New York Journal of Mathematics

New York J. Math. **14** (2008) 325–378.

Koszul equivalences in A_∞-algebras

Di Ming Lu, John H. Palmieri, Quan Shui Wu and James J. Zhang

Abstract. We prove a version of Koszul duality and the induced derived equivalence for Adams connected A_∞-algebras that generalizes the classical Beilinson–Ginzburg–Soergel Koszul duality. As an immediate consequence, we give a version of the Bernšteĭn–Gel'fand–Gel'fand correspondence for Adams connected A_∞-algebras.

We give various applications. For example, a connected graded algebra A is Artin–Schelter regular if and only if its Ext-algebra $\mathrm{Ext}^*_A(k,k)$ is Frobenius. This generalizes a result of Smith in the Koszul case. If A is Koszul and if both A and its Koszul dual $A^!$ are noetherian satisfying a polynomial identity, then A is Gorenstein if and only if $A^!$ is. The last statement implies that a certain Calabi–Yau property is preserved under Koszul duality.

Contents

Received July 28, 2008.

Mathematics Subject Classification. 16A03,16A62,16E65.

Key words and phrases. A_∞-algebra, graded algebra, Artin–Schelter regular algebra, Koszul duality, derived equivalence, Gorenstein property.

The first author is supported by the NSFC (project 10571152) of China and the NSF of Zhejiang Province (J20080154).

The third author is supported by the NSFC (key project 10331030) of China and Doctorate Foundation (No. 20060246003), Ministry of Education of China.

The fourth author is supported by the NSF of USA and the Royalty Research Fund of the University of Washington.

ISSN 1076-9803/08

Introduction

Koszul duality is an incredibly powerful tool used in many areas of mathematics. One aim of this paper is to unify some generalizations by using A_∞-algebras. Our version is comprehensive enough to recover the original version of Koszul duality and the induced derived equivalences due to Beilinson–Ginzburg–Soergel [7] and most of the generalizations in ring theory and algebraic geometry. Although we will restrict ourselves to Adams connected A_∞-algebras (a natural extension of a connected graded algebras — see Definition 2.1), we have set up a framework that will work for other classes of algebras arising from representation theory and algebraic geometry.

We fix a commutative field k and work throughout with vector spaces over k. We define A_∞-algebras over k in Definition 1.1.

Similar to [24, Section 11] we define the Koszul dual of an A_∞-algebra A to be the vector space dual of the bar construction of A — see Section 2 for details. This idea is not new and dates back at least to Beilinson–Ginzburg–Schechtman [6] for graded algebras. Keller also took this approach in [19] for differential graded algebras. Our first result is a generalization of [7, Theorem 2.10.2].

Theorem A. *Let A be an augmented A_∞-algebra. Suppose that the Koszul dual of A is locally finite. Then the double Koszul dual of A is A_∞-isomorphic to A.*

This is proved as Theorem 2.4. A special case of the above theorem was proved in [24, Theorem 11.2].

As in [7] we prove several versions of equivalences of derived categories induced by the Koszul duality. Let $\mathsf{D}^\infty(A)$ be the derived category of right A_∞-modules over A. Let $\mathsf{D}^\infty_{\mathrm{per}}(A)$ (respectively, $\mathsf{D}^\infty_{\mathrm{fd}}(A)$) denote the full triangulated subcategory of $\mathsf{D}^\infty(A)$ generated by all perfect complexes (respectively, all right A_∞-modules whose homology is finite-dimensional) over A. The next result is a generalization of [7, Theorem 2.12.6].

Theorem B. *Let A be an Adams connected A_∞-algebra and E its Koszul dual. If HE is finite-dimensional, then there is an equivalence of triangulated categories $\mathsf{D}^\infty_{\mathrm{per}}(A) \cong \mathsf{D}^\infty_{\mathrm{fd}}(E)$.*

This is proved as Corollary 7.2(b)

Other equivalences of triangulated categories can be found in Sections 4 and 5. If A is either Artin–Schelter regular (Definitions 9.1(c) and 9.2(c)) or right noetherian with finite global dimension, then HE is finite-dimensional and hence Theorem B applies.

Koszul duality has many applications in ring theory, representation theory, algebraic geometry, and other areas. The next result is a generalization of the Bernšteĭn–Gel'fand–Gel'fand correspondence that follows from Theorem B. Let $\underline{\mathsf{D}^\infty_{\mathrm{fg}}}(A)$ be the stable derived category of A_∞-modules over A whose homology is finitely generated over HA, and let $\mathsf{D}^\infty(\operatorname{proj} A)$ be the derived category of the projective scheme of A. These categories are defined in Section 10, and the following theorem is part of Theorem 10.2.

Theorem C. *Let A be an Adams connected A_∞-algebra that is noetherian Artin–Schelter regular. Let E be the Koszul dual of A. Then HE is finite-dimensional and there is an equivalence of triangulated categories $\mathsf{D}^\infty(\operatorname{proj} A) \cong \underline{\mathsf{D}^\infty_{\mathrm{fg}}}(E)$.*

Applications of Koszul duality in ring theory are surprising and useful. We will mention a few results that are related to the Gorenstein property. In the rest of this introduction we let R be a connected graded associative algebra over a base field k.

Corollary D. *Let R be a connected graded algebra. Then R is Artin–Schelter regular if and only if the* Ext*-algebra $\bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^i_R(k_R, k_R)$ is Frobenius.*

This result generalizes a theorem of Smith [32, Theorem 4.3 and Proposition 5.10] that was proved for Koszul algebras. It is proved in Section 9.3. Corollary D is a fundamental result and the project [25] was based on it.

The Gorenstein property plays an important role in commutative algebra and algebraic geometry. We prove that the Gorenstein property is preserved under Koszul duality; see Section 9.4 for details.

Corollary E. *Let R be a Koszul algebra and let $R^!$ be the Koszul dual of R in the sense of Beilinson–Ginzburg–Soergel* [7]. *If R and $R^!$ are both noetherian having balanced dualizing complexes, then R is Gorenstein if and only if $R^!$ is.*

The technical hypothesis about the existence of balanced dualizing complexes can be checked when the rings are close to being commutative. For example, Corollary E holds when R and $R^!$ are noetherian and satisfy a polynomial identity. This technical hypothesis is presented because we do not understand noncommutative rings well enough. We do not know any example in which the technical hypothesis is necessary; however, Corollary E does fail for nonnoetherian rings — for example, the free algebra $R = k\langle x, y\rangle$ is Gorenstein, but $R^! \cong k\langle x, y\rangle/(x^2, xy, yx, y^2)$ is not.

Note that Koszul duality preserves the Artin–Schelter condition (Proposition 9.3). Under the technical hypothesis of Corollary E the Artin–Schelter condition is equivalent to the Gorenstein property. Therefore Corollary E follows. We can restate Corollary E for A_∞-algebras in a way that may be useful for studying the Calabi–Yau property of the derived category $\mathsf{D}^\infty(A)$ (see the discussion in Section 9.4).

The following is proved in Section 9.4.

Corollary F. *Let A be an Adams connected commutative differential graded algebra such that $\mathrm{RHom}_A(k, A)$ is not quasi-isomorphic to zero. If the Ext-algebra $\bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_A(k_A, k_A)$ is noetherian, then A satisfies the Artin–Schelter condition.*

The hypothesis on $\mathrm{RHom}_A(k, A)$ is a version of finite depth condition which is very mild in commutative ring theory and can be checked under some finiteness conditions. This is automatic if A is a finitely generated associative commutative algebra. As said before, the Artin–Schelter condition is equivalent to the Gorenstein property under appropriate hypotheses. Hence Corollary F relates the Gorenstein property of R with the noetherian property of $R^!$ and partially explains why Corollary E holds, and at the same time it suggests that Corollary E should be a consequence of a more basic statement that relates the noetherian property of R with the Gorenstein property of $R^!$. On the other hand, we believe that there should be a version of Corollary E without using the noetherian property. As we commented for Corollary E, Corollary F should hold in a class of noncommutative rings R. Corollary F is also a variation of a result of Bøgvad and Halperin about commutative complete intersection rings [9].

This paper is part four of our A_∞-algebra project and is a sequel to [24, 25, 26]. Some results were announced in [24]. For example, Theorem B and Corollary D were stated in [24] without proof. We also give a proof of [24, Theorem 11.4] in Section 5.

The paper is divided into three parts: Koszul duality for algebras, Koszul duality for modules, and applications in ring theory.

Part I consists of Sections 1 and 2. Section 1 gives background material on A_∞-algebras and their morphisms. The reader may wish to skim it to see the conventions and notation used throughout the paper. Theorem A is proved in Section 2, and we use it to recover the classical Koszul duality of Beilinson, Ginzburg, and Soergel. We also discuss a few examples.

Part II consists of Sections 3–8. Section 3 gives background material on A_∞-modules; most of this is standard, but the results on opposites are new. Section 4 sets up a framework for proving equivalences of various derived categories of DG modules over a DG algebra. Section 5 uses this framework to prove DG and A_∞ versions of the results of Beilinson, Ginzburg, and Soergel which establish equivalences between certain derived categories of modules over a Koszul algebra and over its Koszul dual. The point of Section 6 is a technical theorem which allows us, in Section 7, to rederive the classical results from the A_∞-algebra results. Theorem B is proved in Section 7, also. We discuss a couple of examples in Section 8.

Part III consists of Sections 9–10. Section 9 discusses Artin–Schelter regular algebras and Frobenius algebras, from the A_∞-algebra point of view, and includes proofs of Corollaries D, E, and F. Section 10 gives an A_∞-version of the BGG correspondence; Theorem C is proved there.

Part I. Koszul duality for algebras

1. Background on A_∞-algebras

In this section, we describe background material necessary for the rest of the paper. There are several subsections: grading conventions and related issues; A_∞-algebras and morphisms between them; the bar construction; and homotopy for morphisms of A_∞-algebras.

1.1. Conventions. Throughout we fix a commutative base field k. Unless otherwise stated, every chain complex, vector space, or algebra will be over k. The unadorned tensor product $\otimes$ is over k also.

Vector spaces (and the like) under consideration in this paper are bigraded, and for any bihomogeneous element a, we write

$$\deg a = (\deg_1(a), \deg_2(a)) \in \mathbb{Z} \times G$$

for some abelian group G. The second grading is called the *Adams grading.* In the classical setting G is trivial, but in this paper we have $G = \mathbb{Z}$; many of the abstract assertions in this paper hold for any abelian group G. If V is a bigraded vector space, then the degree (i, j) component of V is denoted by V^i_j. Usually we work with bihomogeneous elements, with the possibility of ignoring the second grading. All chain complexes will have a differential of degree $(1, 0)$. The Koszul sign convention is in force throughout the paper, but one should ignore the second grading when using it: when interchanging elements of degree (i, s) and (j, t), multiply by $(-1)^{ij}$.

Given a bigraded vector space V, we write $V^\sharp$ for its graded dual. Its *suspension* SV is the bigraded space with $(SV)^i_j = V^{i+1}_j$: suspension shifts the first grading down by one, and ignores the second grading. Write $s : V \to SV$ for the obvious map of degree -1. If V has a differential d_V, then define a differential d_{SV} on SV by $d_{SV}(sv) = -sd_V(v)$. The *Adams shift* of V is ΣV with $(\Sigma V)^i_j = V^i_{j+1}$. If V has a differential, then there is no sign in the differential for ΣV: $d_{\Sigma V}$ is just the shift of d_V.

If (M, d) and (N, d) are complexes, so are $\mathrm{Hom}_k(M, N)$ and $M \otimes N (= M \otimes_k N)$, with differentials given by

$$d(f) = df - (-1)^{\deg_1(f)} fd, \quad \forall\, f \in \mathrm{Hom}_k(M, N),$$

$$d(m \otimes n) = dm \otimes n + (-1)^{\deg_1(m)} m \otimes dn, \quad \forall\, m \otimes n \in M \otimes N,$$

respectively.

If C is a category, we write $\mathsf{C}(X, Y)$ for morphisms in C from X to Y. We reserve Hom to denote the chain complex with differential as in the previous paragraph.

1.2. A_∞-algebras and morphisms. In this paper, we will frequently work in the category of augmented A_∞-algebras; in this subsection and the next, we define the objects and morphisms of this category. Keller's paper [20] provides a nice introduction to A_∞-algebras; it also has references for many of the results which we cite here and in later subsections. Lefèvre-Hasegawa's thesis [23] provides more details for a lot of this; although it has not been published, it is available online. Another reference is [24] which contains some easy examples coming from ring theory. The following definition is originally due to Stasheff [34].

Definition 1.1. An *A_∞-algebra* over k is a $\mathbb{Z} \times \mathbb{Z}$-graded vector space A endowed with a family of graded k-linear maps

$$m_n : A^{\otimes n} \to A, \quad n \geq 1,$$

of degree $(2 - n, 0)$ satisfying the following *Stasheff identities*: for all $n \geq 1$,

$$\mathrm{SI}(n) \qquad \sum (-1)^{r+st} m_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0,$$

where the sum runs over all decompositions $n = r + s + t$, with $r, t \geq 0$ and $s \geq 1$, and where $u = r + 1 + t$. Here $\mathbf{1}$ denotes the identity map of A. Note that when these formulas are applied to elements, additional signs appear due to the Koszul sign rule.

A *DG (differential graded) algebra* is an A_∞-algebra with $m_n = 0$ for all $n \geq 3$.

The reader should perhaps be warned that there are several different sign conventions in the A_∞-algebra literature. We have chosen to follow Keller [20], who is following Getzler and Jones [14]. Stasheff [34] and Lefèvre-Hasegawa [23] use different signs: they have the sign $(-1)^{rs+t}$ in $\mathrm{SI}(n)$, and this requires sign changes in other formulas (such as $\mathrm{MI}(n)$ below).

As remarked above, we work with bigraded spaces throughout, and this requires a (very mild) modification of the standard definitions: ordinarily, an A_∞-algebra is singly graded and $\deg m_n = 2-n$; in our bigraded case, we have put $\deg m_n = (2-n, 0)$. Thus if one wants to work in the singly graded setting, one can just work with objects concentrated in degrees $(*, 0) = \mathbb{Z} \times \{0\}$.

Definition 1.2. An A_∞-algebra A is *strictly unital* if A contains an element 1 which acts as a two-sided identity with respect to m_2, and for $n \neq 2$, $m_n(a_1 \otimes \cdots \otimes a_n) = 0$ if $a_i = 1$ for some i.

In this paper we assume that A_∞-algebras (including DG algebras) are strictly unital.

Definition 1.3. A *morphism* of A_∞-algebras $f : A \to B$ is a family of k-linear graded maps

$$f_n : A^{\otimes n} \to B, \quad n \geq 1,$$

of degree $(1-n, 0)$ satisfying the following *Stasheff morphism identities*: for all $n \geq 1$,

$$\mathrm{MI}(n) \qquad \sum (-1)^{r+st} f_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = \sum (-1)^w m_q(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_q}),$$

where the first sum runs over all decompositions $n = r+s+t$ with $r, t \geq 0$, $s \geq 1$, and where we put $u = r+1+t$; and the second sum runs over all $1 \leq q \leq n$ and all decompositions $n = i_1 + \cdots + i_q$ with all $i_s \geq 1$. The sign on the right-hand side is given by

$$w = (q-1)(i_1-1) + (q-2)(i_2-1) + \cdots + 2(i_{q-2}-1) + (i_{q-1}-1).$$

When A_∞-algebras have a strict unit (as we usually assume), an A_∞-morphism between them is also required to be *strictly unital*, which means that it must satisfy these *unital morphism conditions*: $f_1(1_A) = 1_B$ where 1_A and 1_B are strict units of A and B respectively, and $f_n(a_1 \otimes \cdots \otimes a_n) = 0$ if some $a_i = 1_A$ and $n \geq 2$ (see [20, 3.5], [24, Section 4]).

As with A_∞-algebras, we have modified the grading on morphisms: we have changed the usual grading of $\deg f_n = 1-n$ to $\deg f_n = (1-n, 0)$. The composite of two morphisms is given by a formula similar to the morphism identities MI(n); see [20] or [23] for details.

Definition 1.4. A morphism $f : A \to B$ of A_∞-algebras is *strict* if $f_n = 0$ for $n \neq 1$. The *identity morphism* is the strict morphism f with $f_1 = \mathbf{1}$. A morphism f is a *quasi-isomorphism* or an A_∞-*isomorphism* if f_1 is a quasi-isomorphism of chain complexes.

Note that quasi-isomorphisms of A_∞-algebras have inverses: a morphism is a quasi-isomorphism if and only if it is a homotopy equivalence — see Theorem 1.16 below.

We write Alg for the category of associative $\mathbb{Z} \times \mathbb{Z}$-graded algebras with morphisms being the usual graded algebra morphisms, and we write Alg^∞ for the category of A_∞-algebras with A_∞-morphisms.

Let A and B be associative algebras, and view them as A_∞-algebras with $m_n = 0$ when $n \neq 2$. We point out that there may be nonstrict A_∞-algebra morphisms between them. That is, the function

$$\mathsf{Alg}(A, B) \to \mathsf{Alg}^\infty(A, B),$$

sending an algebra map to the corresponding strict A_∞-morphism, need not be a bijection. See Example 2.8 for an illustration of this.

The following theorem is important and useful.

Theorem 1.5 ([17]). *Let A be an A_∞-algebra and let HA be its cohomology ring. There is an A_∞-algebra structure on HA with $m_1 = 0$ and m_2 equal to its usual associative product, and with the higher multiplications constructed from the A_∞-structure of A, such that there is a quasi-isomorphism of A_∞-algebras $HA \to A$ lifting the identity of HA.*

This theorem was originally proved for $\mathbb{Z}$-graded A_∞-algebras, and holds true in our $\mathbb{Z} \times \mathbb{Z}$-setting.

1.3. Augmented A_∞-algebras. A strictly unital A_∞-algebra A comes equipped with a strict, strictly unital morphism $\eta : k \to A$.

Definition 1.6. (a) A strictly unital A_∞-algebra A is *augmented* if there is a strictly unital A_∞-algebra morphism $\varepsilon : A \to k$ so that $\varepsilon \circ \eta = \mathbf{1}_k$.
(b) If A is an augmented A_∞-algebra with augmentation $\varepsilon : A \to k$, its *augmentation ideal* is defined to be $\ker(\varepsilon_1)$.
(c) A *morphism* of augmented A_∞-algebras $f : A \to B$ must be strictly unital and must respect the augmentations: $\varepsilon_A = \varepsilon_B \circ f$.

We write $\mathsf{Alg}^\infty_{\mathrm{aug}}$ for the resulting category of augmented A_∞-algebras.

Proposition 1.7 (Section 3.5 in [20]). *The functor $\mathsf{Alg}^\infty_{\mathrm{aug}} \to \mathsf{Alg}^\infty$ sending an augmented A_∞-algebra to its augmentation ideal is an equivalence of categories. The quasi-inverse sends an A_∞-algebra A to $k \oplus A$ with the apparent augmentation.*

Using this equivalence, one can translate results and constructions for A_∞-algebras to the augmented case. The bar construction is an application of this.

1.4. The bar construction. The bar construction $B(-)$ is of central importance in this paper, since we define the Koszul dual of A to be the vector space dual of its bar construction. In this subsection, we describe it. We also discuss the cobar construction $\Omega(-)$, the composite $\Omega(B(-))$, and other related issues.

The following definition is a slight variant on that in [20, Section 3.6].

Definition 1.8. Let A be an augmented A_∞-algebra and let I be its augmentation ideal. The *bar construction* $B^\infty_{\mathrm{aug}}A$ on A is a coaugmented differential graded (DG) coalgebra defined as follows: as a coaugmented coalgebra, it is the tensor coalgebra $T(SI)$ on SI:

$$T(SI) = k \oplus SI \oplus (SI)^{\otimes 2} \oplus (SI)^{\otimes 3} \oplus \cdots .$$

As is standard, we use bars rather than tensors, and we also conceal the suspension s, writing $[a_1|\cdots|a_m]$ for the element $sa_1 \otimes \cdots \otimes sa_m$, where

$a_i \in I$ for each i. The degree of this element is

$$\deg[a_1|\cdots|a_m] = \left(\sum(-1+\deg_1 a_i), \sum \deg_2 a_i\right).$$

The differential b on $B^\infty_{\mathrm{aug}}A$ is the degree $(1,0)$ map given as follows: its component $b_m : (SI)^{\otimes m} \to T(SI)$ is given by

$$(1.9)\quad b_m([a_1|\cdots|a_m]) = \sum_{j,n}(-1)^{w_{j,n}}[a_1|\cdots|a_j|\overline{m}_n(a_{j+1},\cdots,a_{j+n})|a_{j+n+1}|\cdots|a_m],$$

where $\overline{m}_n = (-1)^n m_n$ and

$$w_{j,n} = \sum_{1\le s\le j}(-1+\deg_1 a_s) + \sum_{1\le t<n}(n-t)(-1+\deg_1 a_{j+t}).$$

That is, its component mapping $(SI)^{\otimes m}$ to $(SI)^{\otimes u}$ is

$$\sum 1^{\otimes j} \otimes (s\circ \overline{m}_n \circ (s^{-1})^{\otimes n}) \otimes 1^{\otimes m-j-n},$$

where the sum is over pairs (j,n) with $m \ge j+n$, and where $u = m-n+1$. See [24, p. 115] for a discussion of the sign involved in $\overline{m}_n$.

If A is an augmented DG algebra, then the above bar construction is the original bar construction and it is also denoted by BA.

Note that, with this definition, the bar construction of a bigraded algebra is again bigraded.

Remark 1.10. In [20, 3.6], Keller describes the bar construction in the nonaugmented situation. Aside from grading issues, the relation between his version and ours is as follows: if we write B^∞ for Keller's version, then B^∞_{aug} is the composite

$$\mathsf{Alg}^\infty_{\mathrm{aug}} \to \mathsf{Alg}^\infty \xrightarrow{B^\infty} \mathsf{DGC} \to \mathsf{DGC}_{\mathrm{coaug}},$$

where the first arrow is the equivalence from Proposition 1.7, and the last arrow takes a coalgebra C to $k \oplus C$, with the apparent coaugmentation.

The coderivation b encodes all of the higher multiplications of A into a single operation. Keller [20, 3.6] notes that if A and A' are augmented A_∞-algebras, then there is a bijection between Hom sets

$$(1.11)\qquad \mathsf{Alg}^\infty_{\mathrm{aug}}(A, A') \longleftrightarrow \mathsf{DGC}_{\mathrm{coaug}}(B^\infty_{\mathrm{aug}}A, B^\infty_{\mathrm{aug}}A').$$

(Again, he is working with nonaugmented A_∞-algebras, but Proposition 1.7 allows us to translate his result to this setting.)

