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Measures of sum-free intersecting families

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Abstract. Let α be the supremum of all δ such that there is a sequence $\langle A_n \rangle_{n=1}^{\infty}$ of measurable subsets of $(0,1)$ with the property that each A_n has measure at least δ and for all $n, m \in \mathbb{N}$, $A_n \cap A_m \cap A_{n+m} = \emptyset$. For $k \in \mathbb{N}$, let α_k be the corresponding supremum for finite sequences $\langle A_n \rangle_{n=1}^{k}$. We show that $\alpha = \lim_{k\to\infty} \alpha_k$ and find the exact value of α_k for $k \leq 41$. In the process of finding these exact values, we also determine exactly the number of maximal sum free subsets of $\{1, 2, \ldots, k\}$ for $k \leq 41$. We also investigate the size of sets $\langle A_x \rangle_{x\in S}$ with $A_x \cap A_y \cap A_{x+y} = \emptyset$ where S is a subsemigroup of $\big((0,\infty), +\big)$.

Contents

1. Introduction

Recall that Schur's Theorem [5] says that whenever the set $\mathbb{N}$ of positive integers is partitioned into finitely many cells, one of these must contain n, m, and $n + m$ for some n and m. And van der Waerden's Theorem [8] says that whenever $\mathbb{N}$ is partitioned into finitely many cells, one of these contains arbitrarily long arithmetic progressions. And Szemerédi's Theorem [7] says that in fact any set with positive upper density contains arbitrarily long arithmetic progressions.

In his mathematical youth, when he was a student of Furstenberg (who had provided an ergodic theoretic proof [3] of Szemerédi's Theorem) Vitaly Bergelson was thinking of possible density versions of Schur's Theorem. Of course, the obvious density version is false because the set of odd integers has density $\frac{1}{2}$. However, he was able to establish a strong density statement which does have Schur's Theorem

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as a corollary, namely that whenever $\mathbb{N}$ is finitely partitioned, there is one cell C of the partition such that $\overline{d}\big(\{n \in C : \overline{d}\big(C \cap (C-n)\big) > 0\}\big) > 0$, where $\overline{d}$ is upper asymptotic density. (The statement of [1, Theorem 1.1] is actually stronger than this.)

At about the same time, Bergelson thought it would be nice if one could show that whenever $\langle A_n\rangle_{n=1}^{\infty}$ is a sequence of measurable subsets of $(0,1)$ each with positive measure, there would have to exist some m and n such that the measure of $A_n \cap A_m \cap A_{n+m}$ is positive. Unfortunately, as he quickly noticed, this is easily seen to be false. In fact, if for each $n \in \mathbb{N}$ one lets

$$A_n = \{x \in (0,1) : \lfloor nx \rfloor + \tfrac{1}{3} < nx < \lfloor nx \rfloor + \tfrac{2}{3}\}\,,$$

then for all $n, m \in \mathbb{N}$, $A_n \cap A_m \cap A_{n+m} = \emptyset$. Further, these sets are all quite simple, each being the finite union of open intervals.

This curious fact, called to the attention of the authors in conversation with Bergelson, is the starting point of this paper. One naturally wants to know whether measure $\frac{1}{3}$ is as good as one can do. That is, if α is as defined below, is $\alpha = \frac{1}{3}$? We write $\mu(A)$ for the Lebesgue measure of a subset of $(0,1)$.

Definition 1.1.

(a) $\alpha = \sup\{\delta$: there exists a sequence $\langle A_n\rangle_{n=1}^{\infty}$ of measurable subsets of $(0,1)$ such that for each $n \in \mathbb{N}$, $\mu(A_n) \geq \delta$ and for all $n, m \in \mathbb{N}$, $A_n \cap A_m \cap A_{n+m} = \emptyset\}$.

(b) Let $k \in \mathbb{N}$. Then $\alpha_k = \sup\{\delta$: there exists a sequence $\langle A_n\rangle_{n=1}^{k}$ of measurable subsets of $(0,1)$ such that for each $n \in \{1, 2, \ldots, k\}$, $\mu(A_n) \geq \delta$ and for all $n, m \in \{1, 2, \ldots, k\}$, if $n + m \leq k$, then $A_n \cap A_m \cap A_{n+m} = \emptyset\}$.

One of course has trivially that for each $k \in \mathbb{N}$, $\alpha_k \geq \alpha_{k+1} \geq \alpha$. We in fact hoped that we could find some reasonably small k for which $\alpha_k = \frac{1}{3}$. This turns out not to be the case (assuming that "reasonably small" means some number less than 42). We do however have the following theorem. Note that in particular, it tells us that the supremum in Definition 1.1 is obtained, and in a reasonably simple fashion.

Theorem 1.2. $\alpha = \lim_{k\to\infty} \alpha_k$. *There exist* $\langle B_t\rangle_{t=1}^{\infty}$ *such that for each* $t \in \mathbb{N}$, B_t *is a union of at most* 2^{t-1} *open intervals in* $(0,1)$, *for each* $t \in \mathbb{N}$, $\mu(B_t) \geq \alpha$, *and for all* $n, m \in \mathbb{N}$, $B_n \cap B_m \cap B_{n+m} = \emptyset$.

We shall prove Theorem 1.2 in Section 3. Section 2 is devoted to the computation of α_k for $k \leq 41$. In Section 4 we consider the problem of finding $\langle A_x\rangle_{x\in S}$ for other subsemigroups S of $\big((0,\infty),+\big)$ such that $A_x \cap A_y \cap A_{x+y} = \emptyset$ for all $x, y \in S$. We show for example that we can find $\langle A_x\rangle_{x\in\mathbb{Q}^+}$ with this property such that each $\mu(A_x) = \frac{1}{3}$.

