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Centered densities and fractal measures

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Abstract. We have collected definitions and basic results for the (centered ball) density in metric space with respect to an arbitrary Hausdorff function. We have kept the definitions general: we do not assume the Hausdorff functions are continuous or blanketed, and we do not assume the metric space is a subset of Euclidean space. We discuss the covering measure (= centered Hausdorff measure) and packing measure defined from these densities.

Contents

Introduction

The Hausdorff measure and the packing measure have been used in the mathematical study of fractal geometry. In Euclidean space, and with the classical Hausdorff functions, there are certain basic facts related to them. Here we will consider their generalization to other metric spaces and other Hausdorff functions. We must take extra care with the definitions for Hausdorff functions that are not continuous, or not blanketed. (We say φ is *blanketed* iff $\limsup_{t\to 0} \varphi(2t)/\varphi(t) < \infty$.) In many cases we will need to consider two or more variants of definitions.

Why do we even consider such general Hausdorff functions?

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(a) *Discontinuous.* When defining a Hausdorff function to fit a particular situation, it is sometimes artificial to impose continuity. For example when a metric takes only a discrete set of values, the natural definitions may yield Hausdorff functions that are piecewise constant.
(b) *Unblanketed.* An infinite-dimensional metric space (that is, a space with Hausdorff dimension $+\infty$) may still admit a Hausdorff function φ for which the Hausdorff measure (or covering measure, packing measure, etc.) is finite. These "infinite-dimensional" Hausdorff functions φ satisfy $\lim_{t\to 0}\varphi(t)/t^a = 0$ for all real a. They are typically unblanketed [23, 3, 13, 14, 27].

The packing and covering measures complement each other nicely. So instead of the usual Hausdorff measure, we have used primarily the covering measure. See Proposition 4.24 for the relation to the Hausdorff measure. When generalizing statements and definitions from Euclidean space to arbitrary metric space, there may be multiple alternative versions which are equivalent in Euclidean space, but not in metric spaces. For example, we will use $\varphi(r)$ rather than $\varphi(2r)$ or $\varphi\big(\operatorname{diam} B_r(x_0)\big)$.

We begin with statements of the results to be considered for generalization. More complete definitions are given below. Let $s > 0$ be real and $d \geq 1$ an integer.

0.1. Density theorem for covering measure ([30, Theorem 1.1(i)])**.** Write $\mathcal{C}^s$ for the s-dimensional covering measure (= centered Hausdorff measure). Let μ be a finite Borel measure on $\mathbb{R}^d$ and write

$$\overline{D}^s_\mu(x) = \limsup_{r\to 0}\frac{\mu\big(B_r(x)\big)}{(2r)^s}$$

for the s-dimensional upper density of μ at $x \in \mathbb{R}^d$. If $E \subseteq \mathbb{R}^d$ is a Borel set, then

$$\mathcal{C}^s(E)\inf_{x\in E}\overline{D}^s_\mu(x) \leq \mu(E) \leq \mathcal{C}^s(E)\sup_{x\in E}\overline{D}^s_\mu(x),$$

provided the products are not 0 times ∞.

0.2. Density theorem for packing measure ([30, Theorem 1.1(ii)], [4])**.** Write $\mathcal{P}^s$ for the s-dimensional packing measure. Let μ be a finite Borel measure on $\mathbb{R}^d$ and write

$$\underline{D}^s_\mu(x) = \liminf_{r\to 0}\frac{\mu\big(B_r(x)\big)}{(2r)^s}$$

for the s-dimensional lower density of μ at $x \in \mathbb{R}^d$. If $E \subseteq \mathbb{R}^d$ is a Borel set, then

$$\mathcal{P}^s(E)\inf_{x\in E}\underline{D}^s_\mu(x) \leq \mu(E) \leq \mathcal{P}^s(E)\sup_{x\in E}\underline{D}^s_\mu(x),$$

provided the products are not 0 times ∞.

0.3. Covering measure as fine variation. Let $C(x,r) = (2r)^s$, and write v^C for the fine variation of this constituent function using the centered-ball basis. Then for all Borel sets $E \subseteq \mathbb{R}^d$ we have

$$\mathcal{C}^s(E) = v^C(E).$$

Reference: [10].

0.4. Packing measure as full variation. Let $C(x,r) = (2r)^s$, and write V^C for the full variation of this constituent function using the centered-ball basis. Then for all Borel sets $E \subseteq \mathbb{R}^d$ we have

$$\mathcal{P}^s(E) = V^C(E).$$

References: [28, 9].

0.5. Product inequalities. Let $k, l \geq 1$ be integers, and let $s, t > 0$ be real numbers. There exists a constant $c > 0$ such that for all Borel sets $E \subseteq \mathbb{R}^k$, $F \subseteq \mathbb{R}^l$,

$$\begin{aligned}
\mathcal{C}^s(E)\mathcal{C}^t(F) &\leq c\,\mathcal{C}^{s+t}(E \times F),\\
\mathcal{C}^{s+t}(E \times F) &\leq c\,\mathcal{C}^s(E)\mathcal{P}^t(F),\\
\mathcal{C}^s(E)\mathcal{P}^t(F) &\leq c\,\mathcal{P}^{s+t}(E \times F),\\
\mathcal{P}^{s+t}(E \times F) &\leq c\,\mathcal{P}^s(E)\mathcal{P}^t(F).
\end{aligned}$$

Under good conditions, these inequalities hold with $c = 1$. References: [2, 26, 20, 29, 15, 16, 21, 8]. Howroyd [20] discusses the generalization to metric space.

We will be interested in proofs using densities.

1. Basic definitions

We begin with definitions and basic results for the (centered ball) density in metric space and an arbitrary Hausdorff function. In particular, we make an effort to use the definitions that will apply in the case of metric space other than Euclidean space, and Hausdorff functions other than simple powers. Sometimes our proofs may seem overly pedantic, because there are many details to take care of. This is particularly true when we attempt to use discontinuous Hausdorff functions.

Hausdorff function. A *Hausdorff function* is a function φ defined on an interval $(0, \delta)$ for some $\delta > 0$ such that:

- $\varphi(t) > 0$ for all $t > 0$.
- If $t_1 < t_2$, then $\varphi(t_1) \leq \varphi(t_2)$.

The examples most often used are the Hausdorff functions of the form

$$\varphi_s(t) = (2t)^s \tag{1}$$

for a constant $s \geq 0$. This is the one we use to discuss "dimension s" in the fractal sense.

Write $\varphi(r+)$ for the right limit $\varphi(r+) = \lim_{t \searrow r} \varphi(t)$. Sometimes we may extend the definition with the convention $\varphi(0) = 0$. But the official definition includes only $(0, \delta)$; so, for example, when we say φ is "right-continuous" we mean to assert that it is right-continuous at positive t, and not that it is right-continuous at 0. The properties of Hausdorff functions that come into play [for densities or for fractal measures] are those that depend on the values $\varphi(t)$ for t near 0. But we may always assume φ has domain $(0, \infty)$ by choosing some $t_0 > 0$ in the domain, and stipulating $\varphi(t) = \varphi(t_0)$ for all $t > t_0$.

Other common examples of Hausdorff functions:

$$\varphi(t) = t^{s_0} \left(\log \frac{1}{t}\right)^{-s_1}, \tag{2}$$

$$\varphi(t) = t^{s_0} \left(\log \frac{1}{t}\right)^{-s_1} \left(\log\log \frac{1}{t}\right)^{-s_2}, \tag{3}$$

$$\varphi(t) = 2^{-M/t^\alpha}. \tag{4}$$

A Hausdorff function φ is called *blanketed* iff

$$\limsup_{t\to 0} \frac{\varphi(2t)}{\varphi(t)} < \infty,$$

or, equivalently, $\sup \{\, \varphi(2t)/\varphi(t) : 0 < t < \delta \,\} < \infty$. And of course

$$\limsup \frac{\varphi(at)}{\varphi(t)} < \infty$$

is true for one constant $a > 1$ if and only if it is true for all constants $a > 1$. Note the Hausdorff functions of the forms (1), (2), (3) are blanketed, but (4) is not blanketed. [The term "blanketed" is from Larman [24]—other terminology can be found in the later literature.]

A Hausdorff function φ will be called *right moderate* iff

$$\limsup_{r\to 0} \frac{\varphi(r+)}{\varphi(r)} < \infty.$$

Note that, in particular, if φ is right-continuous, then φ is right moderate. And if φ is blanketed, then φ is right moderate.

We allow discontinuous Hausdorff functions. But the discontinuity is important only in the unblanketed case. When φ is blanketed, the possibility of discontinuities only changes our fractal measures by at most a constant factor. Indeed, if $\varphi(2t)/\varphi(t) \le M$, then define

$$\varphi_0(t) = \frac{1}{t} \int_t^{2t} \varphi(s)\, ds$$

to get a *continuous* Hausdorff function φ_0 satisfying $\varphi(t) \le \varphi_0(t) \le \varphi(2t) \le M\varphi(t)$. So our fractal measures such as $\mathcal{C}^\varphi$ all satisfy inequalities of the type $\mathcal{C}^\varphi(E) \le \mathcal{C}^{\varphi_0}(E) \le M\mathcal{C}^\varphi(E)$.

Metric space. We will usually write ρ for the metric in any metric space. Notation for open and closed balls in the metric space X:

$$B_r(a) = \{\, x \in X : \rho(x,a) < r \,\}, \qquad \overline{B}_r(a) = \{\, x \in X : \rho(x,a) \le r \,\}.$$

We will assume whenever convenient that our metric space is separable and complete. Therefore, if μ is any finite Borel measure on X and $E \subseteq X$ is a Borel set,

$$\begin{aligned} \mu(E) &= \sup \{\, \mu(F) : F \subseteq E, F \text{ compact} \,\} \\ &= \inf \{\, \mu(V) : V \supseteq E, V \text{ open} \,\}. \end{aligned}$$

Uncountable limit points. A simple variant of the lim sup and lim inf will be used below. They are the limsup and liminf if we ignore countable sets.

Suppose $q(r) \in \mathbb{R}$ is defined for each $r > 0$. Then:

- $\operatorname{u}\limsup_{r\to 0} q(r)$ is the infimum of all α such that, for some $\eta > 0$, we have $q(r) < \alpha$ for all but countably many r with $0 < r < \eta$.
- $\operatorname{u}\limsup_{r\to 0} q(r) \geq \alpha$ means: for all $\varepsilon > 0$ and all $\eta > 0$, there are uncountably many r with $0 < r < \eta$ and $q(r) > \alpha - \varepsilon$.
- $\operatorname{u}\liminf_{r\to 0} q(r)$ is the supremum of all α such that, for some $\eta > 0$, we have $q(r) > \alpha$ for all but countably many r with $0 < r < \eta$.
- $\operatorname{u}\liminf_{r\to 0} q(r) \leq \alpha$ means: for all $\varepsilon > 0$ and all $\eta > 0$, there are uncountably many r with $0 < r < \eta$ and $q(r) < \alpha + \varepsilon$.

Of course when q is continuous, $\operatorname{u}\limsup q(r) = \limsup q(r)$ and $\operatorname{u}\liminf q(r) = \liminf q(r)$.

2. Densities

Let X be a metric space, let $a \in X$, let μ be a finite Borel measure on X, and let φ be a Hausdorff function. The *upper φ-density of μ at a* is

$$\overline{D}^{\varphi}_{\mu}(a) = \limsup_{r\to 0} \frac{\mu\big(B_r(a)\big)}{\varphi(r)}.$$

The *lower φ-density of μ at a* is

$$\underline{D}^{\varphi}_{\mu}(a) = \liminf_{r\to 0} \frac{\mu\big(\overline{B}_r(a)\big)}{\varphi(r)}.$$

If a is an isolated point, then $\overline{B}_r(a) = B_r(a) = \{a\}$ for small enough r, so we have:

- If $\mu(\{a\}) = 0$, then $\overline{D}^{\varphi}_{\mu}(a) = \underline{D}^{\varphi}_{\mu}(a) = 0$.
- If $\mu(\{a\}) > 0$ and $\varphi(0+) = 0$, then $\overline{D}^{\varphi}_{\mu}(a) = \underline{D}^{\varphi}_{\mu}(a) = \infty$.
- If $\mu(\{a\}) > 0$ and $\varphi(0+) > 0$, then $\overline{D}^{\varphi}_{\mu}(a) = \underline{D}^{\varphi}_{\mu}(a) = \mu(\{a\})/\varphi(0+)$.

Although it is not immediate from the definition, we do have (Corollary 2.3) $\underline{D}^{\varphi}_{\mu}(a) \leq \overline{D}^{\varphi}_{\mu}(a)$.

In most cases it won't matter whether we use open or closed balls; for example it does not matter in cases when φ is continuous. But there are simple counterexamples showing that open and closed balls need not yield the same value for the density when φ is discontinuous. Take $X = \mathbb{R}$, and define μ with point-mass 2^{-k} at point 2^{-k} for $k = 1, 2, 3, \dots$. For one example, take $\varphi(r) = \mu(B_r(0))$ to get

$$\limsup_{r\to 0} \frac{\mu\big(\overline{B}_r(0)\big)}{\varphi(r)} = 2 > 1 = \limsup_{r\to 0} \frac{\mu\big(B_r(0)\big)}{\varphi(r)}.$$

For the other example, take $\varphi(r) = \mu(\overline{B}_r(0))$ to get

$$\liminf_{r\to 0} \frac{\mu\big(\overline{B}_r(0)\big)}{\varphi(r)} = 1 > \frac{1}{2} = \liminf_{r\to 0} \frac{\mu\big(B_r(0)\big)}{\varphi(r)}.$$

In both cases, the 1 is what we want. To ignore the countably many bad values, we can use the "uncountable" liminf and limsup. More precisely:

Theorem 2.1. *Let X be a metric space, let $a \in X$, and let μ be a finite Borel measure on X. Write:*

$$\begin{aligned} D_1 &= \limsup_{r\to 0} \frac{\mu\big(B_r(a)\big)}{\varphi(r)} = \overline{D}^{\varphi}_{\mu}(a) \\ D_2 &= \operatorname{u}\limsup_{r\to 0} \frac{\mu\big(B_r(a)\big)}{\varphi(r)} \\ D_3 &= \limsup_{r\to 0} \frac{\mu\big(\overline{B}_r(a)\big)}{\varphi(r)} \\ D_4 &= \operatorname{u}\limsup_{r\to 0} \frac{\mu\big(\overline{B}_r(a)\big)}{\varphi(r)}. \end{aligned}$$

Then $D_1 = D_2 = D_4 \le D_3$. All of them are equal provided φ is right-continuous.

Proof. Comparing $\operatorname{u}\limsup$ to $\limsup$, we get $D_1 \ge D_2$ and $D_3 \ge D_4$. Also, $\overline{B}_r(a) \supseteq B_r(a)$, so $D_3 \ge D_1$ and $D_4 \ge D_2$. And $\overline{B}_r(a) = B_r(a)$ for all but countably many r, so $D_4 = D_2$.

Now we claim $D_4 \ge D_1$. Let $\alpha < D_1$. Let $\eta > 0$ be given. Then there exists $r < \eta$ such that $\mu(B_r(a))/\varphi(r) > \alpha$. Now since $\lim_{s\nearrow r} \mu(\overline{B}_s(a)) = \mu(B_r(a))$, for all s greater than r but sufficiently close to r we have

$$\mu\big(\overline{B}_s(a)\big) > \alpha\varphi(r) \ge \alpha\varphi(s).$$

Thus, there are uncountably many s with $0 < s < \eta$ and $\mu(\overline{B}_s(a))/\varphi(s) > \alpha$. So $D_4 \ge \alpha$. And therefore we conclude $D_4 \ge D_1$.

So we have: $D_4 \ge D_1 \ge D_2 = D_4$, so $D_1 = D_2 = D_4 \le D_3$.

Assume φ is right-continuous. Let $\alpha > D_1$. There is $\eta > 0$ so that for all $r < \eta$, we have $\mu(B_r(a)) < \alpha\varphi(r)$. Taking the limit of this from the right, for all $r < \eta$ we have $\mu(\overline{B}_r(a)) \le \alpha\varphi(r)$. Thus $\limsup_{r\to 0} \mu(\overline{B}_r(a))/\varphi(r) \le \alpha$. So $D_3 \le \alpha$. This shows $D_3 \le D_1$, so that D_3 agrees with the other three values. □

Theorem 2.2. *Let X be a metric space, let $a \in X$, and let μ be a finite Borel measure on X. Write:*

$$\begin{aligned} D_1 &= \liminf_{r\to 0} \frac{\mu\big(B_r(a)\big)}{\varphi(r)} \\ D_2 &= \operatorname{u}\liminf_{r\to 0} \frac{\mu\big(B_r(a)\big)}{\varphi(r)} \\ D_3 &= \liminf_{r\to 0} \frac{\mu\big(\overline{B}_r(a)\big)}{\varphi(r)} = \underline{D}^{\varphi}_{\mu}(a) \\ D_4 &= \operatorname{u}\liminf_{r\to 0} \frac{\mu\big(\overline{B}_r(a)\big)}{\varphi(r)}. \end{aligned}$$

Then $D_1 \le D_2 = D_3 = D_4$. All of them are equal provided φ is left-continuous.

Proof. Comparing $\operatorname{u}\liminf$ to $\liminf$, we get $D_1 \le D_2$ and $D_3 \le D_4$. Also, $\overline{B}_r(a) \supseteq B_r(a)$, so $D_3 \ge D_1$ and $D_4 \ge D_2$. And $\overline{B}_r(a) = B_r(a)$ for all but countably many r, so $D_4 = D_2$.

Now we claim $D_3 \ge D_2$. Let $\alpha > D_3$. Let $\eta > 0$ be given. Then there exists $r < \eta$ such that $\mu(\overline{B}_r(a))/\varphi(r) < \alpha$. Now since $\lim_{s\searrow r} \mu(B_s(a)) = \mu(\overline{B}_r(a))$, for all s less than r but sufficiently close to r we have

$$\mu\big(B_s(a)\big) < \alpha\varphi(r) \le \alpha\varphi(s).$$

Thus, there are uncountably many s with $0 < s < \eta$ and $\mu(B_s(a))/\varphi(s) < \alpha$. So $D_2 \le \alpha$. And therefore we conclude $D_2 \le D_3$.

So we have: $D_2 \le D_3 \le D_4 = D_2$, so $D_2 = D_3 = D_4 \ge D_1$.

