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Vertex tensor category structure on a category of Kazhdan–Lusztig

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Abstract. We incorporate a category considered by Kazhdan and Lusztig of certain modules (of a fixed level ℓ, not a positive integer) for an affine Lie algebra, into the representation theory of vertex operator algebras. We do this using the logarithmic tensor product theory for generalized modules for a vertex operator algebra developed by Huang, Lepowsky and the author; we prove that the conditions for applying this general logarithmic tensor product theory hold. As a consequence, we prove that this category has a natural vertex tensor category structure, and in particular we obtain a new, vertex-algebraic, construction of the natural associativity isomorphisms and proof of their properties.

Contents

1. Introduction

In a series of papers [KL2]–[KL5] (see also [KL1]), Kazhdan and Lusztig constructed a braided tensor category structure on a category $\mathcal{O}_\kappa$ of certain modules for an affine Lie algebra of a fixed level $\kappa - h$, h the dual Coxeter number of the Lie algebra and κ a complex number not in $\mathbb{Q}_{\geq 0}$, the set of nonnegative rational numbers, and showed that this braided tensor

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category is equivalent to a tensor category of modules for a quantum group constructed from the same Lie algebra. The most interesting cases are those of the allowable negative levels. Their construction of the tensor product uses ideas in the important work [MS] by Moore and Seiberg. While the category in [MS] consists essentially of the integrable highest weight modules for an affine Lie algebra of a fixed positive integral level rather than modules of a fixed negative level, for example, it was first discovered by Moore and Seiberg in that paper that this positive-level category has a braided tensor category structure; however, this was based on the strong assumption of the axioms for conformal field theory and in particular, the assumption of the existence of the "operator product expansion" for "intertwining operators."

Suitable modules for affine Lie algebras give rise to an important family of vertex operator algebras and their modules. In [HL1]–[HL4] and [H2], Huang and Lepowsky developed a substantial tensor product theory for modules for a "rational" vertex operator algebra, under certain conditions. As one of the applications of this theory, they proved in [HL5] that the conditions required for this theory are satisfied for the module category of the vertex operator algebra constructed from the category of integrable highest weight modules for an affine Lie algebra of a fixed positive integral level. As a result they directly constructed a braided tensor category structure, and further, a "vertex tensor category" structure, on this category.

It has been expected that the category $\mathcal{O}_\kappa$ considered by Kazhdan and Lusztig should also be covered by a suitable generalization of the tensor product theory developed by Huang and Lepowsky, even though $\kappa - h$ cannot be positive integral. However, this category is very different from the one associated with the positive integral level case. For example, the objects of this category in general are only direct sums of generalized eigenspaces, rather than eigenspaces, for the operator L_0 that defines the conformal weights. Therefore, such a generalization, if it exists, should include categories of these more general modules. Recently, such a generalization has been achieved in [HLZ1] and [HLZ2] by Huang, Lepowsky and the author. A fundamental subtlety in this generalization is that the corresponding intertwining operators involve logarithms of the variables. The questions are now whether the category of Kazhdan and Lusztig satisfies the required conditions for the generalized tensor product theory of [HLZ1] and [HLZ2], and if the answer is affirmative, whether the resulting tensor product construction is equivalent to the original construction given by Kazhdan and Lusztig.

In this paper, we prove that category $\mathcal{O}_\kappa$ indeed satisfies all necessary conditions in [HLZ1] and [HLZ2]. We establish an equivalent condition for the subtle "compatibility condition" in the construction of the tensor product and use it to prove that the two constructions of the tensor product functor are identical. We then use the methods in [H2] and [H3] and their generalizations in [HLZ1] and [HLZ2] to obtain a new construction, very different from the original one by Kazhdan and Lusztig, of the natural associativity isomorphisms. As a result, we incorporate the tensor category theory of $\mathcal{O}_\kappa$ into the theory of vertex operator algebras and more importantly, prove that $\mathcal{O}_\kappa$ has a natural vertex tensor category structure.

The contents of this paper are as follows: In Section 2 we recall the construction of the tensor product by Kazhdan and Lusztig and some of their results on the category $\mathcal{O}_\kappa$. In Section 3 we recall from [HLZ1] and [HLZ2] the construction of tensor product of generalized modules for a vertex operator algebra. In Section 4 we prove an equivalent condition for the "compatibility condition." Then in Section 5 we first apply the general theory to the case of $\mathcal{O}_\kappa$ and prove the equivalence of the two constructions of the tensor product functor; then we show that the objects of the category $\mathcal{O}_\kappa$ are C_1-cofinite and quasi-finite-dimensional. These results imply that the conditions for applying the results in [HLZ1] to the category $\mathcal{O}_\kappa$ are satisfied, and thus $\mathcal{O}_\kappa$ has a vertex tensor category structure. This paper is heavily based on the generalized tensor product theory developed in [HLZ1] and [HLZ2].

In this paper, $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}_+$ are the complex numbers, the nonnegative integers and the positive integers, respectively.

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2. Kazhdan–Lusztig's tensor product and the category $\mathcal{O}_\kappa$

In this section we recall the "double dual" construction of tensor product of certain modules for an affine Lie algebra of a fixed level, given by Kazhdan and Lusztig in [KL2] (see also [KL1]). We also recall from their papers the category $\mathcal{O}_\kappa$ for a complex number $\kappa \notin \mathbb{Q}_{\geq 0}$ and the result on closedness of tensor product on this category (see also [Y]).

Let $\mathfrak{g}$ be a complex semisimple finite-dimensional Lie algebra equipped with a nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)$. The (*untwisted*) *affine Lie algebra associated with* $\mathfrak{g}$ is the vector space

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\mathbf{k},$$

equipped with the bilinear bracket operations

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m(a, b)\delta_{m+n,0}\mathbf{k}, \tag{2.1}$$

$$[\cdot, \mathbf{k}] = 0 = [\mathbf{k}, \cdot] \tag{2.2}$$

for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. We will also need its algebraic completion $\mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathbf{k}$, which satisfies the same bracket relations (2.1) and (2.2), and more generally, for $a, b \in \mathfrak{g}$ and $g_1, g_2 \in \mathbb{C}((t))$,

$$[a \otimes g_1, b \otimes g_2] = [a, b] \otimes g_1 g_2 + \{g_1, g_2\}(a, b)\mathbf{k} \tag{2.3}$$

where $\{g_1, g_2\} = \operatorname{Res} g_2 \frac{d}{dt} g_1$, the coefficient of t^{-1} in the formal Laurent series $g_2 \frac{d}{dt} g_1$.

Equipped with the $\mathbb{Z}$-grading

$$\hat{\mathfrak{g}} = \coprod_{n \in \mathbb{Z}} \hat{\mathfrak{g}}_{(n)},$$

where

$$\hat{\mathfrak{g}}_{(0)} = \mathfrak{g} \oplus \mathbb{C}\mathbf{k} \ \text{ and } \ \hat{\mathfrak{g}}_{(n)} = \mathfrak{g} \otimes t^{-n} \text{ for } n \neq 0,$$

$\hat{\mathfrak{g}}$ becomes a $\mathbb{Z}$-graded Lie algebra. We have the following graded subalgebras of $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}}_{(\pm)} = \coprod_{n>0} \mathfrak{g} \otimes t^{\mp n},$$

$$\hat{\mathfrak{g}}_{(\leq 0)} = \hat{\mathfrak{g}}_{(-)} \oplus \mathfrak{g} \oplus \mathbb{C}\mathbf{k}.$$

A $\hat{\mathfrak{g}}$-module W is said to be of *level* $\ell \in \mathbb{C}$ if $\mathbf{k}$ acts on W as scalar ℓ. A module W for $\hat{\mathfrak{g}}$ or $\hat{\mathfrak{g}}_{(\leq 0)}$ is said to be *restricted* if for any $a \in \mathfrak{g}$ and $w \in W$, $(a \otimes t^n)w = 0$ for n sufficiently large. Note that a restricted module W for $\hat{\mathfrak{g}}$ is naturally a module for $\mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathbf{k}$ by letting $a \otimes \sum_{n \in \mathbb{Z}} c_n t^n$, $a \in \mathfrak{g}$, $c_n \in \mathbb{C}$, act on $w \in W$ as $\sum_{n \in \mathbb{Z}} c_n (a \otimes t^n) w$.

A $\hat{\mathfrak{g}}$-module W is said to be *smooth* if for any $w \in W$, there is $N \in \mathbb{N}$ such that for any $a_1, \ldots, a_N \in \mathfrak{g}$, $(a_1 \otimes t) \cdots (a_N \otimes t)w = 0$. By (2.1) and the fact that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ it is clear that a smooth $\hat{\mathfrak{g}}$-module must be restricted.

Let m be a positive integer. First recall that the direct sum $\mathfrak{g}^{\oplus m}$ of m copies of $\mathfrak{g}$ is a semisimple Lie algebra with nondegenerate invariant symmetric bilinear form $(\cdot, \cdot)$ given by

$$((a_1, \cdots, a_m), (b_1, \cdots, b_m)) = (a_1, b_1) + \cdots + (a_m, b_m)$$

for $a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathfrak{g}$.

