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Convexity of the first eigenfunction of the drifting Laplacian operator and its applications

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Abstract. In this short note, we prove the convexity of the first eigenfunction of the drifting Laplacian operator with zero Dirichlet boundary value provided a suitable assumption to the drifting term is added. After giving a gradient estimate, we then use the convexity of the first eigenfunction to get a lower bound of the difference of the first and second eigenvalues of the drifting Laplacian.

Contents

1. Introduction

Given a smooth bounded convex domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and h a smooth function on the closure of Ω. We consider the following Dirichlet eigenvalue problem on the domain $\Omega \subset \mathbb{R}^n$.

$$\begin{cases} -\Delta_h u = \lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $\Delta_h = \Delta - \nabla h \cdot \nabla$ is the drifting Laplacian (also called the h-Laplacian since $\Delta_h u = e^h \mathrm{div}(e^{-h}\nabla u)$) and Ω is a convex bounded regular domain in $\mathbf{R}^n$. Here we assume $\Delta u = u^{''}$ on the real line $\mathbf{R}$. People may refer to papers of J. Lott [L03] and G. Perelman [P02] for more geometric background of the h-Laplacian. We point out that the problem (1) is closely related to eigenvalue problems on warped product space (see [CM04], [Ma06] and [MZ07]). In fact, associated to the function h, we let

$$f(x) = \frac{1}{4}(2\Delta h - |\nabla h|^2).$$

Then $f(x)$ is the scalar curvature of the metric $g_0 + e^{-h}d\theta^2$ on the product $\Omega \times S^1$ with g_0 being the Euclidean metric in the domain Ω. We mention that the modified scalar curvature of a metric g and dilation function h, as introduced by Perelman in [P02, Remark 1.3], is $R^m = R + 4f$, where R is the scalar curvature of g and $f = f(x)$ was defined above. Just as in the case when $h = 0$, we need to show the convexity of the first eigenfunction of the h-Laplacian; however, the known results developed by others can not be used directly when h is nontrivial. We find a trick to overcome this difficulty. Once we obtain the required convexity for the first eigenfunction, we can proceed as in [SWYY85] to find the lower bound of the gap between the first eigenvalue and the second eigenvalue of the h-Laplacian.

Let us state these results precisely. Denote the first and second eigenvalues of (1) by λ_1 and λ_2, respectively, and denote the corresponding first and second eigenfunctions by f_1 and f_2.

Following the works of Caffarelli and Spruck [CS82] and Korevaar [K83], who were in turn independently generalizing results of Brascamp and Lieb [BL76], we first show:

Theorem 1. *Assume that h is a smooth concave function on the closure of the domain Ω. Assume that*

$$f = \frac{1}{4}(2\Delta h - |\nabla h|^2)$$

is concave on $\overline{\Omega}$. Let $u_1 = f_1$ be the first eigenfunction to the first eigenvalue problem

$$-\Delta u + \nabla h \cdot \nabla u = \lambda_1 u, \ u > 0 \ \text{in} \ \Omega$$

with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Let $v = -\log u_1$. Then v is convex in Ω.

Note that there are many h's satisfying the assumptions in Theorem 1. In fact, it is clear that the linear function $h(x) = \sum_j a_j x_j$, where all a_j are real constants, and the more geometrically natural quadratic function

$$h(x) = -\sum_j \lambda_j x_j^2,$$

where $\lambda_j > 0$ are positive constants, satisfy the assumptions in Theorem 1. The special case when $h(x) = -|x|^2$ corresponds to the metric

$$g = dx^2 + \exp(|x|^2)d\theta^2$$

with its Riemannian measure $dv = \exp(|x|^2)dxd\theta$. Note that when $h(r) = 2\log r$ $(r > 0)$, the assumptions in Theorem 1 are also satisfied and the metric $g = dr^2 + e^{-2\log r}d\theta^2 = dr^2 + r^{-2}d\theta^2$ is the dual metric to the flat metric $dr^2 + r^2 d\theta^2$.

Another nontrivial example is $h(x) = -e^{-x}$ in the real line.

Let $u = \frac{f_2}{f_1}$ and $\lambda = \lambda_2 - \lambda_1$.

Theorem 2. *Let Ω be a smooth, bounded, convex domain in $\mathbb{R}^n$, h and f be as in Theorem 1. Then we have*

$$|\nabla u|^2 + \lambda(\mu - u)^2 \leq \sup_{\Omega} \lambda(\mu - u)^2,$$

where μ is a constant no less than $\sup_\Omega u$.

Using this gradient estimate, we can give a lower bound for the difference of eigenvalues λ.