Assume φ is left-continuous. Let $\alpha < D_3$. There is $\eta > 0$ so that for all $r < \eta$, we have $\mu(\overline{B}_r(a)) > \alpha\varphi(r)$. Taking the limit of this from the left, for all $r < \eta$ we have $\mu(B_r(a)) \ge \alpha\varphi(r)$. Thus $\liminf_{r\to 0} \mu(B_r(a))/\varphi(r) \ge \alpha$. So $D_1 \ge \alpha$. This shows $D_1 \ge D_3$, so that D_1 agrees with the other three values. □

Corollary 2.3. *For all $a \in X$, $\underline{D}^\varphi_\mu(a) \le \overline{D}^\varphi_\mu(a)$.*

Proof. $\operatorname{u}\liminf \le \operatorname{u}\limsup$. □

Comparing the two theorems, we see a reason for using open balls in the definition of the upper density and closed balls in the definition of the lower density. For the nonstandard densities, write

$$\overline{\Delta}^\varphi_\mu(a) = \limsup_{r\to 0} \frac{\mu\big(\overline{B}_r(a)\big)}{\varphi(r)},$$

$$\underline{\Delta}^\varphi_\mu(a) = \liminf_{r\to 0} \frac{\mu\big(B_r(a)\big)}{\varphi(r)}.$$

Our densities satisfy

$$\underline{\Delta}^\varphi_\mu(a) \le \underline{D}^\varphi_\mu(a) \le \overline{D}^\varphi_\mu(a) \le \overline{\Delta}^\varphi_\mu(a).$$

Proposition 2.4. *The densities $\overline{D}^\varphi_\mu(x)$, $\underline{D}^\varphi_\mu(x)$, $\overline{\Delta}^\varphi_\mu(x)$, and $\underline{\Delta}^\varphi_\mu(x)$ are Borel-measurable functions of x.*

Proof. [10, (1.1)] First we claim: for fixed $r > 0$, the function $x \mapsto \mu(B_r(x))$ is Borel measurable. Indeed, for any $t \in \mathbb{R}$, we claim that

$$V = \big\{\, x \in X : \mu\big(B_r(x)\big) > t \,\big\}$$

is an open set. Let $x_0 \in V$, so that $\mu(B_r(x_0)) > t$. Now $\mu(B_{r-1/n}(x_0)) \nearrow \mu(B_r(x_0))$, so there is n with $\mu(B_{r-1/n}(x_0)) > t$. Then for any $x \in B_{1/n}(x_0)$, we have $B_r(x) \supseteq B_{r-1/n}(x_0)$, so $\mu(B_r(x)) \ge \mu(B_{r-1/n}(x_0)) > t$. So $x \in V$. This shows that V is an open set. Thus, $x \mapsto \mu(B_r(x))$ is Borel measurable (in fact, lower semicontinuous).

Next we claim: for fixed $r > 0$, the function $x \mapsto \mu(\overline{B}_r(x))$ is Borel measurable. This could be proved similarly to the above (in fact, it is upper semicontinuous). Or it can be deduced from the above, since

$$\lim_{n\to\infty} \mu\big(B_{r+1/n}(x)\big) = \mu\big(\overline{B}_r(x)\big).$$

Next we claim: for all $x \in X$ and all $\eta > 0$,

$$\sup\left\{ \frac{\mu\big(B_r(x)\big)}{\varphi(r)} : 0 < r < \eta \right\} = \sup\left\{ \frac{\mu\big(B_r(x)\big)}{\varphi(r)} : 0 < r < \eta, r \in \mathbb{Q} \right\}.$$

Inequality $\ge$ is clear. Let $\alpha < \sup\{\, \mu(B_r(x))/\varphi(r) : 0 < r < \eta \,\}$. So there exists $r_0 \in (0, \eta)$ with $\mu(B_{r_0}(x))/\varphi(r_0) > \alpha$. Now for all $r < r_0$ sufficiently close to r_0 we have $\mu(B_r(x)) > \alpha\varphi(r_0) \geq \alpha\varphi(r)$; in particular there is a rational r that satisfies this. So $\alpha < \sup\{\,\mu(B_r(x))/\varphi(r) : 0 < r < \eta, r \in \mathbb{Q}\,\}$. This proves inequality $\leq$.

For each fixed $r \in \mathbb{Q}$, the function $x \mapsto \mu(B_r(x))/\varphi(r)$ is Borel measurable, so the supremum over all $r \in \mathbb{Q} \cap (0,\eta)$ is Borel measurable. And the value $\sup\{\,\mu(B_r(x))/\varphi(r) : 0 < r < \eta\,\}$ decreases as $\eta > 0$ decreases, so the limit may be taken over rational η. So we conclude that $x \mapsto \overline{D}^{\varphi}_{\mu}(x)$ is Borel measurable.

Next we claim: for all $x \in X$ and all $\eta > 0$,

$$\inf\left\{\frac{\mu\big(\overline{B}_r(x)\big)}{\varphi(r)} : 0 < r < \eta\right\} = \inf\left\{\frac{\mu\big(\overline{B}_r(x)\big)}{\varphi(r)} : 0 < r < \eta, r \in \mathbb{Q}\right\}.$$

Inequality $\leq$ is clear. Let $\alpha > \inf\big\{\,\mu(\overline{B}_r(x))/\varphi(r) : 0 < r < \eta\,\big\}$. So there exists $r_0 \in (0,\eta)$ such that $\mu(\overline{B}_{r_0}(x))/\varphi(r_0) < \alpha$. Now for all $r > r_0$ sufficiently close to r_0 we have $\mu(\overline{B}_r(x)) < \alpha\varphi(r_0) \leq \alpha\varphi(r)$; in particular there is a rational r that satisfies this. So $\alpha > \inf\big\{\,\mu(\overline{B}_r(x))/\varphi(r) : 0 < r < \eta, r \in \mathbb{Q}\,\big\}$. This proves inequality $\geq$.

For each fixed $r \in \mathbb{Q}$, the function $x \mapsto \mu(\overline{B}_r(x))/\varphi(r)$ is Borel measurable, so the infimum over all $r \in \mathbb{Q} \cap (0,\eta)$ is Borel measurable. And the value $\inf\{\,\mu(B_r(x))/\varphi(r) : 0 < r < \eta\,\}$ increases as $\eta > 0$ decreases, so the limit may be taken over rational η. So we conclude that $x \mapsto \underline{D}^{\varphi}_{\mu}(x)$ is Borel measurable.

Now φ is nondecreasing, so it has only countably many discontinuities. Let J be a countable set, dense in $(0,\infty)$, that includes all of the discontinuities of φ.

Next we claim: for all $x \in X$ and all $\eta > 0$,

$$\sup\left\{\frac{\mu\big(\overline{B}_r(x)\big)}{\varphi(r)} : 0 < r < \eta\right\} = \sup\left\{\frac{\mu\big(\overline{B}_r(x)\big)}{\varphi(r)} : 0 < r < \eta, r \in J\right\}.$$

Inequality $\geq$ is clear. Let $\alpha < \sup\big\{\,\mu\big(\overline{B}_r(x)\big)/\varphi(r) : 0 < r < \eta\,\big\}$. So there exists $r_0 \in (0,\eta)$ such that $\mu(\overline{B}_{r_0}(x))/\varphi(r_0) > \alpha$. If $r_0 \in J$, we are done. So assume $r_0 \notin J$. Then φ is continuous at r_0 and $\alpha\varphi(r_0) < \mu(\overline{B}_{r_0}(x))$. So for all $r > r_0$ sufficiently close to r_0, we have $\alpha\varphi(r) < \mu(\overline{B}_{r_0}(x)) \leq \mu(\overline{B}_r(x))$; in particular there is $r \in J \cap (r_0,\eta)$ that satisfies this. So $\sup\big\{\,\mu\big(\overline{B}_r(x)\big)/\varphi(r) : 0 < r < \eta, r \in J\,\big\} > \alpha$. This proves the inequality $\leq$.

For each fixed $r \in J$, the function $x \mapsto \mu(\overline{B}_r(x))/\varphi(r)$ is Borel measurable, so the supremum over all $r \in J \cap (0,\eta)$ is Borel measurable. And the value $\sup\{\,\mu(B_r(x))/\varphi(r) : 0 < r < \eta\,\}$ decreases as $\eta > 0$ decreases, so the limit may be taken over rational η. So we conclude that $x \mapsto \overline{\Delta}^{\varphi}_{\mu}(x)$ is Borel measurable.

Next we claim: for all $x \in X$ and all $\eta > 0$,

$$\inf\left\{\frac{\mu\big(B_r(x)\big)}{\varphi(r)} : 0 < r < \eta\right\} = \inf\left\{\frac{\mu\big(B_r(x)\big)}{\varphi(r)} : 0 < r < \eta, r \in J\right\}.$$

Inequality $\leq$ is clear. Let $\alpha > \inf\big\{\,\mu\big(B_r(x)\big)/\varphi(r) : 0 < r < \eta\,\big\}$. So there exists $r_0 \in (0,\eta)$ such that $\mu(B_{r_0}(x))/\varphi(r_0) < \alpha$. If $r_0 \in J$, we are done. So assume $r_0 \notin J$. Then φ is continuous at r_0 and $\alpha\varphi(r_0) > \mu(B_{r_0}(x))$. So for all $r < r_0$ sufficiently close to r_0, we have $\alpha\varphi(r) > \mu(B_{r_0}(x)) \geq \mu(B_r(x))$; in particular there is $r \in J \cap (0,r_0)$ that satisfies this. So $\inf\big\{\,\mu\big(B_r(x)\big)/\varphi(r) : 0 < r < \eta, r \in J\,\big\} < \alpha$. This proves the inequality $\geq$.

For each fixed $r \in J$, the function $x \mapsto \mu(B_r(x))/\varphi(r)$ is Borel measurable, so the infimum over all $r \in J \cap (0,\eta)$ is Borel measurable. And the value

$\inf\{\,\mu(B_r(x))/\varphi(r) : 0 < r < \eta\,\}$ increases as $\eta > 0$ decreases, so the limit may be taken over rational η. So we conclude that $x \mapsto \underline{\Delta}^{\varphi}_{\mu}(x)$ is Borel measurable. $\square$

3. Variations

We will use the Thomson–Henstock type "full" and "fine" variations with respect to the centered ball derivation basis ([17, 31, 32]). Sometimes we use open balls and sometimes we use closed balls. As we have seen, the uncountable limit points $\operatorname{u\,lim\,sup}$ and $\operatorname{u\,lim\,inf}$ may be used in this connection. But we have gone to that much trouble only to allow the possibility that the Hausdorff function φ is not continuous.

A *constituent* is an ordered pair (x, r) with $x \in X$ and $r > 0$. It represents the ball centered at x with radius r. In a general metric space the center x and/or radius r are not uniquely determined by the point-set $\overline{B}_r(x)$ or $B_r(x)$.

Let $E \subseteq X$. A *centered closed ball packing* of E is a collection π of constituents such that $x \in E$ for all $(x, r) \in \pi$, and $\rho(x, x') > r + r'$ for all $(x, r), (x', r') \in \pi$ with $(x, r) \neq (x', r')$. Note that this implies that the corresponding closed balls $\overline{B}_r(x)$ are pairwise disjoint. But more than that: if X is embedded isometrically in a larger metric space, and the constituents are interpreted to represent closed balls in that metric space, they are still disjoint.

A *centered closed ball relative packing* of E is a collection π of constituents such that $x \in E$ for all $(x, r) \in \pi$, and $\overline{B}_r(x) \cap \overline{B}_{r'}(x') = \varnothing$ for all $(x, r), (x', r') \in \pi$ with $(x, r) \neq (x', r')$. This is called *pseudo-packing* in [30].

A *centered closed ball weak packing* of E is a collection π of constituents such that $x \in E$ for all $(x, r) \in \pi$, and $\rho(x, x') > r \vee r'$ for all $(x, r), (x', r') \in \pi$ with $(x, r) \neq (x', r')$. Note that this is equivalent to $x' \notin \overline{B}_r(x)$ and $x \notin \overline{B}_{r'}(x')$. This is called *pseudo-packing* in [19].

If we just say "packing", we will mean centered closed ball packing. Of course in Euclidean space, $\rho(x, x') > r + r'$ is equivalent to $\overline{B}_r(x) \cap \overline{B}_r(x') = \varnothing$. So when *packing measure* was defined, it did not matter which of these two definitions was used. Any metric space X may be embedded isometrically into a larger metric space in which $\overline{B}_r(x) \cap \overline{B}_{r'}(x') = \varnothing$ if and only if $\rho(x, x') > r + r'$. Saint Raymond & Tricot [30] used the term *pseudo-packing* for our relative packing, and showed that (for blanketed Hausdorff functions and subsets of Euclidean space) the two packing measures agree. Das [5] examines more general spaces where equality of packing and pseudo-packing measures remains valid.

A *centered open ball packing* of E is a collection π of constituents such that $x \in E$ for all $(x, r) \in \pi$, and $\rho(x, x') \geq r + r'$ for all $(x, r), (x', r') \in \pi$ with $(x, r) \neq (x', r')$. Note that this implies that the corresponding open balls $B_r(x)$ are pairwise disjoint, even when interpreted in a larger metric space.

A *centered open ball relative packing* of E is a collection π of constituents such that $x \in E$ for all $(x, r) \in \pi$, and $B_r(x) \cap B_{r'}(x') = \varnothing$ for all $(x, r), (x', r') \in \pi$ with $(x, r) \neq (x', r')$.

A *centered open ball weak packing* of E is a collection π of constituents such that $x \in E$ for all $(x, r) \in \pi$, and $\rho(x, x') \geq r \vee r'$ for all $(x, r), (x', r') \in \pi$ with $(x, r) \neq (x', r')$. This is equivalent to $x' \notin B_r(x)$ and $x \notin B_{r'}(x')$.

Vitali theorems. Let X be a metric space, and let $E \subseteq X$. A *fine cover* of E is a collection β of constituents such that: $x \in E$ for every $(x, r) \in \beta$, and for every

$x \in E$ and every $\delta > 0$, there exists $r > 0$ such that $r < \delta$ and $(x, r) \in \beta$. A collection β of constituents is a *very fine cover* of E iff: $x \in E$ for every $(x, r) \in \beta$ and for every $\delta > 0$ there are uncountably many r with $0 < r < \delta$ and $(x, r) \in \beta$.

Next is a standard Vitali theorem. But care is taken to make sure the proof allows the packing as defined here.

Theorem 3.1. *Let X be a metric space, let $E \subseteq X$ be a subset, and let β be a fine cover of E. Then there exists either:*

(a) *an infinite (centered closed ball) packing $\{(x_i, r_i)\} \subseteq \beta$ such that* $\inf r_i > 0$, *or*

(b) *a countable (possibly finite) centered closed ball packing $\{(x_i, r_i)\} \subseteq \beta$ such that for all $n \in \mathbb{N}$,*

$$E \setminus \bigcup_{i=1}^{n} \overline{B}_{r_i}(x_i) \subseteq \bigcup_{i=n+1}^{\infty} \overline{B}_{3r_i}(x_i).$$

Proof. We define recursively a sequence (x_n, r_n) of constituents and a decreasing sequence of fine covers $\beta_n \subseteq \beta$.

Let $\beta_1 = \{\, (x, r) \in \beta : r \leq 1 \,\}$. Then β_1 is again a fine cover of E. Define

$$t_1 = \sup \{\, r : (x, r) \in \beta_1 \,\},$$

and then choose $(x_1, r_1) \in \beta_1$ with $r_1 \geq t_1/2$. Now suppose (x_1, r_1), $(x_2, r_2), \dots,$ (x_n, r_n) and $\beta_1, \beta_2, \dots, \beta_n$ have been chosen. Let

$$\beta_{n+1} = \{\, (x, r) \in \beta_n : \rho(x, x_n) > r + r_n \,\}.$$

If β_{n+1} is empty, the construction terminates. If β_{n+1} is not empty, define

$$t_{n+1} = \sup \{\, r : (x, r) \in \beta_{n+1} \,\},$$

and choose $(x_{n+1}, r_{n+1}) \in \beta_{n+1}$ with $r_{n+1} \geq t_{n+1}/2$. This completes the recursive construction.

Consider the case where the construction terminates, say $\beta_{n+1} = \varnothing$. We claim that $E \subseteq \bigcup_{i=1}^{n} \overline{B}_{r_i}(x_i)$. Indeed, if $x \in E \setminus \bigcup_{i=1}^{n} \overline{B}_{r_i}(x_i)$, then $\rho(x, x_i) - r_i > 0$ for $i = 1, \dots, n$, and thus

$$\varepsilon = \min \{\, \rho(x, x_i) - r_i : 1 \leq i \leq n \,\} > 0$$

and there is $(x, r) \in \beta_n$ with $0 < r < \varepsilon$, so $\beta_{n+1} \neq \varnothing$. So in case the construction terminates, (b) holds.

So suppose the construction does not terminate, and (a) is false. We must prove (b). Fix j, and let $x \in E \setminus \bigcup_{i=1}^{j} \overline{B}_{r_i}(x_i)$. We must prove that $x \in \bigcup_{i=j+1}^{\infty} \overline{B}_{3r_i}(x_i)$. Just as before, $\min \{\, \rho(x, x_i) - r_i : 1 \leq i \leq j \,\} > 0$ and β_1 is a fine cover of E, so there is $r_0 > 0$ with $(x, r_0) \in \beta_1$ and $\rho(x, x_i) > r_0 + r_i$ for $i = 1, \dots, j$.

Now $r_n \to 0$, so there exists a least n with $r_n < (1/2)r_0$. We claim that there is $i < n$ with $\rho(x, x_i) \leq r_0 + r_i$. Indeed, if not then $(x, r_0) \in \beta_n$, so $t_n \geq r_0 > 2r_n \geq t_n$, a contradiction. This i satisfies $j < i < n$. Because $i < n$, we have $r_i \geq (1/2)r_0$. Then $\rho(x, x_i) \leq r_0 + r_i \leq 3r_i$. That is, $x \in \overline{B}_{3r_i}(x_i)$ with $i \geq j + 1$ as claimed. $\square$

We will need the following specialized variant later for weak packing. A fine cover β is *upward closed* iff, for every sequence $(x_n, r_n) \in \beta$ such that $x_n \to x$ and $r_n \nearrow r$, it follows that $(x, r) \in \beta$.

Lemma 3.2. *Let X be a metric space, let $E \subseteq X$ be compact subset, let β be a upward closed fine cover of E. Then there is a finite centered open ball weak packing $\pi \subseteq \beta$ such that $E \subseteq \bigcup_{(x,r)\in\pi} B_r(x)$.*

Proof. First, $\beta_1 = \{\, (x,r) \in \beta : r \le 1 \,\}$ is also a upward closed fine cover of E. Because it is upward closed and E is compact, $\{\, r : (x,r) \in \beta_1 \,\}$ achieves a maximum value. Let $(x_1, r_1) \in \beta_1$ be such that $r_1 = \sup\{\, r : (x,r) \in \beta_1 \,\}$. Next, let $\beta_2 = \{\, (x,r) \in \beta_1 : \rho(x_1, r) \ge r_1 \,\}$. Now $E_2 = \{\, x \in E : \rho(x_1, r) \ge r_1 \,\}$ is compact, so β_2 is a upward closed fine cover of E_2. There is $(x_2, r_2) \in \beta_2$ such that $r_2 = \sup\{\, r : (x,r) \in \beta_2 \,\}$. Note $r_2 \le r_1$, and $\{(x_1, r_1), (x_2, r_2)\}$ is a centered open ball weak packing for E. Next $\beta_3 = \{\, (x,r) \in \beta_2 : \rho(x_2, x) \ge r_2 \,\}$ is a upward closed fine cover of $\{\, x \in E : \rho(x, x_1) \ge r_1, \rho(x, x_2) \ge r_2 \,\}$. There is $(x_3, r_3) \in \beta_3$ such that $r_3 = \sup\{\, r : (x,r) \in \beta_3 \,\}$. Note $r_3 \le r_2$, and $\{(x_1, r_1), (x_2, r_2), (x_3, r_3)\}$ is a centered open ball weak packing for E. Suppose that we have defined $\{(x_1, r_1), \dots, (x_n, r_n)\}$ a centered open ball weak packing, $r_1 \ge r_2 \ge \cdots \ge r_n$. Let $\beta_{n+1} = \{\, (x,r) : \rho(x_n, x) \ge r_n \,\}$. If $\beta_{n+1} = \varnothing$, the construction terminates. If not, choose (x_{n+1}, r_{n+1}) so that $r_{n+1} = \sup\{\, r : (x,r) \in \beta_{n+1} \,\}$.