Given m restricted $\hat{\mathfrak{g}}$-modules of a fixed level $\ell \in \mathbb{C}$, the goal is to produce a "tensor product" of these modules that is also a $\hat{\mathfrak{g}}$-module of the *same* level. (Note that the usual tensor product for Lie algebra modules is a module of level $m\ell$.) As we recall from [KL1] and [KL2] below, this can be defined in terms of a Riemann sphere with $m+1$ distinct points and local coordinates at these points.

Let $p_0, p_1, \ldots, p_m$ be distinct points, or *punctures* (see [H1]), on the Riemann sphere $C = \mathbb{C}P^1$ and let $\varphi_s : C \to \mathbb{C}P^1$ be isomorphisms such that

$\varphi_s(p_s) = 0$ for each of $s = 0, 1, \dots, m$, that is, φ_s is the *local coordinate around* p_s for each s. We will still use C for the Riemann sphere equipped with these punctures and local coordinates.

Let R denote the algebra of regular functions on $C \backslash \{p_0, p_1, \dots, p_m\}$. Define $\{f_1, f_2\} = \mathrm{Res}_{p_0} f_2 df_1$, i.e., the residue of the meromorphic 1-form $f_2 df_1$ on C at the point p_0. Then $\{\cdot, \cdot\} : R \times R \to \mathbb{C}$ is a bilinear form satisfying

$$\{f_1, f_2\} + \{f_2, f_1\} = 0, \qquad \{f_1 f_2, f_3\} + \{f_2 f_3, f_1\} + \{f_3 f_1, f_2\} = 0$$

for all $f_1, f_2, f_3 \in R$. As a result the Lie algebra $\mathfrak{g} \otimes R$ has a natural central extension $\Gamma_R = (\mathfrak{g} \otimes R) \oplus \mathbb{C}\mathbf{k}$ with central element $\mathbf{k}$ and bracket relations

$$[a \otimes f_1, b \otimes f_2] = [a, b] \otimes f_1 f_2 + \{f_1, f_2\}(a, b)\mathbf{k}, \tag{2.4}$$

for $a, b \in \mathfrak{g}$ and $f_1, f_2 \in R$.

Remark 2.1. In [KL2], C is allowed to be a smooth curve with k connected components each of which is isomorphic to $\mathbb{C}P^1$, and the construction would give a $\hat{\mathfrak{g}}^{\oplus k}$-module as the tensor product. For the purpose of this paper, however, we need only the $k = 1$ case.

For each $s = 0, 1, \dots, m$, denote by $\iota_{p_s} : R \to \mathbb{C}((t))$ the linear map which sends $f \in R$ to the power series expansion of $f \circ \varphi_s^{-1}$ around 0. Then we have Lie algebra homomorphisms

$$\begin{gathered} \Gamma_R \to \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathbf{k}, \\ a \otimes f \mapsto a \otimes \iota_{p_0} f, \quad \mathbf{k} \mapsto \mathbf{k} \end{gathered} \tag{2.5}$$

for $a \in \mathfrak{g}$, $f \in R$, and

$$\begin{gathered} \Gamma_R \to (\mathfrak{g} \otimes \mathbb{C}((t)))^{\oplus m} \oplus \mathbb{C}\mathbf{k}, \\ a \otimes f \mapsto (a \otimes \iota_{p_1} f, \cdots, a \otimes \iota_{p_m} f), \quad \mathbf{k} \mapsto -\mathbf{k} \end{gathered} \tag{2.6}$$

for $a \in \mathfrak{g}$ and $f \in R$ (cf. Remark 2.2 below). Here we see

$$(\mathfrak{g} \otimes \mathbb{C}((t)))^{\oplus m} \oplus \mathbb{C}\mathbf{k} = \mathfrak{g}^{\oplus m} \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathbf{k}$$

as the algebraic completion of the affine Lie algebra

$$\widehat{\mathfrak{g}^{\oplus m}} = \mathfrak{g}^{\oplus m} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$$

associated with the semisimple Lie algebra $\mathfrak{g}^{\oplus m}$; so in particular,

$$\begin{aligned} &[(a_1 \otimes g_1, \cdots, a_m \otimes g_m), (a_1' \otimes g_1', \cdots, a_m' \otimes g_m')] \\ &\quad = ([a_1, a_1'] \otimes g_1 g_1', \cdots, [a_m, a_m'] \otimes g_m g_m') + \sum_{i=1}^{m} (a_i, a_i')\{g_i, g_i'\}\mathbf{k} \end{aligned} \tag{2.7}$$

for $a_1, \dots, a_m, a_1', \dots, a_m' \in \mathfrak{g}$ and $g_1, \dots, g_m, g_1', \dots, g_m' \in \mathbb{C}((t))$.

Remark 2.2. That (2.5) is a Lie algebra homomorphism is straightforward by (2.4) and (2.3). Formula (2.6) gives a Lie algebra homomorphism due to (2.4), (2.7) and the fact that for any $f_1, f_2 \in R$,

$$\{\iota_{p_1} f_1, \iota_{p_1} f_2\} + \cdots + \{\iota_{p_m} f_1, \iota_{p_m} f_2\} = -\{\iota_{p_0} f_1, \iota_{p_0} f_2\},$$

by the residue theorem.

Now let W_1, W_2, $\ldots$, W_m be $\hat{\mathfrak{g}}$-modules of level ℓ. The vector space tensor product $W_1 \otimes \cdots \otimes W_m$ is naturally a module for the affine Lie algebra $\widehat{\mathfrak{g}^{\oplus m}} = \mathfrak{g}^{\oplus m} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k} = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}])^{\oplus m} \oplus \mathbb{C}\mathbf{k}$ by $\mathbf{k}$ acting as scalar ℓ and

$$(2.8)\qquad \begin{aligned}&(a_1 \otimes t^{n_1}, \cdots, a_m \otimes t^{n_m})(w_{(1)} \otimes \cdots \otimes w_{(m)})\\ &= (a_1 \otimes t^{n_1})w_{(1)} \otimes w_{(2)} \otimes \cdots \otimes w_{(m)} + \cdots\\ &\quad + w_{(1)} \otimes w_{(2)} \otimes \cdots \otimes (a_m \otimes t^{n_m})w_{(m)}\end{aligned}$$

for $a_1, \ldots, a_m \in \mathfrak{g}$, $n_1, \ldots, n_m \in \mathbb{Z}$ and $w_{(i)} \in W_i$, $i = 1, \ldots, m$. This follows from the bracket relations (2.7) where g_i and g_i''s are in $\mathbb{C}[t, t^{-1}]$.

If each $\hat{\mathfrak{g}}$-module W_i is restricted, $i = 1, \ldots, m$, then it is clear from (2.8) that $W_1 \otimes \cdots \otimes W_m$ is a restricted $\widehat{\mathfrak{g}^{\oplus m}}$-module, and hence naturally a module for $\mathfrak{g}^{\oplus m} \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathbf{k}$, satisfying

$$(2.9)\qquad \begin{aligned}&(a_1 \otimes g_1, \cdots, a_m \otimes g_m)(w_{(1)} \otimes \cdots \otimes w_{(m)})\\ &= (a_1 \otimes g_1)w_{(1)} \otimes w_{(2)} \otimes \cdots \otimes w_{(m)} + \cdots\\ &\quad + w_{(1)} \otimes w_{(2)} \otimes \cdots \otimes (a_m \otimes g_m)w_{(m)}.\end{aligned}$$

for $a_1, \ldots, a_m \in \mathfrak{g}$, $g_1, \ldots, g_m \in \mathbb{C}((t))$ and $w_{(i)} \in W_i$, $i = 1, \ldots, m$. Thus by (2.6) we have a Γ_R-module structure on $W_1 \otimes \cdots \otimes W_m$ where $\mathbf{k}$ acts as the scalar $-\ell$. The dual vector space $(W_1 \otimes \cdots \otimes W_m)^* = \mathrm{Hom}(W_1 \otimes \cdots \otimes W_m, \mathbb{C})$ has an induced natural Γ_R-module structure by

$$(2.10)\qquad \langle \xi(\lambda), w\rangle = -\langle \lambda, \xi(w)\rangle$$

for all $\xi \in \Gamma_R$, $\lambda \in (W_1 \otimes \cdots \otimes W_m)^*$ and $w \in W_1 \otimes \cdots \otimes W_m$. Here and below we use $\langle \cdot, \cdot \rangle$ to denote the natural pairing between a vector space and its dual.