We briefly mention the cobar construction. In full generality, this would probably take a coaugmented A_∞-coalgebra as input, and produce an augmented DG algebra. We have no interest in working with A_∞-coalgebras, though, and we do not need this generality.

Definition 1.12. Given a coaugmented DG coalgebra C with coproduct Δ and differential b_C, the *cobar construction* ΩC on C is the augmented DG algebra which as an augmented algebra is the tensor algebra $T(S^{-1}J)$ on the desuspension of the coaugmentation coideal $J = \operatorname{cok}(k \to C)$. It is graded by putting

$$\deg[x_1|\cdots|x_m] = \left(\sum(1+\deg_1 x_i), \sum \deg_2 x_i\right).$$

Its differential is the sum $d = d_0 + d_1$ of the differentials

$$d_0[x_1|\cdots|x_m] = -\sum_{i=1}^{m}(-1)^{n_i}[x_1|\cdots|b_C(x_i)|\cdots|x_m],$$

and

$$d_1[x_1|\cdots|x_m] = \sum_{i=1}^{m}\sum_{j=1}^{k_i}(-1)^{n_i+\deg_1 a_{ij}}[x_1|\cdots|x_{i-1}|a_{ij}|b_{ij}|\cdots|x_m]$$

where $n_i = \sum_{j<i}(1+\deg_1 x_j)$ and $\sum_{j=1}^{k_i} a_{ij}\otimes b_{ij} = \overline{\Delta}(x_i)$. Here $\overline{\Delta}$ is the induced coproduct on J.

Definition 1.13 ([23, Section 2.3.4])**.** If A is an augmented A_∞-algebra, then its *enveloping algebra* UA is defined to be the DG algebra $UA := \Omega(B^\infty_{\mathrm{aug}}A)$.

Thus the enveloping algebra of an augmented A_∞-algebra is an augmented DG algebra.

Proposition 1.14 ([23, 1.3.3.6 and 2.3.4.3])**.** *There is a natural quasi-isomorphism of A_∞-algebras $A \to UA$.*

The map $A \to UA$ arises as follows: between the categories of DG coalgebras and DG algebras, the bar B and cobar Ω constructions are adjoint, with Ω the left adjoint, and thus for any DG coalgebra C, there is a map $C \to B(\Omega C)$. In the case where $C = B^\infty_{\mathrm{aug}}A$, we get a map

$$B^\infty_{\mathrm{aug}}A \to B(\Omega(B^\infty_{\mathrm{aug}}A)) = B^\infty_{\mathrm{aug}}(\Omega(B^\infty_{\mathrm{aug}}A)).$$

(One can view an augmented DG algebra R as an A_∞-algebra with all higher multiplications equal to zero. In this situation, the A_∞-bar construction $B^\infty_{\mathrm{aug}}R$ reduces to the standard DG algebra bar construction $B(R)$.) The bijection (1.11) says that this corresponds to a map

$$A \to \Omega(B^\infty_{\mathrm{aug}}A).$$

This is the map in Proposition 1.14. This proposition says that every augmented A_∞-algebra is quasi-isomorphic to an augmented DG algebra. A similar result is also true in the nonaugmented case, although we will not need this. The quasi-isomorphism between A and $\Omega(BA)$ is also a standard result in the case when A itself is an augmented DG algebra, although the natural map goes the other way in that setting; indeed, there is a chain homotopy equivalence $\Omega B(A) \to A$ which is a map of DG algebras, but its inverse need not be an algebra map. See [13, Section 19], for example. One application of Proposition 1.14 is that in the DG case, there is a quasi-inverse in the category $\mathsf{Alg}^{\infty}_{\mathrm{aug}}$.

Also from [13, Section 19], we have the following result.

Lemma 1.15 ([13, Section 19]). *Let R be an augmented DG algebra. Assume that R is locally finite. Then there is a natural isomorphism $\Omega(R^\sharp) \cong (B^{\infty}_{\mathrm{aug}} R)^\sharp = B^\sharp R$.*

In light of the lemma and Remark 1.10, we point out that if R is a locally finite augmented associative algebra, then the homology of the dual of its bar construction is isomorphic to $\mathrm{Ext}^*_R(k,k) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_R(k,k)$. This is also true when R is an A_∞-algebra; see [24, Lemma 11.1] and its proof. Thus by Theorem 1.5, there is a quasi-isomorphism of A_∞-algebras

$$\mathrm{Ext}^*_R(k,k) \to B^\sharp R.$$

The A_∞-structure on $\mathrm{Ext}^*_R(k,k)$ is studied in [26].

1.5. Homotopy. Earlier, we said that we work in the category of augmented A_∞-algebras. We also need the homotopy category of such algebras, and so we need to discuss the notion of homotopy between A_∞-algebra morphisms. See [20, 3.7] and [23, 1.2.1.7] for the following.

Let A and A' be augmented A_∞-algebras, and suppose that $f, g : A \to A'$ are morphisms of augmented A_∞-algebras. Let F and G denote the corresponding maps $B^{\infty}_{\mathrm{aug}} A \to B^{\infty}_{\mathrm{aug}} A'$. Write b and b' for the differentials on $B^{\infty}_{\mathrm{aug}} A$ and $B^{\infty}_{\mathrm{aug}} A'$, respectively. Then f and g are *homotopic*, written $f \simeq g$, if there is a map $H : B^{\infty}_{\mathrm{aug}} A \to B^{\infty}_{\mathrm{aug}} A'$ of degree -1 such that

$$\Delta H = (F \otimes H + H \otimes G)\Delta \quad \text{and} \quad F - G = b' \circ H + H \circ b.$$

One can also express this in terms of a sequence of maps $h_n : A^{\otimes n} \to A'$ satisfying some identities, but we will not need this formulation. See [23, 1.2.1.7] for details (but note that he uses different sign conventions).

Two A_∞-algebras A and A' are *homotopy equivalent* if there are morphisms $f : A \to A'$ and $g : A' \to A$ such that $f \circ g \simeq \mathbf{1}_{A'}$ and $g \circ f \simeq \mathbf{1}_A$.

We will use the following theorem.

Theorem 1.16 ([18], [29], [23, 1.3.1.3]). (a) *Homotopy is an equivalence relation on the set of morphisms of A_∞-algebras $A \to A'$.*

(b) *An A_∞-algebra morphism is a quasi-isomorphism if and only if it is a homotopy equivalence.*

By part (a), we can define the homotopy category $\mathsf{HoAlg}^{\infty}_{\mathrm{aug}}$ to be the category of augmented A_∞-algebras in which the morphisms are homotopy classes of maps: that is,

$$\mathsf{HoAlg}^{\infty}_{\mathrm{aug}}(A, A') := \left(\mathsf{Alg}^{\infty}_{\mathrm{aug}}(A, A')/\simeq\right).$$

By part (b), in this homotopy category, quasi-isomorphisms are isomorphisms.

2. The Koszul dual of an A_∞-algebra

Let A be an augmented A_∞-algebra. Its *A_∞-Koszul dual*, or *Koszul dual* for short, is defined to be $E(A) := (B^\infty_{\text{aug}} A)^\sharp$. By [24, Section 11 and Lemma 11.1], $E(A)$ is a DG algebra model of the A_∞-Ext-algebra $\bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^i_A(k_A, k_A)$ where k_A is the trivial right A_∞-module over A, and where, by definition, $\operatorname{Ext}^i_A(M,N) = \bigoplus_{j\in\mathbb{Z}} \mathsf{D}^\infty(A)(M, S^i\Sigma^j(N))$ for any right A_∞-modules M and N. See Section 3 for the definitions of A_∞-modules and their derived categories.

In this section, we study some of the basic properties of A_∞-Koszul duality, we connect it to "classical" Koszul duality, and we discuss a few simple examples. The main result is Theorem A, restated as Theorem 2.4.

2.1. Finiteness and connectedness conditions. In this subsection, we introduce some technical conditions related to finite-dimensionality and connectivity of bigraded objects.

Definition 2.1. Let A be an augmented A_∞-algebra and let I be its augmentation ideal. We write I^i_* for the direct sum $I^i_* = \bigoplus_j I^i_j$, and similarly $I^*_j = \bigoplus_i I^i_j$. We say that A is *locally finite* if each bihomogeneous piece A^i_j of A is finite-dimensional. We say that A is *strongly locally finite* if I satisfies the following:

(1) Each bihomogeneous piece I^i_j of I is finite-dimensional (i.e., A is locally finite).
(2) Either for all $j \le 0$, $I^*_j = 0$; or for all $j \ge 0$, $I^*_j = 0$.
(3) Either for all j, there exists an $m = m(j)$ so that for all $i > m(j)$, $I^i_j = 0$; or for all j, there exists an $m' = m'(j)$ so that for all $i < m'(j)$, $I^i_j = 0$.

We say that A is *Adams connected* if, with respect to the Adams grading, A is (either positively or negatively) connected graded and locally finite. That is:

- I^*_j is finite-dimensional for all j.
- Either for all $j \le 0$, $I^*_j = 0$, or for all $j \ge 0$, $I^*_j = 0$.

We say that a DG algebra A is *weakly Adams connected* if:

- The DG bar construction $B(A;A) \cong B(A) \otimes A$ is locally finite.
- The only simple DG A-modules are k and its shifts.
- A is an inverse limit of a family of finite-dimensional left DG A-bimodules.

Lemma 2.2. *We have the following implications.*

(a) *Adams connected $\Rightarrow$ strongly locally finite $\Rightarrow$ weakly Adams connected.*

(b) *A weakly Adams connected $\Rightarrow$ $E(A)$ locally finite.*
(c) *A strongly locally finite $\Rightarrow$ $E(A)$ strongly locally finite, and hence every iterated Koszul dual of A is strongly locally finite.*
(d) *A Adams connected $\Rightarrow$ $E(A)$ Adams connected, and hence every iterated Koszul dual of A is Adams connected.*

Proof. The first implication in part (a) is clear. For the second implication, if A is strongly locally finite, then for connectivity reasons, k and its shifts will be the only simple modules. We defer the proof that $B(A;A)$ is locally finite until after the proof of part (c).

For the inverse limit condition, we assume that $I^*_j = 0$ when $j \leq 0$, and that for each j, there is an $m'(j)$ such that $I^i_j = 0$ when $i < m'(j)$. The other cases are similar. We will describe a sequence of two-sided ideals J_n in A, $n \geq 1$, so that A/J_n is finite-dimensional, and $A = \varprojlim A/J_n$. Define J_n to be

$$\begin{aligned} J_n = I^1_{\geq n} &\oplus I^2_{\geq n+m'(1)} \oplus I^3_{\geq n+\min(2m'(1),m'(2))} \oplus \cdots \\ &\oplus I^s_{\geq n+\min_{\sigma \vdash n-1}(\sum_{a\in\sigma} m'(a))} \oplus \cdots \oplus I^n_{\geq n+\min(\cdots)} \\ &\oplus I^{n+1}_* \oplus I^{n+2}_* \oplus \cdots . \end{aligned}$$

The notation "$\sigma \vdash n-1$" means that σ partitions $n-1$. The idea here is that, for example, if J contains all of the elements in bidegrees $(1, \geq n)$, and since A has elements in bidegrees $(1, \geq m'(1))$, then for J to be an ideal, it should contain all of the elements in bidegrees $(2, \geq n+m'(1))$.

Part (b) is clear: as graded vector spaces, $B(A;A)$ and $B(A) \otimes_k A$ are isomorphic; thus if $B(A;A)$ is locally finite, so is $B(A)$.

(c) Since $E(A)$ is dual to the tensor coalgebra $T(SI)$, we focus on $T(SI)$. If we suppose that $I^*_j = 0$ when $j \leq 0$, then for all $j < n$, we have $(I^{\otimes n})^*_j = 0$. Shifting I by S does not change this: $((SI)^{\otimes n})^*_j$ will be zero if $j < n$. Therefore, $T(SI)$ satisfies condition (2) of Definition 2.1. Dualizing, we see that $E(A)$ satisfies the other version of condition (2): if J is its augmentation ideal, then $J^*_j = 0$ when $j \geq 0$. Similarly, if $I^*_j = 0$ when $j \geq 0$, then $J^*_j = 0$ when $j \leq 0$.

So if I satisfies condition (2), then so does J.

Now suppose that I satisfies (1) and this version of condition (3): for each j, there is an $m'(j)$ such that $I^i_j = 0$ when $i < m'(j)$. Then for fixed n and j:

- $(I^{\otimes n})^i_j$ is zero if i is small enough.
- $(I^{\otimes n})^i_j$ is finite-dimensional for all i.

Therefore $T(SI)$ satisfies condition (3). Furthermore, since for fixed j, $(I^{\otimes n})^*_j$ is zero for all but finitely many values of n, we see that $T(SI)$ satisfies condition (1). Dualizing, we see that the augmentation ideal J of $E(A)$ satisfies (1) and (3), also (although J satisfies the "other version" of (3)). This completes the proof of part (c).

We return to part (a): if A is strongly locally finite, then by part (c), so is the bar construction $B(A)$; more precisely, A and $B(A)$ will satisfy the same version of condition (3). Hence it is easy to verify that their tensor product will be locally finite. This completes the proof of (a).

(d) A being Adams connected is equivalent to I satisfying (1), (2), and both versions of (3): for each j, there are numbers $m(j)$ and $m'(j)$ so that $I^i_j = 0$ unless $m'(j) \le i \le m(j)$. By the proof of (c), this implies that $E(A)$ satisfies the same conditions. □

Remark 2.3. One may interchange the roles of i and j in the definition of strong local finiteness, but the presence of the shift S in $E(A) = T(SI)$ makes the situation asymmetric. Suppose that I satisfies the following:

(1′) Each bihomogeneous piece I^i_j of I of A is finite-dimensional.

(2′) Either for all i there exists an $m = m(i)$ so that for all $j \ge m(i)$, $I^i_j = 0$; or for all i there exists an $m' = m'(i)$ so that for all $j \le m'$, $I^i_j = 0$.

(3′) Either for all $i \ge 1$, $I^i_* = 0$; or for all $i \le 1$, $I^i_* = 0$.

Then by imitating the proof of part (c) of the lemma, one can show that $E(A)$ is locally finite; however, it may not satisfy (2′).

Suppose that I satisfies (1′), (2′), and the following:

(3″) Either for all $i \ge 0$, $I^i_* = 0$; or for all $i \le 1$, $I^i_* = 0$.

Then the same proof shows that the augmentation ideal of $E(A)$ satisfies (1′), (2′), and (3″) as well, and hence the same holds for every iterated Koszul dual of A.

2.2. A_∞-Koszul duality. Here is the main theorem of this section, which is a slight generalization of [24, Theorem 11.2]. This is Theorem A from the introduction.

Theorem 2.4. *Suppose that A is an augmented A_∞-algebra with $E(A)$ locally finite. Then there is a natural quasi-isomorphism of A_∞-algebras $A \cong E(E(A))$.*

If A is weakly Adams connected, then essentially by definition (or see Lemma 2.2), $E(A)$ is locally finite. Hence by Lemma 2.2, $E(A)$ is locally finite if A is Adams connected or strongly locally finite. The summary of the theorem's proof is that the double Koszul dual is the enveloping algebra UA of A (see Definition 1.13 and Proposition 1.14).

Proof. By definition, the double Koszul dual $E(E(A))$ is $(B^\infty_{\mathrm{aug}}((B^\infty_{\mathrm{aug}}A)^\sharp))^\sharp$. Apply Lemma 1.15 to the DG algebra $E(A) = (B^\infty_{\mathrm{aug}}A)^\sharp$, which is locally finite. Then there are natural DG algebra isomorphisms

$$E(E(A)) = (B^\infty_{\mathrm{aug}}((B^\infty_{\mathrm{aug}}A)^\sharp))^\sharp \cong \Omega(((B^\infty_{\mathrm{aug}}A)^\sharp)^\sharp) \cong \Omega(B^\infty_{\mathrm{aug}}A).$$

Proposition 1.14 gives a natural A_∞-isomorphism $A \xrightarrow{\cong} \Omega(B^\infty_{\mathrm{aug}}A)$. □

Koszul duality $E(-)$ is a contravariant functor from A_∞-algebras to DG algebras, and since one can view a DG algebra as being an A_∞-algebra, there are functions

$$\mathsf{Alg}^\infty_{\mathrm{aug}}(A, A') \xrightarrow{E(-)} \mathsf{DGA}_{\mathrm{aug}}(E(A'), E(A)) \to \mathsf{Alg}^\infty_{\mathrm{aug}}(E(A'), E(A)),$$

for augmented A_∞-algebras A and A'.

Corollary 2.5. *Suppose that A and A' are augmented A_∞-algebras with $E(A)$, $E(E(A))$, $E(A')$, and $E(E(A'))$ locally finite.*

(a) *Then $E(-)$ gives a bijection*

$$\mathsf{Alg}^\infty_{\mathrm{aug}}(A, A') \xrightarrow{\sim} \mathsf{DGA}_{\mathrm{aug}}(E(A'), E(A)).$$

(b) *The composite*

$$\mathsf{Alg}^\infty_{\mathrm{aug}}(A, A') \xrightarrow{\sim} \mathsf{DGA}_{\mathrm{aug}}(E(A'), E(A)) \to \mathsf{Alg}^\infty_{\mathrm{aug}}(E(A'), E(A))$$

induces a bijection

$$\mathsf{HoAlg}^\infty_{\mathrm{aug}}(A, A') \xrightarrow{\sim} \mathsf{HoAlg}^\infty_{\mathrm{aug}}(E(A'), E(A)).$$

(c) *Hence every A_∞-algebra map $f : E(A') \to E(A)$ is homotopic to a DG algebra map.*

Proof. (a) From (1.11) we have a bijection

$$\mathsf{Alg}^\infty_{\mathrm{aug}}(A, A') \xrightarrow{\sim} \mathsf{DGC}_{\mathrm{coaug}}(B^\infty_{\mathrm{aug}}A, B^\infty_{\mathrm{aug}}A').$$

We are assuming that $B^\infty_{\mathrm{aug}}A$ and $B^\infty_{\mathrm{aug}}A'$ are locally finite, so the vector space duality maps

$$\begin{aligned}\mathsf{DGC}_{\mathrm{coaug}}(B^\infty_{\mathrm{aug}}A, B^\infty_{\mathrm{aug}}A') &\to \mathsf{DGA}_{\mathrm{aug}}((B^\infty_{\mathrm{aug}}A')^\sharp, (B^\infty_{\mathrm{aug}}A)^\sharp) \\ &\to \mathsf{DGC}_{\mathrm{coaug}}(((B^\infty_{\mathrm{aug}}A)^\sharp)^\sharp, ((B^\infty_{\mathrm{aug}}A')^\sharp)^\sharp)\end{aligned}$$

are bijections. Therefore so is

$$\mathsf{Alg}^\infty_{\mathrm{aug}}(A, A') \xrightarrow{\sim} \mathsf{DGA}_{\mathrm{aug}}(E(A'), E(A)).$$

(b, c) The naturality of the quasi-isomorphism (= homotopy equivalence) in Theorem 2.4 says that the function

$$\begin{aligned}\mathsf{HoAlg}^\infty_{\mathrm{aug}}(A, A') &\xrightarrow{E(E(-))} \mathsf{HoAlg}^\infty_{\mathrm{aug}}(E(E(A)), E(E(A'))) \\ f &\longmapsto E(E(f))\end{aligned}$$

is a bijection. That is, the composite

$$\begin{aligned}\mathsf{HoAlg}^\infty_{\mathrm{aug}}(A, A') &\to \mathsf{HoAlg}^\infty_{\mathrm{aug}}(E(A'), E(A)) \\ &\to \mathsf{HoAlg}^\infty_{\mathrm{aug}}(E(E(A)), E(E(A')))\end{aligned}$$

is a bijection. The first map here is induced by

$$\mathsf{Alg}^\infty_{\mathrm{aug}}(A, A') \xrightarrow{\sim} \mathsf{DGA}_{\mathrm{aug}}(E(A'), E(A)) \xrightarrow{i} \mathsf{Alg}^\infty_{\mathrm{aug}}(E(A'), E(A)),$$

and the second by

$$\begin{aligned}\mathsf{Alg}^{\infty}_{\mathrm{aug}}(E(A'),E(A)) &\xrightarrow{\sim} \mathsf{DGA}_{\mathrm{aug}}(E(E(A)),E(E(A')))\\ &\xrightarrow{j} \mathsf{Alg}^{\infty}_{\mathrm{aug}}(E(E(A)),E(E(A'))).\end{aligned}$$

Since both of the functions i and j are inclusions, the functions $\mathsf{Ho}\, i$ and $\mathsf{Ho}\, j$ must be bijections. This proves (b) and (c). □

2.3. Classical Koszul duality. Classically, a *Koszul algebra* is a connected graded associative algebra R which is locally finite, is generated in degree 1, has quadratic relations, and has its ith graded Ext-group $\mathrm{Ext}^i_R(k,k)$ concentrated in degree $-i$ for each i; see [7, Theorem 2.10.1] or [32, Theorem 5.9(6)], for example. (In those papers, $\mathrm{Ext}^i_R(k,k)$ is actually required to be concentrated in degree i, but that is the result of different grading conventions.) Its (classical) Koszul dual, also denoted by $R^!$, is $\mathrm{Ext}^*_R(k,k)$. One can show that if R is a Koszul algebra, then so is $R^!$ — see [7, 2.9.1], for example.

A standard example is an exterior algebra $R=\Lambda(x_1,\dots,x_n)$ on generators x_i each in degree 1; then its Koszul dual $R^!$ is the polynomial algebra $k[y_1,\dots,y_n]$, with each y_i in degree $(1,-1)$.

We want all of our algebras to be bigraded, though, and we want the double Koszul dual to be isomorphic, as a bigraded algebra, to the original algebra. Thus we might grade $\Lambda(x_1,\dots,x_n)$ by putting each x_i in degree $(0,1)$, in which case $R^!=k[y_1,\dots,y_n]$. The grading for $R^!$ is given as follows: y_i is represented in the dual of the bar construction for R by the dual of $[x_i]$, and since $\deg[x_i]=(-1,1)$, y_i has degree $(1,-1)$. The double Koszul dual is exterior on classes dual to $[y_i]$ in the bar construction on $R^!$, each of which therefore has degree $(0,1)$.