So if the construction never terminates, we end up with an infinite weak packing $\pi = \{(x_i, r_i)\}$. Note that (by compactness or total boundedness) $r_i \to 0$. We claim that $E \subseteq \bigcup_{i=1}^\infty B_{r_i}(x_i)$. Let $x \in E$. There is $r_0 \le 1$ so that $(x, r_0) \in \beta$. Since $r_i \to 0$, there is a least m so that $r_m < r_0$. Then we claim $\rho(x_i, x) < r_i$ for some $i < m$: if not, then $(x, r_0) \in \beta_m$ so $r_m \ge r_0$, a contradiction. But then $\rho(x_i, x) < r_i$ means $x \in B_{r_i}(x_i)$.

Finally, by compactness, this open cover has a finite subcover, so in fact there is a finite weak packing that covers E. (That is, the construction terminates at some finite stage.) □

Vitali properties. Let X be a metric space. Let μ be a Borel measure on X. Then we say that μ has the *Strong Vitali Property* iff, for every Borel set $E \subseteq X$ and every fine cover β of E, there exists a (countable) centered closed ball packing $\pi \subseteq \beta$ such that

$$\mu\left(E \setminus \bigcup_{(x,r)\in\pi} \overline{B}_r(x)\right) = 0.$$

We say that the packing π *almost covers* the set E.

We say that the metric space X has the *Strong Vitali Property* (or SVP) iff every finite Borel measure on X has the SVP. Das [5] argues that what we really want is a property of the support of the measure μ and not a property of the space X itself.

Let X be a metric space. Let μ be a Borel measure on X. Then we say that μ has the *Weak Vitali Property* iff, for every Borel set $E \subseteq X$ and every fine cover β of E, there exists a centered closed ball weak packing $\pi \subseteq \beta$ such that

$$\mu\left(E \setminus \bigcup_{(x,r)\in\pi} \overline{B}_r(x)\right) = 0.$$

We say that the metric space X has the *Weak Vitali Property* (or WVP) iff every finite Borel measure on X has the WVP.

Vitali showed that Lebesgue measure in Euclidean space has the SVP. Besicovitch [1] generalized that to every finite measure so that (as defined here) Euclidean space has the SVP. Davies [7] gave an example of a metric space where the SVP fails. Larman [25] defined a notion of "finite-dimensional" metric space where the Besicovitch proof will establish the SVP. Das [6] formulated a "Besicovitch weak-packing property" that similarly implies the WVP.

The Strong Vitali Property yields a version for *open* ball packings if we use a very fine cover.

Proposition 3.3. *Let X be a metric space. Let μ be a finite Borel measure on X. Let $E \subseteq X$ be a Borel set and let β be a very fine cover of E. Assume μ has the Strong Vitali Property. Then there is a centered open ball packing $\pi \subseteq \beta$ such that*

$$\mu\left(E \setminus \bigcup_{(x,r)\in\pi} B_r(x)\right) = 0.$$

Proof. For a fixed point x, the sets

$$S_r(x) = \{\, y \in X : \rho(x,y) = r \,\}$$

are pairwise disjoint closed sets. Thus, only countably many have positive measure. Therefore

$$\beta_1 = \big\{\, (x,r) \in \beta : \mu\big(S_r(x)\big) = 0 \,\big\}$$

is a fine cover of E. Therefore, by the SVP, there exists a centered closed ball packing $\pi \subseteq \beta_1$ with

$$\mu\left(E \setminus \bigcup_{(x,r)\in\pi} \overline{B}_r(x)\right) = 0.$$

Now $\pi \subseteq \beta$, and π is also a centered open ball packing. Because of the definition of β_1, we have $\mu(S_r(x)) = 0$ for all $(x,r) \in \pi$, and therefore $\mu\big(E \setminus \bigcup B_r(x)\big) = 0$. □

The same proof will show:

Proposition 3.4. *Let X be a metric space. Let μ be a finite Borel measure on X. Let $E \subseteq X$ be a Borel set and let β be a very fine cover of E. Assume μ has the Weak Vitali Property. Then there is a centered open ball weak packing $\pi \subseteq \beta$ such that*

$$\mu\left(E \setminus \bigcup_{(x,r)\in\pi} B_r(x)\right) = 0.$$

We can eliminate the WVP if we add a hypothesis on the fine cover β. Recall that a fine cover β is *upward closed* if, for every sequence $(x_n, r_n) \in \beta$ such that $x_n \to x$ and $r_n \nearrow r$, it follows that $(x,r) \in \beta$.

Proposition 3.5. *Let X be a complete separable metric space, let $E \subseteq X$ be a Borel set, let μ be a finite Borel measure, and let β be a upward closed fine cover of E. Then there is a centered open ball weak packing $\pi \subseteq \beta$ with $\mu\big(E \setminus \bigcup_\pi B_r(x)\big) = 0$.*

Proof. There is a compact $F \subseteq E$ with $\mu(F) > (1/2)\mu(E)$, and by Lemma 3.2 there is a weak packing $\{(x_1, r_1), \dots, (x_{n_1}, r_{n_1})\} \subseteq \beta$ with $F \subseteq \bigcup_{i=1}^{n_1} B_{r_i}(x_i)$, so $\mu\big(E \setminus \bigcup B_{r_i}(x_i)\big) < (1/2)\mu(E)$. Now $E_2 = E \setminus \bigcup_{i-1}^{n_1} B_{r_i}(x_i)$ is compact, and

$$\beta_2 = \{\, (x, r) \in \beta : x \in E_2, r \leq \min(\rho(x, x_1), \dots, \rho(x, x_{n_1}))\,\}$$

is a upward closed fine cover of E_2, so we may repeat to get a weak packing

$$\{(x_{n_1+1}, r_{n_1+1}), \dots, (x_{n_2}, r_{n_2})\}$$

of E_2 with $\mu\big(E \setminus \bigcup_{i=1}^{n_2} B_{r_i}(x_i)\big) < (1/4)\mu(E)$. Continue in this way. □

Example 3.6 (Ultrametric product space)**.** We consider an example. We will use it again many times. First (Example 3.7) it will provide an example showing that the Strong Vitali Property in the sense of centered closed ball packings is not the same as the sense of centered closed ball relative packings.

Begin with positive integers $k_1, k_2, \dots,$ all ≥ 2. For each n let G_n be a finite set with k_n elements. Let $\Omega = \prod_{n=1}^{\infty} G_n$ be the infinite Cartesian product. Let positive numbers ρ_n be given, with $1 \geq \rho_1 > \rho_2 > \cdots$ and $\lim \rho_n = 0$. Cylinders $\Omega(x_1, x_2, \dots, x_n)$ consist of all elements of Ω where the first n coordinates have these fixed values. Define a metric ρ on Ω so that $\rho(x, x) = 0$ and $\rho(x, y) = \rho_n$ if x and y first differ in the nth coordinate. So Ω is a compact ultrametric space.

For $m \in \mathbb{N}$, write $\mathcal{F}_m$ for the collection of all subsets of the product Ω that depend on the first m coordinates. That is, $\mathcal{F}_m$ consists of the sets that may be written as a union cylinders $\Omega(x_1, x_2, \dots, x_m)$ of generation m; or equivalently a union of open balls of radius ρ_m.

Define the *uniform measure* μ on Ω so that

$$\mu\big(\Omega(x_1, \dots, x_n)\big) = \gamma_n,$$

where $\gamma_n = 1/K_n$, $K_n = k_1 k_2 \cdots k_n$. Note that any two cylinders in generation n are isometric to each other, so if any of the common fractal measures happens to be positive and finite on Ω, then it must be a constant multiple of this uniform measure. The open ball $B_r(x)$: for $\rho_{n+1} < r \leq \rho_n$, we have $B_r(x) = B_{\rho_n}(x) = \overline{B}_{\rho_{n+1}}(x)$ and $\mu(B_r(x)) = \gamma_n$. The closed ball $\overline{B}_r(x)$: For $\rho_{n+1} \leq r < \rho_n$, we have $\overline{B}_r(x) = \overline{B}_{\rho_{n+1}}(x) = B_{\rho_n}(x)$ and $\mu(\overline{B}_r(x)) = \gamma_n$. In particular, $\mu(B_{\rho_n}(x)) = \gamma_n$ and $\mu(\overline{B}_{\rho_n}(x)) = \gamma_{n-1}$.

Note. The Davies example, as in [7], can be thought of as an ultrametric product space of this type, with extra points added so that certain balls (disjoint in the relative sense but not in the absolute sense) are made nondisjoint, and therefore so that it fails the SVP even in the relative sense. The set of points with only peripheral coordinates is the product space.

Example 3.7 (Ultrametric product space: Failure of SVP)**.** For each $\delta > 0$, there are only finitely many distinct balls with radius $\geq \delta$. Closed balls are open sets. For any two balls in Ω, either they are disjoint or one contains the other. The SVP in the sense of relative packings follows. But SVP in the sense used here is false in Ω for certain choices of k_n and ρ_n, as we will see below.

In the space Ω with $\rho_n = 1/2^n$, the question of whether the uniform measure μ has the SVP depends on the sequence k_n. Bounded k_n is "finite-dimensional" and unbounded k_n is "infinite-dimensional" in ways we will see.

Proposition 3.8. *Let Ω be an ultrametric product space with $\rho_n = 1/2^n$. Assume $\{k_n\}$ is bounded. Then μ has the SVP.*

Proof. Say $k_n \le k$ for all n. If $1/2^n \le r < 1/2^{n-1}$, then each closed ball of radius r is the union of $k_n \le k$ closed balls of radius $r/2$.

If β is a fine cover of E, apply Theorem 3.1 to get a packing $(x_i, r_i) \subseteq \beta$ such that

$$E \setminus \bigcup_{i=1}^{n} \overline{B}_{r_i}(x_i) \subseteq \bigcup_{i=n+1}^{\infty} \overline{B}_{3r_i}(x_i)$$

for all n. Now the sets $\overline{B}_{r_i}(x_n)$ are disjoint, so $\sum \mu(\overline{B}_{r_i}(x_i)) < \infty$. Each ball of radius $3r_i$ is covered by at most k^2 balls of radius r_i, so $\sum \mu(\overline{B}_{3r_i}(x_i)) < \infty$. So we get $\mu\big(E \setminus \bigcup_{i=1}^{\infty} \overline{B}_{r_i}(x_i)\big) = 0$. $\square$

Remark. Ω itself does not have the SVP. With $k_n = 2$ for all n, we can take a "biased coin" measure ν and for E the set obeying the Strong Law of Large Numbers for that measure. The set β of (x, r) where $\nu(\overline{B}_r(x)) < (1/10)\mu(\overline{B}_r(x))$ is a fine cover, but any packing $\pi \subseteq \beta$ has $\sum \nu(\overline{B}_r(x)) < 1/10$.

Proposition 3.9. *Let Ω be an ultrametric product space with $\rho_n = 1/2^n$. Assume k_n is unbounded. Then the uniform measure μ fails the SVP.*

Proof. Choose a sequence $n_1 < n_2 < n_3 < \cdots$ so that $k_{n_j} > j^2$. Fix $m \in \mathbb{N}$. Define a fine cover

$$\beta_m = \left\{\, (x, r) : r = 1/2^{n_j+1} \text{ for some } j \ge m \,\right\}.$$

Let $\pi \subseteq \beta_m$ be a centered closed ball packing. For a given j, if $(x, 1/2^{n_j+1})$, $(x', 1/2^{n_j+1})$ both belong to π, then $\rho(x, x') > 2/2^{n_j+1} = 1/2^{n_j}$ so $\rho(x, x') \ge 1/2^{n_j-1}$. Therefore x and x' differ in some coordinate from 1 to $n_j - 1$. So for fixed j, there are at most K_{n_j-1} consituents (x, r) in π with $r = 1/2^{n_j+1}$. And the measure $\mu(\overline{B}_r(x))$ is γ_{n_j}. So for the entire packing π we have

$$\mu\left(\bigcup_{(x,r)\in\pi} \overline{B}_r(x)\right) \le \sum_{j=m}^{\infty} K_{n_j-1}\gamma_{n_j} = \sum_{j=m}^{\infty} \frac{1}{k_{n_j}} \le \sum_{j=m}^{\infty} \frac{1}{j^2}.$$

For large m this is < 1, so there is no packing $\subseteq \beta_m$ that almost covers Ω. The SVP fails for the measure μ. $\square$

Recall that if any of the common fractal measures happens to be positive and finite on Ω, then it must be a constant multiple of this uniform measure. So (at least in these "infinite-dimenional" ultrametric product spaces) the Strong Vitali Property fails for all of the measures we use in fractal geometry.

Sometimes the following proposition will be used in place of the SVP.

Proposition 3.10. *Let Ω be an ultrametric product space with $\rho_n = 1/2^n$. Assume $k_n \to \infty$. Let β be a fine cover of Ω. Then there is a centered closed ball packing $\pi \subseteq \beta$ such that $\sum_{(x,r)\in\pi} \mu(\overline{B}_{2r}(x)) = \infty$.*

Proof. If $1/2^{n+1} \le r < 1/2^n$, write $r^+ = 1/2^n$ so $2r^+ = 1/2^{n-1}$, so that $\overline{B}_r(x) = B_{r^+}(x) \in \mathcal{F}_n$ and $\overline{B}_{2r}(x) = B_{2r^+}(x) \in \mathcal{F}_{n-1}$.

Claim. Let $U \in \mathcal{F}_m$ and let $\beta^* \subseteq \beta$ be a fine cover of U with $r < 1/2^m$ for all $(x,r) \in \beta^*$. Then there is a finite set $\pi^* \subseteq \beta^*$ such that

$$U = \bigcup_{(x,r)\in\pi^*} B_{2r^+}(x) \qquad \text{and} \qquad B_{2r^+}(x) \text{ are pairwise disjoint.}$$

Note that $\{\, B_{2r^+}(x) : (x,r) \in \beta^* \,\}$ is an open cover of the compact set U. So there is a finite subcover. By the ultrametric property, for any two balls, either one contains the other or they are disjoint. So there is a further subcover that is pairwise disjoint. This proves the claim.

We proceed recursively. To begin, let $U_0 = \Omega$ and $m_0 \in \mathbb{N}$ so that $k_n \geq 2$ for all $n \geq m_0$. Now

$$\beta_0 = \left\{\, (x,r) \in \beta : r < \frac{1}{2^{m_0+2}} \,\right\}$$

is a fine cover of Ω. Apply the claim to get a finite $\pi_1 \subseteq \beta_0$ with $B_{2r^+}(x)$ pairwise disjoint and union U_0. Let $s_1 = \min\{\, r^+ - r : (x,r) \in \pi_1 \,\} > 0$. Then

$$\sum_{\pi_1} \mu\big(B_{2r^+}(x)\big) = 1.$$

For each $(x,r) \in \pi_1$ we have $r < 1/2^{m_0+2}$, so $\mu(B_{r^+}(x))/\mu(B_{2r^+}(x)) \leq 1/2$. Thus $\sum_{\pi_1} \mu\big(B_{r^+}(x)\big) \leq 1/2$. Let

$$U_1 = U_0 \setminus \bigcup_{(x,r)\in\pi_1} B_{r^+}(x).$$

So $\mu(U_1) \geq 1/2$.

Next, let $m_1 > m_0$ so that: $1/2^{m_1} < s_1$; $U_1 \in \mathcal{F}_{m_1}$; and $k_n \geq 3$ for all $n \geq m_1$. Then

$$\beta_1 = \left\{\, (x,r) \in \beta_0 : x \in U_1, r < \frac{1}{2^{m_1+2}} \,\right\}$$

is a fine cover of U_1. Apply the claim to get a finite $\pi_2 \subseteq \beta_1$ with $B_{2r^+}(x)$ pairwise disjoint and union U_1. Let $s_2 = \min\{\, r^+ - r : (x,r) \in \pi_1 \cup \pi_2 \,\} > 0$. Then

$$\sum_{\pi_2} \mu\big(B_{2r^+}(x)\big) = \mu(U_1) \geq \frac{1}{2}.$$

Let

$$U_2 = U_1 \setminus \bigcup_{(x,r)\in\pi_2} B_{r^+}(x).$$

Then $\mu(U_2) \geq (2/3)\mu(U_1) \geq (2/3)(1/2) = 1/3$.

Continue. Let $m_2 > m_1$ so that: $1/2^{m_2} < s_2$; $U_2 \in \mathcal{F}_{m_2}$; and $k_n \geq 4$ for all $n \geq m_2$. Let

$$\beta_2 = \left\{\, (x,r) \in \beta_1 : x \in U_2, r < \frac{1}{2^{m_2+2}} \,\right\}.$$

As before get finite $\pi_3 \subseteq \beta_2$ and s_3 with

$$\sum_{\pi_3} \mu\big(B_{2r^+}(x)\big) = \mu(U_2) \geq \frac{1}{3}.$$

Let

$$U_3 = U_2 \setminus \bigcup_{(x,r)\in\pi_3} B_{r^+}(x),$$

so $\mu(U_3) \geq (3/4)\mu(U_2) \geq (3/4)(1/3) = 1/4$. And so on.

After defining π_j recursively, let $\pi = \bigcup_{j=1}^{\infty} \pi_j$. We claim π is a centered closed ball packing. Let $(x,r),(x',r') \in \pi$, $r \geq r'$. If they are in the same π_j, then $B_{2r^+}(x) \cap B_{2r'^+}(x') = \varnothing$ so $\rho(x,x') \geq 2r^+ > r+r'$. On the other hand, if $(x,r) \in \pi_j$ and $(x',r') \in \pi_{j'}$ with $j' > j$, then $x' \notin B_{r+}(x)$ so $\rho(x,x') \geq r^+ = r + (r^+ - r) \geq r + s_j > r + r'$. Finally,

$$\sum_{(x,r)\in\pi} \mu\big(\overline{B}_{2r}(x)\big) = \sum_{j=1}^{\infty} \sum_{(x,r)\in\pi_j} \mu\big(B_{2r^+}(x)\big) \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty. \qquad \square$$

Full variation. A *gauge* for E is a function $\Delta\colon E \to (0,\infty)$. A packing π of E is said to be Δ-*fine* iff $r < \Delta(x)$ for all $(x,r) \in \pi$.