Let N be a positive integer. Let G_N be the subspace of $U(\Gamma_R)$ spanned by all products $(a_1 \otimes f_1) \cdots (a_N \otimes f_N)$ with $a_1, \ldots, a_N \in \mathfrak{g}$ and $f_1, \ldots, f_N \in R$ satisfying $\iota_{p_0} f_i \in t\mathbb{C}[[t]]$ for $i = 1, \ldots, N$. Here and below we use notation $U(L)$ for the univeral enveloping algebra of a Lie algebra L. Define $Z^N \subset (W_1 \otimes \cdots \otimes W_m)^*$ to be the annihilator of $G_N(W_1 \otimes \cdots \otimes W_m)$. Then

$$Z^N = \{\lambda \in (W_1 \otimes \cdots \otimes W_m)^* \,|\, G_N \lambda = 0\},$$

and we have an increasing sequence $Z^1 \subset Z^2 \subset \cdots$. Let $Z^\infty = \cup_{N \in \mathbb{Z}_+} Z^N$. It is clear that Z^∞ is a Γ_R-submodule of $(W_1 \otimes \cdots \otimes W_m)^*$.

Define a $\mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathbf{k}$-module structure on Z^∞ as follows: $\mathbf{k}$ acts as scalar ℓ; and for $\lambda \in Z^\infty$, $a \in \mathfrak{g}$ and $g \in \mathbb{C}((t))$, choose $N \in \mathbb{N}$ such that $\lambda \in Z^N$, choose $f \in R$ such that $\iota_{p_0} f - g \in t^N \mathbb{C}[[t]]$, and define

$$(2.11)\qquad (a \otimes g)\lambda = (a \otimes f)\lambda.$$

It is easy to verify that this is independent of the choice of f and gives a $\mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathbf{k}$-module structure on Z^∞ with $\mathbf{k}$ acting as scalar ℓ. Restricted to Lie subalgebra $\hat{\mathfrak{g}}$ of $\mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathbf{k}$, we have on Z^∞ a structure of $\hat{\mathfrak{g}}$-module of level ℓ. We will denote this $\hat{\mathfrak{g}}$-module by $\circ_C(W_1, W_2, \cdots, W_m)$, and when $m = 2$, simply by $W_1 \circ_C W_2$.

Finally we need:

Definition 2.3. Given a $\hat{\mathfrak{g}}$-module W, consider $\mathrm{Hom}(W, \mathbb{C})$ as a $\hat{\mathfrak{g}}$-module with the actions given by

$$((a \otimes t^n)\lambda)(w) = -\lambda((a \otimes (-t)^{-n})w), \quad (\mathbf{k} \cdot \lambda)(w) = \lambda(\mathbf{k}w) \tag{2.12}$$

for $v \in \mathfrak{a}$, $\lambda \in W^*$ and $w \in W$. The *contragredient module* $D(W)$ of W is defined by

$$D(W) = \{\lambda \in \mathrm{Hom}(W, \mathbb{C}) \mid \text{there is } N \in \mathbb{N} \text{ such that for any } \\ a_1, \ldots, a_N \in \mathfrak{g}, (a_1 \otimes t) \cdots (a_N \otimes t)\lambda = 0\},$$

a $\hat{\mathfrak{g}}$-submodule of $\mathrm{Hom}(W, \mathbb{C})$.

The $\hat{\mathfrak{g}}$-module $D(\circ_C(W_1, W_2, \cdots, W_m))$ is defined to be the desired tensor product of $W_1, \ldots, W_m$.

Now we recall the category $\mathcal{O}_\kappa$ for a complex number $\kappa \notin \mathbb{Q}_{\geq 0}$ from [KL2].

Let M be a module for $\hat{\mathfrak{g}}_{(\leq 0)}$ satisfying the condition that $\mathbf{k}$ acts as a scalar ℓ. Then the induced $\hat{\mathfrak{g}}$-module

$$U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{(\leq 0)})} M \tag{2.13}$$

is a $\hat{\mathfrak{g}}$-module of level ℓ. In case M is finite-dimensional, restricted, and the subalgebra $\mathfrak{g}_{(-)}$ acts nilpotently, the corresponding induced module is called a *generalized Weyl module*.

Definition 2.4. Given complex number $\kappa \notin \mathbb{Q}_{\geq 0}$, the category $\mathcal{O}_\kappa$ is defined to be the full subcategory of the category of $\hat{\mathfrak{g}}$-modules whose objects are quotients of some generalized Weyl module of level $\ell = \kappa - h$, where h is the dual Coxeter number of $\mathfrak{g}$.

For an object W of $\mathcal{O}_\kappa$, define the *Segal–Sugawara operator* $L_k : W \to W$ by

$$L_k(w) = \frac{1}{2\kappa} \sum_{j \geq -k/2} \sum_p (c_p \otimes t^{-j})(c_p \otimes t^{j+k})w \\ + \frac{1}{2\kappa} \sum_{j < -k/2} \sum_p (c_p \otimes t^{j+k})(c_p \otimes t^{-j})w \tag{2.14}$$

where $\{c_p\}$ is an orthonormal basis of $\mathfrak{g}$. For any $n \in \mathbb{C}$ denote by $W_{[n]}$ the generalized eigenspace for L_0 with respect to the eigenvalue n. Then it is shown in [KL2] that $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ and $\dim W_{[n]} < \infty$.

The following result is proved in [KL2]

Theorem 2.5. *For W_1 and W_2 in $\mathcal{O}_\kappa$, $W_1 \circ_C W_2$ is again an object of $\mathcal{O}_\kappa$; the functor $D(\cdot)$ is closed in $\mathcal{O}_\kappa$, and furthermore, as a vector space*

$$D(W) = \coprod_{n\in\mathbb{C}} (W_{[n]})^*.$$

Let z be a nonzero complex number. Consider the Riemann sphere C with punctures $p_0 = z$, $p_1 = \infty$ and $p_2 = 0$ and local coordinates given by $\varphi_0(\epsilon) = \epsilon - z$, $\varphi_1(\epsilon) = 1/\epsilon$ and $\varphi_2(\epsilon) = \epsilon$ at p_0, p_1 and p_2, respectively. A Riemann sphere equipped with these punctures and local coordinates is denoted by $Q(z)$ as in [HL2].

By (2.6) and (2.9), the action of Γ_R on $W_1 \otimes W_2$ associated with $Q(z)$ is given by $\mathbf{k}$ acting as $-\ell$ and

$$\begin{aligned}(a \otimes f)(w_{(1)} \otimes w_{(2)}) = (a \otimes \iota_\infty f)(w_{(1)}) \otimes w_{(2)}\\ + w_{(1)} \otimes (a \otimes \iota_0 f)(w_{(2)})\end{aligned} \tag{2.15}$$

for $a \in \mathfrak{g}$, $f \in R$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. Hence by (2.10) the action of Γ_R on $(W_1 \otimes W_2)^*$ is given by $\mathbf{k}$ acting as ℓ and

$$\begin{aligned}\langle (a \otimes f)(\lambda), w_{(1)} \otimes w_{(2)}\rangle &= -\langle \lambda, (a \otimes f)(w_{(1)} \otimes w_{(2)})\rangle\\ &= -\langle \lambda, (a \otimes \iota_\infty f)(w_{(1)}) \otimes w_{(2)} + w_{(1)} \otimes (a \otimes \iota_0 f)(w_{(2)})\rangle\end{aligned}$$

for $a \in \mathfrak{g}$, $f \in R$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. This gives an action of the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}, (z+t)^{-1}] \oplus \mathbb{C}\mathbf{k}$. (Note that $\mathbb{C}[t, t^{-1}, (z+t)^{-1}] = \iota_z R$.) In particular, for $f = t^n$, $n \in \mathbb{Z}$, we have $\iota_z t^n = (z+t)^n$, and

$$\begin{aligned}&\langle (a \otimes (z+t)^n)(\lambda), w_{(1)} \otimes w_{(2)}\rangle\\ &= -\langle \lambda, (a \otimes \iota_\infty t^n)(w_{(1)}) \otimes w_{(2)} + w_{(1)} \otimes (a \otimes \iota_0 t^n)(w_{(2)})\rangle\\ &= -\langle \lambda, (a \otimes t^{-n})(w_{(1)}) \otimes w_{(2)} + w_{(1)} \otimes (a \otimes t^n)(w_{(2)})\rangle\end{aligned} \tag{2.16}$$

for any $a \in \mathfrak{g}$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$; and in case $f = (t-z)^n$, $n \in \mathbb{Z}$, we have $\iota_z (t-z)^n = t^n$, and

$$\begin{aligned}&\langle (a \otimes t^n)(\lambda), w_{(1)} \otimes w_{(2)}\rangle\\ &= -\langle \lambda, (a \otimes \iota_\infty (t-z)^n)(w_{(1)}) \otimes w_{(2)}\\ &\qquad + w_{(1)} \otimes (a \otimes \iota_0 (t-z)^n)(w_{(2)})\rangle\\ &= -\langle \lambda, (a \otimes \iota_+ (t^{-1}-z)^n)(w_{(1)}) \otimes w_{(2)}\\ &\qquad + w_{(1)} \otimes (a \otimes \iota_+ (t-z)^n)(w_{(2)})\rangle\\ &= -\left\langle \lambda, \sum_{i\in\mathbb{N}} \binom{n}{i} (-z)^i (a \otimes t^{i-n})(w_{(1)}) \otimes w_{(2)} + \right.\\ &\qquad \left. + w_{(1)} \otimes \sum_{i\in\mathbb{N}} \binom{n}{i} (-z)^{n-i} (a \otimes t^i)(w_{(2)}) \right\rangle\end{aligned} \tag{2.17}$$

for any $a \in \mathfrak{g}$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

By the general construction we now have

$$\begin{aligned}(2.18)\qquad & W_1 \circ_{Q(z)} W_2 \\ &= \{\lambda \in (W_1 \otimes W_2)^* \mid \text{for some } N \in \mathbb{N},\ \xi_1\xi_2 \cdots \xi_N \lambda = 0 \\ &\qquad \text{for any } \xi_1, \xi_2, \ldots, \xi_N \in \mathfrak{g} \otimes t\mathbb{C}[t, (z+t)^{-1}]\}.\end{aligned}$$

The tensor product is then defined as

$$D(W_1 \circ_{Q(z)} W_2).$$

3. Tensor product for modules for a conformal vertex algebra

In this section we recall the construction and some results in the tensor product theory for suitable module categories for a conformal vertex algebra from [HL2]–[HL4], [H2] and [HLZ1], [HLZ2].