2. Computing α_k and maximal sum free sets

We see immediately that the problem of computing α_k is intimately related to finding all of the maximal sum free subsets of $\{1, 2, \ldots, k\}$. It is a curiosity that while there is a substantial literature on sum free sets, as a quick search of *MathSciNet* will reveal, the only papers we have been able to find on maximal sum free subsets of $\{1, 2, \ldots, k\}$ are [4] and [2]. In particular, the number of maximal sum free subsets of $\{1, 2, \ldots, k\}$ was not on *The online encyclopedia of integer sequences* [6] until we submitted it with this paper as a reference. The connection is given by Theorem 2.3. Before presenting this theorem we introduce some notation. Since we are dealing throughout this paper with subsets of $(0, 1)$ we will adopt the convention that $\bigcap_{n\in\emptyset} A_n = (0, 1)$. We denote the set of finite nonempty subsets of a set X by $\mathcal{P}_f(X)$.

Definition 2.1. Let $H \in \mathcal{P}_f(\mathbb{N})$, let $F \subseteq H$, and let $\langle A_n\rangle_{n\in H}$ be a sequence of subsets of $(0, 1)$.

$$D(F, \langle A_n\rangle_{n\in H}) = \textstyle\bigcap_{t\in F} A_t \setminus \bigcup_{t\in H\setminus F} A_t .$$

If $H = \{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}$, then we write $D(F, \langle A_n\rangle_{n=1}^k)$ for $D(F, \langle A_n\rangle_{n\in H})$. Notice that $D(F, \langle A_n\rangle_{n=1}^k)$ is that cell of the Venn diagram of $\langle A_n\rangle_{n=1}^k$ determined by F. That is, given $x \in (0, 1)$, $x \in D(F, \langle A_n\rangle_{n=1}^k)$ if and only if

$$F = \{n \in \{1, 2, \ldots, k\} : x \in A_n\} .$$

In particular, if $D(F, \langle A_n\rangle_{n=1}^k) \cap D(G, \langle A_n\rangle_{n=1}^k) \neq \emptyset$, then $F = G$.

Definition 2.2. Let $k \in \mathbb{N}$.

(a) $\mathcal{F}_k = \{F \subseteq \{1, 2, \ldots, k\}$: there do not exist n and m such that $\{n, m, n + m\} \subseteq F\}$.
(b) $\mathcal{M}_k$ is the set of maximal members of $\mathcal{F}_k$.

The following theorem tells us that when calculating α_k, we need only be concerned with sequences $\langle A_n\rangle_{n=1}^k$ for which the nonempty portions of the Venn diagram correspond to maximal sum free sets.

Theorem 2.3. *Let $k \in \mathbb{N}$ and let $\delta > 0$. Assume that $\langle A_n\rangle_{n=1}^k$ is a sequence of measurable subsets of $(0, 1)$ such that for each $n \in \{1, 2, \ldots, k\}$, $\mu(A_n) \geq \delta$ and whenever $\{n, m, n + m\} \subseteq \{1, 2, \ldots, k\}$ one has $A_n \cap A_m \cap A_{n+m} = \emptyset$. Then there exists a sequence $\langle B_n\rangle_{n=1}^k$ of measurable subsets of $(0, 1)$ such that for each $n \in \{1, 2, \ldots, k\}$, $\mu(B_n) \geq \delta$, whenever $\{n, m, n + m\} \subseteq \{1, 2, \ldots, k\}$ one has $B_n \cap B_m \cap B_{n+m} = \emptyset$, and for each $F \subseteq \{1, 2, \ldots, k\}$, if $D(F, \langle B_n\rangle_{n=1}^k) \neq \emptyset$, then $F \in \mathcal{M}_k$.*

Proof. For each $F \in \mathcal{F}_k$ pick $H_F \in \mathcal{M}_k$ such that $F \subseteq H_F$. For $n \in \{1, 2, \ldots, k\}$, let $B_n = \bigcup\{D(F, \langle A_t\rangle_{t=1}^k) : F \in \mathcal{F}_k \text{ and } n \in H_F\}$.

First we observe that for each $n \in \{1, 2, \ldots, k\}$, $A_n \subseteq B_n$, so in particular $\mu(B_n) \geq \mu(A_n)$. To see this, let $x \in A_n$ and let $F = \{t \in \{1, 2, \ldots, k\} : x \in A_t\}$. Then $x \in D(F, \langle A_t\rangle_{t=1}^k)$ and $n \in F \subseteq H_F$ so $x \in B_n$.

Next assume that $\{n, m, n + m\} \subseteq \{1, 2, \ldots, k\}$ and suppose that we have $x \in B_n \cap B_m \cap B_{n+m}$. Pick F_1, F_2, and F_3 in $\mathcal{F}_k$ such that $n \in H_{F_1}$, $m \in H_{F_2}$, $n+m \in H_{F_3}$, and $x \in D(F_1, \langle A_t\rangle_{t=1}^k) \cap D(F_2, \langle A_t\rangle_{t=1}^k) \cap D(F_3, \langle A_t\rangle_{t=1}^k)$. But since this intersection is nonempty we have that $F_1 = F_2 = F_3$ and so $\{n, m, n + m\} \subseteq H_{F_1}$, a contradiction.

Now let $G \subseteq \{1, 2, \ldots, k\}$ and assume that we have some $x \in D(G, \langle B_n\rangle_{n=1}^k)$. Then $G = \{n \in \{1, 2, \ldots, k\} : x \in B_n\}$ so by what we have just seen, $G \in \mathcal{F}_k$. Let $F = \{n \in \{1, 2, \ldots, k\} : x \in A_n\}$. We claim that $G = H_F$ for which it suffices to show that $H_F \subseteq G$. So let $n \in H_F$. Since $x \in D(F, \langle A_t\rangle_{t=1}^k)$ we have that $x \in B_n$ so $n \in G$. $\square$

At this stage we pause to establish a result which does not require computer assistance.

