenomena (Noto, 1987), *NATO Adv. Sci. Inst. Ser. B Phys.*, vol. 176, *Plenum, New York*, 1988, pp. 101–110. [MR1109972](#), [Zbl 0707.58038](#).

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