We assume the reader is familiar with the material in [FLM] and [FHL], such as the language of formal calculus, the notion of vertex operator algebra and their modules, etc. Results from [HL2]–[HL4], [H2] and [HLZ1], [HLZ2] will be recalled without proof. We refer the reader to these papers for details.

We will focus on the "$Q(z)$-tensor product" in this section, due to the fact that tensor product constructed in [KL1] corresponds to $Q(1)$.

In [HLZ1] and [HLZ2], for an abelian group A and an abelian group $\widetilde{A}$ containing A as a subgroup, the notions of strongly A-graded conformal vertex algebra and strongly $\widetilde{A}$-graded generalized modules were introduced. The vertex operator algebras and their (ordinary) modules are exactly the conformal vertex algebras and their (ordinary) modules that are strongly graded with respect to the trivial group (see Remark 2.4 in [HLZ1] and Remarks 2.24 and 2.27 in [HLZ2]). In the present paper, we shall work in the special case of [HLZ1] and [HLZ2] in which A and $\widetilde{A}$ are trivial.

Let V be a vertex operator algebra, that is, as a special case of the general theory developed in [HLZ1] and [HLZ2], a strongly A-graded conformal vertex algebra with trivial A. A *generalized V-module* is a strongly $\widetilde{A}$-graded generalized module in the sense of [HLZ1] and [HLZ2] with trivial $\widetilde{A}$. It can also be defined directly in the same way as a V-module except that instead of being the direct sum of eigenspaces for the operator $L(0)$, it is assumed to be the direct sum of *generalized* eigenspaces for $L(0)$.

Recall from Definition 2.5 in [HLZ1] or Definition 3.32 in [HLZ2] that given a generalized V-module (W, Y_W) with

$$W = \coprod_{n \in \mathbb{C}} W_{[n]}$$

where $W_{[n]}$ is the generalized eigenspace for $L(0)$ with respect to the eigenvalue n, its *contragredient module* is the vector space

$$W' = \coprod_{n\in\mathbb{C}} (W_{[n]})^*$$

equipped with the vertex operator map Y' defined by

$$\langle Y'(v,x)w', w\rangle = \langle w', Y_W^o(v,x)w\rangle,$$

for any $v \in V$, $w' \in W'$ and $w \in W$, where

$$Y_W^o(v,x) = Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}),$$

for any $v \in V$, is the *opposite vertex operator* (cf. [FHL]). We will use the standard notation

$$\overline{W} = \prod_{n\in\mathbb{C}} W_{[n]},$$

the formal completion of W with respect to the $\mathbb{C}$-grading.

Fix a nonzero complex number z. The concept of $Q(z)$-intertwining map is defined as follows (see Definition 4.17 of [HLZ1] or Definition 4.32 of [HLZ2]):

Definition 3.1. Let W_1, W_2 and W_3 be generalized modules for a vertex operator algebra V. A *$Q(z)$-intertwining map of type* $\binom{W_3}{W_1\,W_2}$ is a linear map $I : W_1 \otimes W_2 \to \overline{W}_3$ such that the following conditions are satisfied: the *lower truncation condition:* for any elements $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and any $n \in \mathbb{C}$,

$$\pi_{n-m} I(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for} \quad m \in \mathbb{N} \ \text{ sufficiently large;} \tag{3.1}$$

and the *Jacobi identity*:

$$\begin{aligned} z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) & Y_3^o(v,x_0) I(w_{(1)} \otimes w_{(2)}) \\ = x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) & I(Y_1^o(v,x_1)w_{(1)} \otimes w_{(2)}) \\ - x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) & I(w_{(1)} \otimes Y_2(v,x_1)w_{(2)}) \end{aligned} \tag{3.2}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (note that the left-hand side of (3.2) is meaningful because any infinite linear combination of v_n of the form $\sum_{n<N} a_n v_n$ ($a_n \in \mathbb{C}$) acts on any $I(w_{(1)} \otimes w_{(2)})$, due to (3.1)).

Given generalized V-modules W_1 and W_2, we first have the notion of a $Q(z)$-product, as follows (see Definition 4.39 of [HLZ2]):

Definition 3.2. Let W_1 and W_2 be generalized V-modules. A *$Q(z)$-product of W_1 and W_2* is a generalized V-module (W_3, Y_3) together with a $Q(z)$-intertwining map I_3 of type $\binom{W_3}{W_1W_2}$, We denote it by $(W_3, Y_3; I_3)$ or simply by (W_3, I_3). Let $(W_4, Y_4; I_4)$ be another $Q(z)$-product of W_1 and W_2. A

morphism from $(W_3, Y_3; I_3)$ to $(W_4, Y_4; I_4)$ is a module map η from W_3 to W_4 such that
$$I_4 = \overline{\eta} \circ I_3.$$
where $\overline{\eta}$ is the natural map from $\overline{W}_3$ to $\overline{W}_4$ uniquely extending η.

Let $\mathcal{C}$ be a full subcategory of the category of generalized V-modules. The notion of $Q(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$ is defined in term of a universal property as follows (see Definition 4.40 of [HLZ2]):

Definition 3.3. For $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}$, a *$Q(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$* is a $Q(z)$-product $(W_0, Y_0; I_0)$ with $W_0 \in \mathrm{ob}\,\mathcal{C}$ such that for any $Q(z)$-product $(W, Y; I)$ with $W \in \mathrm{ob}\,\mathcal{C}$, there is a unique morphism from $(W_0, Y_0; I_0)$ to $(W, Y; I)$. Clearly, a $Q(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$, if it exists, is unique up to a unique isomorphism. In this case we will denote it as $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ and call the object $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)})$ the *$Q(z)$-tensor product module of W_1 and W_2 in $\mathcal{C}$*. We will skip the term "in $\mathcal{C}$" if the category $\mathcal{C}$ under consideration is clear in context.

Now we recall the construction of the $Q(z)$-tensor product from Section 5.3 of [HLZ2], which generalizes that in [HL2]–[HL4]. Let W_1 and W_2 be generalized V-modules. We first have the following linear action $\tau_{Q(z)}$ of the space
$$V \otimes \iota_+\mathbb{C}[t, t^{-1}, (z+t)^{-1}]$$
on $(W_1 \otimes W_2)^*$:
$$\begin{aligned}
&\left(\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0)\right)\lambda\right)(w_{(1)} \otimes w_{(2)}) \\
&\quad = x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right)\lambda(Y_1^o(v, x_1)w_{(1)} \otimes w_{(2)}) \\
&\qquad - x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right)\lambda(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}).
\end{aligned} \tag{3.3}$$
for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, where
$$Y_t(v, x) = v \otimes t^{-1}\delta\left(\frac{t}{x}\right). \tag{3.4}$$
This includes an action $Y'_{Q(z)}$ of $V \otimes \mathbb{C}[t, t^{-1}]$ defined by
$$Y'_{Q(z)}(v, x) = \tau_{Q(z)}\left(v \otimes t^{-1}\delta\left(\frac{t}{x}\right)\right), \tag{3.5}$$
that is, by taking Res_{x_1} in (3.3),
$$\begin{aligned}
&(Y'_{Q(z)}(v, x_0)\lambda)(w_{(1)} \otimes w_{(2)}) \\
&\quad = \lambda(Y_1^o(v, x_0 + z)w_{(1)} \otimes w_{(2)}) \\
&\qquad - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right)\lambda(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}).
\end{aligned} \tag{3.6}$$

We also have the operators $L'_{Q(z)}(n)$, $n \in \mathbb{Z}$ defined by

$$Y'_{Q(z)}(\omega, x) = \sum_{n\in\mathbb{Z}} L'_{Q(z)}(n)x^{-n-2}.$$

We have the following construction of $W_1 \boxbackslash_{Q(z)} W_2$, a subspace of $(W_1 \otimes W_2)^*$ (see Definition 5.60 and Theorem 5.74 of [HLZ2]):

Definition 3.4. Given W_1 and W_2 as above, the vector space $W_1 \boxbackslash_{Q(z)} W_2$ consists of all the elements $\lambda \in (W_1 \otimes W_2)^*$ satisfying the following two conditions:

(1) ($Q(z)$-compatibility condition)
  (a) (Lower truncation condition) For all $v \in V$, the formal Laurent series $Y'_{Q(z)}(v, x)\lambda$ involves only finitely many negative powers of x.
  (b) The following formula holds:

$$\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0)\right)\lambda = z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y'_{Q(z)}(v, x_0)\lambda \quad \text{for all} \quad v \in V. \tag{3.7}$$

(2) ($Q(z)$-local grading restriction condition)
  (a) (Grading condition) λ is a (finite) sum of generalized eigenvectors of $(W_1 \otimes W_2)^*$ for the operator $L'_{Q(z)}(0)$.
  (b) The smallest subspace W_λ of $(W_1 \otimes W_2)^*$ containing λ and stable under the component operators $\tau_{Q(z)}(v \otimes t^n)$ of the operators $Y'_{Q(z)}(v, x)$ for $v \in V$, $n \in \mathbb{Z}$, have the properties:

$$\dim(W_\lambda)_{[n]} < \infty \tag{3.8}$$

$$(W_\lambda)_{[n+k]} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative;} \tag{3.9}$$

  for any $n \in \mathbb{C}$, where the subscripts denote the $\mathbb{C}$-grading by $L'_{Q(z)}(0)$-eigenvalues.

The importance of the space $W_1 \boxbackslash_{Q(z)} W_2$ is given by the following theorem from [HL2] and its generalization in [HLZ2] (see Theorems 5.70, 5.71, 5.72, 5.73 and 5.74 of [HLZ2]):

Theorem 3.5. *The vector space $W_1 \boxbackslash_{Q(z)} W_2$ is closed under the action $Y'_{Q(z)}$ of V and the Jacobi identity holds on $W_1 \boxbackslash_{Q(z)} W_2$. Furthermore, the $Q(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$ exists if and only if $W_1 \boxbackslash_{Q(z)} W_2$ equipped with $Y'_{Q(z)}$ is an object of $\mathcal{C}$, and in this case, this $Q(z)$-tensor product is the contragredient module of $(W_1 \boxbackslash_{Q(z)} W_2, Y'_{Q(z)})$.*

4. Strong lower truncation condition

In this section we define what we shall call the strong lower truncation condition and prove its equivalence to the compatibility condition.

By using (3.4), (3.5) and the delta function identity

$$z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)=x_1^{-1}\delta\left(\frac{z+x_0}{x_1}\right),$$

formula (3.7) in the compatibility condition can be written as

$$\tau_{Q(z)}\left(x_1^{-1}\delta\left(\frac{z+x_0}{x_1}\right)Y_t(v,x_0)\right)\lambda \;=\; x_1^{-1}\delta\left(\frac{z+x_0}{x_1}\right)\tau_{Q(z)}(Y_t(v,x_0))\lambda.$$

Taking the coefficient of x_1^n for $n\in\mathbb{Z}$ of both sides we get

$$\tau_{Q(z)}((z+x_0)^{-n-1}Y_t(v,x_0))\lambda=(z+x_0)^{-n-1}Y'_{Q(z)}(v,x_0)\lambda.$$

or, by using (3.4) and the property of the δ-function

$$t^{-1}\delta\left(\frac{t}{x_0}\right)f(x_0)=t^{-1}\delta\left(\frac{t}{x_0}\right)f(t)$$

for formal series $f(x_0)$ we have

$$\tau_{Q(z)}((z+t)^{-n-1}Y_t(v,x_0))\lambda=(z+x_0)^{-n-1}Y'_{Q(z)}(v,x_0)\lambda. \tag{4.1}$$

Further taking the coefficient of x_0^{-m-1} for $m\in\mathbb{Z}$ this becomes

$$\tau_{Q(z)}(v\otimes(z+t)^{-n-1}t^m)\lambda=\sum_{i\in\mathbb{N}}\binom{-n-1}{i}z^{-n-1-i}\tau_{Q(z)}(v\otimes t^{m+i})\lambda. \tag{4.2}$$

We have:

Proposition 4.1. *Let W_1 and W_2 be modules for V as a vertex algebra, $\lambda\in(W_1\otimes W_2)^*$, $v\in V$ and $n\in\mathbb{N}$. Then*

$$\tau_{Q(z)}((z+t)^{-n-1}Y_t(v,x_0))\lambda$$

is lower truncated in x_0 if and only if $Y'_{Q(z)}(v,x_0)\lambda$ is lower truncated in x_0 and (4.1) *holds.*

Proof. The "if" part is obvious. For the "only if" part, suppose that

$$\tau_{Q(z)}((z+t)^{-n-1}Y_t(v,x_0))\lambda=\tau_{Q(z)}((z+x_0)^{-n-1}Y_t(v,x_0))\lambda$$

is lower truncated in x_0, then so is

$$\begin{aligned}(z+x_0)^{n+1}&\tau_{Q(z)}((z+x_0)^{-n-1}Y_t(v,x_0))\lambda\\ &=\tau_{Q(z)}((z+x_0)^{n+1}(z+x_0)^{-n-1}Y_t(v,x_0))\lambda\\ &=\tau_{Q(z)}(Y_t(v,x_0))\lambda\\ &=Y'_{Q(z)}(v,x_0)\lambda.\end{aligned} \tag{4.3}$$

(Note that here we need the existence of the triple product

$$(z+x_0)^{n+1}(z+x_0)^{-n-1}Y_t(v,x_0)$$

which can be seen by, for example, observing that the coefficient of each power of t exists.) That is, $Y'_{Q(z)}(v,x_0)\lambda$ is lower truncated in x_0.

By (4.3) we also have

$$\begin{aligned}(4.4)\quad &(z+x_0)^{n+1}\\ &\quad\cdot\big(\tau_{Q(z)}((z+x_0)^{-n-1}Y_t(v,x_0))\lambda-(z+x_0)^{-n-1}Y'_{Q(z)}(v,x_0)\lambda\big)\\ &=(z+x_0)^{n+1}\tau_{Q(z)}((z+x_0)^{-n-1}Y_t(v,x_0))\lambda-Y'_{Q(z)}(v,x_0)\lambda\\ &=Y'_{Q(z)}(v,x_0)\lambda-Y'_{Q(z)}(v,x_0)\lambda\\ &=0.\end{aligned}$$

Since both terms of

$$\tau_{Q(z)}((z+x_0)^{-n-1}Y_t(v,x_0))\lambda-(z+x_0)^{-n-1}Y'_{Q(z)}(v,x_0)\lambda \tag{4.5}$$

are lower truncated in x_0, (4.4) implies that (4.5) is equal to 0, as desired. □

Now we define the strong lower truncation condition:

Definition 4.2. Let $v\in V$. An element λ in $(W_1\otimes W_2)^*$ is said to satisfy the *strong lower truncation condition with respect to* $v\in V$ if there exists $N\in\mathbb{N}$ depending on v and λ such that

$$(\tau_{Q(z)}(v\otimes t^m(z+t)^{-n-1}))\lambda=0 \tag{4.6}$$

for all $m\geq N$ and $n\in\mathbb{Z}$. We say that λ satisfies the *strong lower truncation condition* if it satisfies the strong lower truncation condition with respect to every vector in V.

As a consequence of Proposition 4.1 we have the following equivalent condition for the $Q(x)$-compatibility condition:

Proposition 4.3. *Let $\lambda\in(W_1\otimes W_2)^*$. Then λ satisfies the $Q(z)$-compatibility condition if and only if λ satisfies the strong lower truncation condition.*

Proof. Suppose that λ satisfies the $Q(z)$-compatibility condition. Then (4.2) holds for any $v\in V$ and $m,n\in\mathbb{Z}$. But then by part (a) of the compatibility condition we see that the right-hand side of (4.2) is 0 when m is large enough, independent of n. This proves half of the statement. The other half follows directly from Proposition 4.1. □

We will need:

Lemma 4.4. *Let $N\in\mathbb{N}$. For $v\in V$ and $\lambda\in(W_1\otimes W_2)^*$, (4.6) holds for any $m\geq N$ and $n\in\mathbb{Z}$ if and only if for any $w_{(1)}\in W_1$ and $w_{(2)}\in W_2$,*

$$(x_1-z)^N\lambda(Y_1^o(v,x_1)w_{(1)}\otimes w_{(2)}-w_{(1)}\otimes Y_2(v,x_1)w_{(2)})=0.$$

Proof. By definition, (4.6) holds for any $m \geq N$ and $n \in \mathbb{Z}$ if and only if all powers of x_0 with nonzero coefficients in

$$\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0)\right)\lambda$$

are at least $-N$. By definition of the action $\tau_{Q(z)}$ in (3.3) this in turn is equivalent to the condition that $\mathrm{Res}_{x_0} x_0^m$ of the right-hand side of (3.3) is 0 for any $m \geq N$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. The statement now follows from this. □

The following result in formal calculus will be handy for us:

Lemma 4.5. *Let x and y be formal variables, ξ be either a formal variable or a complex number. Let $K \in \mathbb{N}$ and let $f_k(x, \xi)$, $k \in \mathbb{Z}$ be a sequence of formal series with coefficients in some vector space satisfying the condition that $f_k(x, \xi) = 0$ for any $k \geq K$. Suppose that for $N_1, N_2 \in \mathbb{N}$ we have*

$$(x + \xi)^{N_1}(x + y + \xi)^{N_2} \sum_{n \in \mathbb{Z}} f_n(x, \xi) y^{-n-1} = 0. \tag{4.7}$$

Then

$$(x + \xi)^{N_1+N_2+s} f_{K-1-s}(x, \xi) = 0 \tag{4.8}$$

for any $s \in \mathbb{N}$.