Note that these are graded in such a way that there are no possible nonzero higher multiplications m_n on them. This absence of higher multiplications is typical for a Koszul algebra, as Keller [20, 3.3] and the authors [24, Section 11] point out. Conversely, if we grade our algebras in such a way that there are no possible higher multiplications, we can recover classical Koszul duality.

Definition 2.6. Fix a pair of integers (a,b) with $b\neq 0$. A bigraded associative algebra A is an *(a,b)-generated Koszul algebra* if it satisfies these conditions:

(a) $A_{0,0}=k$.
(b) A is locally finite.
(c) A is generated in bidegree (a,b).
(d) The relations in A are generated in bidegree $(2a,2b)$.
(e) For each i, the graded vector space $\mathrm{Ext}^i_A(k,k)$ is concentrated in degree $(i(a-1),-ib)$.

In fact, conditions (c) and (d) should follow from condition (e): one should be able to imitate the proofs of [7, 2.3.1 and 2.3.2].

If A is a bigraded associative algebra, the classical Koszul dual of A, denoted by $A^!$, is defined to be $HE(A)$ — the *homology* of the A_∞-Koszul dual $E(A) = (B^\infty_{\mathrm{aug}} A)^\sharp$. Forgetting grading issues, $A^!$ is isomorphic to $\mathrm{Ext}^*_A(k,k)$. In classical ring theory, we often consider the classical Koszul dual as an associative algebra — an A_∞-algebra with $m_n = 0$ if $n \neq 2$.

Corollary 2.7. *Fix a pair of integers (a,b) with $b \neq 0$. If A is an (a,b)-generated Koszul algebra, then $E(A)$ and $E(E(A))$ are quasi-isomorphic to associative algebras $A^!$ and $(A^!)^!$, respectively, and there is an isomorphism of bigraded algebras $A \cong (A^!)^!$.*

This result is known [7] so we only give a sketch of proof.

Sketch of proof of Corollary 2.7. By Theorem 1.5, the A_∞-Koszul dual $E(A)$ is quasi-isomorphic to $A^!$ with some possible higher multiplications. We need to show, among other things, that in this case, the higher multiplications on $A^!$ are zero.

Since $b \neq 0$, both $E(A)$ and $E(E(A))$ will be locally finite.

The bidegree $(\deg_1 x, \deg_2 x)$ of a nonzero nonunit element $x \in \mathrm{Ext}^*_A(k,k)$ satisfies

$$(*) \qquad \frac{\deg_1 x}{\deg_2 x} = -\frac{a-1}{b},$$

and this fraction makes sense since $b \neq 0$. The same is true for any tensor product of such elements. Since the higher multiplication m_n has degree $(2-n, 0)$, one can see that if $n \neq 2$, the bidegree of $m_n(x_1 \otimes \cdots \otimes x_n)$ will not satisfy $(*)$, and so will be zero. Thus there is no nonzero higher multiplications on $A^!$ which is compatible with the bigrading. This implies that the A_∞-algebra $E(A)$ is quasi-isomorphic to the associative algebra $A^!$.

Now we claim that $A^!$ is $(1-a, -b)$-generated Koszul. There is an obvious equivalence of categories between $\mathbb{Z}$-graded algebras and $\mathbb{Z} \times \mathbb{Z}$-graded algebras concentrated in degrees (na, nb) for $n \in \mathbb{Z}$; under that equivalence, (a,b)-generated Koszul algebras correspond to Koszul algebras in the sense of [7]. Koszul duality takes $\mathbb{Z}$-graded algebras generated in degree 1 to $\mathbb{Z} \times \mathbb{Z}$-graded algebras generated in degree $(1,-1)$. It takes $\mathbb{Z} \times \mathbb{Z}$-graded algebras generated in degree (a,b) to $\mathbb{Z} \times \mathbb{Z}$-graded algebras generated in degree $(1-a, -b)$. The proof of [7, 2.9.1] carries over to show that since A is (a,b)-generated Koszul, its dual $A^!$ is $(1-a,-b)$-generated Koszul.

Since $A^!$ is Koszul, its Koszul dual $(A^!)^!$ is associative (or there is no nonzero higher multiplications on $(A^!)^!$, by the first part of the proof). A similar grading argument shows that any morphism $f : A \to (A^!)^!$ of A_∞-algebras must be strict; thus the isomorphism of A_∞-algebras $A \to (A^!)^!$ is just an isomorphism of associative algebras. $\square$

2.4. Examples: exterior and polynomial algebras. In this subsection, we consider some simple examples involving exterior algebras and polynomial algebras. The first example shows that in the classical setting, it is crucial that a Koszul algebra be generated in a single degree.

Example 2.8. Assume that the ground field k has characteristic 2, and consider the exterior algebra $\Lambda = \Lambda(x_1, x_2)$ with $\deg x_i = (0, i)$: this is *not* a classical Koszul algebra, nor is it an (a, b)-generated Koszul algebra, since there are generators in multiple degrees. The same goes for $\Lambda \otimes \Lambda$. The Ext-algebra for Λ is the polynomial algebra $\Lambda^! = k[y_1, y_2]$ with $\deg y_i = (1, -i)$. Similarly, the Ext-algebra for $\Lambda \otimes \Lambda$ is isomorphic to $\Lambda^! \otimes \Lambda^!$. Although it is true that $(\Lambda^!)^! \cong \Lambda$, there are naturality problems. In particular, the map

$$\mathsf{Alg}(\Lambda, \Lambda \otimes \Lambda) \to \mathsf{Alg}(\Lambda^! \otimes \Lambda^!, \Lambda^!)$$

is not injective: one can show that the following two maps induce the same map on Ext:

$$\begin{aligned} f &: x_1 \mapsto x_1 \otimes 1 + 1 \otimes x_1, & g &: x_1 \mapsto x_1 \otimes 1 + 1 \otimes x_1, \\ f &: x_2 \mapsto x_2 \otimes 1 + 1 \otimes x_2, & g &: x_2 \mapsto x_2 \otimes 1 + 1 \otimes x_2 + x_1 \otimes x_1. \end{aligned}$$

(Indeed, any coassociative coproduct on Λ induces the Yoneda product on $\mathrm{Ext}^*_\Lambda(k, k)$.) This shows the importance of the requirement that Koszul algebras be generated in a single degree.

Now, Theorem 2.4 and Corollary 2.5 apply here. The A_∞-version of the Koszul dual of Λ is quasi-isomorphic to $\Lambda^!$: $E(\Lambda) \cong \Lambda^! = k[y_1, y_2]$, where the A_∞-structure on $\Lambda^!$ is given by $m_n = 0$ when $n \neq 2$. Corollary 2.5 says that there is a bijection

$$\mathsf{HoAlg}^\infty_{\mathrm{aug}}(\Lambda, \Lambda \otimes \Lambda) \xrightarrow{\sim} \mathsf{HoAlg}^\infty_{\mathrm{aug}}(\Lambda^! \otimes \Lambda^!, \Lambda^!).$$

This fixes the flaw above; the two (strict) maps f and g correspond to A_∞-algebra morphisms $E(f)$ and $E(g)$, and while $E(f)_1 = E(g)_1$, the morphisms must differ in some higher component. (Even though the algebras involved here have A_∞-structures with zero higher multiplications, there are nonstrict A_∞-algebra morphisms between them. Also, $k[y_1, y_2]$ is quasi-isomorphic, not equal, to the A_∞-Koszul dual of Λ, so Corollary 2.5(c), which says that every nonstrict map on Koszul duals is homotopic to a strict one, does not apply here.)

Example 2.9. Let $A = \Lambda(x)$ with $\deg x = (a, 0)$; Corollary 2.7 does not apply in this case. The Koszul dual is $E(A) = k[y]$ with $\deg y = (1 - a, 0)$ and with $m_n = 0$ for $n \neq 2$. Assume that $a \neq 0, 1$; then $E(A)$ is locally finite, as is $E(E(A))$ by Remark 2.3. Thus Corollary 2.5 says that there is a bijection

$$\mathsf{HoAlg}^\infty_{\mathrm{aug}}(\Lambda(x), \Lambda(x)) \xrightarrow{\sim} \mathsf{HoAlg}^\infty_{\mathrm{aug}}(k[y], k[y]).$$

For degree reasons, every map $\Lambda(x) \to \Lambda(x)$ must be strict, so each map is given by the image of x: $x \longmapsto cx$ for any scalar $c \in k$.

On the other hand, degree reasons do not rule out nonstrict maps $k[y] \to k[y]$. Strict maps will correspond to those from $\Lambda(x)$ to itself, with the map given by the scalar c corresponding to the map $y \mapsto cy$. Thus, as pointed out in Corollary 2.5(c), if there are any nonstrict maps, then they are homotopic to strict ones. In particular, one can see that if two A_∞-algebra maps $f, g : k[y] \to k[y]$ are homotopic, then $f_1 = g_1$. So if $f = (f_1, f_2, \dots)$ is such a map, then it will be homotopic to the strict map $(f_1, 0, 0, \dots)$. None of this is immediately clear from the morphism and homotopy identities, so Koszul duality, in the form of Corollary 2.5, gives some insight into A_∞-maps from $k[y]$ to itself.

Example 2.10. Now consider $A = \Lambda(x)$ with $\deg x = (1, 0)$. As in the previous example, every map $\Lambda(x) \to \Lambda(x)$ must be strict. In this case, $B^\infty_{\text{aug}}A$ is the vector space spanned by the classes $[x|\cdots|x]$, all of which are in bidegree $(0, 0)$. Since $B^\infty_{\text{aug}}A$ is not locally finite, Theorem 2.4 does not apply, and the Koszul dual $E(A)$ ends up being the power series ring $k[[y]]$ instead of the polynomial ring $k[y]$. Consider the composite

$$\mathsf{Alg}^\infty_{\text{aug}}(A, A) \xrightarrow{\sim} \mathsf{DGC}_{\text{coaug}}(B^\infty_{\text{aug}}A, B^\infty_{\text{aug}}A) \to \mathsf{DGA}^{\text{aug}}(E(A), E(A)).$$

The first map is a bijection by (1.11), but the second map is not, essentially since the map is given by vector space duality and the vector spaces involved are not finite-dimensional. Since strict A_∞-maps are homotopic if and only if they are equal, we get a proper inclusion

$$\mathsf{HoAlg}^\infty_{\text{aug}}(A, A) \overset{\neq}{\hookrightarrow} \mathsf{HoAlg}^\infty_{\text{aug}}(E(A), E(A)).$$

Thus Corollary 2.5 fails here.

Example 2.11. We also mention the case when $A = \Lambda(x)$ with $\deg x = (a, b)$ with $b \neq 0$. In this case, $E(A) = k[y]$ with $\deg y = (1 - a, -b)$. Also, A_∞-morphisms $\Lambda(x) \to \Lambda(x)$ and $k[y] \to k[y]$ must be strict, and it is easy to show that the strict maps are in bijection, as Corollary 2.5 says they should be. Indeed in this case, classical Koszul duality (Corollary 2.7) applies, since $\Lambda(x)$ is (a, b)-generated Koszul with $b \neq 0$.

Part II. Koszul duality for modules

3. Background on A_∞-modules

Koszul duality relates not just to algebras, but also to modules over them. In this section, we briefly review the relevant categories of modules over an A_∞-algebra. See Keller [20, 4.2] for a few more details, keeping in mind that since he is not working in the augmented setting, a little translation is required, especially in regards to the bar construction. The paper [24, Section 6] also has some relevant information, as does Lefèvre-Hasegawa's thesis [23].

There are several subsections here: the definition of A_∞-module; the bar construction; derived categories; and several sections about "opposites."

3.1. A_∞-modules. Let A be an A_∞-algebra. A bigraded vector space M is a *right A_∞-module over A* if there are graded maps

$$m_n : M \otimes A^{\otimes n-1} \to M, \quad n \geq 1,$$

of degree $(2-n, 0)$ satisfying the Stasheff identities $\mathrm{SI}(n)$, interpreted appropriately. Similarly, a bigraded vector space N is a *left A_∞-module over A* if there are graded maps

$$m_n : A^{\otimes n-1} \otimes N \to N, \quad n \geq 1,$$

of degree $(2-n, 0)$ satisfying the Stasheff identities $\mathrm{SI}(n)$, interpreted appropriately.

Morphisms of right A_∞-modules are defined in a similar way: a *morphism $f : M \to M'$ of right A_∞-modules over A* is a sequence of graded maps

$$f_n : M \otimes A^{\otimes n-1} \to M', \quad n \geq 1,$$

of degree $(1-n, 0)$ satisfying the Stasheff morphism identities $\mathrm{MI}(n)$. Morphisms of left A_∞-modules are defined analogously, and so are homotopies in both the right and left module settings.

Now suppose that A is an augmented A_∞-algebra. A right A_∞-module M over A is *strictly unital* if for all $x \in M$ and for all $a_i \in A$, $m_2(x \otimes 1) = x$ and

$$m_n(x \otimes a_2 \otimes \cdots \otimes a_n) = 0, \quad \text{for all } n \geq 3$$

if $a_i = 1$ for some i. A morphism f of such is *strictly unital* if for all $n \geq 2$, we have $f_n(x \otimes a_2 \otimes \cdots \otimes a_n) = 0$ if $a_i = 1$ for some i.

Given an augmented A_∞-algebra A, let $\mathsf{Mod}^\infty(A)$ denote the category of strictly unital right A_∞-modules with strictly unital morphisms over A.

Suppose A is an augmented A_∞-algebra. The morphism $\varepsilon : A \to k$ makes the vector space k into a left A_∞-module over A. It is called the *trivial left A_∞-module over A* and is denoted by ${}_A k$. The *trivial right A_∞-module over A* is defined similarly, and is denoted by k_A.

3.2. The bar construction for modules. The bar construction is as useful for A_∞-modules as it is for A_∞-algebras: recall that the bar construction on A is $B^\infty_{\mathrm{aug}} A = T(SI)$. A strictly unital right A_∞-module structure on a bigraded vector space M gives a comodule differential on the right $B^\infty_{\mathrm{aug}} A$-comodule

$$B^\infty_{\mathrm{aug}}(M; A) := SM \otimes T(SI),$$

as in Definition 1.8. Also, morphisms of right modules $M \to M'$ are in bijection with morphisms of right DG comodules $B^\infty_{\mathrm{aug}}(M; A) \to B^\infty_{\mathrm{aug}}(M'; A)$ as in (1.11), and the notion of homotopy translates as well.

Similarly, one has a bar construction for left A_∞-modules, defined by

$$B^\infty_{\mathrm{aug}}(A; N) := T(SI) \otimes SN,$$

with the same formula for the differential.

3.3. Derived categories. Let $\mathsf{D}^\infty(A)$ be the derived category associated to the module category $\mathsf{Mod}^\infty(A)$. By [20, 4.2] (see also [23, 2.4.2]), this derived category is the same as the homotopy category of $\mathsf{Mod}^\infty(A)$: in the homotopy category for A_∞-modules, quasi-isomorphisms have already been inverted. This statement is the module version of Theorem 1.16.

Given a DG algebra R, write $\mathsf{Mod}\, R$ for the category of unital DG right R-modules, and write $\mathsf{D}(R)$ for its derived category. A good reference for the derived category $\mathsf{D}(R)$ is [19]; also see [21, 22]. See [23, 2.4.3 and 4.1.3] and [24, 7.2 and 7.3] for the following.

Proposition 3.1. (a) *Suppose that A and B are augmented A_∞-algebras. If $f : A \to B$ is an A_∞-isomorphism, then the induced functor*

$$f^* : \mathsf{D}^\infty(B) \to \mathsf{D}^\infty(A)$$

is a triangulated equivalence.

(b) *Suppose that R is an augmented DG algebra. Then the inclusion $\mathsf{Mod}\, R \hookrightarrow \mathsf{Mod}^\infty(R)$ induces a triangulated equivalence*

$$\mathsf{D}(R) \to \mathsf{D}^\infty(R).$$

(c) *Hence if A is an augmented A_∞-algebra and R is an augmented DG algebra with an A_∞-isomorphism $A \to R$, there is a triangulated equivalence $F : \mathsf{D}(R) \to \mathsf{D}^\infty(A)$. Under this equivalence, $F(k_R) \cong k_A$ and $F(R) \cong A$.*

Hence in the category $\mathsf{D}^\infty(A)$ one can perform many of the usual constructions, by first working in the derived category $\mathsf{D}(UA)$ of its enveloping algebra and then applying the equivalence of categories $\mathsf{D}(UA) \to \mathsf{D}^\infty(A)$. See Section 5 for an application of this idea. We note that the Adams shift is an automorphism of $\mathsf{D}^\infty(A)$.

3.4. The opposite of a DG algebra. If $A = (A, m_1, m_2)$ is a DG algebra with differential m_1 and multiplication m_2, we define the *opposite algebra* of A to be $(A^{\mathrm{op}}, m_1^{\mathrm{op}}, m_2^{\mathrm{op}})$, where $A^{\mathrm{op}} = A$, $m_1^{\mathrm{op}} = -m_1$, and

$$m_2^{\mathrm{op}}(a \otimes b) = (-1)^{(\deg a)(\deg b)} m_2(b \otimes a).$$

That is, $m_2^{\mathrm{op}} = m_2 \circ \tau$, where τ is the twist function, which interchanges tensor factors at the expense of the appropriate Koszul sign. One can verify that this is a DG algebra: m_1^{op} and m_2^{op} satisfy a Leibniz formula. (Choosing $m_1^{\mathrm{op}} = m_1$ also works, but is not compatible with the bar construction: see below.)

If $f : A \to A'$ is a map of DG algebras, define $f^{\mathrm{op}} : A^{\mathrm{op}} \to (A')^{\mathrm{op}}$ by $f^{\mathrm{op}} = f$. Then f^{op} is also a DG algebra map, so op defines an automorphism of the category of DG algebras, and it is clearly its own inverse.

Dually, given a DG coalgebra $C = (C, d, \Delta)$, we define its *opposite coalgebra* to be $(C^{\mathrm{op}}, d^{\mathrm{op}}, \Delta^{\mathrm{op}})$, where $C^{\mathrm{op}} = C$, $d^{\mathrm{op}} = -d$, and $\Delta^{\mathrm{op}} = \tau \circ \Delta$. With these definitions, for any DG algebra A there is an isomorphism

$$\begin{aligned} B(A^{\mathrm{op}}) &\xrightarrow{\Phi} B(A)^{\mathrm{op}}, \\ [a_1|\dots|a_m] &\longmapsto (-1)^{\sum_{i<j}(-1+\deg a_i)(-1+\deg a_j)}[a_m|\dots|a_1], \end{aligned}$$

of DG coalgebras. Note that, had we defined the differential m_1^{op} in A^{op} by $m_1^{\mathrm{op}} = m_1$, this map Φ would not be compatible with the differentials, and so would not be a map of DG coalgebras.

If $F : (C, d, \Delta) \to (C', d', \Delta')$ is a morphism of DG coalgebras, then define F^{op} to be equal to F. One can check that $F^{\mathrm{op}} : C^{\mathrm{op}} \to (C')^{\mathrm{op}}$ is also a map of DG coalgebras, making op into a functor op : $\mathsf{DGC} \to \mathsf{DGC}$. As above, it is an automorphism which is its own inverse.

The definition of homotopy in Subsection 1.5 works in general for DG coalgebra morphisms: if F and G are DG coalgebra morphisms $C \to C'$, then a *homotopy from F to G* is a map $H : C \to C'$ of degree -1 such that

$$\Delta H = (F \otimes H + H \otimes G)\Delta, \quad F - G = d' \circ H + H \circ d.$$

We write $H = H(F \to G)$ to indicate the "direction" of the homotopy. One can check that, in this situation, the map H also defines a homotopy from G^{op} to F^{op}, as maps $C^{\mathrm{op}} \to (C')^{\mathrm{op}}$. Therefore, we may define $H^{\mathrm{op}}(G^{\mathrm{op}} \to F^{\mathrm{op}})$ to be $H(F \to G)$. As a consequence, op induces an automorphism on the homotopy category of DG coalgebras.

3.5. The opposite of an A_∞-algebra. Now we define a functor

$$\mathrm{op} : \mathsf{Alg}^\infty \to \mathsf{Alg}^\infty$$

which generalizes the opposite functor on the category of DG algebras. Given an A_∞-algebra $(A, m_1, m_2, m_3, \dots)$, define $(A^{\mathrm{op}}, m_1^{\mathrm{op}}, m_2^{\mathrm{op}}, m_3^{\mathrm{op}}, \dots)$ as follows: as a bigraded vector space, A^{op} is the same as A. The map $m_n^{\mathrm{op}} : (A^{\mathrm{op}})^{\otimes n} \to A^{\mathrm{op}}$ is defined by $m_n^{\mathrm{op}} = (-1)^{\varepsilon(n)} m_n \circ (\text{twist})$, where "twist" is the map which reverses the factors in a tensor product, with the appropriate Koszul sign, and

$$\varepsilon(n) = \begin{cases} 1, & \text{if } n \equiv 0, 1 \pmod 4, \\ 0, & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

Equivalently, since only the parity of $\varepsilon(n)$ is important, $\varepsilon(n) = \binom{n+2}{2}$, or $\varepsilon(n) = \binom{n}{2} + 1$. Thus when applied to elements,

$$m_n^{\mathrm{op}}(a_1 \otimes \cdots \otimes a_n) = (-1)^{\varepsilon(n) + \sum_{i<j}(\deg a_i)(\deg a_j)} m_n(a_n \otimes \cdots \otimes a_1).$$

Lemma 3.2. *The function ε satisfies the following additivity formula: for any $q \geq 1$ and any $i_s \geq 1$, $s = 1, 2, \dots, q$,*

$$\sum_{1 \leq s \leq q} \varepsilon(i_s) + \varepsilon\left(\sum_{1 \leq s \leq q} i_s - q + 1\right) + \sum_{1 \leq s < t \leq q} (i_s - 1)(i_t - 1) \equiv q + 1 \pmod 2.$$

Proof. The $q = 1$ case is trivial, the $q = 2$ case may be established by (for example) considering the different congruence classes of i_1 mod 4, and for larger q, one can use a simple induction argument. □

Lemma 3.3. $(A^{\mathrm{op}}, m_1^{\mathrm{op}}, m_2^{\mathrm{op}}, m_3^{\mathrm{op}}, \dots)$ *is an A_∞-algebra.*

Proof. We need to check that $(A^{\mathrm{op}}, m_1^{\mathrm{op}}, m_2^{\mathrm{op}}, m_3^{\mathrm{op}}, \dots)$ satisfies the Stasheff identities. This is a tedious, but straightforward, verification, which we leave to the reader. The $q = 2$ case of Lemma 3.2 is useful. □

We also need to specify what happens to morphisms. Given a morphism $f : A \to B$, we define $f^{\mathrm{op}} : A^{\mathrm{op}} \to B^{\mathrm{op}}$ by defining

$$f_n^{\mathrm{op}} : (A^{\mathrm{op}})^{\otimes n} \to B^{\mathrm{op}}$$

to be $f_n^{\mathrm{op}} = (-1)^{1+\varepsilon(n)} f_n \circ (\text{twist})$; that is,

$$f_n^{\mathrm{op}}(a_1 \otimes \cdots \otimes a_n) = (-1)^{1+\varepsilon(n)+\sum_{i<j} \deg a_i \deg a_j} f_n(a_n \otimes \cdots \otimes a_1).$$

Lemma 3.4. *The family $f^{\mathrm{op}} = (f_n^{\mathrm{op}})$ is a morphism of A_∞-algebras.*

Proof. This is another tedious verification. Lemma 3.2 is used here. □

To complete this circle of ideas, we should consider the bar construction. That is, consider the following diagram of functors:

$$\begin{tikzcd}
\mathsf{Alg}^\infty \arrow[rr, "\mathrm{op}", "\sim"'] \arrow[d, "B(-)"] & & \mathsf{Alg}^\infty \arrow[d, "B(-)"] \\
\mathsf{DGC} \arrow[r, "\mathrm{op}", "\sim"'] & \mathsf{DGC} \arrow[r, dotted, "\sim ?"] & \mathsf{DGC}
\end{tikzcd}$$

The horizontal arrows are equivalences of categories. The vertical arrows are fully faithful embeddings. Starting with an A_∞-algebra A in the upper left corner, mapping down and then to the right gives $B(A)^{\mathrm{op}}$, while mapping to the right and then down gives $B(A^{\mathrm{op}})$. It would be nice if these two DG coalgebras agreed, and indeed they do. The proof of the following is left to the reader.