Theorem 2.4. $\alpha_5 = \alpha_6 = \alpha_7 = \frac{4}{9}$.

Proof. Let
$$\begin{aligned}
A_1 &= (0, \tfrac{1}{3}) \cup (\tfrac{8}{9}, 1)\\
A_2 &= (\tfrac{1}{3}, \tfrac{7}{9})\\
A_3 &= (0, \tfrac{1}{9}) \cup (\tfrac{1}{3}, \tfrac{5}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9})\\
A_4 &= (\tfrac{1}{9}, \tfrac{1}{3}) \cup (\tfrac{7}{9}, 1)\\
A_5 &= (0, \tfrac{1}{9}) \cup (\tfrac{5}{9}, \tfrac{8}{9})\\
A_6 &= (\tfrac{1}{9}, \tfrac{1}{3}) \cup (\tfrac{5}{9}, \tfrac{7}{9})\\
A_7 &= (0, \tfrac{1}{9}) \cup (\tfrac{1}{3}, \tfrac{5}{9}) \cup (\tfrac{8}{9}, 1)\,.
\end{aligned}$$
This assignment establishes that $\alpha_7 \geq \frac{4}{9}$.

Now assume that we have A_1, A_2, A_3, A_4, and A_5 with each $\mu(A_n) \geq \delta$ and $A_n \cap A_m \cap A_{n+m} = \emptyset$ whenever $\{n, m, n+m\} \subseteq \{1,2,3,4,5\}$. By Theorem 2.3 we may assume that $D(F, \langle A_n \rangle_{n=1}^5) = \emptyset$ whenever $F \notin \mathcal{M}_5$.

Note that $\mathcal{M}_5 = \big\{\{1,4\},\{2,3\},\{2,5\},\{1,3,5\},\{3,4,5\}\big\}$. Let
$$\begin{aligned}
x_1 &= \mu\big(D(\{1,4\}, \langle A_n \rangle_{n=1}^5)\big)\,,\\
x_2 &= \mu\big(D(\{2,3\}, \langle A_n \rangle_{n=1}^5)\big)\,,\\
x_3 &= \mu\big(D(\{2,5\}, \langle A_n \rangle_{n=1}^5)\big)\,,\\
x_4 &= \mu\big(D(\{1,3,5\}, \langle A_n \rangle_{n=1}^5)\big), \text{ and}\\
x_5 &= \mu\big(D(\{3,4,5\}, \langle A_n \rangle_{n=1}^5)\big)\,.
\end{aligned}$$
Then we have that
$$\begin{aligned}
\delta &\leq \mu(A_1) = x_1 + x_4\,,\\
\delta &\leq \mu(A_2) = x_2 + x_3\,,\\
\delta &\leq \mu(A_3) = x_2 + x_4 + x_5\,,\\
\delta &\leq \mu(A_4) = x_1 + x_5, \qquad \text{and}\\
\delta &\leq \mu(A_5) = x_3 + x_4 + x_5\,.
\end{aligned}$$
Therefore we have
$$\begin{aligned}
1 &\geq x_1 + x_2 + x_3 + x_4 + x_5\\
&= \tfrac{1}{2}(x_1 + x_4) + \tfrac{3}{4}(x_2 + x_3) + \tfrac{1}{4}(x_2 + x_4 + x_5) + \tfrac{1}{2}(x_1 + x_5) + \tfrac{1}{4}(x_3 + x_4 + x_5)\\
&\geq \tfrac{1}{2}\delta + \tfrac{3}{4}\delta + \tfrac{1}{4}\delta + \tfrac{1}{2}\delta + \tfrac{1}{4}\delta\\
&= \tfrac{9}{4}\delta
\end{aligned}$$
so $\delta \leq \frac{4}{9}$. □

We now show that evaluating α_k can be reduced to a linear programing problem. We also see that α_k, which is defined as a supremum, is actually obtained as a value.

Theorem 2.5. *Let $k \in \mathbb{N}$ and let $\langle H_n \rangle_{n=1}^m$ enumerate $\mathcal{M}_k$. For $i \in \{1,2,\ldots,k\}$ let $G_i = \{n \in \{1,2,\ldots,m\} : i \in H_n\}$. Let d be the minimum value of $\sum_{n=1}^m x_n$ subject to the constraints $x_n \geq 0$ for each $n \in \{1,2,\ldots,m\}$ and $\sum_{n \in G_i} x_n \geq 1$*

for each $i \in \{1,2,\ldots,k\}$. *Then* $\alpha_k = \frac{1}{d}$. *Further, there exist* $\langle A_t\rangle_{t=1}^k$ *such that each* A_t *is a finite union of open intervals contained in* $(0,1)$, $\mu(A_t) \geq \alpha_k$ *for each* $t \in \{1,2,\ldots,k\}$, *and whenever* $\{n,m,n+m\} \subseteq \{1,2,\ldots,k\}$, $A_n \cap A_m \cap A_{n+m} = \emptyset$.

Proof. We first show that $\alpha_k \leq \frac{1}{d}$. Suppose instead that $\alpha_k > \frac{1}{d}$ and pick γ such that $\alpha_k > \gamma > \frac{1}{d}$. Pick $\langle A_t\rangle_{t=1}^k$ such that each A_t is a measurable subset of $(0,1)$, $\mu(A_t) \geq \gamma$ for each $t \in \{1,2,\ldots,k\}$, and whenever $\{n,m,n+m\} \subseteq \{1,2,\ldots,k\}$, $A_n \cap A_m \cap A_{n+m} = \emptyset$. By Theorem 2.3 we may presume that $D(F,\langle A_t\rangle_{t=1}^k) = \emptyset$ whenever $F \subseteq \{1,2,\ldots,k\}$ and $F \notin \mathcal{M}_k$.