Theorem 3. *Assume the same conditions as in Theorem 1. Then we have*

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{4d^2},$$

where d is the diameter of Ω .

We remark that when $h =$ constant, Theorem 3 was obtained by I. M. Singer, Bun Wong, Shing-Tung Yau and Stephen S.-T. Yau in [SWYY85]. Note that, some of our results may be extended to more general divergence-form elliptic operators. For example, let Δ_g denote the Laplace–Beltrami operator associated to a Riemannian metric g on the closure of the domain Ω. Then Theorem 1 should apply with Δ replaced by Δ_g. Theorem 2 should apply as long as the Ricci curvature of g is nonnegative, in which case the well-known Bochner formula for $\Delta_g|\nabla u|^2$ preserves the necessary inequality in (2) below. The trouble is that it is not easy to formulate the convexity property of the domain. It is not easy to extend our results to the case where a more general divergence-form elliptic operator

$$Lu = \sum_{ij} \partial_i(a_{ij}(x)\partial_j u)$$

is used in place of the Laplacian operator. The difficult part is to find a suitable corresponding Bochner formula (see (2) below). We leave these as questions. However, using a trick from the recent work [Y03] of S.-T. Yau, we can extend part of our results here to more general elliptic operators (see [ML08]).

The plan of the paper is as follows: we prove Theorem 1 in Section 2. Theorem 2 will be proved in Section 3, and Theorem 3 will be proved in Section 4.

2. Convexity of the first eigenfunction

We shall follow the method introduced by Korevaar [K83], see also [CS82] and [Ka85] for improvements. Their results can not be used directly. So we need a transformation to overcome this difficulty.

Proof of Theorem 1. By direct computation, we have

$$\Delta v = \nabla h \cdot \nabla v + \lambda_1 + |\nabla v|^2.$$

Let

$$w = v + \frac{h}{2}.$$

Then we have

$$\Delta w = |\nabla w|^2 + \lambda_1 + f(x),$$

where $f(x) = \frac{1}{4}(2\Delta h - |\nabla h|^2)$ as before. Then by the result of Korevaar (see Theorem 1.3 in [K83] and Lemma 3.2 in [CS82]), we know that w is convex, and then we know that v is convex. □

3. Gradient estimate for the quotient of eigenfunctions

We first set two lemmata to get a gradient estimate for u.

Lemma 1. *Let $\Omega \subset \mathbb{R}^n$ be smooth, bounded and convex domain. Then $u = \frac{f_2}{f_1}$ is smooth up to the boundary $\partial\Omega$.*

Proof. For $\forall p \in \partial\Omega$, choose local coordinates $\{x_1, x_2, \dots, x_n\}$ on a sufficiently small neighborhood U such that $U \cap \partial\Omega = U \cap \{x_1 = 0\}$.
Since

$$\begin{cases} f_1 = 0 & \text{on } \partial\Omega \\ f_1 > 0 & \text{in } \Omega, \end{cases}$$

by Hopf's lemma we have $\frac{\partial f_1}{\partial x_1} < 0$ on $\partial\Omega$. Using the Malgrange preparation theorem, we have $f_1 = g_1 x_1$ in $\overline{\Omega} \cap U$, where g_1 satisfies $g_1 \neq 0$ and smooth on $\overline{\Omega} \cap U$.

For $f_2 = 0$ on $\partial\Omega$, we can write $f_2 = g_2 x_1$, where g_2 is smooth on $\overline{\Omega} \cap U$. Thus, $u = \frac{f_2}{f_1} = \frac{g_2}{g_1}$ is smooth on $\overline{\Omega} \cap U$. □

Lemma 2.

$$\Delta_h u = -\lambda u - 2\nabla u \cdot \nabla \log f_1.$$

Proof. By direct computation, we have

$$\begin{aligned}
\Delta u &= \frac{\Delta f_2}{f_1} - 2\frac{\nabla f_1 \cdot \nabla f_2}{f_1^2} - \frac{f_2}{f_1^2}\Delta f_1 + 2\frac{f_2}{f_1^3}|\nabla f_1|^2 \\
&= \frac{1}{f_1^2}(-\lambda_2 f_1 f_2 + \lambda_1 f_1 f_2) + \frac{1}{f_1^2}(f_1 \nabla h \cdot \nabla f_2 - f_2 \nabla h \cdot \nabla f_1) \\
&\quad - 2\frac{\nabla f_1 \cdot \nabla f_2}{f_1^2} + 2f_2\frac{|\nabla f_1|^2}{f_1^3} \\
&= -\lambda u + \frac{\nabla h \cdot \nabla f_2}{f_1} - \frac{f_2}{f_1^2}\nabla h \cdot \nabla f_1 - 2\frac{\nabla f_1 \cdot \nabla f_2}{f_1^2} + 2f_2\frac{|\nabla f_1|^2}{f_1^3}.
\end{aligned}$$