Proof. By taking coefficient of powers y, we see that (4.7) is equivalent to

$$\sum_{i=0}^{N_2} \binom{N_2}{i} (x + \xi)^{N_1+N_2-i} f_{i+k}(x, \xi) = 0 \tag{4.9}$$

for any $k \in \mathbb{Z}$. Since $f_k(x, \xi) = 0$ for any $k \geq K$, setting $k = K-1$ in (4.9) we obtain $(x + \xi)^{N_1+N_2} f_{K-1}(x, \xi) = 0$. Now (4.8) follows by induction on s, as follows: If the case $0, 1, \ldots, s-1$ holds, then by setting $k = K-1-s$ in (4.9) and multiplying both sides by $(x+\xi)^s$ we see that only the $i = 0$ term remains and must equal the right-hand side 0, that is, $(x+\xi)^{N_1+N_2+s} f_{K-1-s}(x, \xi) = 0$ also holds. □

We now have:

Proposition 4.6. *Let $u, v \in V$ and $\lambda \in (W_1 \otimes W_2)^*$. Suppose that λ satisfies the strong lower truncation condition with respect to both u and v. Then for any $k \in \mathbb{Z}$, λ satisfies the strong lower truncation condition with respect to $u_k v$. More precisely, let N_1 be the integer such that (4.6) holds with v replaced by u for any $m \geq N_1$ and $n \in \mathbb{Z}$ and let N_2 be the corresponding number for v, then (4.6) holds with v replaced by $u_k v$ for any $m \geq N_1 + N_2 + K - 1 - k$ and $n \in \mathbb{Z}$, where $K \in \mathbb{N}$ is such that $u_n v = 0$ for any $n \geq K$.*

Proof. By assumption and Lemma 4.4 we have

$$(x_1 - z)^{N_1} \lambda(Y_1^o(u, x_1)w_{(1)} \otimes w_{(2)} - w_{(1)} \otimes Y_2(u, x_1)w_{(2)}) = 0.$$

and

$$(x_1 - z)^{N_2} \lambda(Y_1^o(v, x_1)w_{(1)} \otimes w_{(2)} - w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) = 0.$$

for any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. For $k \in \mathbb{Z}$, we need a formula similar to either of these, with u or v replaced by $u_k v$. We derive as follows: First, for formal variables $x - 1$, y_0 and y_1, using the above identities we have:

$$\begin{aligned}
&(x_1 - z)^{N_2}(x_1 + y_0 - z)^{N_1}\lambda\bigg(y_0^{-1}\delta\bigg(\frac{y_1 - x_1}{y_0}\bigg)Y_1^o(v, x_1)Y_1^o(u, y_1)w_{(1)} \otimes w_{(2)}\bigg)\\
&= y_0^{-1}\delta\left(\frac{y_1 - x_1}{y_0}\right)(x_1 - z)^{N_2}(x_1 + y_0 - z)^{N_1}\\
&\qquad \cdot \lambda\big(Y_1^o(v, x_1)Y_1^o(u, y_1)w_{(1)} \otimes w_{(2)}\big)\\
&= y_0^{-1}\delta\left(\frac{y_1 - x_1}{y_0}\right)(x_1 - z)^{N_2}(x_1 + y_0 - z)^{N_1}\\
&\qquad \cdot \lambda\big(Y_1^o(u, y_1)w_{(1)} \otimes Y_2(v, x_1)w_{(2)}\big)\\
&= y_0^{-1}\delta\left(\frac{y_1 - x_1}{y_0}\right)(x_1 - z)^{N_2}(y_1 - z)^{N_1}\lambda\big(Y_1^o(u, y_1)w_{(1)} \otimes Y_2(v, x_1)w_{(2)}\big)\\
&= y_0^{-1}\delta\left(\frac{y_1 - x_1}{y_0}\right)(x_1 - z)^{N_2}(y_1 - z)^{N_1}\lambda\big(w_{(1)} \otimes Y_2(u, y_1)Y_2(v, x_1)w_{(2)}\big)\\
&= y_0^{-1}\delta\left(\frac{y_1 - x_1}{y_0}\right)(x_1 - z)^{N_2}(x_1 + y_0 - z)^{N_1}\\
&\qquad \cdot \lambda\big(w_{(1)} \otimes Y_2(u, y_1)Y_2(v, x_1)w_{(2)}\big)\\
&= (x_1 - z)^{N_2}(x_1 + y_0 - z)^{N_1}\\
&\qquad \cdot \lambda\left(w_{(1)} \otimes y_0^{-1}\delta\left(\frac{y_1 - x_1}{y_0}\right)Y_2(u, y_1)Y_2(v, x_1)w_{(2)}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(x_1 - z)^{N_2}(x_1 + y_0 - z)^{N_1}\lambda\bigg(y_0^{-1}\delta\bigg(\frac{x_1 - y_1}{-y_0}\bigg)Y_1^o(u, y_1)Y_1^o(v, x_1)w_{(1)} \otimes w_{(2)}\bigg)\\
&= y_0^{-1}\delta\left(\frac{x_1 - y_1}{-y_0}\right)(x_1 - z)^{N_2}(x_1 + y_0 - z)^{N_1}\\
&\qquad \cdot \lambda\big(Y_1^o(u, y_1)Y_1^o(v, x_1)w_{(1)} \otimes w_{(2)}\big)\\
&= y_0^{-1}\delta\left(\frac{x_1 - y_1}{-y_0}\right)(x_1 - z)^{N_2}(y_1 - z)^{N_1}\lambda\big(Y_1^o(u, y_1)Y_1^o(v, x_1)w_{(1)} \otimes w_{(2)}\big)\\
&= y_0^{-1}\delta\left(\frac{x_1 - y_1}{-y_0}\right)(x_1 - z)^{N_2}(y_1 - z)^{N_1}\lambda\big(Y_1^o(v, x_1)w_{(1)} \otimes Y_2(u, y_1)w_{(2)}\big)
\end{aligned}$$

$$= y_0^{-1}\delta\left(\frac{x_1-y_1}{-y_0}\right)(x_1-z)^{N_2}(x_1+y_0-z)^{N_1}$$
$$\cdot\,\lambda\big(Y_1^o(v,x_1)w_{(1)}\otimes Y_2(u,y_1)w_{(2)}\big)$$
$$= y_0^{-1}\delta\left(\frac{x_1-y_1}{-y_0}\right)(x_1-z)^{N_2}(x_1+y_0-z)^{N_1}$$
$$\cdot\,\lambda\big(w_{(1)}\otimes Y_2(v,x_1)Y_2(u,y_1)w_{(2)}\big)$$
$$= (x_1-z)^{N_2}(x_1+y_0-z)^{N_1}$$
$$\cdot\,\lambda\left(w_{(1)}\otimes y_0^{-1}\delta\left(\frac{x_1-y_1}{-y_0}\right)Y_2(v,x_1)Y_2(u,y_1)w_{(2)}\right).$$

Taking difference of these two equalities, using the Jacobi identity and the opposite Jacobi identity we have

$$(x_1-z)^{N_2}(x_1+y_0-z)^{N_1}\lambda(x_1^{-1}\delta\left(\frac{y_1-y_0}{x_1}\right)Y_1^o(Y(u,y_0)v,x_1)w_{(1)}\otimes w_{(2)}) =$$
$$(x_1-z)^{N_2}(x_1+y_0-z)^{N_1}\lambda(w_{(1)}\otimes x_1^{-1}\delta\left(\frac{y_1-y_0}{x_1}\right)Y_2(Y(u,y_0)v,x_1)w_{(2)}).$$

Applying Res_{y_1} we have

$$(x_1-z)^{N_2}(x_1+y_0-z)^{N_1}\cdot$$
$$\lambda(Y_1^o(Y(u,y_0)v,x_1)w_{(1)}\otimes w_{(2)}-w_{(1)}\otimes Y_2(Y(u,y_0)v,x_1)w_{(2)})=0.$$

Let K be a number such that $u_kv=0$ for all $k\geq K$. Then Lemma 4.5 applies with ξ being $-z$ and $f_k(x,\xi)$ being

$$\lambda(Y_1^o(u_kv,x_1)w_{(1)}\otimes w_{(2)}-w_{(1)}\otimes Y_2(u_kv,x_1)w_{(2)}).$$

As a result we have

$$(x_1-z)^{N_2+N_2+s}\lambda(Y_1^o(u_{K-1-s}v,x_1)w_{(1)}\otimes w_{(2)}-w_{(1)}\otimes Y_2(u_{K-1-s}v,x_1)w_{(2)}) = 0,$$

for any $s\in\mathbb{N}$. This is exactly what we need. □

Combining Proposition 4.3 and Proposition 4.6 we obtain:

Theorem 4.7. *Let V be a vertex algebra and S a generating set for V in the sense that*

$$V = \mathrm{span}\{a^{(k)}_{n_k}\cdots a^{(2)}_{n_2}a^{(1)}_{n_1}a^{(0)}\,|\,a^{(0)},a^{(1)},\ldots,a^{(k)}\in S,\ n_1,n_1,\ldots,n_k\in\mathbb{Z},k\in\mathbb{N}\}.$$

Let W_1 and W_2 be generalized V-modules. Then $\lambda\in(W_1\otimes W_2)^*$ satisfies the $Q(z)$-compatibility condition if and only if λ satisfies the strong lower truncation condition with respect to all elements of S.