For each $n \in \{1,2,\ldots,m\}$, let $x_n = \frac{1}{\gamma}\cdot\mu\big(D(H_n,\langle A_t\rangle_{t=1}^k)\big)$. Then each $x_n \geq 0$ and, given $i \in \{1,2,\ldots,k\}$, $\gamma \leq \mu(A_i) = \sum_{n\in G_i}\mu\big(D(H_n,\langle A_t\rangle_{t=1}^k)\big) = \sum_{n\in G_i}\gamma\cdot x_n$ so $\sum_{n\in G_i} x_n \geq 1$. Also, $\sum_{n=1}^m x_n = \frac{1}{\gamma}\cdot\sum_{n=1}^m \mu\big(D(H_n,\langle A_t\rangle_{t=1}^k)\big) \leq \frac{1}{\gamma}$, so $d \leq \frac{1}{\gamma}$, a contradiction.

Now we show that $\alpha_k \geq \frac{1}{d}$ and simultaneously that the last sentence of the theorem is valid. Choose $\langle x_n\rangle_{n=1}^m$ such that $\sum_{n=1}^m x_n = d$, $\sum_{n\in G_i} x_n \geq 1$ for each $i \in \{1,2,\ldots,k\}$, and $x_n \geq 0$ for each $n \in \{1,2,\ldots,m\}$.

Let $a_0 = 0$ and for $n \in \{1,2,\ldots,m\}$, let $a_n = \frac{1}{d}\cdot\sum_{t=1}^n x_t$, so that $a_n - a_{n-1} = \frac{x_n}{d}$. For $i \in \{1,2,\ldots,k\}$, let $A_t = \bigcup\{(a_{n-1},a_n) : t \in H_n\}$, so that $D(H_n,\langle A_t\rangle_{t=1}^k) = (a_{n-1},a_n)$ and if $F \subseteq \{1,2,\ldots,k\}$ and $F \notin \mathcal{M}_k$, then $D(F,\langle A_t\rangle_{t=1}^k) = \emptyset$. Then for each $t \in \{1,2,\ldots,k\}$, $\mu(A_t) = \sum_{n\in G_t}(a_n - a_{n-1}) = \sum_{n\in G_t}\frac{x_n}{d} \geq \frac{1}{d}$. □

We computed α_k for $8 \leq k \leq 41$ as follows. We ran a Pascal program which found all maximal sumfree subsets of $\{1,2,\ldots,k\}$ and prepared a text file as input for the Mathematica *LinearProgramming* program. We took the Mathematica results and processed them with another Pascal program. We present the results in the Table 1. As we indicated earlier, the exact value of $|\mathcal{M}_k|$ is of independent interest so we record that count. Further it is known that $\lfloor\frac{k}{4}\rfloor \leq \log_2(|\mathcal{M}_k|) \leq \frac{k}{2} - 2^{-28}k$. (The first inequality is from [2]. The second inequality is from [4], where it is said to hold for all sufficiently large k, though it almost certainly holds for all k.) For $5 \leq k \leq 41$, $\frac{\log_2(|\mathcal{M}_k|)}{k}$ varies from approximately .4644 to approximately .3974.

It is interesting to note that the sets $\langle A_t\rangle_{t=1}^k$ obtained from the linear programing problem have a relatively small number of sets $H \in \mathcal{M}_k$ with $D(H,\langle A_t\rangle_{t=1}^k) \neq \emptyset$. For example, $|\mathcal{M}_{41}| = 80256$ and $|\{H \in \mathcal{M}_{41} : D(H,\langle A_t\rangle_{t=1}^k) \neq \emptyset\}| = 39$.

3. Proof of Theorem 1.2

As we have noted, it is trivial that $\alpha \leq \alpha_{k+1} \leq \alpha_k$ for each $k \in \mathbb{N}$. In this section we set out to show that $\alpha = \lim_{k\to\infty}\alpha_k$ and, furthermore, that α is actually achieved in a quite simple fashion via sets $\langle A_n\rangle_{n=1}^\infty$ each with measure at least α and each consisting of a finite union of intervals.

Our approach is via assignments of weight to $\bigcap_{t\in F} A_t$ for each $F \in \mathcal{P}_f(\mathbb{N})$. Recall that we are interpreting $\bigcap_{t\in\emptyset} A_t$ as $(0,1)$.

Lemma 3.1. *For all* $H \in \mathcal{P}_f(\mathbb{N})$, *every sequence* $\langle A_t\rangle_{t\in H}$ *of subsets of* $(0,1)$, *and all* $F \subseteq H$,

$$\mu\big(D(F,\langle A_t\rangle_{t\in H})\big) = \textstyle\sum_{G\subseteq H\setminus F}(-1)^{|G|}\cdot\mu\big(\bigcap_{t\in F\cup G} A_t\big)\,.$$