Lemma 3.5. *For any A_∞-algebra A, the map*

$$\begin{aligned}
\Phi : B(A^{\mathrm{op}}) &\longrightarrow B(A)^{\mathrm{op}}, \\
[a_1|\dots|a_m] &\longmapsto (-1)^{\sum_{i<j}(-1+\deg a_i)(-1+\deg a_j)}[a_m|\dots|a_1]
\end{aligned}$$

is an isomorphism of DG coalgebras.

Note that $B(A)^{\mathrm{op}}$ is the opposite coalgebra to $B(A)$, as defined in Subsection 3.4.

This result gives us a second way to prove Lemma 3.4, that the definition $f_n^{\mathrm{op}} = (-1)^{1+\varepsilon(n)} f_n \circ (\text{twist})$ defines a morphism of A_∞-algebras: one just has to check that if the A_∞-algebra morphism $f : A \to A'$ corresponds to the DG coalgebra morphism $B(f) : B(A) \to B(A')$, then the composite

$$B(A)^{\mathrm{op}} \xrightarrow[\cong]{\Phi^{-1}} B(A^{\mathrm{op}}) \xrightarrow{B(f^{\mathrm{op}})} B((A')^{\mathrm{op}}) \xrightarrow[\cong]{\Phi} B(A')^{\mathrm{op}}$$

is equal to $B(f)^{\mathrm{op}} = B(f)$. This is straightforward.

Once we know that the bar construction works well with opposites, we can define the opposite of a homotopy between A_∞-algebra maps in terms of the bar construction. Thus op defines an automorphism on the homotopy category of A_∞-algebras.

3.6. The opposite of an A_∞-module. Since modules over an A_∞-algebra are defined using exactly the same identities $\mathrm{SI}(n)$ as for A_∞-algebras, and since morphisms between modules satisfy only slight variants on the identities $\mathrm{MI}(n)$, essentially the same proofs show that the opposite of a right A-module is a left A^{op}-module, etc. That is, there are equivalences of categories

$$(\text{left } A_\infty\text{-modules over } A) \xrightarrow{\mathrm{op}} \mathsf{Mod}^\infty(A^{\mathrm{op}}),$$
$$\mathsf{D}^\infty(\text{left } A_\infty\text{-modules over } A) \xrightarrow{\mathrm{op}} \mathsf{D}^\infty(A^{\mathrm{op}}).$$

So whenever left A_∞-modules arise, we may easily convert them to right A_∞-modules, and vice versa.

4. Adjunctions and equivalences

This section lays more groundwork: generalities for establishing equivalences between categories via Auslander and Bass classes, results about derived functors for DG modules, and $\otimes$-Hom adjointness. Two of the main results of the section are Propositions 4.10 and 4.11, which describe when certain subcategories of derived categories of DG-modules are equivalent.

4.1. Auslander and Bass classes. Let C and D be two categories. Let $F : \mathsf{C} \to \mathsf{D}$ be left adjoint to a functor $G : \mathsf{D} \to \mathsf{C}$. Then there are natural transformations

$$\eta : \mathbf{1}_{\mathsf{C}} \to GF,$$
$$\varepsilon : FG \to \mathbf{1}_{\mathsf{D}}.$$

We define two full subcategories as follows. The *Auslander class* associated to (F, G) is the subcategory of C whose objects are

$$\{M \mid \eta_M : M \to GF(M) \text{ is an isomorphism}\}.$$

The Auslander class is denoted by $\mathbb{A}$. The *Bass class* associated to (F, G) is the subcategory of D whose objects are

$$\{N \mid \varepsilon_N : FG(N) \to N \text{ is an isomorphism}\}.$$

The Bass class is denoted by $\mathbb{B}$. These definitions are abstractions of ideas of Avramov and Foxby [2, Section 3]. The following lemma is proved by imitating [2, Theorem 3.2].

Lemma 4.1. *Let (F, G) be a pair of adjoint functors between C and D.*

(a) *The functors F and G restrict to an equivalence of categories between $\mathbb{A}$ and $\mathbb{B}$.*

(b) *If C and D are additive and F and G are additive functors, then $\mathbb{A}$ and $\mathbb{B}$ are additive subcategories. If C and D are triangulated and F and G are triangulated functors, then $\mathbb{A}$ and $\mathbb{B}$ are triangulated subcategories.*

4.2. Derived functors over a DG algebra. The derived category and derived functors over a DG algebra are well-understood constructions nowadays. See [33], [19], and [13], for example. We review some details in this subsection. As with A_∞-algebras and modules, every DG module in this paper is $\mathbb{Z} \times \mathbb{Z}$-graded.

Let R be a DG algebra and let M be a DG R-module. Then M is called *acyclic* if $HM = 0$; it is called *free* if it is isomorphic to a direct sum of shifts of R; and it is called *semifree* if there is a sequence of DG submodules

$$0 = M_{-1} \subset M_0 \subset \cdots \subset M_n \subset \cdots$$

such that $M = \bigcup_n M_n$ and that each M_n/M_{n-1} is free on a basis of cycles. Semifree modules are a replacement for free complexes over an associative algebra.

Notation. If R is a DG algebra and M and N are DG R-modules, we write $\mathrm{Hom}_R(M, N)$ for the DG k-module whose degree n elements are degree n R-module maps $M \to N$, ignoring the differential; see Subsection 1.1 for the formula for the differential in $\mathrm{Hom}_R(M, N)$. Similarly, $\mathrm{End}_R(M)$ means the complex $\mathrm{Hom}_R(M, M)$.

In DG homological algebra, K-projective and K-injective DG modules are used to define derived functors. A DG R-module M is called *K-projective* if the functor $\mathrm{Hom}_R(M, -)$ preserves quasi-isomorphisms, or equivalently, $\mathrm{Hom}_R(M, -)$ maps acyclic DG R-modules to acyclic DG k-modules. For example, a semifree DG R-module is always K-projective. A DG R-module M is called *K-flat* if the functor $M \otimes_R -$ preserves quasi-isomorphisms; every K-projective DG R-module is K-flat. A DG R-module N is called *K-injective* if the functor $\mathrm{Hom}_R(-, N)$ preserves quasi-isomorphisms, or equivalently, $\mathrm{Hom}_R(-, N)$ maps acyclic DG R-modules to acyclic DG k-modules.

Given a DG R-module M, a map $f : L \to M$ is called a *semifree* (or *K-projective* or *K-flat*, respectively) *resolution* of M if $f : L \to M$ is a quasi-isomorphism and L is semifree (or K-projective or K-flat, respectively).

Similarly, a K*-injective* resolution of M is a quasi-isomorphism $M \to L$ where L is K-injective. In all of these cases, we will also abuse notation slightly and refer to L itself as the resolution, omitting mention of the map f.

The right derived functor of $\mathrm{Hom}_R(-,-)$ is $\mathrm{RHom}_R(-,-)$, defined by

$$\mathrm{RHom}_R(M,N) := \mathrm{Hom}_R(P,N) \quad \text{or} \quad \mathrm{RHom}_R(M,N) := \mathrm{Hom}_R(M,I)$$

where P is a K-projective resolution of M and I is a K-injective resolution of N. The left derived functor of $- \otimes_R -$ is $- \otimes^L_R -$, defined by

$$M \otimes^L_R N := M \otimes_R Q \quad \text{or} \quad M \otimes^L_R N := S \otimes_R N$$

where S is a K-flat resolution of M and Q is a K-flat resolution of N.

4.3. Tensor-Hom and Hom-Hom adjunctions. We discuss $\otimes$-Hom and Hom-Hom adjointness, both basic and derived. These are well-known, at least in the case of modules over an associative algebra. The DG case may not be as familiar, so we provide some details. Here is the basic version.

Lemma 4.2. *Let A, B, C, and D be DG algebras.*

(a) [Hom-Hom adjointness] *Let ${}_AL_C$, ${}_DM_B$ and ${}_AN_B$ be DG bimodules. Then*

$$\mathrm{Hom}_{A^{\mathrm{op}}}({}_AL_C, \mathrm{Hom}_B({}_DM_B, {}_AN_B)) \cong \mathrm{Hom}_B({}_DM_B, \mathrm{Hom}_{A^{\mathrm{op}}}({}_AL_C, {}_AN_B))$$

as DG (C,D)-bimodules.

(b) [$\otimes$-Hom adjointness] *Let ${}_DL_B$, ${}_BM_A$ and ${}_CN_A$ be DG bimodules. Then*

$$\mathrm{Hom}_A({}_DL \otimes_B M_A, {}_CN_A) \cong \mathrm{Hom}_B({}_DL_B, \mathrm{Hom}_A({}_BM_A, {}_CN_A))$$

as DG (C,D)-bimodules.

The isomorphism in part (a) gives a pair of adjoint functors

$$A\text{-}\mathsf{Mod} \rightleftarrows (\mathsf{Mod}\, B)^{\mathrm{op}},$$

namely $\mathrm{Hom}_{A^{\mathrm{op}}}(-, {}_AN_B)$ (left adjoint) and $\mathrm{Hom}_B(-, {}_AN_B)$ (right adjoint). This explains the label, "Hom-Hom adjointness."

Proof. (a) The desired isomorphism

$$\phi : \mathrm{Hom}_{A^{\mathrm{op}}}({}_AL_C, \mathrm{Hom}_B({}_DM_B, {}_AN_B)) \\ \to \mathrm{Hom}_B({}_DM_B, \mathrm{Hom}_{A^{\mathrm{op}}}({}_AL_C, {}_AN_B))$$

is defined by the following rule. Let $f \in \mathrm{Hom}_{A^{\mathrm{op}}}({}_AL_C, \mathrm{Hom}_B({}_DM_B, {}_AN_B))$, and $l \in L$ and $m \in M$, write $f(l) \in \mathrm{Hom}_B({}_DM_B, {}_AN_B)$ and $f(l)(m) \in N$; then $\phi(f) : M \to \mathrm{Hom}_{A^{\mathrm{op}}}({}_AL_C, {}_AN_B)$ is determined by

$$\phi(f)(m)(l) = (-1)^{|l||m|} f(l)(m)$$

for all $l \in L, m \in M$. It is straightforward to check that ϕ is an isomorphism of DG bimodules.

The above construction is given in the unpublished manuscript [3, (3.4.2), p. 27]. The isomorphism ϕ is called the *swap isomorphism* [3, Sect. 3.4].

(b) This is standard and a proof is given in [3, (3.4.3), p. 28]. A non-DG version is in [31, Theorem 2.11, p. 37]. $\square$

To get derived versions of these, we need information about bimodules, semifree resolutions, K-projectives, etc.

Lemma 4.3. *Let A and B be DG algebras. Let M and L be DG (B, A)-bimodules, or equivalently, DG $B^{\mathrm{op}} \otimes A$-modules.*

(a) *Let N be a DG A-module. If there is a sequence of DG submodules*

$$0 = N_{-1} \subset N_0 \subset \cdots \subset N_n \subset \cdots$$

such that $N = \bigcup_n N_n$, $C_n := N_n/N_{n-1}$ is K-projective, and the underlying graded module C_n is projective, then N is K-projective.

(b) *If M is semifree (B, A)-bimodule, then it is K-projective over A. As a consequence, if $M \to L$ is a semifree resolution of L, then restricted to the right-hand side, it is a K-projective resolution of L_A.*

(c) *If M is K-injective (B, A)-bimodule, then it is K-injective over A. As a consequence, if $L \to M$ is a K-injective resolution of L, then restricted to the right-hand side, it is a K-injective resolution of L_A.*

Proof. (a) First consider the sequence

$$0 \to N_0 \to N_1 \to C_1 \to 0$$

of DG A-modules. This is a split (hence exact) sequence after omitting the differentials, as the underlying graded module C_1 is projective. Let X be an acyclic DG A-module. Then we have an exact sequence

$$0 \to \mathrm{Hom}_A(C_1, X) \to \mathrm{Hom}_A(N_1, X) \to \mathrm{Hom}_A(N_0, X) \to 0.$$

If the two ends $\mathrm{Hom}_A(C_1, X)$ and $\mathrm{Hom}_A(N_0, X)$ are acyclic, so is the middle term $\mathrm{Hom}_A(N_1, X)$. This shows that if $C_0(= N_0)$ and C_1 are K-projective, so is N_1. By induction on n we see that N_n is K-projective and projective for all n. Since every sequence

$$0 \to N_{n-1} \to N_n \to C_n \to 0$$

splits, the map $\mathrm{Hom}_A(N_n, X) \to \mathrm{Hom}_A(N_{n-1}, X)$ is surjective. This means that the inverse system $\{\mathrm{Hom}_A(N_n, X)\}_n$ satisfies Mittag-Leffler condition. Since each $\mathrm{Hom}_A(N_n, X)$ is acyclic,

$$\mathrm{Hom}_A(N, X) = \varprojlim \mathrm{Hom}_A(N_n, X)$$

is acyclic by [37, Theorem 3.5.8].

(b) The second assertion follows from the first one.

By part (a) and the definition of a semifree module, we may assume M is free. Since a free module is a direct sum of shifts of $B^{\mathrm{op}} \otimes A$, we may assume M is a copy of $B^{\mathrm{op}} \otimes A$. Therefore M is free over A, and hence K-projective over A.

(c) The second assertion follows from the first one.

Let N_A be an acyclic DG A-module. By $\otimes$-Hom adjointness (Lemma 4.2(b)), we have

$$\begin{aligned}\operatorname{Hom}_A(N, {}_BM_A) &= \operatorname{Hom}_A(N, \operatorname{Hom}_{B^{\mathrm{op}}\otimes A}(B^{\mathrm{op}}\otimes A, {}_BM_A))\\ &\cong \operatorname{Hom}_{B^{\mathrm{op}}\otimes A}(N\otimes_A (B^{\mathrm{op}}\otimes A), {}_BM_A)\\ &\cong \operatorname{Hom}_{B^{\mathrm{op}}\otimes A}(N\otimes B^{\mathrm{op}}, {}_BM_A).\end{aligned}$$

Since N is acyclic, so is $N\otimes B^{\mathrm{op}}$. Since ${}_BM_A$ is K-injective, the above formula implies that $\operatorname{Hom}_A(N, M_A)$ is acyclic. Hence M_A is K-injective. □

We can combine the previous two lemmas to get derived Hom-Hom and $\otimes$-Hom adjointness.

Lemma 4.4 (Derived Hom-Hom adjointness)**.** *Let A, B, C, and D be DG algebras. Let ${}_AL_B$, ${}_BM_C$ and ${}_AN_D$ be DG bimodules.*

(a) *There is an isomorphism of complexes*

$$\begin{aligned}&\operatorname{RHom}_{A^{\mathrm{op}}}({}_AL_C, \operatorname{RHom}_B({}_DM_B, {}_AN_B))\\ &\qquad\qquad\cong \operatorname{RHom}_B({}_DM_B, \operatorname{RHom}_{A^{\mathrm{op}}}({}_AL_C, {}_AN_B))\end{aligned}$$

in $\mathsf{D}(C^{\mathrm{op}}\otimes D)$.

(b) *There is an isomorphism of k-vector spaces*

$$\mathsf{D}(A^{\mathrm{op}})({}_AL, \operatorname{RHom}_B(M_B, {}_AN_B)) \cong \mathsf{D}(B)(M_B, \operatorname{RHom}_{A^{\mathrm{op}}}({}_AL, {}_AN_B)).$$

Proof. (a) This follows from Lemmas 4.2 and 4.3, and by taking semifree resolutions of the DG bimodules M and L.

(b) This follows from (a) by taking H^0. □

Lemma 4.5 (Derived $\otimes$-Hom adjointness)**.** *Let A, B, C and D be DG algebras. Let ${}_DL_B$, ${}_BM_A$ and ${}_CN_A$ be DG bimodules.*

(a) *There is an isomorphism of complexes*

$$\operatorname{RHom}_A({}_DL\otimes^L_B M_A, {}_CN_A) \cong \operatorname{RHom}_B({}_DL_B, \operatorname{RHom}_A({}_BM_A, {}_CN_A))$$

in $\mathsf{D}(C^{\mathrm{op}}\otimes D)$.

(b) *There is an isomorphism of k-vector spaces*

$$\mathsf{D}(A)(L\otimes^L_B M_A, N_A) \cong \mathsf{D}(B)(L_B, \operatorname{RHom}_A({}_BM_A, N_A)).$$

Proof. (a) This follows from Lemmas 4.2 and 4.3, and by taking semifree resolutions of the DG bimodules M and L and a K-injective resolution of the bimodule N.

(b) This follows from (a) by taking H^0. □

4.4. Balanced bimodules and equivalences. In this subsection we prove Propositions 4.10 and 4.11, the main results in this section; these establish a framework for proving the derived equivalences in Section 5.

Let A and E be two DG algebras. A DG A-module M is a DG (E, A)-bimodule if and only if there is a map of DG algebras $E \to \mathrm{Hom}_A(M, M)$.

Definition 4.6. Let $\mathcal{B}$ be a DG (E, A)-bimodule. We call $\mathcal{B}$ *left balanced* if there is a quasi-isomorphism $\mathcal{B} \to N$ of DG (E, A)-bimodules such that N is K-injective over $E^{\mathrm{op}} \otimes A$ and the canonical map $E \to \mathrm{Hom}_A(N, N)$ is a quasi-isomorphism of DG algebras. The *right balanced* property is defined in a similar way, in terms of the map $A^{\mathrm{op}} \to \mathrm{Hom}_{E^{\mathrm{op}}}(N, N)$.

Lemma 4.7. *Let $\mathcal{B}$ be a DG (E, A)-bimodule. Then the following conditions are equivalent:*

(i) *$\mathcal{B}$ is left balanced.*

(ii) *If $P \to \mathcal{B}$ is a quasi-isomorphism of DG (E, A)-bimodules with P_A being K-projective, then the canonical map $E \to \mathrm{Hom}_A(P, P)$ is a quasi-isomorphism of DG algebras.*

(iii) *If $\mathcal{B} \to I$ is a quasi-isomorphism of DG (E, A)-bimodules with I_A being K-injective, then the canonical map $E \to \mathrm{Hom}_A(I, I)$ is a quasi-isomorphism of DG algebras.*

Proof. (i) $\Leftrightarrow$ (ii). Suppose that there is a quasi-isomorphism $\mathcal{B} \to N$ where N is K-injective over $E^{\mathrm{op}} \otimes A$. (By a result of Spaltenstein [33, Corollary 3.9], there always is such a map.) By Lemma 4.3(c), N is K-injective over A. The quasi-isomorphism $f : P \to \mathcal{B} \to N$ induces two maps

$$E \xrightarrow{i_N} \mathrm{Hom}_A(N, N) \xrightarrow{f^*} \mathrm{Hom}_A(P, N)$$

and

$$E \xrightarrow{i_P} \mathrm{Hom}_A(P, P) \xrightarrow{f_*} \mathrm{Hom}_A(P, N)$$

of (E, E)-bimodules. Since N is K-injective over A and P is K-projective over A, both g and h are quasi-isomorphisms. It is easy to see that $f^* i_N = f_* i_P$: they both map $e \in E$ to $ef = fe \in \mathrm{Hom}_A(P, N)$. Therefore i_N is a quasi-isomorphism if and only if i_P is.

(ii) $\Leftrightarrow$ (iii). This proof is similar. $\square$

By the above lemma, we can construct plenty of left balanced bimodules. For example, let M be a K-injective (or K-projective) DG A-module and let $E = \mathrm{Hom}_A(M, M)$. It follows from the lemma that M becomes a left balanced DG (E, A)-bimodule with its natural left E-module structure.

We now recall a few definitions.

Definition 4.8. An object M in an additive category C with infinite direct sums is called *small* if $\mathsf{C}(M, -)$ commutes with arbitrary direct sums.

Let A be an A_∞-algebra and M be a right A_∞-module over A. Let $\mathsf{triang}^\infty_A(M)$ denote the triangulated subcategory of $\mathsf{D}^\infty(A)$ generated by

M and its Adams shifts. (Recall that every subcategory in this paper is full.) Let $\mathsf{thick}_A^\infty(M)$ be the thick subcategory generated by $\mathsf{triang}_A^\infty(M)$: the smallest triangulated subcategory, closed under summands, which contains $\mathsf{triang}_A^\infty(M)$. Similarly, if R is a DG algebra and N is a DG R-module, let $\mathsf{triang}_R(N)$ denote the triangulated subcategory of $\mathsf{D}(R)$ generated by N and its shifts, and let $\mathsf{thick}_R(N)$ be the thick subcategory generated by $\mathsf{triang}_R(N)$. Let $\mathsf{D}_{\mathrm{per}}^\infty(A) = \mathsf{thick}_A^\infty(A)$, and let $\mathsf{D}_{\mathrm{per}}(R) = \mathsf{thick}_R(R)$. We call objects in $\mathsf{D}_{\mathrm{per}}^\infty(A)$ and $\mathsf{D}_{\mathrm{per}}(R)$ *perfect complexes.* Let $\mathsf{loc}_A^\infty(M)$ denote the localizing subcategory (= triangulated and closed under arbitrary direct sums) generated by $\mathsf{triang}_A^\infty(M)$, and similarly for $\mathsf{loc}_R(N)$. We can also define each of these with M or N replaced by a collection of modules. It is well-known that $\mathsf{loc}_R(R) = \mathsf{D}(R)$.

