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Corrigendum to "Knotted Hamiltonian cycles in spatial embeddings of complete graphs"

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ABSTRACT. We state and prove a correct version of a theorem presented in Blain, Bowlin, Foisy, Hendricks and LaCombe, 2007.

Professor Masakazu Teragaito has recently pointed out that Theorem 3.3 of [1] is incorrect as stated. The fact that $\mu_f(G, \Gamma; 6) = 3$ is independent of embedding of K_8 does not necessarily imply that there are at least 3 knotted Hamiltonian cycles. For example, there could be exactly one knotted Hamiltonian cycle with $a_2(K) = 3$.

Professor Kouki Taniyama has further pointed out that Lemma 2, in [2], has a gap in its proof. He has proposed a rigorous proof of a weaker version Lemma 2. In this short paper, we will state and prove this weaker version of Lemma 2 in [2], and then apply it to obtain a weaker version of Theorem 3.3 of [1]. For definitions of terms, see [2] and [1]. Here is the modified version of Shimabara's Lemma 2 that we will prove:

Lemma 1. *Let Γ be a set of cycles in an undirected graph G. The invariant $\mu_f(G, \Gamma; n)$ does not depend on the spatial embedding f of G if the following two conditions hold:*

(1) *For any edges A, B, E such that A is adjacent to B,*

$$\nu_1(\Gamma; A, B, E) \equiv 0 \mod n.$$

(2) *For any pairs of nonadjacent edges (A, B) and (E, F),*

$$\nu_2(\Gamma; A, B; E, F) \equiv 0 \mod 2n.$$

The difference between this new lemma and the original comes in the second condition, where the equivalence is mod $2n$, not n. For the proof of Lemma 2, case 2, on p. 410 of [2], the definition of linking number used does

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not work. For the version of linking number used, $\zeta(A, B)$ depends on the order of A and B, whereas the equality

$$\sum_{E,F}\sum_{\gamma\in\Gamma_1} \epsilon(c)\zeta(f_\gamma(E), f_\gamma(F)) = \sum_{E,F}(n_3 - n_4)\zeta(f(E), f(F))$$

implicitly uses the assumption that $\zeta(A, B) = \zeta(B, A)$.

A proof of Lemma 1 is possible if one uses a different (but equivalent) version of ζ and linking number. For A and B two disjoint oriented arcs or circles in $\mathbb{R}^3$, define

$$\zeta'(A, B) = \frac{1}{2}\sum_{c} \epsilon(c),$$

with the summation being over all crossings between A and B. If A and B are circles, then $\zeta'(A, B)$ gives the linking number of A and B, $\mathrm{lk}(A, B)$. It then follows immediately that $\zeta'(A, B) = \zeta'(B, A)$. The proof of Lemma 1 proceeds as in the proof of Lemma 2 in [2], but now ends with:

$$\delta(u) = \sum_{E,F}\sum_{\gamma\in\Gamma_1} \epsilon(c)\zeta(f_\gamma(E), f_\gamma(F)) = \sum_{E,F}(n_3 - n_4)\zeta'(f(E), f(F)).$$

It then follows, since each $\zeta'(f(E), f(F))$ is either an integer or an integer divided by 2, that, $\delta(u)$ is congruent to 0 mod n if $|n_3 - n_4| \equiv 0 \bmod 2n$.

Now, in [1], it was shown that $\nu_2 \equiv 0 \bmod 6$. It was also shown that there exists an embedding of K_8 with exactly 21 knotted Hamiltonian cycles, each with Arf invariant 1. One can also verify that each of these knotted cycles is a trefoil, with $a_2 = 1$. This embedding, together with Lemma 1, implies that $\mu_f(K_8, \Gamma, 3) \equiv 0$ for every spatial embedding of K_8. By Theorem 2.2 of [1], there is at least one Hamiltonian cycle with Arf invariant 1, in every spatial embedding of K_8.

We thus have the following corrected version of Theorem 3.3 from [1].

Theorem 2. *Given an embedding of K_8, at least one of the following must occur in that embedding:*

(1) *At least* 3 *knotted Hamiltonian cycles.*
(2) *Exactly* 2 *knotted Hamiltonian cycles C_1 and C_2, with*

$$a_2(C_1) \equiv 1 \mod 3 \quad \textit{and} \quad a_2(C_2) \equiv 2 \mod 3,$$

or

$$a_2(C_1) \equiv 0 \mod 3 \quad \textit{and} \quad a_2(C_2) \equiv 0 \mod 3.$$

Either C_1 or C_2 has nonzero Arf invariant.
(3) *Exactly* 1 *knotted Hamiltonian cycle, C with*

$$a_2(C) \equiv 1 \mod 2 \quad \textit{and} \quad a_2(C) \equiv 0 \mod 3.$$

(*Equivalently:* $a_2(C) \equiv 3 \mod 6$.)

It thus remains an open question to determine if 1 is the best lower bound for the minimum number of knotted Hamiltonian cycles in every spatial embedding of K_8.

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