TABLE 1

k	α_k	$\lvert\mathcal{M}_k\rvert$	k	α_k	$\lvert\mathcal{M}_k\rvert$
1	1	1	21	$\frac{3623}{9483} \approx .38205$	471
2	$\frac{1}{2} = .50000$	2	22	$\frac{26393}{69755} \approx .37837$	598
3	$\frac{1}{2} = .50000$	2	23	$\frac{26393}{69755} \approx .37837$	797
4	$\frac{1}{2} = .50000$	4	24	$\frac{26393}{69755} \approx .37837$	1043
5	$\frac{4}{9} \approx .44444$	5	25	$\frac{3748414}{9981749} \approx .37553$	1378
6	$\frac{4}{9} \approx .44444$	6	26	$\frac{3517985}{9376079} \approx .37521$	1765
7	$\frac{4}{9} \approx .44444$	8	27	$\frac{3517985}{9376079} \approx .37521$	2311
8	$\frac{3}{7} \approx .42857$	13	28	$\frac{2130392}{5683115} \approx .37486$	3064
9	$\frac{3}{7} \approx .42857$	17	29	$\frac{115338484}{307691643} \approx .37485$	3970
10	$\frac{2}{5} = .40000$	23	30	$\frac{6023718}{16172311} \approx .37247$	5017
11	$\frac{90}{227} \approx .39648$	29	31	$\frac{6023718}{16172311} \approx .37247$	6537
12	$\frac{90}{227} \approx .39648$	37	32	$\frac{69394069}{186601165} \approx .37188$	8547
13	$\frac{112}{283} \approx .39576$	51	33	$\frac{35492659}{95630427} \approx .37114$	11020
14	$\frac{453}{1150} \approx .39391$	66	34	$\frac{9351744259}{25226495604} \approx .37071$	14007
15	$\frac{453}{1150} \approx .39391$	86	35	$\frac{598841590}{1617567693} \approx .37021$	18026
16	$\frac{551}{1403} \approx .39273$	118	36	$\frac{598841590}{1617567693} \approx .37021$	23404
17	$\frac{203}{517} \approx .39265$	158	37	$\frac{598841590}{1617567693} \approx .37021$	30026
18	$\frac{435}{1108} \approx .39260$	201	38	$\frac{4962276726}{13411951105} \approx .36999$	37989
19	$\frac{435}{1108} \approx .39260$	265	39	$\frac{4962276726}{13411951105} \approx .36999$	48945
20	$\frac{3623}{9483} \approx .38205$	359	40	$\frac{428070802153}{1167464566698} \approx .36667$	62759
			41	$\frac{428070802153}{1167464566698} \approx .36667$	80256

Proof. We proceed by induction on $|H \setminus F|$. If $F = H$, then both sides are equal to $\mu(\bigcap_{t\in H} A_t)$. So assume that $|H \setminus F| > 0$ and the lemma is true for smaller values.

Pick $k \in H \setminus F$. Then $D(F, \langle A_t\rangle_{t\in H}) \cup D(F \cup \{k\}, \langle A_t\rangle_{t\in H}) = D(F, \langle A_t\rangle_{t\in H\setminus\{k\}})$ and $D(F, \langle A_t\rangle_{t\in H}) \cap D(F \cup \{k\}, \langle A_t\rangle_{t\in H}) = \emptyset$. So

$$\begin{aligned}
\mu\big(D(F, \langle A_t\rangle_{t\in H})\big) &= \mu\big(D(F, \langle A_t\rangle_{t\in H\setminus\{k\}})\big) - \mu\big(D(F \cup \{k\}, \langle A_t\rangle_{t\in H})\big) \\
&= \textstyle\sum_{G\subseteq(H\setminus\{k\})\setminus F} (-1)^{|G|} \cdot \mu(\bigcap_{t\in F\cup G} A_t) \\
&\quad - \textstyle\sum_{G\subseteq H\setminus(F\cup\{k\})} (-1)^{|G|} \cdot \mu(\bigcap_{t\in F\cup\{k\}\cup G} A_t) \\
&= \textstyle\sum_{G\subseteq(H\setminus\{k\})\setminus F} (-1)^{|G|} \cdot \mu(\bigcap_{t\in F\cup G} A_t) \\
&\quad + \textstyle\sum_{G\cup\{k\}\subseteq H\setminus F} (-1)^{|G|+1} \cdot \mu(\bigcap_{t\in F\cup(G\cup\{k\})} A_t) \\
&= \textstyle\sum_{G\subseteq H\setminus F} (-1)^{|G|} \cdot \mu(\bigcap_{t\in F\cup G} A_t).
\end{aligned}$$

□

We shall be interested in the following lemma in the case $M = \{1, 2, \ldots, k\}$ and $t = k$ for some $k \in \mathbb{N}$. The more general version is needed for the proof.

Lemma 3.2. *Let $M \in \mathcal{P}_f(\mathbb{N})$ and let $\gamma : \mathcal{P}(M) \to [0, 1]$. For $F \subseteq M$ define $\delta(M, F) = \sum_{G \subseteq M \setminus F} (-1)^{|G|} \cdot \gamma(F \cup G)$.*

(a) *If $F \subsetneq M$ and $t \in M \setminus F$, then $\delta(M, F) + \delta(M, F \cup \{t\}) = \delta(M \setminus \{t\}, F)$.*
(b) *If $H \subseteq M$, then $\sum_{H \subseteq F \subseteq M} \delta(M, F) = \gamma(H)$.*

Proof. (a) We have

$$\begin{aligned}\delta(M, F) &= \textstyle\sum_{G \subseteq (M \setminus \{t\}) \setminus F} (-1)^{|G|} \cdot \gamma(F \cup G) \\ &\quad + \textstyle\sum_{G \subseteq (M \setminus \{t\}) \setminus F} (-1)^{|G|+1} \cdot \gamma(F \cup G \cup \{t\}) \\ &= \delta(M \setminus \{t\}, F) - \textstyle\sum_{G \subseteq M \setminus (F \cup \{t\})} (-1)^{|G|} \cdot \gamma(F \cup \{t\} \cup G) \\ &= \delta(M \setminus \{t\}, F) - \delta(M, F \cup \{t\}) \,.\end{aligned}$$