Lemma 4.9. (a) *If R is a DG algebra and N is a right DG R-module, then N is small in $\mathsf{D}(R)$ if and only if $\mathrm{RHom}_R(N, -)$ commutes with arbitrary colimits, and if and only if $N \in \mathsf{D}_{\mathrm{per}}(R)$.*

(b) *If A is an A_∞-algebra and M is a right A_∞-module over A, then M is small in $\mathsf{D}^\infty(A)$ if and only if $M \in \mathsf{D}_{\mathrm{per}}^\infty(A)$.*

Proof. (a) The equivalence that N is small if and only if N is in $\mathsf{D}_{\mathrm{per}}(R)$ is somewhat standard; see Keller [19, 5.3], for example. If $\mathrm{RHom}_R(N, -)$ commutes with arbitrary colimits, then since homology also commutes with colimits, $\mathsf{D}(R)(N, -) = H\,\mathrm{RHom}_R(N, -)$ does as well, and so N is small. Finally, $\mathrm{RHom}_R(R, -)$ commutes with arbitrary colimits, and hence so does $\mathrm{RHom}_R(N, -)$ for any object N in $\mathsf{D}_{\mathrm{per}}(R) = \mathsf{thick}_R(R)$.

(b) This follows from part (a) and Propositions 1.14 and 3.1(c). $\square$

Let $\mathcal{B}$ be a DG (E, A)-bimodule. By Lemma 4.5(b),

$$F_{\mathcal{B}} := - \otimes_E^L \mathcal{B} : \mathsf{D}(E) \to \mathsf{D}(A) \quad \text{and}$$
$$G_{\mathcal{B}} := \mathrm{RHom}_A(\mathcal{B}, -) : \mathsf{D}(A) \to \mathsf{D}(E)$$

form a pair of adjoint functors. Let $\mathbb{A}_{\mathcal{B}}$ and $\mathbb{B}_{\mathcal{B}}$ be the Auslander and Bass classes associated to the pair $(F_{\mathcal{B}}, G_{\mathcal{B}})$. By Lemma 4.1, $(F_{\mathcal{B}}, G_{\mathcal{B}})$ induces a triangulated equivalence between $\mathbb{A}_{\mathcal{B}}$ and $\mathbb{B}_{\mathcal{B}}$.

The next two results are precursors of the derived equivalences in the next section.

Proposition 4.10. *Let A and E be DG algebras, and suppose that $\mathcal{B}$ is a left balanced DG (E, A)-module. Define adjoint functors $F = F_{\mathcal{B}} = - \otimes_E^L \mathcal{B}$ and $G = G_{\mathcal{B}} = \mathrm{RHom}_A(\mathcal{B}, -)$, as above.*

(a) *Then $F_{\mathcal{B}}$ and $G_{\mathcal{B}}$ induce an equivalence of categories $\mathbb{A}_{\mathcal{B}} \cong \mathbb{B}_{\mathcal{B}}$. Furthermore, $E_E \in \mathbb{A}_{\mathcal{B}}$, $\mathcal{B}_A \in \mathbb{B}_{\mathcal{B}}$, and $F_{\mathcal{B}}(E_E) = \mathcal{B}_A$.*

(b) *There is an equivalence of triangulated categories*

$$\mathsf{triang}_E(E) \cong \mathsf{triang}_A(\mathcal{B}).$$

(c) *There is an equivalence of triangulated categories*

$$\mathsf{D}_{\mathrm{per}}(E) = \mathsf{thick}_E(E) \cong \mathsf{thick}_A(\mathcal{B}).$$

(d) *If $\mathcal{B}_A$ is small, then there is an equivalence of triangulated categories*

$$\mathsf{D}(E) \cong \mathsf{loc}_A(\mathcal{B}).$$

Proof. (a) The first assertion is Lemma 4.1. Without loss of generality, we assume that $\mathcal{B}$ is K-injective over $E^{\mathrm{op}} \otimes A$. Since $\mathcal{B}$ is left balanced,

$$E \to \mathrm{Hom}_A(\mathcal{B}, \mathcal{B}) \cong \mathrm{RHom}_A(\mathcal{B}, \mathcal{B}) = \mathrm{RHom}_A(\mathcal{B}, E \otimes^L_E \mathcal{B}) = GF(E)$$

is a quasi-isomorphism of DG A-modules. Hence $E \in \mathbb{A}_{\mathcal{B}}$. Clearly $F(E) = \mathcal{B}$ and $G(\mathcal{B}) = E$. Consequently, $\mathcal{B} \in \mathbb{B}_{\mathcal{B}}$.

(b,c) These follow from (a).

(d) By definition, F commutes with arbitrary colimits. If $\mathcal{B}$ is small, G commutes with arbitrary colimits. In this case, $\mathbb{A}_{\mathcal{B}}$ and $\mathbb{B}_{\mathcal{B}}$ have arbitrary colimits. Since $E \in \mathbb{A}_{\mathcal{B}}$, $\mathbb{A}_{\mathcal{B}} = \mathsf{D}(E)$. Since F is an equivalence, $\mathbb{B}_{\mathcal{B}} = \mathsf{loc}_A(\mathcal{B})$. $\square$

Now we consider two other functors $\widetilde{F}^{\mathcal{B}} = \mathrm{RHom}_{E^{\mathrm{op}}}(-, \mathcal{B}) : \mathsf{D}(E^{\mathrm{op}}) \to \mathsf{D}(A)$ and $\widetilde{G}^{\mathcal{B}} = \mathrm{RHom}_A(-, \mathcal{B}) : \mathsf{D}(A) \to \mathsf{D}(E^{\mathrm{op}})$. Both of them are contravariant; however, if we view them as $F^{\mathcal{B}} : \mathsf{D}(E^{\mathrm{op}}) \to \mathsf{D}(A)^{\mathrm{op}}$ and $G^{\mathcal{B}} : \mathsf{D}(A)^{\mathrm{op}} \to \mathsf{D}(E^{\mathrm{op}})$, then they become covariant. By Lemma 4.4(a), $(F^{\mathcal{B}}, G^{\mathcal{B}})$ is an adjoint pair. Let $\mathbb{A}^{\mathcal{B}}$ and $\mathbb{B}^{\mathcal{B}}$ be the Auslander and Bass classes associated to the pair $(F^{\mathcal{B}}, G^{\mathcal{B}})$.

Proposition 4.11. *Let A and E be DG algebras, and suppose that $\mathcal{B}$ is a left balanced DG (E, A)-module. Define adjoint functors $F = F^{\mathcal{B}}$ and $G = G^{\mathcal{B}}$, as above.*

(a) *Then $F^{\mathcal{B}}$ and $G^{\mathcal{B}}$ induce an equivalence of categories $\mathbb{A}^{\mathcal{B}} \cong \mathbb{B}^{\mathcal{B}}$. Furthermore, ${}_E E \in \mathbb{B}^{\mathcal{B}}$, $\mathcal{B}_A \in \mathbb{A}^{\mathcal{B}}$, and $F^{\mathcal{B}}({}_E E) = \mathcal{B}_A$.*

(b) *If $\mathcal{B}$ is also right balanced, then $A_A \in \mathbb{A}^{\mathcal{B}}$, ${}_E\mathcal{B} \in \mathbb{B}^{\mathcal{B}}$, and $F^{\mathcal{B}}({}_E\mathcal{B}) = A_A$.*

(c) *There is an equivalence of triangulated categories*

$$\mathsf{triang}_{E^{\mathrm{op}}}(E) \cong \mathsf{triang}_A(\mathcal{B})^{\mathrm{op}}.$$

If $\mathcal{B}$ is also right balanced, then

$$\mathsf{triang}_{E^{\mathrm{op}}}(E, \mathcal{B}) \cong \mathsf{triang}_A(A, \mathcal{B})^{\mathrm{op}}.$$

(d) *There is an equivalence of triangulated categories*

$$\mathsf{D}_{\mathrm{per}}(E^{\mathrm{op}}) = \mathsf{thick}_{E^{\mathrm{op}}}(E) \cong \mathsf{thick}_A(\mathcal{B})^{\mathrm{op}}.$$

If $\mathcal{B}$ is also right balanced, then

$$\mathsf{thick}_{E^{\mathrm{op}}}(E, \mathcal{B}) \cong \mathsf{thick}_A(A, \mathcal{B})^{\mathrm{op}}.$$

(e) *If ${}_E\mathcal{B}$ is small and $\mathcal{B}$ is right balanced, then there is an equivalence of triangulated categories*

$$\mathsf{D}_{\mathrm{per}}(E^{\mathrm{op}}) \cong \mathsf{thick}_A(A,\mathcal{B})^{\mathrm{op}} = \mathsf{thick}_A(\mathcal{B})^{\mathrm{op}}.$$

As a consequence, $A \in \mathsf{thick}_A(\mathcal{B})^{\mathrm{op}}$*.*

Proof. (a) The first assertion follows from Lemma 4.1. We may assume that $\mathcal{B}$ is K-injective over $E^{\mathrm{op}} \otimes A$. Since $\mathcal{B}$ is left balanced,

$$G^{\mathcal{B}} F^{\mathcal{B}}({}_E E) = \mathrm{RHom}_A(\mathrm{RHom}_{E^{\mathrm{op}}}(E,\mathcal{B}),\mathcal{B}) = \mathrm{RHom}_A(\mathcal{B},\mathcal{B}) \longleftarrow E$$

is a quasi-isomorphism. This shows that $E \in \mathbb{B}^{\mathcal{B}}$. Since $F^{\mathcal{B}}(E) = \mathcal{B}$, we have $\mathcal{B} \in \mathbb{A}^{\mathcal{B}}$.

(b) This is the right-hand version of (a).

(c,d,e) These follow from (a,b). □

Proposition 4.10 also implies the following easy fact.

Corollary 4.12. *Let M be an object in $\mathsf{D}^{\infty}(A)$ for some A_∞-algebra A. Then $\mathsf{thick}_A(M)$ is triangulated equivalent to $\mathsf{D}_{\mathrm{per}}(E)$ for some DG algebra E.*

Proof. By Proposition 3.1 we may assume that A is a DG algebra, and then we may replace $\mathsf{D}^{\infty}(A)$ by $\mathsf{D}(A)$. Hence we may assume that M is a right DG A-module.

Let $\mathcal{B}_A$ be a K-projective resolution of M and let $E = \mathrm{End}_A(\mathcal{B}_A)$. Then $\mathcal{B}$ is a left balanced (E,A)-bimodule. Note that $M \cong \mathcal{B}$ in $\mathsf{D}(A)$. The assertion follows from Proposition 4.10(c). □

5. Koszul equivalences and dualities

In the setting of classical Koszul duality [7], there is an equivalence between certain subcategories of the derived categories of a Koszul algebra A and of its Koszul dual; the subcategories consist of objects satisfying certain finiteness conditions. In this section, we explore the analogous results for non-Koszul algebras, DG algebras, and A_∞-algebras. The main results are Theorems 5.4 and 5.5 in the DG setting, and Theorems 5.7 and 5.8 in the A_∞ setting.

5.1. Koszul equivalence and duality in the DG case. Let A be an augmented DG algebra and let $E(A) = (B^{\infty}_{\mathrm{aug}}A)^{\sharp}$ be its Koszul dual, as defined in Section 2.2. The usual bar construction $B(A;A)$ [13, p. 269], where the second A is viewed as a DG A-bimodule, agrees with the A_∞-module version $B^{\infty}_{\mathrm{aug}}(A;A)$ from Section 3.2. By [13, Proposition 19.2(b)], $B(A;A)$ is a semifree resolution of the right A-module k. Thus to define derived functors we may replace k_A with $B(A;A)_A$.

The following lemma can be viewed as a dual version of [13, Proposition 19.2].

Lemma 5.1. *Let $\mathcal{B} = B(A; A)$ and let $E = E(A)$.*

(a) *The natural embedding $i : {}_E k \to {}_E\mathcal{B}$ is a quasi-isomorphism of left DG E-modules.*

(b) *If A is weakly Adams connected (Definition 2.1), then $\mathcal{B}$ is a K-injective DG left E-module.*

A left DG E-module is called *semiinjective* if it is an injective left graded E-module and a K-injective left DG E-module.

Proof. (a) By [13, Proposition 19.2(a)], the augmentations in BA and A define a quasi-isomorphism $\varepsilon \otimes \varepsilon : \mathcal{B}_A \to k_A$ of right DG A-modules. The map i is a quasi-isomorphism because the composition $k \xrightarrow{i} \mathcal{B} \xrightarrow{\varepsilon\otimes\varepsilon} k$ is the identity. It is easy to see that the map i is a left DG E-module homomorphism.

(b) Let $E^\sharp$ be the E-bimodule $\operatorname{Hom}_k(E, k)$. It follows from the adjunction formula in Lemma 4.2(b) that $E^\sharp$ is semiinjective as a left and a right DG E-module. If V is a finite-dimensional k-vector space, then $E^\sharp \otimes V \cong \operatorname{Hom}_k(E, V)$ and this is also semiinjective as a left and a right DG E-module.

Since $B(A; A)$ is locally finite, $E = (BA)^\sharp$ is locally finite and $E^\sharp = (BA^\sharp)^\sharp \cong BA$. Hence $B(A; {}_A k) \cong BA$ is a semiinjective left DG E-module. By induction one can easily show that if M is a finite-dimensional left DG A-module, then the bar construction $B(A; M)$ is a semiinjective left DG E-module. Since A is weakly Adams connected, $A = \varprojlim N_n$ where the $\{N_n\}_{n\geq 0}$ are finite-dimensional left DG A-modules. Since each N_n is finite-dimensional, we may further assume that the map $N_n \to N_{n-1}$ is surjective for all n. By the assertion just proved, $B(A; N_n)$ is a semiinjective left DG E-module for each n, as is $B(A; N_n/N_{n-1})$.

Since $A = \varprojlim N_n$ and since $B(A; A)$ is locally finite,

$$B(A; A) = \varprojlim B(A; N_n).$$

A result of Spaltenstein [33, Corollary 2.5] says that such an inverse limit of K-injectives is again K-injective, and this finishes the proof. (Spaltenstein's result is for inverse limits of K-injectives in the category of chain complexes over an abelian category, but the proof is formal enough that it extends to the category of DG modules over a DG algebra.) $\square$

Remark 5.2. By the above lemma, ${}_E\mathcal{B}$ is isomorphic to ${}_E k$ in $\mathsf{D}(E^{\mathrm{op}})$. By [13, Proposition 19.2(a)], $\mathcal{B}_A$ is isomorphic to k_A in $\mathsf{D}(A)$. However, $\mathcal{B}$ is not isomorphic to k in $\mathsf{D}(E^{\mathrm{op}} \otimes A)$ in general.

Lemma 5.3. *Let $\mathcal{B}$ be the right DG A-module $B(A; A)$ and let $C = \operatorname{End}_A(\mathcal{B})$.*

(a) *$\mathcal{B}$ is a left balanced (C, A)-bimodule.*

(b) *If $E := E(A)$ is locally finite, then $\mathcal{B}$ is a left balanced (E, A)-bimodule via the natural isomorphism $E \to C$. As a consequence,*

$$HE \cong H\,\mathrm{RHom}_A(k, k).$$

(c) *If A is weakly Adams connected, then $\mathcal{B}$ is a right balanced (E, A)-bimodule.*

Proof. (a) Since $\mathcal{B}$ is semifree over A, the assertion follows from Lemma 4.7.

(b) The first assertion follows from Lemma 4.7 and the fact that $E \to \operatorname{End}_A(\mathcal{B})$ is a quasi-isomorphism [13, Ex 4, p. 272].

Since $\mathcal{B}_A$ is a K-projective resolution of k_A,

$$HE \cong H \operatorname{End}_A(\mathcal{B}) \cong H \operatorname{RHom}_A(k, k).$$

(c) By Lemma 5.1(b), $\mathcal{B}$ is a K-injective left DG E-module. To show that $\mathcal{B}$ is right balanced, we must show that the canonical map $\phi : A^{\text{op}} \to \operatorname{End}_{E^{\text{op}}}(\mathcal{B})$ is a quasi-isomorphism. This canonical map sends $a \in A$ to the endomorphism $y \mapsto ya$. Since ${}_E\mathcal{B}$ is K-injective, $H \operatorname{End}_{E^{\text{op}}}(\mathcal{B}) \cong H \operatorname{RHom}_{E^{\text{op}}}(k, k)$. By part (b),

$$H \operatorname{RHom}_{E^{\text{op}}}(k, k) \cong H(E(E))^{\text{op}} \cong HA^{\text{op}},$$

where the last isomorphism follows from Theorem 2.4. Therefore, since HA is locally finite, it suffices to show that $H\phi$ is injective. Let $a \in A^{\text{op}}$ be a cocycle such that $\phi(a) = 0$ in $H \operatorname{End}_{E^{\text{op}}}(\mathcal{B})$. Then $a \neq 1$ and there is an $f \in \operatorname{End}_{E^{\text{op}}}(\mathcal{B})$ such that $\phi(a) = d(f)$. Applying this equation to $x = [\,] \otimes 1 \in \mathcal{B} = B(A; A)$, we obtain

$$\begin{aligned}
[\,] \otimes a &= \phi(a)(x) = d(f)(x) \\
&= (d \circ f - (-1)^{\deg_1 f} f \circ d)(x) \\
&= d \circ f([\,] \otimes 1) \pm f \circ d([\,] \otimes 1).
\end{aligned}$$

Since f is a left E-module homomorphism, $f([\,] \otimes 1) = [\,] \otimes w$ for some $w \in A$. By definition, $d([\,] \otimes 1) = 0$. Therefore

$$[\,] \otimes a = d \circ f([\,] \otimes 1) = d([\,] \otimes w) = [\,] \otimes dw,$$

and hence $a = dw$ as required. □

Here is a version of [7, 1.2.6] for DG algebras.

Theorem 5.4. *Let A be an augmented DG algebra and let $E = E(A)$ be the Koszul dual of A. Assume that E is locally finite. The functors $\operatorname{RHom}_A(k, -)$ and $- \otimes^L_E k$ induce the following equivalences.*

(a) *The category $\mathsf{triang}_A(k)$ is triangulated equivalent to $\mathsf{triang}_E(E)$.*
(b) *The category $\mathsf{thick}_A(k)$ is triangulated equivalent to $\mathsf{D}_{\text{per}}(E)$.*
(c) *Suppose that k_A is small in $\mathsf{D}(A)$. Then $\mathsf{loc}_A(k)$ is triangulated equivalent to $\mathsf{D}(E)$.*

Proof. Note that $\mathcal{B} = B(A; A) \cong k_A$ as a right DG A-module. Then the assertions follow from Proposition 4.10 and Lemma 5.3(b). □

Theorem 5.5. *Let A be an augmented DG algebra and let $E = E(A)$ be the Koszul dual of A. Assume that A is weakly Adams connected. The functors $\operatorname{RHom}_A(-, k)$ and $\operatorname{RHom}_{E^{\text{op}}}(-, k)$ induce the following equivalences.*

(a) $\mathsf{triang}_A(k,A)^{\mathrm{op}}$ *is triangulated equivalent to* $\mathsf{triang}_{E^{\mathrm{op}}}(E,k)$.
(b) $\mathsf{thick}_A(k,A)^{\mathrm{op}}$ *is triangulated equivalent to* $\mathsf{thick}_{E^{\mathrm{op}}}(E,k)$.
(c) *Suppose that* k_A *is small in* $\mathsf{D}(A)$. *Then*

$$\mathsf{D}_{\mathrm{per}}(A)^{\mathrm{op}} = \mathsf{thick}_A(k,A)^{\mathrm{op}} \cong \mathsf{thick}_{E^{\mathrm{op}}}(E,k) = \mathsf{thick}_{E^{\mathrm{op}}}(k).$$

Proof. This follows from Proposition 4.11 and Lemmas 5.1 and 5.3. □

Corollary 5.6. *Let A be a weakly Adams connected augmented DG algebra and let $E = E(A)$ be its Koszul dual. If k_A is small in $\mathsf{D}(A)$, then HE is finite-dimensional.*

Proof. By Theorem 5.5(c), E is in the thick subcategory generated by k, and every object in $\mathsf{thick}_{E^{\mathrm{op}}}(k)$ has finite-dimensional homology. □

See Corollary 6.2 for a related result for DG algebras, and Corollaries 5.9 and 7.2 for similar results about A_∞-algebras.

5.2. Koszul equivalence and duality in the A_∞ case. Now suppose that A is an A_∞-algebra. By Proposition 1.14, A is quasi-isomorphic to the DG algebra UA, so by Proposition 3.1, the derived category $\mathsf{D}^\infty(A)$ is equivalent to $\mathsf{D}(UA)$. We can use this to prove the following, which is a version of [7, 1.2.6] for A_∞-algebras.

Theorem 5.7. *Let A be an augmented A_∞-algebra and let $E = E(A)$ be the Koszul dual of A. Assume that A is strongly locally finite (Definition 2.1).*
(a) *The category* $\mathsf{triang}^\infty_A(k)$ *is triangulated equivalent to* $\mathsf{triang}^\infty_E(E)$.
(b) *The category* $\mathsf{thick}^\infty_A(k)$ *is triangulated equivalent to* $\mathsf{D}^\infty_{\mathrm{per}}(E)$.
(c) *Suppose that* k_A *is small in* $\mathsf{D}^\infty(A)$. *Then* $\mathsf{loc}^\infty_A(k)$ *is triangulated equivalent to* $\mathsf{D}^\infty(E)$.

Proof. We can replace A by UA and E by $E(UA) = E(E(E(A)))$. The assertions follow from Proposition 3.1 and Theorem 5.4. □

Similarly, Proposition 3.1 combined with 5.5 give the following.