We begin with a "constituent function" $C\colon X \times (0,\infty) \to [0,\infty)$. Define

$$\overline{V}^C_\Delta(E) = \sup \sum_{(x,r)\in\pi} C(x,r),$$

where the supremum is over all Δ-fine (centered closed ball) packings π of E. Note that when Δ decreases, the value $\overline{V}^C_\Delta(E)$ decreases. The (centered closed ball) *full variation* of C on E is defined as the limit as $\Delta \to 0$:

$$\overline{V}^C(E) = \inf \left\{ \overline{V}^C_\Delta(E) : \Delta \text{ is a gauge on } E \right\}.$$

Similar definitions may be given for $V^C(E)$ using centered open ball packings and $\widetilde{V}^C(E)$ using centered open ball weak packings. (Also centered closed ball weak packings, but we do not use them here.) When the constituent function C is of the form $C(x,r) = \varphi(r)$ for some Hausdorff function φ, we may write $\overline{V}^C = \overline{V}^\varphi$, etc.

Proposition 3.11. *$\overline{V}^C$, V^C, and $\widetilde{V}^C$ are metric outer measures.*

Proof. [11, (1.1.16)]. The proof works in all three cases. $\square$

Remark. $\overline{V}^C \leq V^C \leq \widetilde{V}^C$, since every centered closed ball packing is a centered open ball packing, and every centered open ball packing is a centered open ball weak packing. If $C(x,r)$ is left-continuous in r for every x, then then $V^C = \overline{V}^C$, since in any centered open ball packing, we may approximate all of the balls from inside by open balls as closely as we like.

Example 3.12 (Ultrametric product space: $\overline{V}^\varphi(\Omega) < V^\varphi(\Omega)$)**.** Let $\Omega = \prod G_n$ be the ultrametric product space with $k_n = n^2$, $K_n = (n!)^2$, $\gamma_n = 1/(n!)^2$, and $\rho_n = 1/2^n$. Let φ be the discontinuous Hausdorff function defined by $\varphi(r) = \gamma_n$ for $1/2^{n+1} \leq r < 1/2^n$. We will show that $\overline{V}^\varphi(\Omega) = 0$, $V^\varphi(\Omega) = 1$ and $\widetilde{V}^\varphi(\Omega) = \infty$.

We first show $V^\varphi(\Omega) \geq 1$. Let Δ be a gauge on Ω. Then

$$\beta = \left\{ (x,r) : x \in \Omega, r = \frac{1}{2^n} < \Delta(x) \text{ for some } n \right\}$$

is a fine cover of Ω. Then $\{B_{2r}(x) : (x,r) \in \beta\}$ is an open cover of the compact set Ω, and there is a finite subcover. By the ultrametric property, there is a further subcover such that the sets are disjoint. So we get a finite set $\pi \subseteq \beta$ such that $B_{2r}(x)$ are disjoint with union Ω. We claim π is a centered open ball cover of Ω. If $(x,r),(x',r') \in \pi$, $r \geq r'$, then $x' \notin B_{2r}(x)$, so $\rho(x,x') \geq 2r \geq r + r'$. Thus π is a

Δ-fine centered open ball packing of Ω. If $r = 1/2^n$, then $\mu(B_{2r}(x)) = \gamma_{n-1} = \varphi(r)$. Now $\sum_{(x,r)\in\pi} \varphi(r) = \sum \mu(B_{2r}(x)) = 1$. So $V^\varphi_\Delta(\Omega) \geq 1$. This is true for all Δ, so $V^\varphi(\Omega) \geq 1$.

Next we show $V^\varphi(\Omega) \leq 1$. If $1/2^{n+1} \leq r < 1/2^n$, write $r^+ = 1/2^n$. Then $\varphi(r) = \gamma_n = \mu(B_{r^+}(x))$. Let π be a centered open ball packing. If $(x,r),(x',r') \in \pi$, $r \geq r'$, then $\rho(x,x') \geq r + r' > r$, so $\rho(x,x') \geq r^+$. So the balls $B_{r^+}(x)$ are pairwise disjoint. Therefore

$$\sum_{(x,r)\in\pi} \varphi(r) = \sum_\pi \mu\big(B_{r^+}(x)\big) \leq 1.$$

Thus for all gauges Δ we have $V^\varphi_\Delta(\Omega) \leq 1$, so $V^\varphi(\Omega) \leq 1$.

Next we show $\overline{V}^\varphi(\Omega) = 0$. Let $\Delta = 1/2^m$, constant. Let π be a Δ-fine centered closed ball packing. For $n \geq m$, let $\pi_n = \left\{ (x,r) : 1/2^{n+1} \leq r < 1/2^n \right\}$, so that $\pi = \bigcup_{n=m}^\infty \pi_n$. If $(x,r),(x',r') \in \pi_n$ for the same n, then $\rho(x,x') > r + r' \geq 2/2^{n+1} = 1/2^n$ so $\rho(x,x') \geq 1/2^{n-1}$. So x, x' differ in some coordinate between 1 and $n-1$. So π_n has at most K_{n-1} elements. For $1/2^{n+1} \leq r < 1/2^n$, we have $\varphi(r) = \gamma_n = 1/(n!)^2$. Thus

$$\sum_{(x,r)\in\pi} \varphi(r) = \sum_{n=m}^\infty \sum_{(x,r)\in\pi_n} \varphi(r) \leq \sum_{n=m}^\infty K_{n-1}\gamma_n = \sum_{n=m}^\infty \frac{((n-1)!)^2}{(n!)^2} = \sum_{n=m}^\infty \frac{1}{n^2} = \alpha_m.$$

So $\overline{V}^\varphi(\Omega) \leq \overline{V}^\varphi_\Delta(\Omega) \leq \alpha_m$. Take the limit on m to get $\overline{V}^\varphi(\Omega) = 0$.

From the following example, we get $\widetilde{V}^\varphi(\Omega) = \infty$.

Example 3.13 (Ultrametric product space: $\widetilde{V}^\varphi(\Omega)$)**.** Now consider the ultrametric product space with $k_n \to \infty$ and $\rho_n = 1/2^n$, but no other restrictions. We claim that

$$\widetilde{V}^\varphi(\Omega) = \limsup_{n\to\infty} K_n \varphi\left(\frac{1}{2^n}\right). \tag{5}$$

First we prove the upper bound. Let $\alpha > \limsup K_n\varphi(1/2^n)$. There exists m so that for all $n \geq m$, we have $K_n\varphi(1/2^n) \leq \alpha$. For $1/2^{n+1} < r \leq 1/2^n$, write $r^+ = 1/2^n$. Then $\varphi(r) \leq \varphi(1/2^n) \leq \alpha\gamma_n = \alpha\mu(B_{r^+}(x))$. Let $\Delta = 1/2^m$ be a constant gauge. Let π be a Δ-fine centered open ball weak packing. For $(x,r),(x',r') \in \pi$ we have $\rho(x,x') \geq r$, so $\rho(x,x') \geq r^+$ and thus $x' \notin B_{r^+}(x)$. In an ultrametric space, this means that the balls $B_{r^+}(x)$ are pairwise disjoint. So

$$\sum_{(x,r)\in\pi} \varphi(r) \leq \alpha \sum_\pi \mu\big(B_{r^+}(x)\big) \leq \alpha.$$

So $\widetilde{V}^\varphi(\Omega) \leq \widetilde{V}^\varphi_\Delta(\Omega) \leq \alpha$.

Now we prove the lower bound. Let $\alpha < \limsup K_n\varphi(1/2^n)$. Let a gauge Δ be given. Now

$$\beta = \left\{ (x,r) : r < \Delta(x), K_n\varphi\left(\frac{1}{2^n}\right) \geq \alpha, r = \frac{1}{2^n} \text{ for some } n \right\}$$

is a fine cover of Ω. So $\{ B_r(x) : (x,r) \in \beta \}$ is an open cover of the compact set Ω. So there is a finite subcover. By the ultrametric property, there is a further subcover where the sets are disjoint. So we get a finite set $\pi \subseteq \beta$ with $\bigcup B_r(x) = \Omega$ as a disjoint union. This means that π is a centered open ball weak packing. And

$$\sum_{(x,r)\in\pi} \varphi(r) \geq \alpha \sum_{\pi} \mu\big(B_r(x)\big) = \alpha\mu(\Omega) = \alpha.$$

So $\widetilde{V}^{\varphi}_{\Delta}(\Omega) \geq \alpha$. This is true for any gauge, so $\widetilde{V}^{\varphi}(\Omega) \geq \alpha$.

Fine variation. Let X be a metric space, let $E \subseteq X$, and let C be a constituent function. If β is a fine cover of E, define

$$v^C_\beta = \sup \sum_{(x,r)\in\pi} C(x,r),$$

where the supremum is over all centered open ball packings $\pi \subseteq \beta$, and

$$\overline{v}^C_\beta = \sup \sum_{(x,r)\in\pi} C(x,r),$$

where the supremum is over all centered *closed* ball packings $\pi \subseteq \beta$.

In order to deal with discontinous constituent functions, we must be able to disregard countably many radii. Recall that a collection β of constituents is a *very fine cover* of E iff:

(i) $x \in E$ for every $(x,r) \in \beta$.
(ii) For every $x \in E$ and every $\delta > 0$ there are uncountably many r with $0 < r < \delta$ and $(x,r) \in \beta$.

Proposition 3.14. *Let X be a metric space, $E \subseteq X$, and $C\colon X \times (0,\infty) \to [0,\infty)$. Assume $C(x,r)$ is nondecreasing in r for each x. Write:*

$$\begin{aligned} S_1 &= \inf\left\{\, v^C_\beta : \beta \textit{ is a fine cover of } E \,\right\}, \\ S_2 &= \inf\left\{\, v^C_\beta : \beta \textit{ is a very fine cover of } E \,\right\}, \\ S_3 &= \inf\left\{\, \overline{v}^C_\beta : \beta \textit{ is a fine cover of } E \,\right\}, \\ S_4 &= \inf\left\{\, \overline{v}^C_\beta : \beta \textit{ is a very fine cover of } E \,\right\}. \end{aligned}$$

Then $S_3 \leq S_1 \leq S_2 = S_4$. All of them are equal provided $C(x,r)$ is right-continuous in r for every x.

Proof. A very fine cover is fine, so $S_1 \leq S_2$ and $S_3 \leq S_4$. If π is a centered closed ball packing, then it is also a centered open ball packing, so $v^C_\beta \geq \overline{v}^C_\beta$, and therefore $S_1 \geq S_3$ and $S_2 \geq S_4$.

Next we show $S_2 \leq S_4$. Let $t > S_4$. We will show that $t \geq S_2$. There is a very fine cover β such that $\overline{v}^C_\beta < t$. Let $\alpha > 1$ be such that $\alpha\overline{v}^C_\beta < t$. Now for each x, the function $C(x,r)$ is nondecreasing in r, so

$$\beta' = \left\{\, (x,r') : \text{there exists } (x,r) \in \beta \text{ with } r < r', C(x,r') < \alpha C(x,r) \,\right\}$$

is a very fine cover of E. In more detail: Fix x. Write $T = \{\, r : (x,r) \in \beta \,\}$. Then $T \cap (0,\varepsilon)$ is uncountable for every $\varepsilon > 0$. But

$$\{\, r \in T : C(x,\cdot) \text{ is not right-continuous at } r \,\} \quad \text{is countable.}$$

And for any r remaining after this countable set has been removed, there exist uncountably many $r' > r$, with $C(x,r') < \alpha C(x,r)$. Now suppose $\pi' = \{(x_n, r'_n)\} \subseteq \beta'$ is a centered open ball packing. Then there exist $r_n < r'_n$ so that $(x_n, r_n) \in \beta$

and $C(x_n, r'_n) < \alpha C(x_n, r_n)$ for all n. Then $\pi = \{(x_n, r_n)\} \subseteq \beta$ is a centered closed ball packing. So

$$\sum_n C(x_n, r'_n) < \alpha \sum_n C(x_n, r_n) \leq \alpha \overline{v}_\beta^C < t.$$

This is true for all π', so $v_{\beta'}^C \leq t$. Therefore $S_2 \leq t$ as claimed.

Finally, assume C is right-continuous (in r). Then we claim $S_2 \leq S_3$. Let $t > S_3$. We will show that $t \geq S_2$. There is a fine cover β such that $\overline{v}_\beta^C < t$. Let $\alpha > 1$ be such that $\alpha \overline{v}_\beta^C < t$. Now for each x, the function $C(x, r)$ is right-continuous in r. Thus

$$\beta' = \{\, (x, r') : \text{there exists } (x, r) \in \beta \text{ with } r < r', C(x, r') < \alpha C(x, r) \,\}$$

is a very fine cover of E. The rest of this case is the same as the previous one, and we conclude $S_2 \leq t$ as claimed. □

For technical reasons, we adopt $S_2 = S_4$ as our definition, and recall that when C is right continuous, they all agree.

Definition. The centered ball *fine variation* of C on E is

$$\begin{aligned} v^C(E) &= \inf \left\{\, v_\beta^C : \beta \text{ is a very fine cover of } E \,\right\} \\ &= \inf \left\{\, \overline{v}_\beta^C : \beta \text{ is a very fine cover of } E \,\right\}. \end{aligned}$$

For the closed ball version with fine covers, we use S_3:

$$\overline{v}^C(E) = \inf \left\{\, \overline{v}_\beta^C : \beta \text{ is a fine cover of } E \,\right\}.$$

And for S_1:

$$\mathring{v}^C(E) = \inf \left\{\, v_\beta^C : \beta \text{ is a fine cover of } E \,\right\}.$$

When the constituent function is of the form $C(x, r) = \varphi(r)$ for some Hausdorff function φ, we may write $v^C = v^\varphi$, and so on.

Proposition 3.15. *v^C, $\overline{v}^C$, and $\mathring{v}^C$ are metric outer measures.*

Proof. [11, (1.1.19)]. □

Proposition 3.16. *Let X be a metric space, let C be a constituent function, and let $E \subseteq X$. Then $v^C(E) \leq \overline{V}^C(E)$.*

Proof. Let Δ be a gauge on E. Then $\beta = \{\, (x, r) : r < \Delta(x) \,\}$ is a very fine cover of E. If $\pi \subseteq \beta$ is a centered closed ball packing, then $\sum_\pi C(x, r) \leq \overline{V}_\Delta^C(E)$. Take the supremum on π to get $\overline{v}_\beta^C \leq \overline{V}_\Delta^C(E)$. Therefore $v^C(E) \leq \overline{V}_\Delta^C(E)$. Take the infimum on Δ to get $v^C(E) \leq \overline{V}^C(E)$. □

Example 3.17 (Ultrametric product space: A computation of v^φ)**.** Consider an ultrametric product space $\Omega = \prod G_n$. Assume $k_n \to \infty$ and $\rho_n = 1/2^n$. Let φ be a Hausdorff function. Then the value of $v^\varphi(\Omega)$ is determined by whether

$$\inf_n K_{n-1} \varphi \left(\frac{1}{2^{n+1}} + \right) \tag{6}$$

is positive or zero. Note that (in this "infinite-dimensional" case) there is no φ such that $0 < v^\varphi(\Omega) < \infty$.

Proposition 3.18. *Let Ω be an ultrametric Cartesian product space with $k_n \to \infty$ and $\rho_n = 1/2^n$. Assume* (6) *is* 0*. Then $v^\varphi(\Omega) = 0$.*

Proof. Fix $\varepsilon > 0$. Define $n_1 < n_2 < \cdots$ in $\mathbb{N}$ so that

$$K_{n_j-1}\varphi\left(\frac{1}{2^{n_j+1}}+\right) < \frac{\varepsilon}{2^j}.$$

Let $a_j = 1/2^{n_j+1}$ and $b_j > a_j$ so close that $K_{n_j-1}\varphi(b_j) < \varepsilon/2^j$. Then

$$\beta = \left\{ (x,r) : r \in \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}$$

is a very fine cover of Ω. Let $\pi \subseteq \beta$ be a centered open ball packing. Write $\pi_j = \{ (x,r) : r \in (a_j,b_j) \}$ so that $\pi = \bigcup_{j=1}^{\infty} \pi_j$. Now for a given j, if $(x,r), (x',r') \in \pi_j$, then $\rho(x,x') \geq r + r' > a_j + a_j = 1/2^{n_j}$, so $\rho(x,x') \geq 1/2^{n_j-1}$. So the open balls $B_{1/2^{n_j-1}}(x)$ are pairwise disjoint. Each of them has measure $\gamma_{n_j-1} = 1/K_{n_j-1}$, so the number of elements of π_j is at most K_{n_j-1}. Now

$$\sum_{(x,r)\in\pi} \varphi(r) = \sum_{j=1}^{\infty} \sum_{(x,r)\in\pi_j} \varphi(r) \leq \sum_{j=1}^{\infty} K_{n_j-1}\varphi(b_j) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

So, for this β, we have $v_\beta^\varphi \leq \varepsilon$. Take the infimum on β to get $v^\varphi(\Omega) = 0$. $\square$

Proposition 3.19. *Let Ω be an ultrametric Cartesian product space with $k_n \to \infty$ and $\rho_n = 1/2^n$. Assume* (6) *is positive. Then $v^\varphi(\Omega) = \infty$.*

Proof. Write α for the positive infimum in (6). Let β be a very fine cover of Ω. Then

$$\beta' = \left\{ (x,r) \in \beta : r \neq \frac{1}{2^n}, n = 1, 2, \dots \right\}$$

is also a very fine cover of Ω. Apply Proposition 3.10 to get a centered closed ball packing $\pi \subseteq \beta'$ so that $\sum \mu(\overline{B}_{2r}(x)) = \infty$. Now if $1/2^{n+1} < r < 1/2^n$, write $r^+ = 1/2^n$ so $2r^+ = 1/2^{n-1}$. Then by the assumption

$$\varphi(r) \geq \varphi\left(\frac{1}{2^{n+1}}+\right) \geq \alpha\gamma_{n-1} = \alpha\mu\big(B_{2r^+}(x)\big) = \alpha\mu\big(\overline{B}_{2r}(x)\big).$$

Therefore $\sum \varphi(r) = \infty$. Thus $\overline{v}_\beta^\varphi = \infty$. This holds for all very fine β, so $v^\varphi(\Omega) = \infty$. $\square$

Example 3.20 (Ultrametric product space: A computation of $\overline{v}^\varphi$)**.** Consider an ultrametric product space $\Omega = \prod G_n$. Assume $k_n \to \infty$ and $\rho_n = 1/2^n$. Let φ be a Hausdorff function. Then the value of $v^\varphi(\Omega)$ is determined by whether

$$\inf_n K_{n-1}\varphi\left(\frac{1}{2^{n+1}}\right) \tag{7}$$

is positive or zero. Note again that (in this "infinite-dimensional" case) there is no φ such that $0 < \overline{v}^\varphi(\Omega) < \infty$.