Because

$$\nabla u \cdot \nabla \log f_1 = \frac{\nabla f_1 \cdot \nabla f_2}{f_1^2} - f_2\frac{|\nabla f_1|^2}{f_1^3}$$

and

$$\nabla h \cdot \nabla u = \frac{\nabla h \cdot \nabla f_2}{f_1} - \frac{f_2}{f_1^2}\nabla h \cdot \nabla f_1,$$

we obtain

$$\Delta u = -\lambda u + \nabla h \cdot \nabla u - 2\nabla u \cdot \nabla \log f_1,$$

which proves the lemma. □

Proof of Theorem 2. Define a function G by

$$G = |\nabla u|^2 + \lambda(\mu - u)^2.$$

Since u is smooth by Lemma 1, so is G.

Now, we compute

$$G_i = 2\sum_{j=1}^{n} u_j u_{ji} - 2\lambda(\mu - u)u_i,$$

$$\begin{aligned}
\Delta G &= 2|D^2 u|^2 + 2\nabla u \cdot \nabla \Delta u - 2\lambda(\mu - u)\Delta u + 2\lambda|\nabla u|^2 \\
&= 2|D^2 u|^2 + 2\nabla u \cdot \nabla(-\lambda u + \nabla h \cdot \nabla u - 2\nabla u \cdot \nabla \log f_1) \\
&\quad - 2\lambda(\mu - u)(-\lambda u + \nabla h \cdot \nabla u - 2\nabla u \cdot \nabla \log f_1) + 2\lambda|\nabla u|^2 \\
&= 2|D^2 u|^2 - 2\lambda|\nabla u|^2 + 2\nabla u \cdot \nabla(\nabla h \cdot \nabla u) \\
&\quad - 4\nabla u \cdot \nabla(\nabla u \cdot \nabla \log f_1) + 2\lambda^2(\mu - u)u \\
&\quad - 2\lambda(\mu - u)\nabla h \cdot \nabla u + 4\lambda(\mu - u)\nabla u \cdot \nabla \log f_1 \\
&\quad + 2\lambda|\nabla u|^2.
\end{aligned} \tag{2}$$

Let $G(p) = \max_{x \in \overline{\Omega}} G(x)$. All we need to prove is $|\nabla u|(p) = 0$.

Suppose $|\nabla u|(p) \neq 0$.

If $p \in \Omega$, then we can choose coordinates such that

$$u_1(p) \neq 0, \quad u_i(p) = 0, \quad \text{for} \quad 2 \leq i \leq n.$$

Applying the maximum principle, we have

$$\nabla G(p) = 0, \quad \Delta G(p) \le 0. \tag{3}$$

We now have

$$0 = G_i(p) = 2(u_1 u_{1i} - \lambda(\mu - u)u_i)(p),$$

and

$$u_{11}(p) = \lambda(\mu - u(p)), \quad u_{1i}(p) = 0, \quad for \quad 2 \le i \le n.$$

Therefore, at the point p,

$$\begin{aligned}(\nabla u \cdot \nabla \log f_1)_1 &= u_{11}(\nabla \log f_1)_1 + u_1(\nabla \log f_1)_{11} \\ &= \lambda(\mu - u)(\nabla \log f_1)_1 + u_1(\nabla \log f_1)_{11},\end{aligned} \tag{4}$$

and

$$\begin{aligned}(\nabla h \cdot \nabla u)_1 &= h_{11}u_1 + h_1 u_{11} \\ &= u_1 h_{11} + h_1 \lambda(\mu - u).\end{aligned} \tag{5}$$

A computation that uses (2), (3), (4), (5) yields

$$\begin{aligned}0 &\ge \Delta G(p) \\ &= 2|D^2 u| + 2\lambda^2(\mu - u)u \\ &\quad + 2u_1(\nabla h \cdot \nabla u)_1 - 2\lambda(\mu - u)u_1 h_1 \\ &\quad + 4\lambda(\mu - u)u_1(\nabla \log f_1)_1 - 4u_1(\nabla u \cdot \nabla \log f_1)_1 \\ &= 2(u_{11}^2 + \lambda^2(\mu - u)u) + 2u_1^2(h_{11} - 2(\log f_1)_{11}) \\ &\ge 2((\lambda(\mu - u))^2 + \lambda^2(\mu - u)u) \\ &= 2\lambda^2\mu(\mu - u),\end{aligned} \tag{6}$$

where we have used the fact that $h - 2\log f_1$ is convex (see the proof of Theorem 1). For $\mu \ge \sup_\Omega u > 0$, (6) gives rise to a contradiction.