(b) We prove this statement by induction on $|M \setminus H|$. If $H = M$, this is the definition of $\delta(M, H)$, so assume that $H \subsetneq M$, $t \in M \setminus H$, and $\sum_{H \subseteq F \subseteq M \setminus \{t\}} \delta(M \setminus \{t\}, H) = \gamma(H)$. Then

$$\begin{aligned}\textstyle\sum_{H \subseteq F \subseteq M} \delta(M, F) &= \textstyle\sum_{H \subseteq F \subseteq M \setminus \{t\}} \delta(M, F) + \sum_{H \cup \{t\} \subseteq F \subseteq M} \delta(M, F) \\ &= \textstyle\sum_{H \subseteq F \subseteq M \setminus \{t\}} \big(\delta(M \setminus \{t\}, F) - \delta(M, F \cup \{t\})\big) \\ &\quad + \textstyle\sum_{H \cup \{t\} \subseteq F \subseteq M} \delta(M, F) \\ &= \textstyle\sum_{H \subseteq F \subseteq M \setminus \{t\}} \delta(M \setminus \{t\}, F) \\ &= \gamma(H).\end{aligned}$$

□

In the following we write $[\mathbb{N}]^{<\omega}$ for the set of finite subsets of $\mathbb{N}$. Thus $[\mathbb{N}]^{<\omega} = \mathcal{P}_f(\mathbb{N}) \cup \{\emptyset\}$.

Lemma 3.3. *Let $\gamma : [\mathbb{N}]^{<\omega} \to [0, 1]$ and for $k \in \mathbb{N}$ and $F \subseteq \{1, 2, \ldots, k\}$ let $\delta(\{1, 2, \ldots, k\}, F) = \sum_{G \subseteq \{1,2,\ldots,k\} \setminus F} (-1)^{|G|} \cdot \gamma(F \cup G)$. Assume that for all $k \in \mathbb{N}$ and all $F \subseteq \{1, 2, \ldots, k\}$, $\delta(\{1, 2, \ldots, k\}, F) \geq 0$. Then we can choose for each $k \in \mathbb{N}$ and each $F \subseteq \{1, 2, \ldots, k\}$, $a(k, F)$ and $b(k, F)$ in $[0, 1]$ such that:*

(a) *$a(1, \emptyset) = 0$, $b(1, \emptyset) = a(1, \{1\}) = \delta(\{1\}, \emptyset)$, and $b(1, \{1\}) = \delta(\{1\}, \emptyset) + \delta(\{1\}, \{1\}) = \gamma(\emptyset)$.*
(b) *For $k \in \mathbb{N}$ and $F \subseteq \{1, 2, \ldots, k\}$, $b(k, F) = a(k, F) + \delta(\{1, 2, \ldots, k\}, F)$.*
(c) *For $k \in \mathbb{N}$ and $F \subseteq \{1, 2, \ldots, k\}$, $a(k + 1, F) = a(k, F)$, $b(k + 1, F) = a(k + 1, F \cup \{k + 1\})$, and $b(k + 1, F \cup \{k + 1\}) = b(k, F)$.*

If for each $t \in \mathbb{N}$, $B_t = \bigcup_{\{t\} \subseteq F \subseteq \{1,2,\ldots,t\}} \big(a(t, F), b(t, F)\big)$, then for each $H \in \mathcal{P}_f(\mathbb{N})$ with $\max H = k$, $\bigcap_{t \in H} B_t = \bigcup_{H \subseteq F \subseteq \{1,2,\ldots,k\}} \big(a(k, F), b(k, F)\big)$ so $\mu(\bigcap_{t \in H} B_t) = \gamma(H)$.

Proof. To see that one can make the assignments as in (a), (b), and (c), observe that by Lemma 3.2(a), for any $k \in \mathbb{N}$ and any $F \subseteq \{1, 2, \ldots, k\}$, we have that

$$\delta(\{1, 2, \ldots, k + 1\}, F) + \delta(\{1, 2, \ldots, k + 1\}, F \cup \{k + 1\}) = \delta(\{1, 2, \ldots, k\}, F).$$

Next observe that if $F, G \in \mathcal{P}_f(\mathbb{N})$, $k = \max F$, $l = \max G$, and $l \leq k$, then:

(i) If $k = l$ and $F \neq G$, then $\big(a(k, F), b(k, F)\big) \cap \big(a(k, G), b(k, G)\big) = \emptyset$.

(ii) If $l < k$ and $\big(a(k,F),b(k,F)\big) \cap \big(a(l,G),b(l,G)\big) \neq \emptyset$, then

$$G = F \cap \{1,2,\ldots,l\} \text{ and } \big(a(k,F),b(k,F)\big) \subseteq \big(a(l,G),b(l,G)\big).$$

We show that $\bigcap_{t\in H} B_t = \bigcup_{H\subseteq F\subseteq\{1,2,\ldots,k\}} \big(a(k,F),b(k,F)\big)$ for $H \in \mathcal{P}_f(\mathbb{N})$ with $\max H = k$ by induction on $|H|$. If $H = \{k\}$ this is the definition of B_k. So assume that $|H| > 1$, let $L = H \setminus \{k\}$, and let $l = \max L$. Then

$$\begin{aligned}&\textstyle\bigcap_{t\in H} B_t = B_k \cap \bigcap_{t\in L} B_t\\ &= \Big(\textstyle\bigcup_{\{k\}\subseteq F\subseteq\{1,2,\ldots,k\}} \big(a(k,F),b(k,F)\big)\Big) \cap \Big(\textstyle\bigcup_{L\subseteq G\subseteq\{1,2,\ldots,l\}} \big(a(l,G),b(l,G)\big)\Big)\\ &= \textstyle\bigcup_{H\subseteq F\subseteq\{1,2,\ldots,k\}} \big(a(k,F),b(k,F)\big).\end{aligned}$$

To verify the last equality, notice that if $\{k\} \subseteq F \subseteq \{1,2,\ldots,k\}$, $L \subseteq G \subseteq \{1,2,\ldots,l\}$, and $x \in \big(a(k,F),b(k,F)\big) \cap \big(a(l,G),b(l,G)\big)$, then $G = F \cap \{1,2,\ldots,l\}$ so $H \subseteq F$. Also, if $H \subseteq F \subseteq \{1,2,\ldots,k\}$ and $G = F \cap \{1,2,\ldots,l\}$, then $L \subseteq G$.