Proposition 3.21. *Let Ω be an ultrametric Cartesian product space with $k_n \to \infty$ and $\rho_n = 1/2^n$. Assume* (7) *is* 0*. Then $\overline{v}^\varphi(\Omega) = 0$.*

Proof. Fix $\varepsilon > 0$. Define $n_1 < n_2 < \cdots$ in $\mathbb{N}$ so that

$$K_{n_j-2}\varphi\left(\frac{1}{2^{n_j}}\right) < \frac{\varepsilon}{2^j}.$$

Then

$$\beta = \left\{ (x,r) : r = \frac{1}{2^{n_j}} \text{ for some } j \right\}$$

is a fine cover of Ω. Let $\pi \subseteq \beta$ be a centered closed ball packing. Write $\pi_j = \{ (x,r) : r = 1/2^{n_j} \}$ so that $\pi = \bigcup_{j=1}^{\infty} \pi_j$. Now for a given j, if $(x,r),(x',r') \in \pi_j$, then $\rho(x,x') > r + r' = 2/2^{n_j} = 1/2^{n_j-1}$, so $\rho(x,x') \geq 1/2^{n_j-2}$. So the open balls $B_{1/2^{n_j-2}}(x)$ are pairwise disjoint. Each of them has measure $\gamma_{n_j-2} = 1/K_{n_j-2}$, so the number of elements of π_j is at most K_{n_j-2}. Now

$$\sum_{(x,r)\in\pi} \varphi(r) = \sum_{j=1}^{\infty} \sum_{(x,r)\in\pi_j} \varphi(r) \leq \sum_{j=1}^{\infty} K_{n_j-2}\varphi\left(\frac{1}{2^{n_j}}\right) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

So, for this β, we have $\overline{v}^{\varphi}_{\beta} \leq \varepsilon$. Take the infimum on β to get $\overline{v}^{\varphi}(\Omega) = 0$. $\square$

Proposition 3.22. *Let Ω be an ultrametric Cartesian product space with $k_n \to \infty$ and $\rho_n = 1/2^n$. Assume (7) is positive. Then $\overline{v}^{\varphi}(\Omega) = \infty$.*

Proof. Fix α with $0 < \alpha < \inf K_{n-1}\varphi(1/2^{n+1})$. Let β be a fine cover of Ω. By Proposition 3.10 there is a centered closed-ball packing $\pi \subseteq \beta$ with $\sum_{\pi} \mu\big(\overline{B}_{2r}(x)\big) = \infty$. But for $1/2^{n+1} \leq r < 1/2^n$ we have $\overline{B}_{2r}(x) = \overline{B}_{1/2^n}(x)$ and $\varphi(r) \geq \varphi(1/2^{n+1}) > \alpha\gamma_{n-1} = \alpha\mu\big(\overline{B}_{1/2^n}(x)\big) = \alpha\mu\big(\overline{B}_{2r}(x)\big)$. Therefore

$$\sum_{(x,r)\in\pi} \varphi(r) \geq \alpha \sum_{(x,r)\in\pi} \mu\big(\overline{B}_{2r}(x)\big) = \infty.$$

This shows $\overline{v}^{\varphi}_{\beta} = \infty$. The infimum on β gives us $\overline{v}^{\varphi}(\Omega) = \infty$. $\square$

Example 3.23 (Ultrametric product space: Estimates for $\overset{\circ}{v}{}^{\varphi}$)**.** Consider an ultrametric product space $\Omega = \prod G_n$. Assume $k_n \to \infty$ and $\rho_n = 1/2^n$. Let φ be a Hausdorff function. Then

$$\liminf_{n\to\infty} K_{n-1}\varphi\left(\frac{1}{2^{n+1}}+\right) \leq \overset{\circ}{v}{}^{\varphi}(\Omega) \leq \liminf_{n\to\infty} K_{n-1}\varphi\left(\frac{1}{2^{n}}\right). \tag{8}$$

Note the upper and lower bounds are not equal in general. But this case can have a positive finite value.

Lower bound. Let $\alpha < \liminf K_{n-1}\varphi(1/2^{n+1}+)$. There is $m \in \mathbb{N}$ so that for all $n \geq m$, we have $K_{n-1}\varphi(1/2^{n+1}+) \geq \alpha$. Let β be any fine cover of Ω. Then

$$\{ B_{2r}(x) : (x,r) \in \beta, r < 1/2^m \}$$

is an open cover of Ω. So it has a finite subcover $\{ B_{2r}(x) : (x,r) \in \beta_1 \}$. By the ultrametric property, for any two balls, either one contains the other or they are disjoint. So there is a further subcover $\{ B_{2r}(x) : (x,r) \in \beta_2 \}$ such that the balls $B_{2r}(x)$ are pairwise disjoint. We claim that β_2 is a centered open ball packing. Indeed, if $(x,r),(x',r') \in \beta_3$, $r \geq r'$, then $B_{2r}(x) \cap B_{2r'}(x') = \varnothing$, so $\rho(x,x') \geq 2r \geq$

$r + r'$ as required. Now for $(x, r) \in \beta_2$, say $1/2^{n+1} < r \le 1/2^n$, $n \ge m$, we have $\varphi(r) \ge \varphi(1/2^{n+1}+) \ge \alpha\gamma_{n-1} \ge \alpha\mu\big(B_{1/2^{n-1}}(x)\big) = \alpha\mu\big(B_{2r}(x)\big)$. So

$$\sum_{(x,r)\in\beta_2} \varphi(r) \ge \alpha \sum_{(x,r)\in\beta_2} \mu\big(B_{2r}(x)\big) \ge \alpha.$$

Therefore $\overset{\circ}{v}{}^{\varphi}_{\beta} \ge \alpha$. Taking infimum on β, we get $\overset{\circ}{v}{}^{\varphi}(\Omega) \ge \alpha$.

Upper bound. Let $\alpha > \liminf K_{n-1}\varphi(1/2^n)$. So

$$\beta = \left\{ (x, r) : r = \frac{1}{2^n}, K_{n-1}\varphi\left(\frac{1}{2^n}\right) \ge \alpha \right\}$$

is a fine cover of Ω. Let $\pi \subseteq \beta$ be a centered open ball packing. If $(x, r), (x', r') \in \pi$, then $\rho(x, x') \ge r + r' > r = 1/2^n$, say, so $\rho(x, x') \ge 1/2^{n-1} = 2r$. Therefore $B_{2r}(x) \cap B_{2r'}(x') = \varnothing$. Now if $r = 1/2^n$, then $\varphi(r) = \varphi(1/2^n) \le \alpha\gamma_{n-1} = \alpha\mu\big(B_{1/2^{n-1}}(x)\big) = \alpha\mu\big(B_{2r}(x)\big)$. So we have

$$\sum_{(x,r)\in\pi} \varphi(r) \le \alpha \sum_{(x,r)\in\pi} \mu\big(B_{2r}(x)\big) \le \alpha.$$

So $\overset{\circ}{v}{}^{\varphi}_{\beta} \le \alpha$. This shows that $\overset{\circ}{v}{}^{\varphi}(\Omega) \le \alpha$.

Example 3.24 (Ultrametric product space: Example for $\overline{v}^{\varphi}(\Omega) < \overset{\circ}{v}{}^{\varphi}(\Omega) < v^{\varphi}(\Omega)$)**.** Let Ω be the ultrametric product space, $\Omega = \prod G_n$, with $k_n = n$, $K_n = n!$, $\gamma_n = 1/n!$, $\rho_n = 1/2^n$. Define a discontinuous Hausdorff function φ: For all $n \in \mathbb{N}$, let $\varphi(r) = \gamma_{n-1} = 1/(n-1)!$ for $1/2^{n+1} < r \le 1/2^n$. Let $C(x, r) = \varphi(r)$ be the corresponding constituent function. Then by the last three examples, we get

$$\begin{aligned} \overline{v}^{\varphi}(\Omega) &= 0, \\ \overset{\circ}{v}{}^{\varphi}(\Omega) &= 1, \\ v^{\varphi}(\Omega) &= \infty. \end{aligned}$$

I will almost never use $\overset{\circ}{v}$, but I kept it in the discussion because of the extra possibility of positive finite value. But recall that for continuous φ it agrees with v^{φ} and $\overline{v}^{\varphi}$, and therefore can have only values 0 and ∞ in the examples discussed above.

In the blanketed case (or more generally the right moderate case) $\overline{v}^{\varphi}$ and v^{φ} differ by at most a constant factor. So in the situations where they are either 0 or ∞, they are forced to be equal.

Proposition 3.25. *Let X be a metric space and φ a right moderate Hausdorff function. Then for all Borel sets $E \subseteq X$, we have $v^{\varphi}(E) \le M\overline{v}^{\varphi}(E)$, where $M = \limsup_{r\to 0} \varphi(r+)/\varphi(r)$.*

Proof. If $\overline{v}^{\varphi}(E) = \infty$, there is nothing to prove, so assume $\overline{v}^{\varphi}(E) < \infty$. Let $t > \overline{v}^{\varphi}(E)$. There is a fine cover β of E so that $\overline{v}^{\varphi}_{\beta} < t$. Choose $\alpha > 1$ so that $\alpha\overline{v}^{\varphi}_{\beta} < t$. Then by the definition of M,

$$\beta_0 = \{ (x, r) : \text{there exists } r' \text{ with } (x, r') \in \beta, r' \le r, \varphi(r) \le M\alpha\varphi(r') \}$$

is a very fine cover of E. Let $\{(x_1, r_1), (x_2, r_2), \dots\} \subseteq \beta_0$ be a centered closed ball packing. For each (x_j, r_j), choose r'_j so that $(x_j, r'_j) \in \beta$, $r'_j \le r_j$, $\varphi(r_j) \le M\alpha\varphi(r'_j)$. Then $\{(x_1, r'_1), (x_2, r'_2), \dots\} \subseteq \beta$ is a centered closed ball packing, and $\sum \varphi(r_j) \le$

$M\alpha \sum \varphi(r'_j) \leq M\alpha \overline{v}^\varphi_\beta < Mt$. This shows $v^\varphi_{\beta_0} \leq Mt$, and therefore $v^\varphi(E) \leq Mt$. Finally t was arbitrary, so $v^\varphi(E) \leq M\overline{v}^\varphi(E)$. □

4. Covering measure

The covering measure (or centered Hausdorff measure) is a variant of the Hausdorff measure. In classical cases, it is a fine variation for the centered open ball base. Reference: [10]. We will discuss the extent to which that remains true in the generality used here.

Let X be a metric space, and let $A \subseteq X$. A (centered open ball) *cover* of A is a set β of constituents such that

$$A \subseteq \bigcup_{(x,r)\in\beta} B_r(x), \qquad \text{and} \qquad x \in A \text{ for all } (x,r) \in \beta.$$

If $\delta > 0$, then we say the cover β is *δ-fine* provided $r < \delta$ for all $(x,r) \in \beta$. [Do not confuse two similar-sounding definitions: "δ-fine cover" and "fine cover".] Define

$$\begin{aligned}
\mathcal{C}^\varphi_\delta(A) &= \inf \left\{ \sum_{(x,r)\in\beta} \varphi(r) : \beta \text{ is a } \delta\text{-fine cover of } A \right\},\\
\mathcal{C}^\varphi_0(A) &= \sup_{\delta>0} \mathcal{C}^\varphi_\delta(A) = \lim_{\delta\to 0} \mathcal{C}^\varphi_\delta(A),\\
\mathcal{C}^\varphi(A) &= \sup \left\{ \mathcal{C}^\varphi_0(E) : E \subseteq A \right\}.
\end{aligned}$$

Outer measure $\mathcal{C}^\varphi$ is called the *φ-covering outer measure*. When the Hausdorff function has the special form $\varphi(t) = (2t)^s$ for all t, then $\mathcal{C}^\varphi$ is called the *s-dimensional covering outer measure* and written $\mathcal{C}^\varphi = \mathcal{C}^s$.

If A is totally bounded, then there is a finite δ-fine cover of A for all $\delta > 0$. If A is separable (so the Lindelöf property holds), then there is a countable δ-fine cover of A for all $\delta > 0$. But if A is not separable, then for small δ we have $\mathcal{C}^\varphi_\delta(A) = \infty$, so $\mathcal{C}^\varphi_0(A) = \infty$, so $\mathcal{C}^\varphi(A) = \infty$. For nonseparable sets, Hausdorff measure is always infinite, so such sets are too large to be classified in this way.

Proposition 4.1. *$\mathcal{C}^\varphi$ is a metric outer measure; $\mathcal{C}^\varphi_\delta$ and $\mathcal{C}^\varphi_0$ are countably subadditive.*

Proof. [30] The only (centered ball) cover of the empty set is the empty cover, so $\mathcal{C}^\varphi_\delta(\varnothing) = 0$, $\mathcal{C}^\varphi_0(\varnothing) = 0$, and $\mathcal{C}^\varphi(\varnothing) = 0$.

Suppose $E \subseteq F$. We claim that $\mathcal{C}^\varphi(E) \leq \mathcal{C}^\varphi(F)$. if $A \subseteq E$, then $A \subseteq F$, so $\mathcal{C}^\varphi(F) \geq \mathcal{C}^\varphi_0(A)$. Take the supremum over all subsets A of E to obtain $\mathcal{C}^\varphi(F) \geq \mathcal{C}^\varphi(E)$.

Fix $\delta > 0$. We claim that $\mathcal{C}^\varphi_\delta$ is countably subadditive. Suppose $E = \bigcup_{n=1}^\infty E_n$. We claim that $\mathcal{C}^\varphi_\delta(E) \leq \sum_n \mathcal{C}^\varphi_\delta(E_n)$. If $\sum_n \mathcal{C}^\varphi_\delta(E_n) = \infty$, we are done. So assume $\sum_n \mathcal{C}^\varphi_\delta(E_n) < \infty$.

Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, let β_n be a δ-fine centered open ball cover of E_n with

$$\sum_{(x,r)\in\beta_n} \varphi(r) < \mathcal{C}^\varphi_\delta(E_n) + \frac{\varepsilon}{2^n}.$$

Then $\beta = \bigcup_n \beta_n$ is a δ-fine cover of E. And

$$\begin{aligned}\mathcal{C}^\varphi_\delta(E) &\le \sum_{(x,r)\in\beta} \varphi(r) = \sum_{n=1}^\infty \sum_{(x,r)\in\beta_n} \varphi(r) \\ &< \sum_{n=1}^\infty \left(\mathcal{C}^\varphi_\delta(E_n) + \frac{\varepsilon}{2^n}\right) = \left(\sum_{n=1}^\infty \mathcal{C}^\varphi_\delta(E_n)\right) + \varepsilon.\end{aligned}$$

This is true for all $\varepsilon > 0$. So $\mathcal{C}^\varphi_\delta(E) \le \sum_n \mathcal{C}^\varphi_\delta(E_n)$, as claimed.

Next we prove countable subadditivity for $\mathcal{C}^\varphi_0$. Let $E = \bigcup_{n=1}^\infty E_n$. For each $\delta > 0$,

$$\mathcal{C}^\varphi_\delta(E) \le \sum_{n=1}^\infty \mathcal{C}^\varphi_\delta(E_n) \le \sum_{n=1}^\infty \mathcal{C}^\varphi_0(E_n).$$

Let $\delta \to 0$ to obtain $\mathcal{C}^\varphi_0(E) \le \sum_n \mathcal{C}^\varphi_0(E_n)$.

Now subadditivity for $\mathcal{C}^\varphi$. Suppose $E = \bigcup_{n=1}^\infty E_n$. If $A \subseteq E$, then $A \cap E_n \subseteq E_n$ for each n and $A = \bigcup_n (A \cap E_n)$. So

$$\mathcal{C}^\varphi_0(A) \le \sum_{n=1}^\infty \mathcal{C}^\varphi_0(A \cap E_n) \le \sum_{n=1}^\infty \mathcal{C}^\varphi(E_n).$$

Take the supremum over all $A \subseteq E$ to obtain $\mathcal{C}^\varphi(E) \le \sum_n \mathcal{C}^\varphi(E_n)$.

Finally, we prove the metric property. Suppose E, F are sets with $\operatorname{dist}(E,F) = \varepsilon > 0$. If $A \subseteq E$ and $B \subseteq F$, then also $\operatorname{dist}(A,B) \ge \varepsilon$. Now for $\delta < \varepsilon/2$, any δ-fine cover of $A \cup B$ is the disjoint union of a δ-fine cover of A with a δ-fine cover of B. So $\mathcal{C}^\varphi_\delta(A \cup B) = \mathcal{C}^\varphi_\delta(A) + \mathcal{C}^\varphi_\delta(B)$. Let $\delta \to 0$ to obtain $\mathcal{C}^\varphi_0(A \cup B) = \mathcal{C}^\varphi_0(A) + \mathcal{C}^\varphi_0(B)$. Then take the supremum over all $A \subseteq E$ and $B \subseteq F$ to obtain $\mathcal{C}^\varphi(E \cup F) = \mathcal{C}^\varphi(E) + \mathcal{C}^\varphi(F)$. ☐

The extra step at the end of the definition, obtaining $\mathcal{C}^\varphi$ from $\mathcal{C}^\varphi_0$, is awkward. Use of the variation, below, avoids this. When φ is blanketed, the difference is at most a constant factor.

Proposition 4.2. *Assume that φ is blanketed. Then there is a constant M so that for all Borel sets $E \subseteq X$, we have $\mathcal{C}^\varphi_0(E) \le \mathcal{C}^\varphi(E) \le M\mathcal{C}^\varphi_0(E)$.*

Proof. From the definition, $\mathcal{C}^\varphi_0(E) \le \mathcal{C}^\varphi(E)$. If φ is blanketed, then there is $\delta_0 > 0$ and a constant M so that $\varphi(2r)/\varphi(r) \le M$ for all $r < \delta_0$. If X is finite-dimensional, then there is $\delta_0 > 0$ and a constant M so that for any $r < \delta$, every ball of radus $2r$ is contained in a union of at most M balls of radius r.

Let $A \subseteq E$ and let $\delta < \delta_0$. Let β be a δ-fine (centered open ball) cover of E.

Then $\beta_1 = \{\, (x,r) \in \beta : B_r(x) \cap A \ne \varnothing \,\}$ satisfies $A \subseteq \bigcup_{\beta_1} B_r(x)$. For each $(x,r) \in \beta_1$, choose some point $y \in B_r(x) \cap A$, call it $y(x)$. Then $B_{2r}(y(x)) \supseteq B_r(x)$, so $\beta_2 = \{\, (y(x), 2r) : (x,r) \in \beta_1 \,\}$ is a 2δ-fine cover of A. So we get

$$\sum_{\beta_2} \varphi(2r) \le M \sum_\beta \varphi(r).$$

Therefore $\mathcal{C}^\varphi_{2\delta}(A) \le M \sum_\beta \varphi(r)$. Take the infimum on β to get $\mathcal{C}^\varphi_{2\delta}(A) \le M\mathcal{C}^\varphi_\delta(E)$, take the limit on δ to get $\mathcal{C}^\varphi_0(A) \le M\mathcal{C}^\varphi_0(E)$, then take the supremum on A to get $\mathcal{C}^\varphi(E) \le M\mathcal{C}^\varphi_0(E)$. ☐

Corollary 4.3. *Assume that φ is blanketed. Then $\mathcal{C}^\varphi(E) = 0$ if and only if $\mathcal{C}_0^\varphi(E) = 0$.*

In Example 4.5, below, we see that this may fail for unblanketed φ.