Else, i.e., $p \in \partial\Omega$, we can choose an orthonormal frame $\{l_1, \dots, l_n\}$ around p such that $\frac{\partial}{\partial x_1} = l_1|_{\partial\Omega}$ is the outward normal vector. By computation, we have

$$\frac{\partial G}{\partial x_1}(p) = 2u_1 u_{11}(p) + 2\sum_{j=2}^{n} u_j u_{j1}(p) - 2\lambda(\mu - u)u_1(p) \ge 0. \tag{7}$$

Consider

$$\Delta u = -\lambda u + \nabla h \cdot \nabla u - 2\nabla u \cdot \nabla \log f_1, \tag{8}$$

where $\Delta u, u, \nabla h \cdot \nabla u$ are smooth up to the boundary and achieves finite values on $\partial\Omega$. Therefore, $\nabla u \cdot \nabla \log f_1 = \frac{u_1(f_1)_1}{f_1} + \frac{\sum_{i=2}^n u_i(f_1)_i}{f_1}$ also attains finite values on $\partial\Omega$. For $f_1 = 0$, on $\partial\Omega$, we have $(f_1)_i = 0$, $2 \le i \le n$. Then, on $\partial\Omega$,

$$0 = f_1(\nabla u \cdot \nabla \log f_1) = u_1(f_1)_1 + \sum_{i=2}^{n} u_i(f_1)_i = u_1(f_1)_1.$$

Therefore, $u_1(f_1)_1 = 0$ on $\partial\Omega$. Since $(f_1)_1 \neq 0$ on $\partial\Omega$ by Hopf's lemma and $f_1 = 0$ on $\partial\Omega$, we get $u_1 = 0$ on $\partial\Omega$. (7) is simplified to

$$\frac{\partial G}{\partial x_1}(p) = 2\sum_{j=2}^{n} u_j u_{j1}(p) \ge 0.$$

The definition of Hessian gives us that

$$u_{j1} = l_j l_1 u - (\nabla_{l_j} l_1)u.$$

Since $u_1 = 0$ on $\partial\Omega$, we have

$$u_{j1} = -(\nabla_{l_j} l_1)u.$$

Using the second fundamental form (I_{ij}) of the hypersurface $\partial\Omega$ in $\mathbb{R}^n$, we have

$$u_{i1} = -\sum_{j=2}^{n} I_{ij}u_j, \quad i \neq 1.$$

Then we obtain

$$\frac{\partial G}{\partial x_1}(p) = -2\sum_{i,j=2}^{n} u_j I_{ij} u_j(p) \ge 0.$$

From the convexity of $\partial\Omega$ and $u_1(p) = 0$, we get $|\nabla u|(p) = 0$. □

Remark. Since we only need the convexity of $h - 2\log f_1$, we can use it to replace the assumption that h is concave from the assumptions of Theorem 2 and Theorem 3.

4. Lower bound for the eigenvalue difference

We follow the method used in [SWYY85] to prove Theorem 3.

Proof of Theorem 3. Choosing $\mu = \sup_\Omega u$ in Theorem 2, we have

$$\sqrt{\lambda} \ge \frac{|\nabla u|}{\sqrt{(\sup u - \inf u)^2 - (\sup u - u)^2}}. \tag{9}$$

Let $A = \sup u - \inf u$ and $W = \sup u - u$, (9) can be written as

$$\sqrt{\lambda} \ge \frac{|\nabla W|}{\sqrt{A^2 - W^2}}. \tag{10}$$

Let $u(q_1) = \sup_\Omega u, u(q_2) = \inf_\Omega u$. For the convexity of Ω, q_1 and q_2 can be connected by a line $\sigma \subset \overline{\Omega}$. Integrate (10) along σ from q_1 to q_2, we obtain

$$\int_0^A \frac{|dW|}{\sqrt{A^2 - W^2}} \geq \sqrt{\lambda} ds.$$

By calculation, we have

$$\frac{\pi}{2} \leq \sqrt{\lambda} l(\sigma) \leq \sqrt{\lambda} d,$$

where $l(\sigma)$ is the length of σ, d is the diameter of Ω.

Then we get $\lambda_2 - \lambda_1 = \lambda \geq \frac{\pi^2}{4d^2}$. □

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