Finally,

$$\mu\left(\textstyle\bigcap_{t\in H} B_t\right) = \textstyle\sum_{H\subseteq F\subseteq\{1,2,\ldots,k\}} \delta(\{1,2,\ldots,k\},F) = \gamma(H)$$

by Lemma 3.2(b). □

We now restate Theorem 1.2.

Theorem 3.4. $\alpha = \lim\limits_{k\to\infty} \alpha_k$. *There exist* $\langle B_t\rangle_{t=1}^{\infty}$ *such that for each* $t \in \mathbb{N}$, B_t *is a union of at most* 2^{t-1} *open intervals in* $(0,1)$, *for each* $t \in \mathbb{N}$, $\mu(B_t) \geq \alpha$, *and for all* $n,m \in \mathbb{N}$, $B_n \cap B_m \cap B_{n+m} = \emptyset$.

Proof. Since for each $k \in \mathbb{N}$, $\alpha_k \geq \alpha_{k+1} \geq \alpha$, we have that $\eta = \lim\limits_{k\to\infty} \alpha_k$ exists and $\eta \geq \alpha$. It thus suffices to show that there exist $\langle B_t\rangle_{t=1}^{\infty}$ such that for each $t \in \mathbb{N}$, B_t is a union of at most 2^{t-1} open intervals in $(0,1)$, for each $t \in \mathbb{N}$, $\mu(B_t) \geq \eta$, and for all $n,m \in \mathbb{N}$, $B_n \cap B_m \cap B_{n+m} = \emptyset$.

For each $k \in \mathbb{N}$, pick a sequence $\langle A_{k,t}\rangle_{t=1}^{k}$ of measurable subsets of $(0,1)$ such that for each $t \in \{1,2,\ldots,k\}$, $\mu(A_{k,t}) \geq \alpha_k$ and for all n and m, if $\{n,m,n+m\} \subseteq \{1,2,\ldots,k\}$, then $A_{k,n} \cap A_{k,m} \cap A_{k,n+m} = \emptyset$. (By Theorem 2.5 this can be done.) If $t > k$, let $A_{k,t} = \emptyset$.

Since $\mathcal{P}_f(\mathbb{N})$ is countable, we may choose an increasing sequence $\langle m(n)\rangle_{n=1}^{\infty}$ such that for each $F \in \mathcal{P}_f(\mathbb{N})$, $\lim\limits_{n\to\infty} \mu(\bigcap_{t\in F} A_{m(n),t})$ exists. For $F \in \mathcal{P}_f(\mathbb{N})$ let $\gamma(F) = \lim\limits_{n\to\infty} \mu(\bigcap_{t\in F} A_{m(n),t})$. Since whenever $m(n) \geq t$, $\mu(A_{m(n),t}) \geq \alpha_{m(n)} \geq \eta$, we have that for each t, $\gamma(\{t\}) \geq \eta$.

As in Lemma 3.3 let for each $k \in \mathbb{N}$ and each $F \subseteq \{1,2,\ldots,k\}$,

$$\delta(\{1,2,\ldots,k\},F) = \textstyle\sum_{G\subseteq\{1,2,\ldots,k\}\setminus F} (-1)^{|G|} \cdot \gamma(F \cup G)\,.$$

Let $k \in \mathbb{N}$ and let $F \subseteq \{1,2,\ldots,k\}$. We claim that $\delta(\{1,2,\ldots,k\},F) \geq 0$. Suppose instead that $\delta(\{1,2,\ldots,k\},F) < 0$ and let $\epsilon = -\delta(\{1,2,\ldots,k\},F)$. Let $s = k - |F|$. Pick $n \in \mathbb{N}$ such that $m(n) > k$ and for each $G \subseteq \{1,2,\ldots,k\} \setminus F$,

$|\mu(\bigcap_{t\in F\cup G} A_{m(n),t}) - \gamma(F\cup G)| < \dfrac{\epsilon}{2^s}$. Then by Lemma 3.1

$$\begin{aligned}0 &\le \mu\big(D(F,\langle A_{m(n),t}\rangle_{t=1}^{k})\big)\\ &= \textstyle\sum_{G\subseteq\{1,2,\ldots,k\}\setminus F} (-1)^{|G|}\cdot\mu(\bigcap_{t\in F\cup G} A_{m(n),t})\\ &< \textstyle\sum_{G\subseteq\{1,2,\ldots,k\}\setminus F} \Big((-1)^{|G|}\cdot\gamma(F\cup G)+\dfrac{\epsilon}{2^s}\Big)\\ &= \delta(\{1,2,\ldots,k\},F)+\epsilon\\ &= 0\,,\end{aligned}$$

a contradiction.

For $k\in\mathbb{N}$ and $F\subseteq\{1,2,\ldots,k\}$, choose $a(k,F)$, $b(k,F)$, and B_k as in Lemma 3.3. Then given $H\in\mathcal{P}_f(\mathbb{N})$, if $k=\max H$, then $\mu(\bigcap_{t\in H} B_t)=\gamma(H)$. Then for each $t\in\mathbb{N}$, $\mu(B_t)=\gamma(\{t\})\ge\eta$. Now let $r,s\in\mathbb{N}$. If $m(n)\ge r+s$, then $\bigcap_{t\in\{r,s,r+s\}} A_{m(n),t}=\emptyset$ so $\gamma(\{r,s,r+s\})=0$ and so $\mu\big(\bigcap_{t\in\{r,s,r+s\}} B_t\big)=0$. Since each B_t is a finite union of open intervals, $B_r\cap B_s\cap B_{r+s}=\emptyset$. □

4. Larger semigroups

One can ask whether one can get $\langle A_t\rangle_{t\in\mathbb{R}^+}$ with each A_t a relatively large measurable subset of $(0,1)$ such that for all $t,r\in\mathbb{R}^+$, $A_t\cap A_r\cap A_{t+r}=\emptyset$. We cannot answer this question. However, we shall see in Theorem 4.2 that one can find such sets for $t\in\mathbb{Q}^+$. And we see now that we can come close for $t\in\mathbb{R}^+$.