Example 4.4 (Ultrametric product space: Computation for $\mathcal{C}^\varphi(\Omega)$)**.** Let $\Omega = \prod G_n$ be an ultrametric product space with general ρ_n strictly decreasing to 0. Let φ be a Hausdorff function. Then we claim

$$\mathcal{C}^\varphi(\Omega) = \liminf_n K_n \varphi\left(\rho_{n+1}+\right). \tag{9}$$

Lower bound. Let $\alpha < \liminf K_n\varphi(\rho_{n+1}+)$. There is m so that

$$K_n\varphi(\rho_{n+1}+) \geq \alpha$$

for all $n \geq m$. Let $\delta < \rho_m$. Then for any $r < \delta$ there is $n \geq m$ with $\rho_{n+1} < r \leq \rho_n$, and we have $B_r(x) = B_{\rho_n}(x)$, so $\mu(B_r(x)) = \gamma_n \leq (1/\alpha)\varphi(\rho_{n+1}+) \leq (1/\alpha)\varphi(r)$. Now if β is any δ-fine centered open ball cover of Ω, then

$$\sum_{(x,r)\in\beta} \varphi(r) \geq \alpha \sum_\beta \mu\big(B_r(x)\big) \geq \alpha.$$

So $\mathcal{C}_\delta^\varphi(\Omega) \geq \alpha$. Take the limit as $\delta \to 0$ to get $\mathcal{C}^\varphi(\Omega) \geq \mathcal{C}_0^\varphi(\Omega) \geq \alpha$.

Upper bound. Let $\alpha > \liminf K_n\varphi(\rho_{n+1}+)$. Let $A \subseteq \Omega$. Let $\delta > 0$. There are $n \in \mathbb{N}$ and $r > \rho_{n+1}$ so that $\rho_n < \delta$ and $K_n\varphi(r) < \alpha$. There is a finite open cover $\{\, B_r(x_i) : 1 \leq i \leq k \,\}$ consisting of $k \leq K_n$ open balls all with the same radius. So $\mathcal{C}_\delta^\varphi(A) \leq K_n\varphi(r) \leq \alpha$. Take the supremum on δ to get $\mathcal{C}_0^\varphi(A) \leq \alpha$. Take the supremum on A to get $\mathcal{C}^\varphi(\Omega) \leq \alpha$. $\square$

Example 4.5 (Example with different nullsets for $\mathcal{C}^\varphi$ and $\mathcal{C}_0^\varphi$)**.** This example cannot be ultrametric, since in an ultrametric space every point of a ball is a center, and thus we get $\mathcal{C}_0^\varphi = \mathcal{C}^\varphi$.

For $n \in \mathbb{N}$, let G_n be a set with n elements. Write $Q_n = \prod_{j=1}^n G_j$, so that Q_n has $n!$ elements, write $Q_\infty = \prod_{j=1}^\infty Q_j$, and

$$Q = Q_\infty \cup \bigcup_{n=0}^\infty Q_n.$$

If $x_1 \in G_1, \ldots, x_n \in G_n$, then let $Q(x_1,\ldots,x_n)$ be the "cylinder" consisting of those elements of Q that begin with these coordinates. [If $k < n$, then Q_k is disjoint from all $Q(x_1,\ldots,x_n)$.] Let $\rho_n = 1/4^{n-1}$. Define a metric ρ on Q as follows: If $x = y$, then $\rho(x,y) = 0$. If x and y both have length at least n, and first differ in coordinate n, then $\rho(x,y) = \rho_n$. If x has length at least n and $y = (x_1,\ldots,x_{n-1})$ is the first $n-1$ coordinates of x, then $\rho(x,y) = \rho_n/2$. Note that the restriction of this metric to the subset Q_∞ is an ultrametric product space of the type we have considered before, with $\rho_n = 1/4^{n-1}$, $k_n = n$, $K_n = n!$, $\gamma_n = 1/n!$. The difference between Q and Q_∞ is that we have added countably many points to Q_∞ to serve as centers of balls.

Let

$$\varphi(t) = \Gamma\left(\frac{\log(4/t)}{\log 4}\right)^{-1},\ t \leq 1/2,$$

using the Gamma function $\Gamma(t) = \int_0^\infty v^{t-1}e^{-v}\,dv$. So $\varphi(1/4^n) = 1/\Gamma(n+1) = 1/n!$ and φ is a Hausdorff function.

Proposition 4.6. *Let Q be the metric space described above, and φ the Hausdorff function defined above. Then $\mathcal{C}^\varphi(Q) \ge 1$ but $\mathcal{C}^\varphi_0(Q) = 0$.*

Proof. The subset Q_∞ has $\mathcal{C}^\varphi(Q_\infty) = 1$ by Example 4.4. So $\mathcal{C}^\varphi(Q) \ge 1$.

On the other hand, for $\delta > 0$, and any n with $\delta > \rho_n/2$, if $\delta > r > \rho_n/2$, then the space Q is covered by the K_{n-1} balls with centers in Q_{n-1} and radius r. So $\mathcal{C}^\varphi_\delta(Q) \le K_{n-1}\varphi(r)$. Take the infimum over $r > \rho_n/2$ to get

$$\mathcal{C}^\varphi_\delta(Q) \le K_{n-1}\varphi\left(\frac{\rho_n}{2}\right) = \frac{\Gamma(n)}{\Gamma(n+1/2)},$$

and the limit on n to get $\mathcal{C}^\varphi_\delta(Q) = 0$. This is true for all δ, so $\mathcal{C}^\varphi_0(Q) = 0$. □

There is a variant of the covering measure with centered closed ball covers. A collection β of constituents is a *centered closed ball cover* of E if: $x \in E$ for all $(x,r) \in \beta$ and $E \subseteq \bigcup_{(x,r)\in\beta} \overline{B}_r(x)$. Define

$$\overline{\mathcal{C}}^\varphi_\delta(A) = \inf\left\{ \sum_{(x,r)\in\beta} \varphi(r) : \beta \text{ is a } \delta\text{-fine centered closed ball cover of } A \right\},$$
$$\overline{\mathcal{C}}^\varphi_0(A) = \sup_{\delta>0} \overline{\mathcal{C}}^\varphi_\delta(A) = \lim_{\delta\to 0} \overline{\mathcal{C}}^\varphi_\delta(A),$$
$$\overline{\mathcal{C}}^\varphi(A) = \sup\left\{ \overline{\mathcal{C}}^\varphi_0(E) : E \subseteq A \right\}.$$

Example 4.7 (Ultrametric product space: Computation for $\overline{\mathcal{C}}^\varphi(\Omega)$)**.** Let $\Omega = \prod G_n$ be an ultrametric product space with $\rho_n = 1/2^n$. Let φ be a Hausdorff function. Then we claim

$$\overline{\mathcal{C}}^\varphi(\Omega) = \liminf_n K_n \varphi\left(\frac{1}{2^{n+1}}\right). \tag{10}$$

Lower bound. Let $\alpha < \liminf K_n\varphi(1/2^{n+1})$. There is m with $K_n\varphi(1/2^{n+1}) \ge \alpha$ for all $n \ge m$. Let $\delta < 1/2^m$. Then for any $r < \delta$ there is $n \ge m$ with $1/2^{n+1} \le r < 1/2^n$, and we have $\overline{B}_r(x) = \overline{B}_{1/2^{n+1}}(x)$, so $\mu(\overline{B}_r(x)) = \gamma_n \le (1/\alpha)\varphi(1/2^{n+1}) \le (1/\alpha)\varphi(r)$. Now if β is any δ-fine centered closed ball cover of Ω, then

$$\sum_{(x,r)\in\beta} \varphi(r) \ge \alpha \sum_\beta \mu\big(\overline{B}_r(x)\big) \ge \alpha.$$

So $\overline{\mathcal{C}}^\varphi_\delta(\Omega) \ge \alpha$. Take the limit as $\delta \to 0$ to get $\overline{\mathcal{C}}^\varphi(\Omega) \ge \overline{\mathcal{C}}^\varphi_0(\Omega) \ge \alpha$.

Upper bound. Let $\alpha > \liminf K_n\varphi(1/2^{n+1})$. Let $A \subseteq \Omega$. Let $\delta > 0$. There exists $n \in \mathbb{N}$ so that $1/2^n < \delta$ and $K_n\varphi(1/2^{n+1}) < \alpha$. There is a finite open cover $\left\{ B_{1/2^{n+1}}(x_i) : 1 \le i \le k \right\}$ consisting of $k \le K_n$ closed balls all with the same radius. So $\overline{\mathcal{C}}^\varphi_\delta(A) \le K_n\varphi(1/2^{n+1}) \le \alpha$. Take the supremum on δ to get $\overline{\mathcal{C}}^\varphi_0(A) \le \alpha$. Take the supremum on A to get $\overline{\mathcal{C}}^\varphi(\Omega) \le \alpha$. □

Recall that a Hausdorff function φ is *right moderate* iff

$$\limsup_{r\to 0} \frac{\varphi(r+)}{\varphi(r)} < \infty.$$

Proposition 4.8. *Let X be a metric space, let φ be a Hausdorff function. For all $E \subseteq X$, we have $\overline{\mathcal{C}}^\varphi(E) \le \mathcal{C}^\varphi(E)$. If φ is right moderate, then there is a constant*

M so that for all $E \subseteq X$, we have $\mathcal{C}^\varphi(E) \le M\overline{\mathcal{C}}^\varphi(E)$. And if φ is right-continuous, then for all $E \subseteq X$ we have $\mathcal{C}^\varphi(E) = \overline{\mathcal{C}}^\varphi(E)$.

Proof. If $\bigcup_{(x,r)\in\pi} B_r(x) \supseteq A$, then $\bigcup_\pi \overline{B}_r(x) \supseteq A$. Therefore $\overline{\mathcal{C}}^\varphi(E) \le \mathcal{C}^\varphi(E)$.

Conversely, assume φ is right moderate. Then there exist M and ε so that $\varphi(r+)/\varphi(r) < M$ for all $r < \varepsilon$. If $\delta < \varepsilon$ and $\{(x_n, r_n)\}$ is a δ-fine centered closed ball cover of A, then there exist $r'_n > r_n$ so that $r'_n < 2\delta$ and $\sum \varphi(r'_n) < M \sum \varphi(r_n)$. If φ is right-continuous, M may be chosen as close to 1 as we like. But $\{(x_n, r'_n)\}$ is a centered open ball cover of A. So $\mathcal{C}_{2\delta}(A) \le M\overline{\mathcal{C}}_\delta(A)$. Therefore $\mathcal{C}^\varphi(E) \le M\overline{\mathcal{C}}^\varphi(E)$. And if φ is right-continuous, $\mathcal{C}^\varphi(E) \le \overline{\mathcal{C}}^\varphi(E)$. □

Example 4.9 (Ultrametric product space: $\overline{\mathcal{C}}^\varphi(\Omega) < \mathcal{C}^\varphi(\Omega)$)**.** Consider the product space Ω with $k_n = n$, $K_n = n!$, $\rho_n = 1/2^n$, $\gamma_n = 1/K_n = 1/n!$. Let φ be the discontinuous Hausdorff function defined by $\varphi(r) = \gamma_n$ for $1/2^{n+1} < r \le 1/2^n$. Then by (9) and (10) we have $\mathcal{C}^\varphi(\Omega) = 1$ and $\overline{\mathcal{C}}^\varphi(\Omega) = 0$.

Covering measure and fine variation. In the classical case, we have $\mathcal{C}^\varphi = v^\varphi$. Now we will consider this in greater generality.

Theorem 4.10. *Let X be a metric space, and let φ be a Hausdorff function. Let $C(x,r) = \varphi(r)$ and let $v^\varphi = v^C$ be its fine variation. Then for all Borel sets E, we have $v^\varphi(E) \le \mathcal{C}^\varphi(E)$.*

Proof. [10, Theorem 3.1] (a) First we prove: If $\mathcal{C}^\varphi(E) = 0$, then $v^\varphi(E) = 0$. Assume $\mathcal{C}^\varphi(E) = 0$. Let $\varepsilon > 0$. Now $\mathcal{C}_0^\varphi(E) = 0$. For each $n \in \mathbb{N}$ we have $\mathcal{C}_{1/n}^\varphi(E) = 0$. So there is a centered open ball cover

$$\beta_n = \{\, (x_{in}, r_{in}) : i \in \mathbb{N} \,\}$$

of E with $r_{in} < 1/n$ and

$$\sum_i \varphi(r_{in}) < \frac{\varepsilon}{2^{n+1}}.$$

Now for each i and n let

$$\beta_{in} = \{\, (y, r) : \rho(y, x_{in}) < r < r_{in} \,\}.$$

Then

$$\beta = \bigcup_{i,n} \beta_{in}$$

is a very fine cover of E. Let $\pi \subseteq \beta$ be a packing. For each i, n, there is at most one element of β_{in} in π, because the balls $B_r(y)$ for $(y,r) \in \beta_{in}$ are disjoint but all contain the point x_{in}. Thus

$$\sum_{(x,r)\in\pi} \varphi(r) \le \sum_{i,n} \varphi(r_{in}) \le \sum_n \frac{\varepsilon}{2^{n+1}} \le \varepsilon.$$

Thus $v_\beta^\varphi \le \varepsilon$. So $v^\varphi(E) \le \varepsilon$. But $\varepsilon > 0$ was arbitrary, so $v^\varphi(E) = 0$.

(b) Now we prove that $v^\varphi(E) \le \mathcal{C}^\varphi(E)$. If $\mathcal{C}^\varphi(E) = \infty$, there is nothing to prove, so assume $\mathcal{C}^\varphi(E) < \infty$. Let μ be the restriction of $\mathcal{C}^\varphi$ to E; that is, $\mu(A) = \mathcal{C}^\varphi(A \cap E)$ for all sets $A \subseteq X$. Then μ is a finite metric outer measure.

We will decompose E using the density $\overline{D}^{\varphi}_{\mu}$. Fix a number $\alpha > 1$. Write

$$E_1 = \left\{ x \in E : \overline{D}^{\varphi}_{\mu}(x) \le \alpha^{-3} \right\},$$
$$E_2 = \left\{ x \in E : \overline{D}^{\varphi}_{\mu}(x) > \alpha^{-3} \right\}.$$

Consider first E_1. For $n \in \mathbb{N}$ write

$$F_n = \left\{ x \in E_1 : \frac{\mu\big(B_r(x)\big)}{\varphi(r)} < \alpha^{-2} \text{ for all } r < \frac{1}{n} \right\}.$$

Then F_n increases to E_1 as $n \to \infty$ since $\alpha^{-2} > \alpha^{-3}$.

We claim that $\mathcal{C}^{\varphi}(F_n) = 0$. If $\delta < 1/n$, then when F_n is covered by a δ-fine cover β, we have

$$\sum_{(x,r)\in\beta} \varphi(r) \ge \alpha^2 \sum_{\beta} \mu\big(B_r(x)\big) \ge \alpha^2 \mu\left(\bigcup_{\beta} B_r(x)\right) \ge \alpha^2 \mu(F_n) = \alpha^2 \mathcal{C}^{\varphi}(F_n).$$

Therefore $\mathcal{C}^{\varphi}_{\delta}(F_n) \ge \alpha^2 \mathcal{C}^{\varphi}(F_n)$. Let $\delta \to 0$ to obtain $\mathcal{C}^{\varphi}_0(F_n) \ge \alpha^2 \mathcal{C}^{\varphi}(F_n)$. Therefore $\mathcal{C}^{\varphi}(F_n) \ge \alpha^2 \mathcal{C}^{\varphi}(F_n)$. Now $\mathcal{C}^{\varphi}(F_n) < \infty$ and $\alpha^2 > 1$, so $\mathcal{C}^{\varphi}(F_n) = 0$.

Thus $\mathcal{C}^{\varphi}(F_n) = 0$ for all n. By countable subadditivity, we conclude that $\mathcal{C}^{\varphi}(E_1) = 0$. By part (a), $v^{\varphi}(E_1) = 0$ as well.

Next consider the set E_2. From Theorem 2.1, $\overline{D}^{\varphi}_{\mu}(x) = \operatorname{u\,lim\,sup} \mu(B_r(x))/\varphi(r)$. Now $\alpha^{-4} < \alpha^{-3}$, so the set

$$\beta = \left\{ (x,r) \text{ consituent} : x \in E_2, \frac{\mu\big(B_r(x)\big)}{\varphi(r)} > \alpha^{-4} \right\}$$

is a very fine cover of E_2. Now, if $\pi \subseteq \beta$ is a packing, then

$$\sum_{(x,r)\in\pi} \varphi(r) < \alpha^4 \sum_{\pi} \mathcal{C}^{\varphi}\big(B_r(x) \cap E\big) \le \alpha^4 \mathcal{C}^{\varphi}(E).$$

This is true for all packings $\pi \subseteq \beta$, so $v^{\varphi}_{\beta} \le \alpha^4 \mathcal{C}^{\varphi}(E)$, and thus $v^{\varphi}(E_2) \le \alpha^4 \mathcal{C}^{\varphi}(E)$.

Combining the two parts, we have

$$v^{\varphi}(E) \le v^{\varphi}(E_1) + v^{\varphi}(E_2) \le 0 + \alpha^4 \mathcal{C}^{\varphi}(E).$$

Take the infimum over all $\alpha > 1$ to obtain $v^{\varphi}(E) \le \mathcal{C}^{\varphi}(E)$. □

Remark. The same argument may be adapted for the centered closed ball covering measure $\overline{\mathcal{C}}^{\varphi}$. It will establish the inequality where $v^{\varphi}(E)$ is replaced by $\overline{v}^{\varphi}(E) = S_3$ of Proposition 3.14. The upper density in the proof is replaced by $\overline{\Delta}^{\varphi}_{\mu} = D_3$ of Theorem 2.1. The result is $\overline{v}^{\varphi}(E) \le \overline{\mathcal{C}}^{\varphi}(E)$.

Theorem 4.11. *Let X be a metric space, and let φ be a Hausdorff function. Let $C(x,r) = \varphi(r)$ and let $v^{\varphi} = v^{C}$ be its fine variation. Assume that φ is blanketed. Then for all Borel sets E, we have $v^{\varphi}(E) = \mathcal{C}^{\varphi}(E)$.*

Proof. [10, Theorem 3.1] Because of the previous theorem, it suffices to show $\mathcal{C}^{\varphi}(E) \le v^{\varphi}(E)$. If $v^{\varphi}(E) = \infty$, there is nothing to prove. So assume $v^{\varphi}(E) < \infty$.