Theorem 5.8. *Let A be an augmented A_∞-algebra and let $E = E(A)$ be the Koszul dual of A. Assume that A is strongly locally finite.*
(a) $\mathsf{triang}^\infty_A(k,A)^{\mathrm{op}}$ *is triangulated equivalent to* $\mathsf{triang}^\infty_{E^{\mathrm{op}}}(E,k)$.
(b) $\mathsf{thick}^\infty_A(k,A)^{\mathrm{op}}$ *is triangulated equivalent to* $\mathsf{thick}^\infty_{E^{\mathrm{op}}}(E,k)$.
(c) *Suppose that* k_A *is small in* $\mathsf{D}^\infty(A)$. *Then*

$$\mathsf{D}^\infty_{\mathrm{per}}(A)^{\mathrm{op}} = \mathsf{thick}^\infty_A(k,A)^{\mathrm{op}} \cong \mathsf{thick}^\infty_{E^{\mathrm{op}}}(E,k) = \mathsf{thick}^\infty_{E^{\mathrm{op}}}(k).$$

Just as Theorem 5.5 implied Corollary 5.6, this result implies the following.

Corollary 5.9. *Let A and E be as in Theorem 5.8. If k_A is small in $\mathsf{D}^\infty(A)$, then HE is finite-dimensional.*

See Corollary 7.2 for the converse of Corollary 5.9.

Corollary 5.10. *Via the equivalence in Theorem* 5.8*, there is an isomorphism of k-vector spaces*

$$H^i_j(\mathrm{RHom}_A(k, A)) \cong H^{-i}_{-j}(\mathrm{RHom}_{E^{\mathrm{op}}}(k, E))$$

for all i, j.

Proof. Again we may assume that A and E are DG algebras. The functor $G^{\mathcal{B}}$ in Proposition 4.11 is defined as $G^{\mathcal{B}} = \mathrm{RHom}_A(-, \mathcal{B}) \cong \mathrm{RHom}_A(-, k)$, and changes S to S^{-1} and Σ to Σ^{-1}; hence the assertion follows from the fact $G^{\mathcal{B}}(k_A) = {}_E E$ and $G^{\mathcal{B}}(A_A) = {}_E k$. $\square$

Remark 5.11. One might hope to prove Theorems 5.7 and 5.8 directly, working in the category $\mathsf{D}^{\infty}(A)$ rather than $\mathsf{D}(UA)$. Keller [20, 6.3] and Lefèvre-Hasegawa [23, 4.1.1] have described the appropriate functors: A_∞-versions of $\mathrm{RHom}_A(-, -)$ and $- \otimes^L_E -$, which they write as $\overset{\infty}{\mathsf{Hom}}{}^{\bullet}_A(-, -)$ and $-\underset{E}{\overset{\infty}{\otimes}}-$. Although we fully expect these to satisfy all of the required properties (such as adjointness), it was easier to use the more standard results in the DG setting.

6. Minimal semifree resolutions

In this short section we consider the existence of a minimal semifree resolution of a DG module over a DG algebra. The main result is Theorem 6.1; this result is needed in several places. There are similar results in the literature — see, for example [4, Section 1.11] or [11, Lemma A.3] — but they require that A be connected graded with respect to the first (non-Adams) grading. We need to use this in other situations, though, so we include a detailed statement and proof.

We say that A is *positively connected graded* in the second (Adams) grading if $A^*_{<0} = 0$ and $A^*_0 = k$; *negatively connected graded* in the Adams grading is defined similarly. Write $\mathfrak{m}$ for the augmentation ideal of A; then a semifree resolution $F \to M$ of a module M is called *minimal* if $d_F(F) \subset F\mathfrak{m}$.

Theorem 6.1. *Let A be a DG algebra and let M be a right DG A-module.*

(a) *Assume that A is positively connected graded in the second grading and that $HM^*_{\leq n} = 0$ for some n, or that A is negatively connected graded in the second grading and that $HM^*_{\geq n} = 0$ for some n. Then M has a minimal semifree resolution $L \to M$ with $L^*_{\leq n} = 0$ (respectively $L^*_{\geq n} = 0$).*

(b) *Assume further that HA and HM are both bounded on the same side with respect to the first grading: assume that for each j, there is an m so that $(HA)^{\leq m}_j = 0$ and $(HM)^{\leq m}_j = 0$, or $(HA)^{\geq m}_j = 0$ and $(HM)^{\geq m}_j = 0$. If HA and HM are locally finite (respectively, locally finite with respect to the second grading), then so is L.*

Proof. Without loss of generality, we assume that $A^*_{<0} = 0$ and $HM^*_{\leq n} = 0$ for some n. After an Adams shift we may further assume that $n = -1$, that is, $HM = HM^*_{\geq 0}$. We will construct a sequence of right DG A-modules $\{L_u\}_{u\geq 0}$ with the following properties:

(1) $0 = L_{-1} \subset L_0 \subset \cdots \subset L_u \subset \cdots$.
(2) L_u/L_{u-1} is a free DG A-module generated by a cycles of Adams-degree u.
(3) $L_u \otimes_A k$ has a trivial differential; that is, $d_{L_u}(L_u) \subset L_u\mathfrak{m}$.
(4) There is a morphism of DG A-modules $\varepsilon_u : L_u \to M$ such that the kernel and coker of $H(\varepsilon_u)$ have Adams degree at least $u + 1$.

If A and M satisfy the hypotheses in part (b), then each L_u will also satisfy

(5) L_u is locally finite (respectively, locally finite with respect to the second grading).

Let $L_{-1} = 0$. We proceed to construct L_u inductively for $u \geq 0$, so suppose that $\{L_{-1}, L_0, \cdots, L_u\}$ have been constructed and satisfy (1)–(4), and if relevant, (5).

Consider the map $H(\varepsilon_u) : HL_u \to HM$; let C be its cokernel and let K be its kernel. We will focus on the parts of these in Adams degree $u + 1$. Choose an embedding i of C_{u+1} into the cycles in M, and let P_u be the free DG A-module $C_{u+1}\otimes A$ on C_{u+1}, equipped with a map $f : P_u \to M$, sending $x\otimes 1$ to $i(x)$ for each $x \in C_{u+1}$. Since A is positively connected graded in the Adams grading, the map f induces an isomorphism in homology in Adams degrees up to and including $u+1$. Similarly, let Q_u be the free DG A-module on K_{u+1}, mapping to L_u by a map $\widetilde{g}$ inducing a homology isomorphism in degrees less than or equal to $u + 1$. Then let L_{u+1} be the mapping cone of

$$Q_u \xrightarrow{g} L_u \oplus P_u,$$

where g maps Q_u to the first summand by the map $\widetilde{g}$. Since Q_u is free and since the composite $(\varepsilon_u + f)g$ induces the zero map on homology, this composite is null-homotopic. Therefore there is a map $\varepsilon_{u+1} : L_{u+1} \to M$. In more detail, since L_{u+1} is the mapping cone of $g : Q_u \to J_u := L_u \oplus P_u$, it may be written as $L_{u+1} = S(Q_u) \oplus J_u$, with differential given by

$$d(q, l) = (-d_{Q_u}(q), g(q) + d_{J_u}(l)).$$

The null-homotopy gives an A-module homomorphism $\theta : Q_u \to M$ of degree $(-1, 0)$ such that

$$\theta d_{Q_u} + d_M\theta = \delta_u g$$

where $\delta_u = \varepsilon_u + f$. The A-module homomorphism $\varepsilon_{u+1} : L_{u+1} \to M$ is defined by

$$\varepsilon_{u+1}(q, l) = \theta(q) + \delta_u(l) \quad \forall q \in Q_u \text{ and } \ l \in J_u.$$

One can check that ε_{u+1} commutes with the differentials and hence is a morphism of DG A-modules. The morphism ε_{u+1} is an extension of $\varepsilon_u + f$, hence ε_u is the restriction of ε_{u+1} to L_u.

Now we claim that the kernel and cokernel of $H(\varepsilon_{u+1})$ are in Adams degree at least $u+2$. There is a long exact sequence in homology

$$\cdots \to H^i_j(Q_u) \xrightarrow{H(g)} H^i_j(L_u \oplus P_u) \to H^i_j(L_{u+1}) \xrightarrow{\delta} H^{i+1}_j(Q_u) \to \cdots .$$

In Adams degrees less than $u+1$, Q_u and P_u are zero, so L_u and L_{u+1} are isomorphic. In Adams degree $u+1$, $H^*(Q_u)$ maps isomorphically to K_u, a vector subspace of $H(L_u \oplus P_u)$, so the boundary map δ is zero, and the above long exact sequence becomes short exact. Indeed, there is a commutative diagram, where the rows are short exact:

$$\begin{tikzcd}
0 \arrow[r] & H^i(Q_u)_{\le u+1} \arrow[r] \arrow[d] & H^i(L_u \oplus P_u)_{\le u+1} \arrow[r] \arrow[d] & H^i(L_{u+1})_{\le u+1} \arrow[r] \arrow[d, "H(\varepsilon_{u+1})"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & H^i(M)_{\le u+1} \arrow[r] & H^i(M)_{\le u+1} \arrow[r] & 0
\end{tikzcd}$$

The snake lemma immediately shows that $H(\varepsilon_{u+1})$ has zero kernel and zero cokernel in Adams degree $\le u+1$, as desired. This verifies property (4) for L_{u+1}.

With property (4), and the fact that L_i/L_{i-1} has a basis of cycles in Adams degree i, (1) and (2) are easy to see. To see (3) we use induction on u. It follows from the construction and induction that $d_{L_u}(L_u) \subset \mathfrak{m}L_u + L_{u-1}$. Since the semibasis of L_{u-1} has Adams degree no more than $u-1$ and the semibasis of $P_u \oplus Q_u$ has Adams degree u, we see that $d_{L_u}(L_u) \subset \mathfrak{m}L_u + \mathfrak{m}L_{u-1} = \mathfrak{m}L_u$.

Let L be the direct limit $\varinjlim L_u$. Then L is semifree and there is a map $\phi : L \to M$ such that the kernel and cokernel of $H(\phi)$ are zero. Such an L is a semifree resolution of M. Property (3) implies that L is minimal.

If the hypotheses of part (2) are satisfied, then the construction of L_u shows that (5) holds. Since $L^i_j = (L_u)^i_j$ for $u \gg 0$, L is also locally finite (respectively, locally finite with respect to the Adams grading). □

We are often interested in the complex $\mathrm{RHom}_A(k,k)$ or in its homology, namely, the Ext-algebra, $\mathrm{Ext}^*_A(k,k)$. As noted in Section 4.2, to compute this, we replace k by a K-projective resolution P, and then $\mathrm{RHom}_A(k,k) = \mathrm{Hom}_A(P,k)$. Since semifree implies K-projective, we can use a minimal semifree resolution, as in the theorem. The construction of L gives the following: for each u, there is a short exact sequence of DG A-modules

$$0 \to L_{u-1} \to L_u \to L_u/L_{u-1} \to 0.$$

where L_u/L_{u-1} is a free DG A-module, and this leads to a short exact sequence

$$0 \to \mathrm{Hom}_A(L_u/L_{u-1}, k) \to \mathrm{Hom}_A(L_u, k) \to \mathrm{Hom}_A(L_{u-1}, k) \to 0.$$

We see that $\mathrm{Hom}_A(L_u/L_{u-1}, k)_i = 0$ when $i < u$, and therefore

$$\mathrm{Hom}_A(L_u, k)_u \cong \mathrm{Hom}_A(L_u/L_{u-1}, k)_u \cong (B^\sharp)_{-u},$$

where B is a graded basis for the free DG module L_u/L_{u-1}. Note that B is concentrated in degrees $(*, u)$, so its graded dual $B^\sharp$ is in degrees $(*, -u)$. Again by the short exact sequence, by induction on u, $\mathrm{Hom}_A(L_{u-1}, k)_i = 0$ when $i \geq u$, so we see that

$$\mathrm{RHom}_A(k,k)_u \cong \mathrm{Hom}_A(L,k)_u \cong \mathrm{Hom}_A(L_u,k)_u \cong (B^\sharp)_{-u}.$$

Furthermore, since L_u/L_{u-1} is free on a basis of cycles, or alternatively because the resolution L is minimal, we see that

$$\mathrm{Ext}_A(k,k)_u \cong (B^\sharp)_{-u}.$$

This leads to the following corollary; see Corollary 5.6 for a related result.

Corollary 6.2. *Let A be a DG algebra which is connected graded, either positively or negatively, in the second grading, and let $E = E(A)$ be its Koszul dual.*

(a) *Then $HE = \mathrm{Ext}^*_A(k,k)$ is finite-dimensional if and only if k_A is small in $\mathsf{D}(A)$.*
(b) *If HE is finite-dimensional (or if k_A is small in $\mathsf{D}(A)$), then HA is Adams connected.*

Proof. (a) If $HE = \mathrm{Ext}^*_A(k,k)$ is finite-dimensional, then by the above computation, the minimal semifree resolution L of k is built from finitely many free pieces, and so L is a perfect complex: it is in $\mathsf{thick}_A(k)$. Therefore k_A is small in $\mathsf{D}(A)$.

Conversely, if k_A is small, then it is isomorphic in $\mathsf{D}(A)$ to a perfect complex, and we claim that if X is a perfect complex, then $H\,\mathrm{RHom}_A(X,k)$ is finite-dimensional as a vector space: this is true if $X = A$, and therefore it is true for every object in the thick subcategory generated by A. Therefore k_A small implies that HE is finite-dimensional.

(b) Now suppose that HE is finite-dimensional. Without loss of generality, suppose that A is positively graded connected in the second grading. We claim that for each i, $(HA)^*_i$ is finite-dimensional.

Note that in the construction of the minimal semifree resolution for k, the first term L_0 is equal to A, and the map $L_0 \to k$ is the augmentation. Consider the short exact sequence

$$0 \to L_{u-1} \to L_u \to L_u/L_{u-1} \to 0,$$

for $u \geq 1$. Since L_u/L_{u-1} is free on finitely many classes in Adams degree u, then in Adams degree i, $H(L_u/L_{u-1})_i$ is isomorphic to a finite sum of copies of HA_{i-u}. Therefore if $i \leq u$, then this is finite-dimensional. Therefore when $i \leq u$, $H(L_{u-1})_i$ is finite-dimensional if and only if $H(L_u)_i$ is. For i fixed and u sufficiently large, $H(L_u)_i$ stabilizes and gives $H(L)_i$. But $HL \cong k$, since L is a semifree resolution of k. Thus $H(L_0)_i = (HA)_i$ is finite-dimensional for each i, as desired. $\square$

7. Towards classical Koszul duality

In this section we recover the classical version of the Koszul duality given by Beilinson–Ginzburg–Soergel [7]. First we give some useful results about A_∞-algebras with finite-dimensional cohomology, and then we use these to recover classical Koszul duality, in Theorem 7.5.

7.1. Finite-dimensional A_∞-algebras. Let A be an A_∞-algebra. Let $\mathsf{D}^\infty_{\mathrm{fd}}(A)$ denote the thick subcategory of $\mathsf{D}^\infty(A)$ generated by all A_∞-modules M over A such that HM is finite-dimensional.

Lemma 7.1. *Let A be a strongly locally finite A_∞-algebra. For parts* (b) *and* (c)*, assume that A is Adams connected and that HA is finite-dimensional.*

(a) $\mathsf{thick}^\infty_A(k) = \mathsf{D}^\infty_{\mathrm{fd}}(A)$.
(b) *A is quasi-isomorphic to a finite-dimensional Adams connected DG algebra.*
(c) $\mathsf{thick}^\infty_A(k) = \mathsf{D}^\infty_{\mathrm{fd}}(A) \supseteq \mathsf{D}^\infty_{\mathrm{per}}(A)$ *and* $\mathsf{loc}^\infty_A(k) = \mathsf{D}^\infty(A)$.

Proof. (a) Clearly $\mathsf{thick}^\infty_A(k) \subset \mathsf{D}^\infty_{\mathrm{fd}}(A)$. To show the converse, we may replace A by the DG algebra $UA \cong E(E(A))$ (Proposition1.14). Since A is strongly locally finite, so is $E(E(A))$, by Lemma 2.2. So we may assume that A is a strongly locally finite DG algebra. In this case every 1-dimensional right DG A-module M is isomorphic to a shift of the trivial module k. As a consequence, $M \in \mathsf{thick}^\infty_A(k)$. Induction shows that $M \in \mathsf{thick}^\infty_A(k)$ if M is finite-dimensional. If M is a right DG A-module with HM being finite-dimensional, then the minimal semifree resolution L of M is Adams locally finite by Theorem 6.1. Thus M is quasi-isomorphic to a finite-dimensional right DG A-module by truncation: replace L by $\bigoplus_{-N\le s\le N} L^*_s$ for N sufficiently large. Therefore M is in $\mathsf{thick}^\infty_A(k)$. This shows that $\mathsf{thick}^\infty_A(k) = \mathsf{D}^\infty_{\mathrm{fd}}(A)$.

(b) By Theorem 2.4, A is quasi-isomorphic to $B := E(E(A))$. Since A is Adams connected, so is B. Since $HA \cong HB$, $H(B^*_{\ge n}) = 0$ for some n. Hence B is quasi-isomorphic to $C := B/B^*_{\ge n}$. Therefore A is quasi-isomorphic to the Adams connected finite-dimensional DG algebra C.

(c) By part (b) we may assume that A is finite-dimensional, which implies that A is in $\mathsf{D}^\infty_{\mathrm{fd}}(A) = \mathsf{thick}^\infty_A(k) \subseteq \mathsf{loc}^\infty_A(k)$. Therefore

$$\mathsf{loc}^\infty_A(A) = \mathsf{D}^\infty(A) \subseteq \mathsf{loc}^\infty_A(k) \subseteq \mathsf{D}^\infty(A).$$

This proves the last statement. □

Corollary 7.2. *Let A be an augmented A_∞-algebra and let $E = E(A)$.*

(a) *Suppose that A is strongly locally finite. k_A is small in $\mathsf{D}^\infty(A)$ if and only if HE is finite-dimensional.*
(b) *Suppose that A is Adams connected. If k_A is small in $\mathsf{D}^\infty(A)$* (*or if HE is finite-dimensional*), *then $\mathsf{D}^\infty_{\mathrm{per}}(A) \cong \mathsf{D}^\infty_{\mathrm{fd}}(E)$.*

Proof. (a) If k_A is small, then by Corollary 5.9, HE is finite-dimensional.

Conversely, suppose that HE is finite-dimensional. By Proposition 1.14, A is quasi-isomorphic to the augmented DG algebra $UA \cong E(E(A))$, so by Proposition 3.1, k_A is small in $\mathsf{D}^\infty(A)$ if and only if k_{UA} is small in $\mathsf{D}(UA)$. According to Corollary 6.2, k_{UA} is small if and only if $HE(UA)$ is finite-dimensional. Since A is strongly locally finite, so is E, and we have a quasi-isomorphism $E(E(E)) \cong E$. This means that $HE(UA) \cong HE(A) = HE$.

(b) This follows from Theorem 5.7(b) (switching A and E) and Lemma 7.1(a) (for E). (Since A is Adams connected, then by Lemma 2.2, E is Adams connected and strongly locally finite, so these results hold for E.) $\square$

Part (b) is Theorem B from the introduction.

7.2. Twisting the grading. To recover the classical derived equivalences induced by Koszul duality, we need to twist the grading of $E(A)$. Recall that if A is Adams connected and is concentrated in degrees $\{0\} \times \mathbb{N}$, then $E(A)$ is in degrees $\{(n, -n) | n \geq 0\}$. The aim of this section is to change the grading of $E(A)$, to put it in degrees $\{0\} \times \mathbb{N}$.

Let A be a DG algebra with zero differential (there are some problems if $d \neq 0$). Let $\overline{A}$ be the DG algebra with zero differential such that:

(a) $\overline{A} = A$ as ungraded associative algebras.

(b) $\overline{A}^i_j = A^{i+j}_{-j}$ (so $A^i_j = \overline{A}^{i+j}_{-j}$).

Note that the sign before j is not essential; for example, we can define a different $\overline{A}$ by $\overline{A}^i_j = A^{i-j}_j$ or $\overline{A}^i_j = A^{i-j}_{-j}$ or $\overline{A}^i_j = A^{i+j}_{+j}$. Since A is a DG algebra with zero differential, so is $\overline{A}$. If A is Koszul, then $A^! = H\overline{E(A)} = \overline{HE(A)}$ which lives in degrees $\{0\} \times \mathbb{Z}$.

Remark 7.3. If A is a DG algebra with nonzero differential d, then we don't know how to make $\overline{A}$ into a DG algebra naturally. That is, we don't see a good way of defining a differential for $\overline{A}$.

For any right DG A-module M, we define a corresponding right DG $\overline{A}$-module $\overline{M}$ by:

(a) $\overline{M} = M$ as ungraded right A-modules.

(b) $\overline{M}^i_j = M^{i+j}_{-j}$.

One can easily check that $\overline{M}$ with the differential of M is a right DG $\overline{A}$-module.

The following is routine.

Lemma 7.4. *The assignment $M \mapsto \overline{M}$ defines an equivalence between $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, \overline{A}$ which induces a triangulated equivalence between $\mathsf{D}(A)$ and $\mathsf{D}(\overline{A})$. If further A is finite-dimensional, then $\mathsf{D}_{\mathrm{fd}}(A) \cong \mathsf{D}_{\mathrm{fd}}(\overline{A})$.*

We now reprove [7, Theorem 2.12.6] (in a slightly more general setting). Let $\mathsf{D}^\infty_{\mathrm{fg}}(A)$ denote the thick subcategory of $\mathsf{D}^\infty(A)$ generated by all A_∞-modules M over A such that HM is finitely generated over HA. If A is a

DG algebra, $\mathsf{D}_{\mathrm{fg}}(A)$ is defined in a similar way. Note that $\mathsf{D}^{\infty}_{\mathrm{fg}}(A) = \mathsf{D}^{\infty}_{\mathrm{fd}}(A)$ if and only if HA is finite-dimensional.

Theorem 7.5. *Suppose A is a connected graded finite-dimensional Koszul algebra. Let $A^! = \overline{HE(A)}$.*

(a) $\mathsf{D}(A) \cong \mathsf{loc}_{A^!}(k)$.

(b) *If $A^!$ is right noetherian, then* $\mathsf{D}_{\mathrm{fg}}(A) = \mathsf{D}_{\mathrm{fd}}(A) \cong \mathsf{D}_{\mathrm{per}}(A^!) = \mathsf{D}_{\mathrm{fg}}(A^!)$.

Proof. Note that $A^!$ is quasi-isomorphic to $\overline{E(A)}$. It follows from Lemma 7.4 that we can replace $A^!$ by $E := E(A)$

(a) This follows from Theorem 5.4(b) (switching A and E).