Theorem 4.1. *Let $\delta,\epsilon>0$. Then there exist sets B_t for all $t\in(\delta,\infty)$ such that each B_t is a finite union of open intervals in $(0,1)$, $B_t\cap B_r\cap B_{t+r}=\emptyset$ whenever $t,r\in(\delta,\infty)$, and $\mu(B_t)>\frac{1}{3}-\epsilon$ whenever $t\in(\delta,\infty)$.*

Proof. For $t\in(1,\infty)$, let $A_t=\{x\in(0,1):\lfloor tx\rfloor+\frac{1}{3}<tx<\lfloor tx\rfloor+\frac{2}{3}\}$. Then for all $t,r\in(1,\infty)$, $A_t\cap A_r\cap A_{t+r}=\emptyset$. Also, if $t\in(1,\infty)$ and $k=\lfloor t-\frac{2}{3}\rfloor$, then

$$\textstyle\bigcup_{s=0}^{k}\frac{1}{t}(s+\frac{1}{3},s+\frac{2}{3})\subseteq A_t\subseteq\bigcup_{s=0}^{k+1}\frac{1}{t}(s+\frac{1}{3},s+\frac{2}{3})$$

and so $\lim\limits_{t\to\infty}\mu(A_t)=\frac{1}{3}$. Pick k such that for all $t\ge k$, $\mu(A_t)>\frac{1}{3}-\epsilon$ and for $t\in(\delta,\infty)$, let $B_t=A_{kt/\delta}$. □

By restricting ourselves to a countable subsemigroup of $(0,\infty)$, we see that we can replace both δ and ϵ by 0 in Theorem 4.1.

Theorem 4.2. *There exist sets B_t for all $t\in\mathbb{Q}^+$ such that each B_t is a finite union of open intervals in $(0,1)$, $B_t\cap B_r\cap B_{t+r}=\emptyset$ whenever $t,r\in\mathbb{Q}^+$, and $\mu(B_t)\ge\frac{1}{3}$ whenever $t\in\mathbb{Q}^+$.*

Proof. Choose $\varphi:\mathbb{N}\xrightarrow[\text{onto}]{\text{1-1}}\mathbb{Q}^+$. For each $n\in\mathbb{N}$, let $\delta(n)=\min\varphi[\{1,2,\ldots,n\}]$. Pick by Theorem 4.1 $\langle B_{n,t}\rangle_{t\in(\delta(n),\infty)}$ such that for all $t\in(\delta(n),\infty)$, $\mu(B_{n,t})>\frac{1}{3}-\frac{1}{n}$ and $B_{n,t}\cap B_{n,r}\cap B_{n,t+r}=\emptyset$ whenever $t,r\in(\delta(n),\infty)$. Choose an increasing sequence $\langle m(n)\rangle_{n=1}^{\infty}$ such that for each $F\in\mathcal{P}_f(\mathbb{N})$, $\lim\limits_{n\to\infty}\mu(\bigcap_{t\in F} B_{m(n),\varphi(t)})$ exists and let $\gamma(F)=\lim\limits_{n\to\infty}\mu(\bigcap_{t\in F} B_{m(n),\varphi(t)})$. Then for each $t\in\mathbb{N}$ one has $\gamma(\{t\})\ge\frac{1}{3}$.

As in the proof of Theorem 1.2 we have that for each $k\in\mathbb{N}$ and each $F\subseteq\{1,2,\ldots,k\}$, $\sum_{G\subseteq\{1,2,\ldots,k\}\setminus F}(-1)^{|G|}\cdot\gamma(F\cup G)\ge 0$. Choose by Lemma 3.3 a sequence $\langle C_t\rangle_{t=1}^{\infty}$ such that each C_t is a finite union of open intervals in $(0,1)$ and for each $F\in\mathcal{P}_f(\mathbb{N})$, $\mu(\bigcap_{t\in F} C_t)=\gamma(F)$. In particular, for each $t\in\mathbb{N}$, $\mu(C_t)\ge\frac{1}{3}$.

For $t \in \mathbb{Q}^+$, let $B_t = C_{\varphi^{-1}(t)}$. To complete the proof, let $t, r \in \mathbb{Q}^+$. Let $F = \{\varphi^{-1}(t), \varphi^{-1}(r), \varphi^{-1}(t+r)\}$. Then $B_t \cap B_r \cap B_{t+r} = \bigcap_{s\in F} C_s$. Also, if $m(n) \geq \max F$, so that for all $s \in F$, $\varphi(s) \geq \delta\big(m(n)\big)$,

$$\bigcap_{s\in F} B_{m(n),\varphi(s)} = B_{m(n),t} \cap B_{m(n),r} \cap B_{m(n),t+r} = \emptyset$$

and so $\gamma(F) = 0$. Therefore $\mu(\bigcap_{s\in F} C_s) = 0$ so $B_t \cap B_r \cap B_{t+r} = \emptyset$. □

We cannot prove that the value $\frac{1}{3}$ in Theorem 4.2 is best possible, but trivially this cannot be raised above α and we know $\alpha \leq \alpha_{41} \approx .36667$

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