5. Generalized modules for vertex operator algebras associated to $\hat{\mathfrak{g}}$

In this section we recall the vertex operator algebra constructed from suitable modules for the affine Lie algebra $\hat{\mathfrak{g}}$. Then we show that the construction of the tensor products in Section 2 and Section 3 are equivalent, for modules in $\mathcal{O}_\kappa$. In particular, this shows that the tensor product constructed by Kazhdan and Lusztig satisfies the universal property in Definition 3.3. Then we show that the objects of the category $\mathcal{O}_\kappa$ are C_1-cofinite and quasi-finite-dimensional. These results together with a result in [H3] imply that the conditions needed for applying the results obtained in [HLZ1] and [HLZ2] are satisfied. Hence we prove the existence of the associativity isomorphisms and we obtain the braided tensor category structure.

Recall the complex semisimple Lie algebra $\mathfrak{g}$, the nondegenerate invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ and the affine Lie algebra $\hat{\mathfrak{g}}$ in Section 2.

Given any $\mathfrak{g}$-module U and any complex number ℓ, consider U as a $\hat{\mathfrak{g}}_{(\leq)}$-module with $\hat{\mathfrak{g}}_{(-)}$ acting trivially and $\mathbf{k}$ acting as the scalar ℓ. Then the induced $\hat{\mathfrak{g}}$-module

$$\mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(U) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{(\leq)})} U$$

is restricted and is of level ℓ. When $U = \mathbb{C}$ is the trivial $\mathfrak{g}$-module, we will write

$$V_{\hat{\mathfrak{g}}}(\ell, 0) = \mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(\mathbb{C}).$$

In particular, fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a set of positive roots, let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the corresponding triangular decomposition. Let $U = L(\lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$. That is, $L(\lambda)$ is the quotient of $V(\lambda)$ by its maximal proper submodule, where $V(\lambda)$ is the $\mathfrak{g}$-module induced by the $(\mathfrak{h} \oplus \mathfrak{n}_+)$-module $\mathbb{C}v_\lambda$ with

$$hv_\lambda = \lambda(h)v_\lambda \quad \text{for } h \in \mathfrak{h}$$

and $\mathfrak{n}_+$ acts on v_λ as 0. $L(\lambda)$ is finite dimensional if and only if λ is dominant integral in the sense that

$$\lambda(h_\alpha) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{N}, \quad \text{for } \alpha \in \Delta_+.$$

In this case,

$$L(\lambda) = U(\mathfrak{n}_-)/U(\mathfrak{n}_-)\alpha_i^{\lambda(h_{\alpha_i})+1}$$

and we will denote the induced $\hat{\mathfrak{g}}$-module by $M(\ell, \lambda)$, called the *Weyl module* for $\hat{\mathfrak{g}}$ with respect to λ. Let $J(\ell, \lambda)$ be the maximal proper $\hat{\mathfrak{g}}$-submodule of $M(\ell, \lambda)$. Then $L(\ell, \lambda) = M(\ell, \lambda)/J(\ell, \lambda)$ is an irreducible $\hat{\mathfrak{g}}$-module of level ℓ.

We have (see [KL2]):

Proposition 5.1. (1) *The operator $L(0)$ acts semisimply on $M(\ell, \lambda)$.*

(2) *The category $\mathcal{O}_\kappa$ consists of all the $\hat{\mathfrak{g}}$-modules of level ℓ having a finite composition series all of whose irreducible subquotients are of the form $L(\ell, \lambda)$ for various λ.*

Note that $V_{\hat{\mathfrak{g}}}(\ell, 0)$ is spanned by the elements of the form

$$a^{(1)}(-n_1)\cdots a^{(r)}(-n_r)1,$$

where $a^{(1)}, \dots, a^{(r)} \in \mathfrak{g}$ and $n_1, \dots, n_r \in \mathbb{Z}_+$; here and below we use $a(-n)$ to denote the representation image of $a \otimes t^{-n}$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$.

The following theorem is well-known:

Theorem 5.2 ([FZ]; cf. [LL])**.** *There is a unique vertex algebra structure $(V_{\hat{\mathfrak{g}}}(\ell, 0), Y, \mathbf{1})$ on $V_{\hat{\mathfrak{g}}}(\ell, 0)$ such that $\mathbf{1} = 1 \in \mathbb{C}$ is the vacuum vector and such that*

$$Y(a(-1)1, x) = \sum_{n \in \mathbb{Z}} a(n) x^{-n-1}$$

for $a \in \mathfrak{g}$. We have

$$\begin{aligned} &Y(a^{(0)}(-n_0)a^{(1)}(-n_1)\cdots a^{(r)}(-n_r)1, x) \\ &= \mathrm{Res}_{x_1}(x_1 - x)^{-n_0} Y(a^{(0)}(-1)1, x_1) Y(a^{(1)}(-n_1)\cdots a^{(r)}(-n_r)1, x) \\ &\quad - \mathrm{Res}_{x_1}(-x + x_1)^{-n_0} Y(a^{(1)}(-n_1)\cdots a^{(r)}(-n_r)1, x) Y(a^{(0)}(-1)1, x_1). \end{aligned}$$

Any restricted $\hat{\mathfrak{g}}$-module W of level ℓ has a unique $V_{\hat{\mathfrak{g}}}(\ell, 0)$-module structure with the same action as above. Furthermore, in case $\ell \neq -h$,

$$\omega = \frac{1}{2(\ell + h)} \sum_{i=1}^{\dim \mathfrak{g}} g^i(-1)^2 1$$

is a conformal element, where $\{g^i\}_{i=1,\dots,\dim \mathfrak{g}}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the form $(\cdot, \cdot)$, and the quadruple $(V_{\hat{\mathfrak{g}}}(\ell, 0), Y, \mathbf{1}, \omega)$ is a vertex operator algebra.

Since every object of $\mathcal{O}_\kappa$ is a restricted $\hat{\mathfrak{g}}$-module of level $\kappa - h$, it is a module for the vertex algebra $V_{\hat{\mathfrak{g}}}(\kappa - h, 0)$, and when $\kappa \neq 0$, it is a generalized module for $V_{\hat{\mathfrak{g}}}(\kappa - h, 0)$ viewed as a vertex operator algebra.

For any element $\lambda \in W_1 \circ_{Q(z)} W_2$ (recall (2.18)), by Corollary 4.3 and the fact that $\mathfrak{g} \otimes t^N \mathbb{C}[t, (z+t)^{-1}] \subset (\mathfrak{g} \otimes t\mathbb{C}[t, (z+t)^{-1}])^N$ in $U(\mathfrak{g} \otimes \mathbb{C}((t)))$ we see that λ satisfies the $Q(z)$-compatibility condition. On the other hand, the closedness of tensor product in $\mathcal{O}_\kappa$ in Theorem 2.5 shows that $(W_1 \circ_{Q(z)} W_2)'$, and hence $W_1 \circ_{Q(z)} W_2$ itself, is an object of $\mathcal{O}_\kappa$. So all elements of $W_1 \circ_{Q(z)} W_2$ also satisfy the $Q(z)$-local grading restriction condition. Hence $W_1 \circ_{Q(z)} W_2 \subseteq W_1 \boxtimes_{Q(z)} W_2$.

Conversely, if $\lambda \in W_1 \boxtimes_{Q(z)} W_2$, then from the $Q(z)$-local grading restriction condition we see that λ generates a generalized V-module. But since for any $a \in \mathfrak{g}$, $a \otimes t = (a(-1)\mathbf{1})_1$ as an operator on $W_1 \boxtimes_{Q(z)} W_2$ reduces generalized weights by 1, we see that when N is large enough we have $\xi_1 \cdots \xi_N \lambda = 0$ for all $\xi_1, \ldots, \xi_N \in \mathfrak{g} \otimes t\mathbb{C}[t, (z+t)^{-1}]$. Hence $W_1 \boxtimes_{Q(z)} W_2 \subseteq W_1 \circ_{Q(z)} W_2$.