Let $\Lambda \subseteq (0, \infty)$ be the set of points r such that φ is not right-continuous at r. so Λ is countable. Let β be a very fine cover of E with $v^{\varphi}_{\beta} < \infty$. Let $\delta > 0$. Then

$$\beta_1 = \left\{ (x, r) \in \beta : r < \frac{\delta}{3}, r \notin \Lambda \right\}$$

is a fine cover of E. Apply Theorem 3.1 to β_1 to get a packing $\{(x_n, r_n)\} \subseteq \beta$. Note that $\limsup_n r_n > 0$ is impossible, since $\sum_n \varphi(r_n) \leq v^{\varphi}_{\beta} < \infty$; so we have for all $n \in \mathbb{N}$

$$E \setminus \bigcup_{i=1}^{n} \overline{B}_{r_i}(x_i) \subseteq \bigcup_{i=n+1}^{\infty} \overline{B}_{3r_i}(x_i).$$

Now φ is right-continuous at each r_i. Let $\alpha > 1$, and choose $r'_i > r_i$ so that $r'_i < \delta/3$ and $\sum \varphi(r'_i) < \alpha \sum \varphi(r_i)$. Thus we get open covers

$$E \subseteq \bigcup_{i=1}^{n} B_{r'_i}(x_i) \cup \bigcup_{i=n+1}^{\infty} B_{3r'_i}(x_i).$$

Now since φ is blanketed, we may write

$$\mathcal{C}^{\varphi}_{\delta}(E) \leq \sum_{i=1}^{n} \varphi(r'_i) + \sum_{i=n+1}^{\infty} \varphi(3r'_i).$$

Now $\sum \varphi(3r'_i) < \infty$, so taking the limit on n, we get $\mathcal{C}^{\varphi}_{\delta}(E) \leq \sum_{i=1}^{\infty} \varphi(r'_i) \leq \alpha v^{\varphi}_{\beta}$. Let $\alpha \to 1$ and $\delta \to 0$ to get $\mathcal{C}^{\varphi}_{0}(E) \leq v^{\varphi}_{\beta}$. Take the infimum over β to get $\mathcal{C}^{\varphi}_{0}(E) \leq v^{\varphi}(E)$. Take the supremum of this over all subsets to get $\mathcal{C}^{\varphi}(E) \leq v^{\varphi}(E)$. □

Theorem 4.12. *Let X be a metric space, and let φ be a Hausdorff function. Let $C(x, r) = \varphi(r)$ and let $\overline{v}^{\varphi} = \overline{v}^{C}$ be its (closed ball) fine variation. Assume that φ is blanketed. Then for all Borel sets E, we have $\overline{v}^{\varphi}(E) = \overline{\mathcal{C}}^{\varphi}(E)$.*

Proof. [10, Theorem 3.1] It suffices to show $\overline{\mathcal{C}}^{\varphi}(E) \leq \overline{v}^{\varphi}(E)$. If $\overline{v}^{\varphi}(E) = \infty$, there is nothing to prove. So assume $\overline{v}^{\varphi}(E) < \infty$. Let β be a fine cover of E with $v^{\varphi}_{\beta} < \infty$. Let $\delta > 0$. Then

$$\beta_1 = \left\{ (x, r) \in \beta : r < \frac{\delta}{3} \right\}$$

is a fine cover of E. Apply Theorem 3.1 to β_1 to get a packing $\{(x_n, r_n)\} \subseteq \beta$. Thus we get centered closed ball covers

$$E \subseteq \bigcup_{i=1}^{n} \overline{B}_{r_i}(x_i) \cup \bigcup_{i=n+1}^{\infty} \overline{B}_{3r_i}(x_i).$$

Now φ is blanketed, so write

$$\overline{\mathcal{C}}^{\varphi}_{\delta}(E) \leq \sum_{i=1}^{n} \varphi(r_i) + \sum_{i=n+1}^{\infty} \varphi(3r_i).$$

Now $\sum \varphi(3r_i) < \infty$, so taking the limit on n, we get $\overline{\mathcal{C}}^{\varphi}_{\delta}(E) \leq \sum_{i=1}^{\infty} \varphi(r_i) \leq \overline{v}^{\varphi}_{\beta}$. Take the infimum over β to get $\overline{\mathcal{C}}^{\varphi}_{0}(E) \leq \overline{v}^{\varphi}(E)$. Take the supremum of this over all subsets to get $\overline{\mathcal{C}}^{\varphi}(E) \leq \overline{v}^{\varphi}(E)$. □

Example 4.13 (Ultrametric product space: $v^\varphi(\Omega) \neq \mathcal{C}^\varphi(\Omega)$)**.** Consider the example Ω with $k_n = n$, $K_n = n!$, $\rho_n = 1/2^n$, $\gamma_n = 1/n!$, with the Hausdorff function $\varphi(r) = \gamma_n$ for $1/2^{n+1} < r \le 1/2^n$. We saw above that $\mathcal{C}^\varphi(\Omega) = 1$. And $v^\varphi(\Omega) = 0$ by Proposition 3.19. So this is an example with $v^\varphi(\Omega) \neq \mathcal{C}^\varphi(\Omega)$. □

Example 4.14 (Ultrametric product space: $\overline{v}^\varphi(\Omega) \neq \overline{\mathcal{C}}^\varphi(\Omega)$)**.** Consider again the example Ω with $k_n = n$, $K_n = n!$, $\rho_n = 1/2^n$, $\gamma_n = 1/n!$, with the Hausdorff function $\varphi(r) = \gamma_{n-1}$ for $1/2^{n+1} < r \le 1/2^n$. Then $\overline{\mathcal{C}}^\varphi(\Omega) = 1$ by (10) and $\overline{v}^\varphi(\Omega) = 0$ by Proposition 3.19. So this is an example with $\overline{v}^\varphi(\Omega) \neq \overline{\mathcal{C}}^\varphi(\Omega)$. □

Note. The failure of $v^\varphi = \mathcal{C}^\varphi$ in the unblanketed case may be an indication that one of the definitions is wrong, and should be altered somehow (in a way that makes no difference in the blanketed case). But which one? Perhaps the density theorem will show whether we should consider v^φ or $\mathcal{C}^\varphi$ to be the "correct" measure.

Density theorem.

Theorem 4.15. *Let X be a metric space, let φ be a Hausdorff function, let μ be a finite Borel measure on X, and let $E \subseteq X$ be a Borel set.*

(a) *Then*

$$\mu(E) \le \mathcal{C}^\varphi(E) \sup_{x\in E} \overline{D}_\mu^\varphi(x),$$

except when the product is 0 *times* ∞*.*

(b) *Assume φ is blanketed. Then*

$$\mathcal{C}^\varphi(E) \inf_{x\in E} \overline{D}_\mu^\varphi(x) \le \mu(E).$$

Proof. [30, Theorem 1.1], [11, Theorem 1.5.13] (a) We will prove

$$\mu(E) \le \mathcal{C}^\varphi(E) \sup \overline{D}_\mu^\varphi(x).$$

If $\sup \overline{D}_\mu^\varphi(x) = \infty$, this is immediate. So assume $\sup \overline{D}_\mu^\varphi(x) < \infty$. Let h be such that $\overline{D}_\mu^\varphi(x) < h < \infty$ for all $x \in E$. We must show that $\mu(E) \le h\mathcal{C}^\varphi(E)$. For $n \in \mathbb{N}$, let

$$E_n = \left\{ x \in E : \frac{\mu\big(B_r(x)\big)}{\varphi(r)} < h \text{ for all } r < \frac{1}{n} \right\}.$$

The sets E_n increase to E. Let $\delta < 1/n$ and let β be a δ-fine cover of E_n. Then

$$\sum_{(x,r)\in\beta} \varphi(r) \ge \frac{1}{h} \sum_\beta \mu\big(B_r(x)\big) \ge \frac{1}{h}\mu\left(\bigcup_\beta B_r(x)\right) \ge \frac{1}{h}\mu(E_n).$$

Therefore $\mathcal{C}_\delta^\varphi(E_n) \ge (1/h)\mu(E_n)$. Let $\delta \to 0$, so $(1/h)\mu(E_n) \le \mathcal{C}_0^\varphi(E_n) \le \mathcal{C}^\varphi(E)$. Let $n \to \infty$ to obtain $(1/h)\mu(E) \le \mathcal{C}^\varphi(E)$, as required.

(b) Apply Theorem 4.18(a), below, and use Theorem 4.11 to replace v^φ by $\mathcal{C}^\varphi$. □

Remark. The closed version is the same, using $\overline{\mathcal{C}}^\varphi$ and density $\overline{\Delta}_\mu^\varphi$.

Corollary 4.16. *Let X be a metric space, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set.*

(a) *If there is a finite Borel measure μ such that $\sup_{x\in E} \overline{D}_\mu^\varphi(x) = k < \infty$, then $\mathcal{C}^\varphi(E) \geq \mu(E)/k$.*

(b) *Assume φ is blanketed. If there is a finite Borel measure μ such that*

$$\inf_{x\in E} \overline{D}_\mu^\varphi(x) = k > 0,$$

then $\mathcal{C}^\varphi(E) \leq \mu(E)/k$.

Corollary 4.17. *Let X be a metric space, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set such that $\mathcal{C}^\varphi(E) < \infty$. Write μ for the restriction of $\mathcal{C}^\varphi$ to E. Then:*

(a) $\mathcal{C}^\varphi \left\{ x \in E : \overline{D}_\mu^\varphi(x) < 1 \right\} = 0.$

(b) *If φ is blanketed, then* $\mathcal{C}^\varphi \left\{ x \in E : \overline{D}_\mu^\varphi(x) > 1 \right\} = 0.$

Proof. (a) Let $\alpha < 1$. Write $E_\alpha = \left\{ x \in E : \overline{D}_\mu^\varphi(x) \leq \alpha \right\}$. Then $\sup_{x\in E_\alpha} \overline{D}_\mu^\varphi(x) \leq \alpha$, so by Theorem 4.15(a), $\mathcal{C}^\varphi(E_\alpha) = \mu(E_\alpha) \leq \alpha\mathcal{C}^\varphi(E_\alpha)$. Now $\mathcal{C}^\varphi(E_\alpha) < \infty$ and $\alpha < 1$, so we have $\mathcal{C}^\varphi(E_\alpha) = 0$. This is true for all $\alpha < 1$, so taking a countable union we have $\mathcal{C}^\varphi \left\{ x \in E : \overline{D}_\mu^\varphi(x) < 1 \right\} = 0$.

(b) Let $\alpha > 1$. Write $E_\alpha = \left\{ x \in E : \overline{D}_\mu^\varphi(x) \geq \alpha \right\}$. Then $\inf_{x\in E_\alpha} \overline{D}_\mu^\varphi(x) \geq \alpha$, so by Theorem 4.15(b), $\mathcal{C}^\varphi(E_\alpha) = \mu(E_\alpha) \geq \alpha\mathcal{C}^\varphi(E_\alpha)$. Now $\mathcal{C}^\varphi(E_\alpha) < \infty$ and $\alpha > 1$, so we have $\mathcal{C}^\varphi(E_\alpha) = 0$. This is true for all $\alpha > 1$, so taking a countable union we have $\mathcal{C}^\varphi \left\{ x \in E : \overline{D}_\mu^\varphi(x) > 1 \right\} = 0$. □

Remark. These corollaries also hold for $\overline{\mathcal{C}}^\varphi$ and density $\overline{\Delta}_\mu^\varphi$.

Use the fine variation. In case φ is not blanketed, maybe it is better to use v^φ itself, instead of $\mathcal{C}^\varphi$. This time the extra condition is needed for the upper bound.

Theorem 4.18. *Let X be a metric space, let φ be a Hausdorff function, let μ be a finite Borel measure on X, and let $E \subseteq X$ be a Borel set.*

(a) *Then*

$$v^\varphi(E) \inf_{x\in E} \overline{D}_\mu^\varphi(x) \leq \mu(E).$$

(b) *Assume μ has the Strong Vitali Property. Then*

$$\mu(E) \leq v^\varphi(E) \sup_{x\in E} \overline{D}_\mu^\varphi(x),$$

except when the product is 0 times ∞.

(c) *Assume that φ is blanketed. Then*

$$\mu(E) \leq v^\varphi(E) \sup_{x\in E} \overline{D}_\mu^\varphi(x),$$

except when the product is 0 times ∞.

Proof. (a) If $\inf \overline{D}^{\varphi}_{\mu}(x) = 0$ we are done. So assume $\inf \overline{D}^{\varphi}_{\mu}(x) > 0$. Let h be a constant such that $0 < h < \overline{D}^{\varphi}_{\mu}(x)$ for all $x \in E$. We must show $hv^{\varphi}(E) \le \mu(E)$. Let $V \supseteq E$ be an open set. Then

$$\beta = \left\{ (x,r) : x \in E, \frac{\mu\big(B_r(x)\big)}{\varphi(r)} > h, 0 < r < \operatorname{dist}(x, X \setminus V) \right\}$$

is a very fine cover of E. Let $\pi \subseteq \beta$ be a centered open ball packing. Then

$$\sum_{(x,r)\in\pi} \varphi(r) < \frac{1}{h} \sum_{\pi} \mu\big(B_r(x)\big) = \frac{1}{h}\, \mu\left(\bigcup_{\pi} B_r(x)\right) \le \frac{1}{h}\mu(V).$$

Take the supremum on π to get $v^{\varphi}_{\beta} \le (1/h)\mu(V)$. Therefore $v^{\varphi}(E) \le (1/h)\mu(V)$. Take the infimum on V to get $v^{\varphi}(E) \le (1/h)\mu(E)$, as claimed.

(b) If $\sup \overline{D}^{\varphi}_{\mu}(x) = \infty$ there is nothing to prove. So assume $\sup \overline{D}^{\varphi}_{\mu}(x) < \infty$. Let h be a constant such that $\overline{D}^{\varphi}_{\mu}(x) < h < \infty$ for all $x \in E$. We must prove $\mu(E) \le hv^{\varphi}(E)$. Let β be a very fine cover of E. Then

$$\beta_1 = \left\{ (x,r) \in \beta : \frac{\mu\big(\overline{B}_r(x)\big)}{\varphi(r)} < h \right\}$$

is also a very fine cover of E. By the Strong Vitali Property, there is a centered closed ball packing $\pi \subseteq \beta_1$ such that $\mu\big(E \setminus \bigcup_{\pi} \overline{B}_r(x)\big) = 0$. Then

$$\sum_{(x,r)\in\pi} \varphi(r) > \frac{1}{h} \sum_{\pi} \mu\big(\overline{B}_r(x)\big) \ge \frac{1}{h}\, \mu\left(\bigcup_{\pi} \overline{B}_r(x)\right) \ge \frac{1}{h}\mu(E).$$

Therefore $\overline{v}^{\varphi}_{\beta} \ge (1/h)\mu(E)$. This holds for all β, so $v^{\varphi}(E) \ge (1/h)\mu(E)$ as required.

(c) Apply Theorems 4.11 and 4.15(a). □

Remark. The same result holds for $\overline{v}^{\varphi}$ with density $\overline{\Delta}^{\varphi}_{\mu}$.

Corollary 4.19. *Let X be a metric space, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set.*

(a) *Assume there is a finite Borel measure μ such that $\sup_{x\in E} \overline{D}^{\varphi}_{\mu}(x) = k < \infty$. Assume φ is blanketed or μ has the SVP. Then $v^{\varphi}(E) \ge \mu(E)/k$.*

(b) *Assume there is a finite Borel measure μ such that $\inf_{x\in E} \overline{D}^{\varphi}_{\mu}(x) = k > 0$. Then $v^{\varphi}(E) \le \mu(E)/k$.*

Corollary 4.20. *Let X be a metric space, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set such that $v^{\varphi}(E) < \infty$. Write μ for the restriction of v^{φ} to E.*

(a) *Then $v^{\varphi}\left\{ x \in E : \overline{D}^{\varphi}_{\mu}(x) > 1 \right\} = 0$.*

(b) *Assume either φ is blanketed or μ has the SVP. Then*

$$v^{\varphi}\left\{ x \in E : \overline{D}^{\varphi}_{\mu}(x) < 1 \right\} = 0.$$

Remark. Corollaries also hold for $\overline{v}^\varphi$ with density $\overline{\Delta}^\varphi_\mu$.

Example 4.21 (Davies example)**.** Inequalities above have been proved with extra assumptions, such as the Strong Vitali Property. In situations where the SVP fails, some of these inequalities also may fail.

The example (due to R. Davies) discussed in [12] is a compact metric space Ω in which the SVP fails. We follow the notation of [12]. The set P is the set of eventually peripheral points. The measure μ is the uniform measure. Numbers γ_n (rapidly decreasing to zero) are measures of cylinders in generation n. The measures μ_1, μ_2 are used to show the failure of the SVP: in fact, $2\mu_1(B_r(u)) = 2\mu_2(B_r(u)) = \mu(B_r(u))$ for all balls with radius $r < 1$, but $2\mu_1(P) = 4/3 > \mu(P) = 1 > 2\mu_2(P) = 2/3$.

Proposition 4.22. *Let φ be a right-continuous Hausdorff function such that*

$$\varphi(1/2^n) = 2\gamma_n.$$

Then $\overline{D}^\varphi_\mu(u) = 1$ for all $u \in P$, $\mathcal{C}^\varphi(P) \geq 4/3$, and $v^\varphi(P) \leq 2/3$.

Proof. Let $u \in P$. Since u is eventually peripheral, there is m so that for all $n \geq m$, the component u_n is peripheral. Now if $r < 1/2^m$, choose n so that

$$\frac{1}{2^n} < r \leq \frac{1}{2^{n-1}}.$$

Then $B_r(u) = \overline{B}_{1/2^n}(u)$ consists of two cylinders in generation n, so $\mu(B_r(u)) = 2\gamma_n$. So

$$\frac{\mu\big(B_r(u)\big)}{\varphi(r)} = \frac{2\gamma_n}{\varphi(r)}.$$

When taking the limsup, among all r in this interval, we want to make the ratio as large as possible, so we should take r as small as possible. Since we have assumed φ is right-continuous, we may let $r = 1/2^n$. So

$$\overline{D}^\varphi_\mu(u) = \limsup_n \frac{2\gamma_n}{\varphi(1/2^n)} = 1.$$

Now $\mu, 2\mu_1, 2\mu_2$ agree on balls, so their upper densities are all the same. Applying Theorem 4.15(a) with measure $2\mu_1$, we get $\mathcal{C}^\varphi(P) \geq 2\mu_1(P) = 4/3$. Applying Theorem 4.18(a) with measure $2\mu_2$, we get $v^\varphi(P) \leq 2\mu_2(P) = 2/3$. $\square$

This example illustrates:

$$\begin{aligned} v^\varphi(P) &\neq \mathcal{C}^\varphi(P) && \text{(Theorem 4.11)},\\ \mathcal{C}^\varphi(P)\ \inf_{x\in P} \overline{D}^\varphi_\mu(x) &\not\leq \mu(P) && \text{(Theorem 4.15(b))},\\ \mu(P) &\not\leq v^\varphi(P)\ \sup_{x\in P} \overline{D}^\varphi_\mu(x) && \text{(Theorem 4.18(b))}. \end{aligned}$$

[In fact, I think it is probably true that $\mathcal{C}^\varphi(P) = \infty$ and $v^\varphi(P) = 0$. Is the proper choice of Hausdorff function in the absense of SVP clear? Should we attempt, for example, to arrange upper density $= 1$? But for which measure?]