(b) By Corollary 7.2(b) (again switching A and E), $\mathsf{D}_{\mathrm{per}}(E) \cong \mathsf{D}_{\mathrm{fd}}(A)$. Since A is finite-dimensional, $\mathsf{D}_{\mathrm{fd}}(A) = \mathsf{D}_{\mathrm{fg}}(A)$. Note that $A^!$ is concentrated in degrees $\{0\} \times \mathbb{N}$. Since $A^!$ is right noetherian and has finite global dimension (because A is finite-dimensional Koszul), it is clear that $\mathsf{D}_{\mathrm{per}}(A^!) = \mathsf{D}_{\mathrm{fg}}(A^!)$. □

8. Some examples

Example 8.1. Let $\Lambda = \Lambda(x_1, x_2)$ with $\deg x_i = (0, i)$. As noted in Example 2.8, this is not a classical Koszul algebra; nonetheless, it satisfies many of the same derived equivalences as Koszul algebras. Let $\Lambda^!$ denote the Ext algebra $\mathrm{Ext}^*_\Lambda(k, k) \cong k[y_1, y_2]$, where $\deg y_i = (1, -i)$; this is an A_∞-algebra with no higher multiplications. Then $E(\Lambda)$ is quasi-isomorphic to the associative algebra $\Lambda^!$, so there is a triangulated equivalence $\mathsf{D}^\infty(E(\Lambda)) \cong \mathsf{D}(\Lambda^!)$. Similarly, we have $\mathsf{D}^\infty(\Lambda) \cong \mathsf{D}(\Lambda)$. Since the homology of $E(\Lambda^!) \cong E(E(\Lambda))$ is isomorphic to Λ, which is finite-dimensional, the trivial module $k_{\Lambda^!}$ is small in $\mathsf{D}(\Lambda^!)$. So Theorems 5.7 and 5.8 give triangulated equivalences

$$\begin{aligned} &\mathsf{thick}_\Lambda(k) \cong \mathsf{D}_{\mathrm{per}}(\Lambda^!), && \mathsf{thick}_\Lambda(k, \Lambda)^{\mathrm{op}} \cong \mathsf{thick}_{\Lambda^!}(\Lambda^!, k), \\ &\mathsf{thick}_{\Lambda^!}(k) \cong \mathsf{D}_{\mathrm{per}}(\Lambda), && \mathsf{D}_{\mathrm{per}}(\Lambda^!)^{\mathrm{op}} \cong \mathsf{thick}_\Lambda(k), \\ &\mathsf{loc}_{\Lambda^!}(k) \cong \mathsf{D}(\Lambda). \end{aligned}$$

Compare to the classical Koszul equivalences of Theorem 7.5: part (b) of that theorem is the first of these equivalences, while part (a) of the theorem is the last of these.

Slightly more generally, the results of Theorem 7.5 hold for exterior algebras on finitely many generators, as long as they are graded so as to be Adams connected: $\Lambda(x_1, \dots, x_n)$, graded by setting $\deg x_i = (a_i, b_i)$ with each b_i positive, or each b_i negative.

Example 8.2. We fix an integer $p \geq 3$ and define two A_∞-algebras, $B(0)$ and $B(p)$, each with $m_1 = 0$. As associative algebras, they are both isomorphic to $\Lambda(y) \otimes k[z]$ with $\deg y = (1, -1)$ and $\deg z = (2, -p)$. The algebra $B(0)$ has no higher multiplications, while $B(p)$ has a single nonzero higher multiplication m_p. This map m_p satisfies $m_p(y^{\otimes p}) = z$; more generally,

$m_p(a_1 \otimes \cdots \otimes a_p)$ is zero unless each a_i has the form $a_i = yz^{j_i}$ for some $j_i \geq 0$, and

$$m_p(yz^{j_1} \otimes \cdots \otimes yz^{j_p}) = z^{1+\sum j_i}.$$

See [24, Example 3.5] for more on $B(p)$.

We claim that $k_{B(0)}$ is not small in $\mathsf{D}^\infty(B(0)) \cong \mathsf{D}(B(0))$, while $k_{B(p)}$ is small in $\mathsf{D}^\infty(B(p))$. The Ext algebra $\operatorname{Ext}^*_{B(0)}(k,k)$ is isomorphic to the associative algebra $k[u] \otimes \Lambda(v)$ with $\deg u = (0,1)$ and $\deg v = (1,p)$. In particular, this algebra is not finite-dimensional, so by Corollary 7.2, $k_{B(0)}$ is not small in $\mathsf{D}(B(0))$. In contrast, by [24, Proposition 12.6], the Koszul dual of $B(p)$ is A_∞-isomorphic to the associative algebra $A(p) = k[x]/(x^p)$, where $\deg x = (0,1)$. Since $A(p)$ is finite-dimensional, $k_{B(p)}$ is small in $\mathsf{D}^\infty(B(p))$. This verifies the claim.

Part III. Applications in ring theory

9. The Artin–Schelter condition

In this section we prove Corollaries D, E and F. We start by discussing Artin–Schelter regularity, for both associative algebras and A_∞-algebras. The Eilenberg–Moore spectral sequence is a useful tool for connecting results about modules over HA to modules over A, if A is a DG algebra or an A_∞-algebra. Then we discuss Frobenius algebras and prove Corollary D, and we discuss dualizing complexes and prove Corollary E. At the end of the section, we prove Corollary F.

9.1. Artin–Schelter regularity.

Definition 9.1. Let R be a connected graded algebra.

(a) R is called *Gorenstein* if $\operatorname{injdim} R_R = \operatorname{injdim} {}_R R < \infty$.
(b) R is called *Artin–Schelter Gorenstein* if R is Gorenstein of injective dimension d and there is an integer l such that

$$\operatorname{Ext}^i_R(k,R) \cong \operatorname{Ext}^i_{R^{\mathrm{op}}}(k,R) \cong \begin{cases} 0 & i \neq d, \\ \Sigma^l(k) & i = d. \end{cases}$$

(c) R is called *Artin–Schelter regular* if R is Artin–Schelter Gorenstein and has global dimension d.

Artin–Schelter regular algebras have been used in many ways in noncommutative algebraic geometry.

Now we consider analogues for A_∞-algebras. When A is an unbounded A_∞-algebra, there is no good definition of global or injective dimension, so we only consider a version of condition (b) in Definition 9.1.

Definition 9.2. Let A be an augmented A_∞-algebra and let k be the trivial module.

(a) We say A satisfies the *right Artin–Schelter condition* if there are integers l and d such that
$$\operatorname{Ext}^i_A(k,A) \cong \begin{cases} 0 & i \neq d, \\ \Sigma^l(k) & i = d. \end{cases}$$
If further k_A is small in $\mathsf{D}^\infty(A)$, then A is called *right A_∞-Artin–Schelter regular*, or just *right Artin–Schelter regular*, if the context is clear.

(b) We say A satisfies the *left Artin–Schelter condition* if there are integers l and d such that
$$\operatorname{Ext}^i_{A^{\mathrm{op}}}(k,A) \cong \begin{cases} 0 & i \neq d, \\ \Sigma^l(k) & i = d. \end{cases}$$
If further ${}_Ak$ is small in $\mathsf{D}^\infty(A^{\mathrm{op}})$, then A is called *left A_∞-Artin–Schelter regular* (or just *left Artin–Schelter regular*).

(c) We say A satisfies the *Artin–Schelter condition* if the conditions in parts (a) and (b) hold for the same pair of integers (l,d). If further k_A is small in $\mathsf{D}^\infty(A)$, then A is called *A_∞-Artin–Schelter regular* (or just *Artin–Schelter regular*).

(d) Finally, if A is A_∞-Artin–Schelter regular, we say that A is *noetherian* if $\mathsf{D}^\infty_{\mathrm{fg}}(A) = \mathsf{D}^\infty_{\mathrm{per}}(A)$.

Suppose that R is a connected graded algebra. If R is Artin–Schelter regular as an associative algebra, then R is A_∞-Artin–Schelter regular. Conversely, if R is A_∞-Artin–Schelter regular, then one can use Corollary 6.2 to show that R is Artin–Schelter regular.

Also, if R is a connected graded algebra, then the cohomology of every object in $\mathsf{D}_{\mathrm{fg}}(R)$ is bounded (with respect to the homological grading), and we see that $\mathsf{D}_{\mathrm{fg}}(R) = \mathsf{D}_{\mathrm{per}}(R)$ if and only if R is noetherian of finite global dimension; this motivates our definition of "noetherian" for A_∞-Artin–Schelter regular algebras.

It is easy to see that A satisfies the left A_∞-Artin–Schelter condition if and only if A^{op} satisfies the right A_∞-Artin–Schelter condition. We conjecture that the left A_∞-Artin–Schelter condition is equivalent to the right A_∞-Artin–Schelter condition. We verify this, with some connectedness and finiteness assumptions, in Theorems 9.8 and 9.11.

Proposition 9.3. *Let A be an augmented A_∞-algebra and let $E = E(A)$ be the Koszul dual of A.*

(a) *Assume that A is strongly locally finite. Then A satisfies the right Artin–Schelter condition if and only if E satisfies the left Artin–Schelter condition.*

(b) *Assume that E is strongly locally finite. Then A satisfies the left Artin–Schelter condition if and only if E satisfies the right Artin–Schelter condition.*

Proof. (a) By Corollary 5.10 A satisfies the right Artin–Schelter condition if and only if E satisfies the left Artin–Schelter condition.

(b) Switch A and E and use the fact that $A \to E(E(A))$ is a quasi-isomorphism: the assertion follows from part (a). □

9.2. The Eilenberg–Moore spectral sequence. In this subsection we recall the Eilenberg–Moore spectral sequence [22, Theorem III.4.7] and [13, p. 280]. This helps to translate homological results for modules over HA to similar results for modules over A.

Suppose that A is a DG algebra and that M and N are right DG A-modules. Because of the $\mathbb{Z}\times\mathbb{Z}$-grading on A, M, and N, $\mathrm{Ext}^*_{HA}(HM, HN)$ is $\mathbb{Z}^3$-graded, and we incorporate the gradings into the notation as follows:

$$\mathrm{Ext}^p_{HA}(HM, HN)^q_s.$$

On the other hand, $\mathrm{Ext}^*_A(M,N)$ is $\mathbb{Z}^2$-graded, since it is defined to be the homology of $\mathrm{RHom}_A(M,N)$. That is,

$$\mathrm{Ext}^i_A(M,N)_j := \mathsf{D}(A)(M, S^i\Sigma^j N) \cong H(\mathrm{RHom}_A(M, S^i\Sigma^j N)).$$

Because of this, each E_r-page of the Eilenberg–Moore spectral sequence is $\mathbb{Z}^3$-graded, while the abutment is $\mathbb{Z}^2$-graded.

Theorem 9.4 ([22, Theorem III.4.7])**.** *Let A be a DG algebra, and let M and N be right DG A-modules. Then there is a spectral sequence of the form*

$$(E_2)^{p,q}_s \cong \mathrm{Ext}^p_{HA}(HM, HN)^q_s \Rightarrow \mathrm{Ext}^{p+q}_A(M,N)_s,$$

natural in M and N. All differentials preserve the lower (Adams) grading s.

This is a spectral sequence of cohomological type, with differential in the E_r-page as follows:

$$d_r : (E_r)^{p,q}_s \to (E_r)^{p+r,q-r+1}_s.$$

Ignoring the Adams grading, the E_2-term is concentrated in the right half-plane (i.e., $p \geq 0$), and it converges strongly if for each (p,q), $d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$ is nonzero for only finitely many values of r.

There is also a Tor version of this spectral sequence, which we do not use. See [22, Theorem III.4.7] and [13, p. 280] for more details.

Note that the above theorem also holds for A_∞-algebras; see [22, Theorem V.7.3]. Another way of obtaining an A_∞-version of this spectral sequence is to use the derived equivalence between $\mathsf{D}^\infty(A)$ and $\mathsf{D}(UA)$.

Corollary 9.5. *Let A be an A_∞-algebra. If HA satisfies the left (respectively, right) Artin–Schelter condition, then so does A.*

Proof. We may assume that A is a DG algebra. Let $M = k$ and $N = A$. Note that $Hk = k$. Since HA satisfies the left Artin–Schelter condition, $\bigoplus_{p,q} \mathrm{Ext}^p_{HA}(HM, HN)^q$ is 1-dimensional. By Theorem 9.4, $\bigoplus_n \mathrm{Ext}^n_A(k, A)$ is 1-dimensional. Therefore A satisfies the left Artin–Schelter condition. □

One naive question is if the converse of Corollary 9.5 holds.

9.3. Frobenius A_∞-algebras. In this subsection we define Frobenius DG algebras and Frobenius A_∞-algebras and then prove Corollary D.

Definition 9.6. An augmented DG algebra A is called *left Frobenius* (respectively, *right Frobenius*) if:

(a) HA is finite-dimensional.
(b) There is a quasi-isomorphism of left (respectively, right) DG A-modules $\alpha : S^l\Sigma^d(A) \to A^\sharp$ for some integers l and d.

An augmented DG algebra A is called *Frobenius* if it is both left and right Frobenius.

Lemma 9.7. *Suppose A is a DG algebra such that there is a quasi-isomorphism of left DG A-modules $\alpha : S^l\Sigma^d(A) \to A^\sharp$ for some integers l and d.*

(a) *There is a quasi-isomorphism of right DG A-modules $\beta : S^l\Sigma^d(A) \to A^\sharp$ for the same integers l and d.*
(b) *If A is connected graded with respect to some grading which is compatible with the $\mathbb{Z}^2$-grading — see below — then HA is finite-dimensional.*
(c) *If HA is finite-dimensional, then HA is Frobenius as an associative algebra.*
(d) *A satisfies the left Artin–Schelter condition and* $\mathrm{RHom}_A(k, A)$ *is quasi-isomorphic to $S^{-l}\Sigma^{-d}(k)$.*

The compatibility requirement for the grading in part (b) means that there should be numbers a and b so that the nth graded piece is equal to $\bigoplus_{ai+bj=n} A^i_j$.

Proof. (a) Let $\beta = S^l\Sigma^d(\alpha^\sharp)$.

(b) Since $S^l\Sigma^d(A) \to A^\sharp$ is an quasi-isomorphism,

$$(HA)^{n-l}_{m-d} \cong (HA^\sharp)^{-n}_{-m} \cong ((HA)^n_m)^\sharp$$

for all n, m. This implies that HA is locally finite. If A is connected graded with respect to some compatible grading, then so is HA. Then the above formula implies that HA is finite-dimensional.

(c) The quasi-isomorphism α gives rise to an isomorphism

$$H(\alpha) : S^l\Sigma^d(HA) \to (HA)^\sharp.$$

If HA is finite-dimensional, then HA is Frobenius.

(d) Since $A \to S^{-l}\Sigma^{-d}A^\sharp$ is a K-injective resolution of A, we can compute $\mathrm{RHom}_A(k, A)$ by $\mathrm{Hom}_A(k, S^{-l}\Sigma^{-d}(A^\sharp))$, which is $S^{-l}\Sigma^{-d}(k)$. □

By Lemma 9.7 above, A is left Frobenius if and only if it is right Frobenius. So we can omit both "left" and "right" before Frobenius. It is therefore easy to see that A is Frobenius if and only if A^{op} is. We show that the Frobenius property is a homological property.

Theorem 9.8. *Let A be an Adams connected DG algebra such that HA is finite-dimensional. Then the following are equivalent.*

(a) *A is Frobenius.*
(b) *HA is Frobenius.*
(c) *A satisfies the left Artin–Schelter condition.*
(d) *A satisfies the right Artin–Schelter condition.*

Proof. By Lemma 7.1, we may assume that A is finite-dimensional.

(a) $\Rightarrow$ (b): Use Lemma 9.7(c).

(b) $\Rightarrow$ (a): Since HA is Frobenius, there is an isomorphism of right HA-modules $f : S^l\Sigma^d(HA) \to HA^\sharp$. Pick $x \in ZA^\sharp$ so that the class of x generates a submodule of $HA^\sharp$ that is isomorphic to $S^l\Sigma^d(HA)$. Hence the map $a \to xa$ is a quasi-isomorphism $A \to A^\sharp$.

(a) $\Rightarrow$ (d): By vector space duality,

$$\mathrm{RHom}_{A^{\mathrm{op}}}(k, A) \cong \mathrm{RHom}_A(A^{\#}, k) \cong \mathrm{RHom}_A(S^l\Sigma^d(A), k) = S^{-l}\Sigma^{-d}(k).$$

Hence A satisfies the right Artin–Schelter condition.

(d) $\Rightarrow$ (a): Suppose $\mathrm{RHom}_{A^{\mathrm{op}}}(k, A) \cong S^{-l}\Sigma^{-d}(k)$ by the right Artin–Schelter condition. Since HA is locally finite, by vector space duality, we obtain that $\mathrm{RHom}_A(A^{\#}, k)$ is quasi-isomorphic to $S^{-l}\Sigma^{-d}(k)$. By Theorem 6.1 $A^\sharp$ has a minimal semifree resolution, say $P \to A^\sharp$. Since P is minimal, it has a semifree basis equal to $(\bigoplus_i \mathrm{Ext}^i_A(A^{\#}, k))^{\#} = \Sigma^d(k)$. Hence $P \cong S^l\Sigma^d(A)$ for some l and d. Thus $S^l\Sigma^d(A) \to A^\sharp$ is a quasi-isomorphism and A is left Frobenius. By Lemma 9.7, A is Frobenius.

Thus we have proved that (a), (b) and (d) are equivalent. By left-right symmetry, (a), (b) and (c) are equivalent. $\square$

We obtain an immediate consequence.

Corollary 9.9. *Let $f : A \to B$ be a quasi-isomorphism of Adams connected DG algebras. Assume that $HA \cong HB$ is finite-dimensional. Then A is Frobenius if and only if B is.*

Since finite-dimensional Hopf algebras are Frobenius, every finite-dimensional DG Hopf algebra is Frobenius.

Suggested by the DG case, we make the following definition.

Definition 9.10. An A_∞-algebra A is called *Frobenius* if HA is finite-dimensional and there is a quasi-isomorphism of right A_∞-modules $S^l\Sigma^d(A) \to A^\sharp$ for some l and d.

The following is similar to Theorem 9.8 and the proof is omitted.

Theorem 9.11. *Let A be an Adams connected A_∞-algebra such that HA is finite-dimensional. The following are equivalent.*

(a) *A is Frobenius.*
(b) *HA is Frobenius.*

(c) *A satisfies the left Artin–Schelter condition.*
(d) *A satisfies the right Artin–Schelter condition.*

Now we are ready to prove Corollary D from the introduction. We restate it for the reader's convenience.

Corollary D. *Let R be a connected graded algebra. Then R is Artin–Schelter regular if and only if the* Ext-*algebra* $\bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_R(k_R, k_R)$ *is Frobenius.*

Proof. Note that R is right Artin–Schelter regular if and only if k_R is small and R satisfies the right A_∞-Artin–Schelter condition [35, Proposition 3.1]. Now we assume, temporarily, that R is Adams connected; by Lemma 2.2, this means that $E(R)$ is, as well. Then the above conditions are in turn equivalent to: $HE(R) = \bigoplus_i \mathrm{Ext}^i(k_R, k_R)$ is finite-dimensional and $E(R)$ satisfies the left Artin–Schelter condition. By Theorem 9.11, this is equivalent to $HE(R)$ being Frobenius.

Now, we use Corollary 6.2 to justify the Adams connected assumption: if R is Artin–Schelter regular, then k_R is small, which implies that $R = HR$ is Adams connected. Conversely, if $HE(R)$ is Frobenius, then it is finite-dimensional, which also implies that R is Adams connected. □

This proof in fact shows that (for associative algebras) left Artin–Shelter regularity is equivalent to right Artin–Shelter regularity.

9.4. Dualizing complexes and the Gorenstein property. The balanced dualizing complex over a graded ring B was introduced by Yekutieli [38]. We refer to [38] for the definition and basic properties. Various noetherian graded rings have balanced dualizing complexes; see [36, 39].

Lemma 9.12. *Suppose R is a noetherian connected graded ring with a balanced dualizing complex. Then R satisfies the right Artin–Schelter condition if and only if R is Gorenstein. In this case, R will be Artin–Schelter Gorenstein.*

Proof. Let B be the balanced dualizing complex over R. Then the functor $F := \mathrm{RHom}_R(-, B)$ induces an equivalence $\mathsf{D}_{\mathrm{fg}}(R) \cong \mathsf{D}_{\mathrm{fg}}(R^{\mathrm{op}})^{\mathrm{op}}$ and satisfies $F(k_R) = {}_Rk$. By the right Artin–Schelter condition,

$$\mathrm{RHom}_R(k, R) \cong S^l \Sigma^d(k)$$

for some l and d. Applying the duality functor F, we have

$$\mathrm{RHom}_{R^{\mathrm{op}}}(B, k) = \mathrm{RHom}_{R^{\mathrm{op}}}(F(R), F(k)) \cong \mathrm{RHom}_R(k, R) \cong S^l \Sigma^d(k).$$

Therefore ${}_RB$ is quasi-isomorphic to $S^{-l}\Sigma^{-d}(R)$. Since ${}_RB$ has finite injective dimension by definition, ${}_RR$ has finite injective dimension. Also it follows from ${}_RB \cong S^{-l}\Sigma^{-d}(R)$ that $B_R \cong S^{-l}\Sigma^{-d}(R)$ by the fact that $R^{\mathrm{op}} = \mathrm{RHom}_{R^{\mathrm{op}}}(B, B)$. So since B_R has finite injective dimension, so does R_R.

For the converse, note that both $H\,\mathrm{RHom}_R(k,R)$ and $H\,\mathrm{RHom}_{R^{op}}(k,R)$ are finite-dimensional since the existence of B implies that R satisfies the χ-condition. The assertion follows from the proof of [40, Lemma 1.1]. $\square$

Now we restate and prove Corollary E.

Corollary E. *Let R be a Koszul algebra and let $R^!$ be the Koszul dual of R. If R and $R^!$ are both noetherian having balanced dualizing complexes, then R is Gorenstein if and only if $R^!$ is.*

Proof. By Lemma 9.12 the Artin–Schelter condition is equivalent to the Gorenstein property. The assertion follows from Proposition 9.3. $\square$

We say a connected graded algebra A *has enough normal elements* if every nonsimple graded prime factor ring A/P contains a homogeneous normal element of positive degree. A noetherian graded ring satisfying a polynomial identity has enough normal elements.