We have proved:

Theorem 5.3. *For W_1 and W_2 in $\mathcal{O}_\kappa$, the two subspaces $W_1 \circ_{Q(z)} W_2$ and $W_1 \boxtimes_{Q(z)} W_2$ of $(W_1 \otimes W_2)^*$ are equal to each other. In particular, the tensor product of two modules constructed in* [KL2] *with respect to $Q(z)$ satisfies the universal property in Definition* 3.3.

Now we proceed to the existence and construction of the associativity isomorphism for this tensor product. For this, we now work in the setting of tensor product associated with another type of Riemann spheres with punctures and local coordinates, namely, the spheres with punctures and local coordinates of type $P(z)$; recall from [H1] that for a nonzero complex number z, $P(z)$ denotes the Riemann sphere with ordered punctures ∞, z, 0 and local coordinates $1/w$, $w-z$, w around these punctures.

Remark 5.4. The reason that we use $P(z)$ here is because it is most convenient for the formulation of the associativity isomorphisms, due to the fact that spheres with punctures and local coordinates of type $P(z)$ are closed under sewing and subsequently decomposing. The corresponding associativity isomorphisms for other types of tensor products can be constructed from those for the type $P(z)$ by natural transformations associated to certain parallel transport over the moduli space of spheres with punctures and local coordinates. The $P(z)$-tensor product of W_1 and W_2 in $\mathcal{O}_\kappa$ exists if and only if their $Q(z)$-tensor product exists; the details are given in [HLZ2].

We need the following notions from [HLZ1] and [HLZ2], which generalize the corresponding notions in [H2] to the logarithmic case:

Definition 5.5. Let V be a vertex operator algebra and $\mathcal{C}$ be a category of generalized V-modules. We say that products of intertwining operators in $\mathcal{C}$ satisfy the *convergence and extension property* if for any objects W_1, W_2, W_3, W_4 and M_1 of $\mathcal{C}$, and intertwining operator $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_4}{W_1\, M_1}$ and $\binom{M_1}{W_2\, W_3}$, respectively, there exists an integer N depending only on $\mathcal{Y}_1$ and $\mathcal{Y}_2$, and for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$, there exist $M \in \mathbb{N}$, $r_k, s_k \in \mathbb{R}$, $i_k, j_k \in \mathbb{N}$, $k = 1, \ldots, M$ and analytic functions $f_{i_k j_k}(z)$ on $|z| < 1$, $k = 1, \ldots, M$, satisfying

$$\mathrm{wt}\ w_{(1)} + \mathrm{wt}\ w_{(2)} + s_k > N, \quad k = 1, \ldots, M,$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_2)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_1=z_1,\, x_2=z_2}$$

is convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multi-valued analytic function

$$\sum_{k=1}^{M} z_2^{r_k}(z_1 - z_2)^{s_k}(\log z_2)^{i_k}(\log(z_1 - z_2))^{j_k} f_{i_k j_k}\left(\frac{z_1 - z_2}{z_2}\right)$$

in the region $|z_2| > |z_1 - z_2| > 0$.

Definition 5.6. Let V be a vertex operator algebra and $\mathcal{C}$ be a category of generalized V-modules. If for any $n \in \mathbb{Z}_+$, any generalized V-modules W_i, $i = 0, \dots, n+1$ and $\widetilde{W}_i$, $i = 1, \dots, n-1$ in $\mathcal{C}$, and intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$, of types $\binom{W_0'}{W_1 \widetilde{W}_1}$, $\binom{\widetilde{W}_1}{W_2 \widetilde{W}_2}, \dots, \binom{\widetilde{W}_{n-2}}{W_{n-1} \widetilde{W}_{n-1}}$, $\binom{\widetilde{W}_{n-1}}{W_n W_{n+1}}$, respectively, the series

$$\langle w_0, \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n) w_{n+1}\rangle \tag{5.1}$$

is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$, then we say that *products of arbitrary number of intertwining operators among objects of $\mathcal{C}$ are convergent.*

The following theorem was proved in [HLZ1] and [HLZ2]:

Theorem 5.7. *Let V be a vertex operator algebra of central charge $c \in \mathbb{C}$ and $\mathcal{C}$ a category of generalized V-modules closed under the contragredient functor $(\cdot)'$ and under taking direct sums and quotients. Assume that the convergence and extension property for products of intertwining operators holds in $\mathcal{C}$ and that products of arbitrary number of intertwining operators among objects of $\mathcal{C}$ are convergent. Further assume that every finitely generated lower truncated generalized V-module is an object of $\mathcal{C}$ and, for any generalized V-modules W_1 and W_2, $W_1 \boxslash_{P(z)} W_2$ is an object of $\mathcal{C}$. Then the category $\mathcal{C}$ has a natural structure of vertex tensor category* (*see* [HL1]) *of central charge equal to the central charge c of V such that for each $z \in \mathbb{C}^\times$, the tensor product bifunctor $\boxtimes_{\psi(P(z))}$ associated to $\psi(P(z)) \in \widetilde{K}^c(2)$ is equal to $\boxtimes_{P(z)}$. In particular, the category $\mathcal{C}$ has a braided tensor category structure.*

By definition it is clear that $\mathcal{O}_\kappa$ is closed under taking direct sums and quotients. By Theorems 2.5 and 5.3 and Remark 5.4, we have that $\mathcal{O}_\kappa$ is closed under the contragredient functor and the $Q(z)$- and $P(z)$-tensor product functors. We now prove:

Proposition 5.8. *A lower truncated, finitely generated generalized $V_{\hat{\mathfrak{g}}}(\ell, 0)$-module is an object of $\mathcal{O}_\kappa$.*

Proof. Let W be a lower truncated generalized V-module generated by a finite set S of elements. That is, we have

$$W = U(\hat{\mathfrak{g}})S = U(\hat{\mathfrak{g}}_{(>)})U(\hat{\mathfrak{g}}_{(\leq)})S.$$

Let $N = U(\mathfrak{g}_{(\le)})S$, the $U(\mathfrak{g}_{(\le)})$-submodule of W generated by S. Then since W is lower truncated and S is finite, N is finite-dimensional. Let N^κ be the $\hat{\mathfrak{g}}$-module induced by N as in (2.13). Then N^κ is a generalized Weyl module and there is a unique $\hat{\mathfrak{g}}$-homomorphism from N^κ to W fixing N. It is clear that this is a surjection and thus W is a quotient module of generalized Weyl module N^κ. Hence W is in $\mathcal{O}_\kappa$. □

Remark 5.9. Note that in general a generalized Weyl module may not be generated by its lowest weight subspace, even if it is indecomposable.

Let V be a vertex operator algebra. A generalized V-module W is C_1-*cofinite* if $W/C_1(W)$ is finite-dimensional, where $C_1(W)$ is the subspace of W spanned by elements of the form $u_{-1}w$ for $u \in V_+ = \coprod_{n\in\mathbb{N}} V_{(n)}$ and $w \in W$.

Proposition 5.10. *The objects of $\mathcal{O}_\kappa$ are C_1-cofinite as generalized $V_{\hat{\mathfrak{g}}}(\ell, 0)$-modules.*

Proof. Let W be an object of $\mathcal{O}_\kappa$. Then W is a quotient of a generalized Weyl module $U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{(\le 0)})} M$. Let $\widetilde{M}$ be the image of M under projection from the generalized Weyl module to W. Then W is spanned by elements of $\widetilde{M}$ together with elements of the form $a(-n)w$ for $w \in W$ and $n \in \mathbb{Z}_+$. By the $L(-1)$-derivative property, we have

$$\begin{aligned} a(-n) &= (a(-1)\mathbf{1})_{-n} \\ &= \frac{(L(-1)a(-1)\mathbf{1})_{-n+1}}{n-1} \end{aligned}$$

when $n \neq 1$. Thus we see that W is spanned by elements of $\widetilde{M}$ together with elements of the form $u_{-1}w$ for $u \in (V_{\hat{\mathfrak{g}}}(\ell, 0))_+$ and $w \in W$. Since $\widetilde{M}$ is finite-dimensional, we see that $W/C_1(W)$ is also finite-dimensional. □

Let V be a vertex operator algebra. A generalized V-module W is *quasi-finite-dimensional* if for any real number N, $\coprod_{\Re(n)<N} W_{[n]}$ is finite-dimensional.

We have the following:

Proposition 5.11. *The objects of $\mathcal{O}_\kappa$ are quasi-finite-dimensional as generalized $V_{\hat{\mathfrak{g}}}(\ell, 0)$-modules.*

Proof. Since generalized Weyl modules are obviously quasi-finite-dimensional, the objects of $\mathcal{O}_\kappa$, as quotients of generalized Weyl modules, are quasi-finite-dimensional. □

By Theorems 2.5 and 5.3, Remark 5.4, Corollary 5.10, Proposition 5.11 and Theorem 7.2 in [HLZ1], we obtain the main conclusion of the present paper:

Theorem 5.12. *The category $\mathcal{O}_\kappa$ has a natural structure of vertex tensor category and in particular, a natural structure of braided tensor category.*

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