Corollary 9.13. *Let R be a Koszul algebra and let $R^!$ the Koszul dual of R. If R and $R^!$ are both noetherian having enough normal elements, then R is (Artin–Schelter) Gorenstein if and only if $R^!$ is.*

Proof. By [39, Theorem 5.13], R and $R^!$ have balanced dualizing complexes. By [40, Proposition 2.3(2)], under the hypothesis, the Artin–Schelter Gorenstein property is equivalent to the Gorenstein property. The assertion follows from Corollary E. $\square$

Let $R^e = R\otimes R^{\mathrm{op}}$. Following the work of Van den Bergh [36], Ginzburg [15] and Etingof–Ginzburg [10], an associative algebra R is called *twisted Calabi–Yau* if

$$\mathrm{Ext}^i_{R^e}(R,R^e)\cong\begin{cases} {}^{\phi}R^1 & \text{if } i=d\\ 0 & \text{if } i\neq d\end{cases}$$

for some d (note that we do not require R to have finite Hochschild dimension). If the above equation holds for $\phi=\mathrm{Id}_R$, then A is called *Calabi–Yau*. If R is connected graded, then ${}^{\phi}R^1$ should be replaced by $\Sigma^l({}^{\phi}R^1)$ for some integer l. It follows from Van den Bergh's result [36, Proposition 8.2] that if R is connected graded noetherian and Artin–Schelter Gorenstein, then R is twisted Calabi–Yau. It is easy to see that if R has finite global dimension and R^e is noetherian, then Artin–Schelter regularity is equivalent to the twisted Calabi–Yau property. It is conjectured that the Artin–Schelter Gorenstein property is equivalent to the twisted Calabi–Yau property for all connected graded noetherian rings.

We end this section with a proof of Corollary F.

Corollary F. *Let A be an Adams connected commutative differential graded algebra such that $\mathrm{RHom}_A(k,A)$ is not quasi-isomorphic to zero. If the Ext-algebra $\bigoplus_{i\in\mathbb{Z}}\mathrm{Ext}^i_A(k_A,k_A)$ is noetherian, then A satisfies the Artin–Schelter condition.*

Proof. Since A is commutative, its Ext algebra $H = HE(A) = \mathrm{Ext}^*_A(k,k)$ is a graded Hopf algebra which is graded cocommutative [12, p. 545]. By hypothesis, H is noetherian, so it satisfies [12, (1.1)]. Since $\mathrm{RHom}_A(k,A) \neq 0$, Corollary 5.10 implies that $\mathrm{RHom}_{E^{\mathrm{op}}}(k,E) \neq 0$ where $E = E(A)$. By Theorem 9.4 $\mathrm{RHom}_{H^{\mathrm{op}}}(k,H) \neq 0$. Since $H^{\mathrm{op}} \cong H$, we have $\mathrm{RHom}_H(k,H) \neq 0$ which says that H has finite depth. By [12, Theorem C], the noetherian property of H implies that H is elliptic, and elliptic Hopf algebras are classified in [12, Theorem B]. It is well-known that the Hopf algebras in [12, Theorem B] are Artin–Schelter Gorenstein. By Corollary 9.5, $E(A)$ satisfies the Artin–Schelter condition, and therefore by Proposition 9.3, A does as well. □

10. The BGG correspondence

The classical Bernšteĭn–Gel'fand–Gel'fand (BGG) correspondence states that the derived category of coherent sheaves over $\mathbb{P}^n$ is equivalent to the stable derived category over the exterior algebra of $(n+1)$-variables [8, Theorem 2]. Some generalizations of this were obtained by Baranovsky [5], He–Wu [16], Mori [27] and so on. In this section we prove a version of the BGG correspondence in the A_∞-algebra setting, as a simple application of Koszul duality.

If R is a right noetherian ring, then the stable bounded derived category over R, denoted by $\underline{\mathsf{D}^b_{\mathrm{fg}}}(R)$, is defined to be the Verdier quotient $\mathsf{D}_{\mathrm{fg}}(R)/\mathsf{D}_{\mathrm{per}}(R)$. With R concentrated in degrees $\{0\}\times\mathbb{Z}$, every complex in $\mathsf{D}_{\mathrm{fg}}(R)$ is bounded. When R is a finite-dimensional Frobenius algebra, then the stable module category over R is equivalent to the stable bounded derived category over R [30].

Recall from Sections 7.1 and 7.2, respectively, that

$$\mathsf{D}^\infty_{\mathrm{fd}}(A) = \mathsf{thick}^\infty_A(M \mid M \in \mathsf{Mod}\,A,\ \dim_k HM < \infty),$$
$$\mathsf{D}^\infty_{\mathrm{fg}}(A) = \mathsf{thick}^\infty_A(M \mid M \in \mathsf{Mod}\,A,\ HM \text{ finitely generated over } HA).$$

If HA is finite-dimensional, then by Lemma 7.1(a), $\mathsf{D}^\infty_{\mathrm{fd}}(A) = \mathsf{thick}^\infty_A(k)$, which is also equal to $\mathsf{thick}^\infty_A(k,A)$. Modelled on the definition of $\underline{\mathsf{D}^b_{\mathrm{fg}}}(R)$, we define the *stable derived category* of an A_∞-algebra A to be

$$\underline{\mathsf{D}^\infty_{\mathrm{fg}}}(A) = \mathsf{D}^\infty_{\mathrm{fg}}(A)/\mathsf{D}^\infty_{\mathrm{per}}(A)$$

where the right-hand side of the equation is a Verdier quotient. Boundedness of complexes does not make sense here since A itself may not be bounded, but for a weakly Adams connected A_∞-algebra we have the following variation. The *small stable derived category* of a weakly Adams connected A_∞-algebra A is defined to be

$$\underline{\mathsf{D}^\infty_{\mathrm{sm}}}(A) = \mathsf{thick}^\infty_A(k,A)/\mathsf{D}^\infty_{\mathrm{per}}(A)$$

where the right-hand side of the equation is a Verdier quotient. If HA is finite-dimensional, then

$$\underline{\mathsf{D}^\infty_{\mathrm{sm}}}(A) = \mathsf{D}^\infty_{\mathrm{fd}}(A)/\mathsf{D}^\infty_{\mathrm{per}}(A) = \underline{\mathsf{D}^\infty_{\mathrm{fg}}}(A).$$

In general, $\underline{\mathsf{D}^\infty_{\mathrm{sm}}}(A) \subset \underline{\mathsf{D}^\infty_{\mathrm{fg}}}(A)$. It is arguable whether or not $\underline{\mathsf{D}^\infty_{\mathrm{sm}}}(A)$ is a good definition. One reason we use the above definition is to make the BGG correspondence easy to prove.

It is easy to see that $\underline{\mathsf{D}^\infty_{\mathrm{sm}}}(A) = 0$ if and only if k_A is small (see Definition 4.8 and Lemma 4.9(b)). In this case we call A *regular*. This is consistent with terminology for associative algebras: Orlov called the triangulated category $\underline{\mathsf{D}^b_{\mathrm{fg}}}(R)$ the *derived category of the singularity of* R [28].

Given any connected graded ring R, we define the projective scheme over R to be the quotient category

$$\operatorname{Proj} R := \mathsf{Mod}\, R/\mathsf{Tor}\, R$$

where $\mathsf{Tor}\, R$ is the Serre localizing subcategory generated by all finite-dimensional graded right R-modules [1]. When R is right noetherian, we denote its noetherian subcategory by $\operatorname{proj} R$. The bounded derived category of $\operatorname{proj} R$ is $\mathsf{D}^b(\operatorname{proj} A)$ which is modelled by the derived category of coherence sheaves over a projective scheme. When A is an A_∞-algebra, we can define the derived category directly without using $\operatorname{Proj} A$. The *derived category of projective schemes over* A is defined to be

$$\mathsf{D}^\infty(\operatorname{Proj} A) = \mathsf{D}^\infty(A)/\mathsf{loc}^\infty_A(k);$$

the *derived category of finite projective schemes over* A is defined to be

$$\mathsf{D}^\infty(\operatorname{proj} A) = \mathsf{D}^\infty_{\mathrm{fg}}(A)/\mathsf{thick}^\infty_A(k);$$

and the *derived category of small projective schemes over* A is defined to be

$$\mathsf{D}^\infty_{\mathrm{sm}}(\operatorname{proj} A) = \mathsf{thick}^\infty_A(k, A)/\mathsf{thick}^\infty_A(k).$$

If A is right noetherian and regular (e.g., A is a commutative polynomial ring), then $\mathsf{D}^\infty(\operatorname{proj} A) = \mathsf{D}^\infty_{\mathrm{sm}}(\operatorname{proj} A)$ and this is equivalent to the derived category of the (noncommutative) projective scheme $\operatorname{proj} A$ [1].

Lemma 10.1. *Let A be an A_∞-algebra satisfying the Artin–Schelter condition. Then* $\underline{\mathsf{D}^\infty_{\mathrm{sm}}}(A)^{\mathrm{op}} \cong \underline{\mathsf{D}^\infty_{\mathrm{sm}}}(A^{\mathrm{op}})$.

Proof. First of all we may assume A is a DG algebra. Let $\mathcal{B}$ be the (A,A)-bimodule A. Clearly $\mathcal{B}$ is a balanced (A,A)-bimodule. The equivalences given in Proposition 4.10 are trivial, but Proposition 4.11 is not trivial. By Proposition 4.11(a), $\mathbb{A}^{\mathcal{B}} \cong \mathbb{B}^{\mathcal{B}}$ and $F^{\mathcal{B}}({}_AA) = A_A \in \mathbb{A}^{\mathcal{B}}$. By the Artin–Schelter condition, $F^{\mathcal{B}}({}_Ak) = \mathrm{RHom}_{A^{\mathrm{op}}}(k, A) = S^l\Sigma^d(k)$ and $G^{\mathcal{B}}(S^l\Sigma^d(k)) = S^{-l}\Sigma^{-d}(G^{\mathcal{B}}(k)) = k$. So ${}_Ak \in \mathbb{A}^{\mathcal{B}}$ and $F^{\mathcal{B}}({}_Ak) = S^l\Sigma^d(k)$. This implies that $F^{\mathcal{B}}$ induces an equivalence between $\mathsf{thick}^\infty_{A^{\mathrm{op}}}(k, A)^{\mathrm{op}}$ and $\mathsf{thick}^\infty_A(k, A)$ which sends ${}_AA$ to A_A. The assertion follows from the definition of $\underline{\mathsf{D}^\infty_{\mathrm{sm}}}(A)$. □

The next theorem is a version of the BGG correspondence.

Theorem 10.2. *Let A be a strongly locally finite A_∞-algebra and let E be its Koszul dual.*

(a) *There is an equivalence of triangulated categories*

$$\mathsf{D}^\infty_{\mathrm{sm}}(\operatorname{proj} A)^{\mathrm{op}} \cong \underline{\mathsf{D}^\infty_{\mathrm{sm}}}(E^{\mathrm{op}}).$$

(b) *If A is Adams connected graded noetherian right Artin–Schelter regular, then there is an equivalence of triangulated categories*

$$\mathsf{D}^\infty(\operatorname{proj} A) \cong \underline{\mathsf{D}^\infty_{\mathrm{fg}}}(E).$$

Proof. (a) By Theorem 5.8(b) there is an equivalence of triangulated categories

$$\mathsf{thick}^\infty_A(k, A)^{\mathrm{op}} \cong \mathsf{thick}^\infty_{E^{\mathrm{op}}}(k, E)$$

which maps k_A to E. Therefore we have

$$\begin{aligned}\mathsf{D}^\infty_{\mathrm{sm}}(\operatorname{proj} A)^{\mathrm{op}} &= (\mathsf{thick}^\infty_A(k, A)/\mathsf{thick}^\infty_A(k))^{\mathrm{op}} \\ &\cong \mathsf{thick}^\infty_{E^{\mathrm{op}}}(k, E)/\mathsf{thick}^\infty_{E^{\mathrm{op}}}(E) = \underline{\mathsf{D}^\infty_{\mathrm{sm}}}(E^{\mathrm{op}}).\end{aligned}$$

(b) Since A is AS regular, Corollary D says that E is Frobenius and hence HE is finite-dimensional. By Lemma 10.1, $\underline{\mathsf{D}^\infty_{\mathrm{sm}}}(E^{\mathrm{op}}) \cong \underline{\mathsf{D}^\infty_{\mathrm{sm}}}(E)^{\mathrm{op}}$. Since HE is finite-dimensional, $\underline{\mathsf{D}^\infty_{\mathrm{sm}}}(E) = \underline{\mathsf{D}^\infty_{\mathrm{fg}}}(E)$. Since A is connected graded Artin–Schelter regular, k_A is small, so

$$\mathsf{thick}^\infty_A(k, A) = \mathsf{thick}^\infty_A(A) = \mathsf{D}^\infty_{\mathrm{per}}(A).$$

By the definition of noetherian (Definition 9.2), $\mathsf{D}^\infty_{\mathrm{fg}}(A) = \mathsf{D}^\infty_{\mathrm{per}}(A)$. Therefore $\mathsf{D}^\infty(\operatorname{proj} A) = \mathsf{D}^\infty_{\mathrm{sm}}(\operatorname{proj} A)$. The assertion follows from part (a). □

Theorem C is part (b) of the above theorem.

References

[1] ARTIN, MICHAEL; ZHANG, JAMES J. Noncommutative projective schemes. *Adv. Math.* **109** (1994), no. 2, 228–287. MR1304753 (96a:14004), Zbl 0833.14002.

[2] AVRAMOV, LUCHEZAR L.; FOXBY, HANS-BJØRN. Ring homomorphisms and finite Gorenstein dimension. *Proc. London Math. Soc.* **75** (1997), no. 2, 241–270. MR1455856 (98d:13014), Zbl 0901.13011.

[3] AVRAMOV, LUCHEZAR L.; FOXBY, HANS-BJØRN; HALPERIN, STEPHEN. Differential graded homological algebra. Manuscript, 2007.

[4] AVRAMOV, LUCHEZAR; HALPERIN, STEPHEN. Through the looking glass: a dictionary between rational homotopy theory and local algebra. *Algebra, algebraic topology and their interactions* (Stockholm, 1983), Lecture Notes in Math., 1183. *Springer, Berlin*, 1986, 1–27. MR0846435 (87k:55015), Zbl 0588.13010.

[5] BARANOVSKY, VLADIMIR. BGG correspondence for projective complete intersections. *Int. Math. Res. Not.* (2005), no. 45, 2759–2774. MR2182470 (2007b:14035).

[6] BEĬLINSON, ALEXANDER; GINZBURG, VICTOR; SCHECHTMAN, VADIM. Koszul duality. *J. Geom. Phys.* **5** (1988), no. 3, 317–350. MR1048505 (91c:18011), Zbl 0695.14009.

[7] BEILINSON, ALEXANDER; GINZBURG, VICTOR; SOERGEL, WOLFGANG. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.* **9** (1996), no. 2, 473–527. MR1322847 (96k:17010), Zbl 0864.17006.

[8] BERNŠTEĬN, JOSEPH; GEL'FAND, ISRAEL M.; GEL'FAND, SERGEI I. Algebraic vector bundles on $\mathbf{P}^n$ and problems of linear algebra *Funktsional. Anal. i Prilozhen.* **12** (1978), no. 3, 66–67. MR0509387 (80c:14010a).

[9] BØGVAD, RIKARD; HALPERIN, STEPHEN. On a conjecture of Roos. *Algebra, algebraic topology and their interactions* (Stockholm, 1983), 120–127, Lecture Notes in Math., 1183. *Springer, Berlin*, 1986. MR0846442 (87h:55006), Zbl 0644.13004.

[10] ETINGOF, PAVEL; GINZBURG, VICTOR. Noncommutative del Pezzo surfaces and Calabi–Yau algebras. Preprint, 2007, arXiv:0709.3593.

[11] FÉLIX, YVES; HALPERIN, STEPHEN; THOMAS, JEAN-CLAUDE. Gorenstein spaces. *Adv. in Math.* **71** (1988), no. 1, 92–112. MR0960364 (89k:55019), Zbl 0659.57011.

[12] FÉLIX, YVES; HALPERIN, STEPHEN; THOMAS, JEAN-CLAUDE. Elliptic Hopf algebras. *J. London Math. Soc.* (2) **43** (1991) 545–555. MR1113392 (92i:57033), Zbl 0755.57018.

[13] FÉLIX, YVES; HALPERIN, STEPHEN; THOMAS, JEAN-CLAUDE. Rational homotopy theory. Graduate Texts in Mathematics, 205. *Springer-Verlag, New York*, 2001. ISBN: 0-387-95068-0. MR1802847 (2002d:55014), Zbl 0961.55002.

[14] GETZLER, EZRA; JONES, JOHN D. S. A_∞-algebras and the cyclic bar complex. *Illinois J. Math.* **34** (1990), no. 2, 256–283. MR1046565 (91e:19001), Zbl 0701.55009.

[15] GINZBURG, VICTOR. Calabi–Yau algebras. Preprint, 2006, arXiv:math.AG/0612139.

[16] HE, JIWEI; WU, QUAN SHUI. Koszul differential graded algebras and BGG correspondence. *J. Algebra* (to appear).

[17] KADEIŠVILI, TORNIKE. On the theory of homology of fiber spaces. *International Topology Conference* (Moscow State Univ., Moscow, 1979). *Uspekhi Mat. Nauk* **35** (1980), no. 3(213), 183–188, Translated in *Russ. Math. Surv.* **35** no. 3 (1980), 231–238. MR0580645 (82a:57041a).

[18] KADEIŠVILI, TORNIKE. The functor D for a category of $A(\infty)$-algebras. *Soobshch. Akad. Nauk Gruzin. SSR* **125** (1987), no. 2, 273–276. MR0921427 (89b:55022).

[19] KELLER, BERNHARD. Deriving DG categories. *Ann. Sci. École Norm. Sup.* (4) **27** (1994), no. 1, 63–102. MR1258406 (95e:18010), Zbl 0799.18007.

[20] KELLER, BERNHARD. Introduction to A-infinity algebras and modules. *Homology Homotopy Appl.* **3** (2001), no. 1, 1–35. MR1854636 (2004a:18008a), Zbl 0989.18009.

[21] KELLER, BERNHARD. On differential graded categories. *International Congress of Mathematicians. Vol.* II, 151–190, *Eur. Math. Soc., Zürich*, 2006. MR2275593 (2008g:18015).

[22] KŘÍŽ, IGOR; MAY, J. PETER. Operads, algebras, modules and motives. *Astérisque* No. 233 (1995). MR1361938 (96j:18006), Zbl 0840.18001.

[23] LEFÈVRE-HASEGAWA, KENJI. Sur les A_∞-catégories. Thèse de Doctorat, Université Paris VII, 2005. http://www.math.jussieu.fr/~keller/lefevre/publ.html.

[24] LU, DI MING; PALMIERI, JOHN H.; WU, QUAN SHUI; ZHANG, JAMES J. A_∞-algebras for ring theorists. Proceedings of the International Conference on Algebra. *Algebra Colloquium* **11** (2004) 91–128. MR2058967 (2005g:16015), Zbl 1066.16049.

[25] LU, DI MING; PALMIERI, JOHN H.; WU, QUAN SHUI; ZHANG, JAMES J. Regular algebras of dimension 4 and their A_∞-Ext-algebras. *Duke Math. J.* **137** (2007), no. 3, 537–584. MR2309153 (2008d:16022).

[26] LU, DI MING; PALMIERI, JOHN H.; WU, QUAN SHUI; ZHANG, JAMES J. A_∞-structure on Ext-algebras. *J. Pure and Applied Algebra* (to appear).

[27] MORI, IZURU. Riemann–Roch like theorem for triangulated categories. *J. Pure Appl. Algebra* **193** (2004), no. 1-3, 263–285. MR2076388 (2005d:14015), Zbl 1075.14006.

[28] ORLOV, DMITRI. Derived categories of coherent sheaves and triangulated categories of singularities. Preprint, 2005, arXiv:math/0503632.

[29] Prouté, Alain. Algébres différentielles fortement homotopiquement associatives. Thése d'Etat, Université Paris VII, 1984.

[30] Rickard, Jeremy. Derived categories and stable equivalence. *J. Pure Appl. Algebra* **61** (1989), no. 3, 303–317. MR1027750 (91a:16004), Zbl 0685.16016.

[31] Rotman, Joseph J. An introduction to homological algebra. Pure and Applied Mathematics, 85. *Academic Press, Inc. New York-London*, 1979. MR0538169 (80k:18001), Zbl 0441.18018.

[32] Smith, S. Paul. Some finite-dimensional algebras related to elliptic curves. *Representation theory of algebras and related topics* (Mexico City, 1994), 315–348, CMS Conf. Proc., 19. *Amer. Math. Soc., Providence, RI*, 1996. MR1388568 (97e:16053), Zbl 0856.16009.

[33] Spaltenstein, Nicolas. Resolutions of unbounded complexes. *Compositio Math.* **65** (1988), no. 2, 121–154. MR0932640 (89m:18013), Zbl 0636.18006.

[34] Stasheff, James D. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc.* **108** (1963), 275-292; ibid., **108** (1963), 293–312. MR0158400 (28 #1623), Zbl 0114.39402.

[35] Stephenson, Darin R.; Zhang, James J. Growth of graded Noetherian rings. *Proc. Amer. Math. Soc.* **125** (1997), no. 6, 1593–1605. MR1371143 (97g:16033), Zbl 0879.16011.

[36] Van den Bergh, Michel. Existence theorems for dualizing complexes over noncommutative graded and filtered rings. *J. Algebra* **195** (1997), no. 2, 662–679. MR1469646 (99b:16010), Zbl 0894.16020.

[37] Weibel, Charles A. An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, 38. *Cambridge University Press, Cambridge*, 1994. MR1269324 (95f:18001), Zbl 0797.18001, Zbl 0834.18001.

[38] Yekutieli, Amnon. Dualizing complexes over noncommutative graded algebras. *J. Algebra* **153** (1992) 41–84. MR1195406 (94a:16077), Zbl 0790.18005.

[39] Yekutieli, Amnon; Zhang, James J. Rings with Auslander dualizing complexes. *J. Algebra* **213** (1999) 1–51. MR1674648 (2000f:16012), Zbl 0948.16006.

[40] Zhang, James J. Connected graded Gorenstein algebras with enough normal elements. *J. Algebra* **189** (1997), no. 2, 390–405. MR1438182 (97m:16018), Zbl 0937.16049.

Department of Mathematics, Zhejiang University, Hangzhou 310027, China
dmlu@zju.edu.cn

Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, USA
palmieri@math.washington.edu zhang@math.washington.edu

Institute of Mathematics, Fudan University, Shanghai, 200433, China
qswu@fudan.edu.cn

This paper is available via http://nyjm.albany.edu/j/2008/14-16.html.