Example 4.23 (Ultrametric product space)**.** Consider again the example Ω with $k_n = n$, $K_n = n!$, $\rho_n = 1/2^n$, $\gamma_n = 1/n!$, with the Hausdorff function $\varphi(r) = \gamma_n$ for $1/2^{n+1} < r \le 1/2^n$. Then

$$
\begin{aligned}
v^\varphi(\Omega) &\ne \mathcal{C}^\varphi(\Omega) && \text{(Theorem 4.11)},\\
\mu(\Omega) &\not\le v^\varphi(\Omega) \sup_{x\in\Omega} \overline{D}^\varphi_\mu(x) && \text{(Theorem 4.18(b))}.
\end{aligned}
$$

Remark. Consider the same example with Hausdorff function $\varphi(r) = \gamma_n$ for $1/2^{n+1} < r \le 1/2^n$. Then

$$
\overline{v}^\varphi(\Omega) \ne \overline{\mathcal{C}}^\varphi(\Omega), \qquad \mu(\Omega) \not\le \overline{v}^\varphi(\Omega) \sup_{x\in\Omega} \overline{\Delta}^\varphi_\mu.
$$

Hausdorff measure. Our "centered Hausdorff measures" $\mathcal{C}^\varphi$ are meant to fulfill the role of the usual Hausdorff measures $\mathcal{H}^\varphi$. When φ is blanketed, the two are within a constant factor. This will be verified next. But when φ is not blanketed, they need not be within a constant factor, and they need not vanish simultaneously.

Let X be a metric space, let φ be a Hausdorff function, and let $E \subseteq X$. For $\delta > 0$, let

$$
\mathcal{H}^\varphi_\delta(E) = \inf \sum_{i=1}^\infty \varphi(\operatorname{diam} A_i),
$$

where the infimum is over all countable families (A_i) such that $E \subseteq \bigcup_{i=1}^\infty A_i$ and $\operatorname{diam} A_i < \delta$. Let

$$
\mathcal{H}^\varphi(E) = \sup_{\delta>0} \mathcal{H}^\varphi_\delta(A).
$$

Proposition 4.24. (a) *$\mathcal{H}^\varphi(E) \ge \overline{\mathcal{C}}^\varphi(E)$.*

(b) *If $\varphi(2r)/\varphi(r) \le M$ for all $r > 0$, then $\mathcal{H}^\varphi(E) \le M\overline{\mathcal{C}}^\varphi(E)$ and $(1/M)\mathcal{C}^\varphi(E) \le \mathcal{H}^\varphi(E) \le M\mathcal{C}^\varphi(E)$.*

Proof. (a) Let $E \subseteq X$ and $\delta > 0$. Let $E \subseteq \bigcup A_i$ with $\operatorname{diam} A_i < \delta$. Because we will take the infimum, we may assume $A_i \cap E \ne \varnothing$ for all i. Choose $x_i \in A_i \cap E$ and $r_i = \operatorname{diam} A_i$. Then $A \subseteq \overline{B}_{r_i}(x_i)$, so $\{(x_i, r_i)\}$ is a δ-fine centered closed ball cover of E. So $\sum \varphi(\operatorname{diam} A_i) = \sum \varphi(r_i) \ge \overline{\mathcal{C}}^\varphi_\delta(E)$. Take the infimum over all (A_i) to get $\mathcal{H}^\varphi_\delta(E) \ge \overline{\mathcal{C}}^\varphi_\delta(E)$. Let $\delta \to 0$ to get $\mathcal{H}^\varphi(E) \ge \overline{\mathcal{C}}^\varphi_0(E)$. And take supremum over subsets to get $\mathcal{H}^\varphi(E) \ge \overline{\mathcal{C}}^\varphi(E)$.

(b) Let $E \subseteq X$ and $\delta > 0$. Let $\{(x_i, r_i)\}$ be a δ-fine centered closed ball cover of E. Now $\operatorname{diam} \overline{B}_{r_i}(x_i) \le 2r_i$, so

$$
\mathcal{H}^\varphi_{2\delta}(E) \le \sum \varphi\big(\operatorname{diam} \overline{B}_{r_i}(x_i)\big) \le \sum \varphi(2r_i) \le M \sum \varphi(r_i).
$$

Therefore $\mathcal{H}^\varphi_{2\delta}(E) \le M\overline{\mathcal{C}}^\varphi_\delta(E)$, so $\mathcal{H}^\varphi(E) \le M\overline{\mathcal{C}}^\varphi_0(E) \le M\overline{\mathcal{C}}^\varphi(E)$. □

Remark. Let Ω be the ultrametric product space with $k_n = n$, $K_n = n!$, $\gamma_n = 1/n!$, $\rho_n = 1/2^n$, and let φ be the Hausdorff function with $\varphi(r) = \gamma_{n-1}$ for $\rho_{n+1} < r \le \rho_n$. Then $\overline{\mathcal{C}}^\varphi(\Omega) = 1$, $\mathcal{H}^\varphi(\Omega) = 1$, but $\mathcal{C}^\varphi(\Omega) = \infty$. Of course φ is not blanketed and not right moderate.

5. Packing measure

The packing measure began with Tricot [33]. (But see Hewitt & Stromberg [18, Exercise (10.51), p. 145].) We will show that it is a full variation for the centered closed ball base.

Let X be a metric space, and $A \subseteq X$. A (centered closed ball) *packing* of A is a set π of constituents such that $x \in A$ for all $(x,r) \in \pi$, and $\rho(x,x') > r + r'$ for all $(x,r),(x',r') \in \pi$ with $(x,r) \neq (x',r')$. If $\delta > 0$, then we say the packing π is *δ-fine* provided $r < \delta$ for all $(x,r) \in \pi$. Define

$$\begin{aligned} \overline{\mathcal{P}}^{\varphi}_{\delta}(A) &= \sup\left\{ \sum_{(x,r)\in\pi} \varphi(r) : \pi \text{ is a } \delta\text{-fine packing of } A \right\}, \\ \overline{\mathcal{P}}^{\varphi}_{0}(A) &= \inf_{\delta>0} \overline{\mathcal{P}}^{\varphi}_{\delta}(A) = \lim_{\delta\to 0} \overline{\mathcal{P}}^{\varphi}_{\delta}(A), \\ \overline{\mathcal{P}}^{\varphi}(A) &= \inf\left\{ \sum_{n=1}^{\infty} \overline{\mathcal{P}}^{\varphi}_{0}(E_n) : A \subseteq \bigcup_{n=1}^{\infty} E_n \right\}. \end{aligned}$$

Outer measure $\overline{\mathcal{P}}^{\varphi}$ is called the *φ-packing outer measure*. When the Hausdorff function has the special form $\varphi(t) = (2t)^s$ for all t, then $\overline{\mathcal{P}}^{\varphi}$ is called the *s-dimensional packing outer measure* and written $\overline{\mathcal{P}}^{\varphi} = \overline{\mathcal{P}}^{s}$.

As before, if $\overline{\mathcal{P}}^{\varphi}(E) < \infty$ for any Hausdorff function, then E must be separable.

Proposition 5.1. *If $\overline{\mathcal{P}}^{\varphi}_{0}(E) < \infty$, then E is totally bounded.*

Proof. Assume E is not totally bounded. There there is $r > 0$ and an infinite r-separated set $\{x_n\}$. Then for all $\delta < r/2$, the set $\{(x_n, \delta/2)\}$ is a δ-fine packing, so $\overline{\mathcal{P}}^{\varphi}_{\delta}(E) = \infty$. Thus $\overline{\mathcal{P}}^{\varphi}_{0}(E) = \infty$. □

Proposition 5.2. *$\overline{\mathcal{P}}^{\varphi}$ is a metric outer measure.*

Proof. [9] The only packing of the empty set is the empty packing, and an empty sum has the value zero. So $\overline{\mathcal{P}}^{\varphi}_{\delta}(\varnothing) = 0$ for all δ, and $\overline{\mathcal{P}}^{\varphi}_{0}(\varnothing) = 0$. Then $\varnothing \subseteq \bigcup_{n=1}^{\infty} E_n$ where all $E_n = \varnothing$, so $\overline{\mathcal{P}}^{\varphi}(\varnothing) = 0$.

If $A \subseteq B$, and $B \subseteq \bigcup_{n=1}^{\infty} E_n$, then also $A \subseteq \bigcup_{n=1}^{\infty} E_n$, so $\overline{\mathcal{P}}^{\varphi}(A) \le \overline{\mathcal{P}}^{\varphi}(B)$.

Suppose $A = \bigcup_{i=1}^{\infty} A_i$. We must show that $\overline{\mathcal{P}}^{\varphi}(A) \le \sum_{i=1}^{\infty} \overline{\mathcal{P}}^{\varphi}(A_i)$. If $\sum_i \overline{\mathcal{P}}^{\varphi}(A_i)$ is infinite, then there is nothing to do. So assume $\sum_i \overline{\mathcal{P}}^{\varphi}(A_i) < \infty$. Let $\varepsilon > 0$ be given. For each i, there exist sets E_{ni}, $n \in \mathbb{N}$, so that $A_i \subseteq \bigcup_n E_{ni}$ and $\sum_n \overline{\mathcal{P}}^{\varphi}_{0}(E_{ni}) < \overline{\mathcal{P}}^{\varphi}(A_i) + \varepsilon/2^i$. Then $A \subseteq \bigcup_i \bigcup_n E_{ni}$, which may be rearranged into a single series, so

$$\overline{\mathcal{P}}^{\varphi}(A) \le \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \overline{\mathcal{P}}^{\varphi}_{0}(E_{ni}) < \sum_{i=1}^{\infty}\left(\overline{\mathcal{P}}^{\varphi}(A_i) + \frac{\varepsilon}{2^i}\right) = \left(\sum_{i=1}^{\infty} \overline{\mathcal{P}}^{\varphi}(A_i)\right) + \varepsilon.$$

This holds for any $\varepsilon > 0$, so $\overline{\mathcal{P}}^{\varphi}(A) \le \sum_i \overline{\mathcal{P}}^{\varphi}(A_i)$.

Let $A, B \subseteq X$ and $\operatorname{dist}(A,B) = \varepsilon > 0$. For any $\delta > 0$ with $\delta < \varepsilon/2$, the union of any packing of A with any packing of B is a packing of $A \cup B$. So $\overline{\mathcal{P}}^{\varphi}_{\delta}(A \cup B) = \overline{\mathcal{P}}^{\varphi}_{\delta}(A) + \overline{\mathcal{P}}^{\varphi}_{\delta}(B)$, so $\overline{\mathcal{P}}^{\varphi}_{0}(A \cup B) = \overline{\mathcal{P}}^{\varphi}_{0}(A) + \overline{\mathcal{P}}^{\varphi}_{0}(B)$. Let $A \cup B \subseteq \bigcup_{n=1}^{\infty} E_n$. Now for each n, we have $\operatorname{dist}(E_n \cap A, E_n \cap B) \ge \varepsilon$, so $\overline{\mathcal{P}}^{\varphi}_{0}(E_n) \ge \overline{\mathcal{P}}^{\varphi}_{0}(E_n \cap (A \cup B)) =$

$\overline{\mathcal{P}}_0^\varphi(E_n \cap A) + \overline{\mathcal{P}}_0^\varphi(E_n \cap B)$. Thus

$$\sum_{n=1}^\infty \overline{\mathcal{P}}_0^\varphi(E_n) \geq \sum_{n=1}^\infty \overline{\mathcal{P}}_0^\varphi(E_n \cap A) + \sum_{n=1}^\infty \overline{\mathcal{P}}_0^\varphi(E_n \cap B) \geq \overline{\mathcal{P}}^\varphi(A) + \overline{\mathcal{P}}^\varphi(B).$$

This is true for all covers E_n, so $\overline{\mathcal{P}}^\varphi(A \cup B) \geq \overline{\mathcal{P}}^\varphi(A) + \overline{\mathcal{P}}^\varphi(B)$. □

Corollary 5.3. *All Borel sets are measurable for the outer measures $\overline{\mathcal{P}}^\varphi$.*

Theorem 5.4 (The Closure Theorem)**.** *Let $E \subseteq X$, and let $\overline{E}$ be the closure of E. Then $\overline{\mathcal{P}}_0^\varphi(E) = \overline{\mathcal{P}}_0^\varphi(\overline{E})$.*

Proof. In the definition of $\overline{\mathcal{P}}_\delta^\varphi(E)$ it is enough to use finite packings because of the sup. (The sum of a series of nonnegative terms is the supremum of the subseries with finitely many terms.) Any packing of E is a packing of $\overline{E}$, so $\overline{\mathcal{P}}_0^\varphi(E) \leq \overline{\mathcal{P}}_0^\varphi(\overline{E})$.

Conversely, let $\delta > 0$ and let π be any finite δ-fine packing for the closure $\overline{E}$. Then

$$\varepsilon = \inf \{ r + r' - \rho(x, x') : (x, r), (x', r') \in \pi, (x, r) \neq (x', r') \} > 0.$$

For every $(x, r) \in \pi$, we have $x \in \overline{E}$, so there is $y \in E$ with $\rho(y, x) < \varepsilon/2$. For each such x, choose such a y and call it $y(x)$. Let $\pi' = \{ (y(x), r) : (x, r) \in \pi \}$. Then π' is a packing for E, since if $(x, r), (x', r') \in \pi$ we have

$$\begin{aligned} \rho\big(y(x), y(x')\big) &\leq \rho\big(y(x), x\big) + \rho\big(y(x'), x'\big) + \rho\big(x, x'\big) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \big(\rho(x, x') - r - r'\big) + \big(r + r'\big) \\ &< \varepsilon - \varepsilon + r + r' = r + r'. \end{aligned}$$

So for every finite δ-fine packing π of $\overline{E}$ there corresponds a δ-fine packing π' of E with the same value for $\sum \varphi(r)$. This shows that $\overline{\mathcal{P}}_\delta^\varphi(E) \geq \overline{\mathcal{P}}_\delta^\varphi(\overline{E})$. Taking the limit as $\delta \to 0$, we obtain $\overline{\mathcal{P}}_0^\varphi(E) \geq \overline{\mathcal{P}}_0^\varphi(\overline{E})$. □

Note in our definition of $\overline{\mathcal{P}}_0^\varphi$ we have used our current definition of packing [if $(x, r), (x', r') \in \pi$, then $\rho(x, x') > r + r'$] and not the "relative packing" [if $(x, r), (x', r') \in \pi$, then $\overline{B}_r(x) \cap \overline{B}_{r'}(x') = \varnothing$]. Example 5.18, below, shows that the Closure Theorem may fail when relative packings are used.

Theorem 5.5. *The outer measure $\overline{\mathcal{P}}^\varphi$ is regular in the sense that: for every set $E \subseteq X$, there is a Borel set $B \supseteq E$ with $\overline{\mathcal{P}}^\varphi(B) = \overline{\mathcal{P}}^\varphi(E)$. In fact, B may be chosen to be an $F_{\sigma\delta}$-set.*

Proof. If $\overline{\mathcal{P}}^\varphi(E) = \infty$, choose $B = X$. Now assume $\overline{\mathcal{P}}^\varphi(E) < \infty$. For any $n \in \mathbb{N}$, there is a countable cover $E \subseteq \bigcup_{i=1}^\infty A_{ni}$ with

$$\sum_{i=1}^\infty \overline{\mathcal{P}}_0^\varphi(A_{ni}) \leq \overline{\mathcal{P}}^\varphi(E) + \frac{1}{n}.$$

Now

$$B = \bigcap_{n=1}^\infty \bigcup_{i=1}^\infty \overline{A}_{ni}$$

is an $F_{\sigma\delta}$-set, $B \supseteq E$, and for every $n \in \mathbb{N}$, we have $B \subseteq \bigcup_{i=1}^\infty \overline{A}_{ni}$, so

$$\overline{\mathcal{P}}^\varphi(B) \le \sum_{i=1}^\infty \overline{\mathcal{P}}_0^\varphi\left(\overline{A}_{ni}\right) = \sum_{i=1}^\infty \overline{\mathcal{P}}_0^\varphi\left(A_{ni}\right) \le \overline{\mathcal{P}}^\varphi(E) + \frac{1}{n}.$$

But n is arbitrary, so $\overline{\mathcal{P}}^\varphi(B) \le \overline{\mathcal{P}}^\varphi(E)$ and therefore $\overline{\mathcal{P}}^\varphi(B) = \overline{\mathcal{P}}^\varphi(E)$. □

Corollary 5.6. *If $E_n \nearrow E$, then $\overline{\mathcal{P}}^\varphi(E_n) \to \overline{\mathcal{P}}^\varphi(E)$.*

Proof. (This is a standard consequence of regularity.) Because Borel sets are measurable, the result is true in the case the sets E_n are Borel sets. Now suppose the E_n are arbitrary sets. Then there exist Borel sets $B_n \supseteq E_n$ with $\overline{\mathcal{P}}^\varphi(B_n) = \overline{\mathcal{P}}^\varphi(E_n)$. Then define

$$C_n = \bigcap_{m=n}^\infty B_m.$$

Then $C_n \supseteq \bigcap_{m=n}^\infty E_m = E_n$, and $\overline{\mathcal{P}}^\varphi(C_n) \le \overline{\mathcal{P}}^\varphi(B_n) = \overline{\mathcal{P}}^\varphi(E_n) \le \overline{\mathcal{P}}^\varphi(C_n)$. But C_n increase, So we have

$$\liminf_{n\to\infty} \overline{\mathcal{P}}^\varphi(E_n) = \liminf_{n\to\infty} \overline{\mathcal{P}}^\varphi(C_n) = \overline{\mathcal{P}}^\varphi\left(\bigcup_{n=1}^\infty C_n\right) \ge \overline{\mathcal{P}}^\varphi(E).$$

And $\overline{\mathcal{P}}^\varphi(E_n) \le \overline{\mathcal{P}}^\varphi(E)$ for all n, so $\lim_{n\to\infty} \overline{\mathcal{P}}^\varphi(E_n) = \overline{\mathcal{P}}^\varphi(E)$. □

Identifying packing measure with the full variation does not require any special assumptions (such as blanketed or Strong Vitali Property).

Theorem 5.7. *Let X be a metric space, let φ be a Hausdorff function. Then $\overline{\mathcal{P}}^\varphi$ is the full variation $\overline{V}^\varphi = \overline{V}^C$ for the constituent function $C(x,r) = \varphi(r)$.*

Proof. Let $E \subseteq X$. Let $\delta > 0$. Then the constant function $\Delta(x) = \delta$ is a gauge, and $\overline{V}^\varphi_\Delta(E) = \overline{\mathcal{P}}^\varphi_\delta(E)$. So $\overline{V}^\varphi(E) \le \overline{\mathcal{P}}^\varphi_0(E)$. Now if $E = \bigcup_{n=1}^\infty E_n$, then we have, since $\overline{V}^\varphi$ is an outer measure,

$$\overline{V}^\varphi(E) \le \sum_{n=1}^\infty \overline{V}^\varphi(E_n) \le \sum_{n=1}^\infty \overline{\mathcal{P}}^\varphi_0(E_n).$$

This is true for all countable covers of E, so $\overline{V}^\varphi(E) \le \overline{\mathcal{P}}^\varphi(E)$.

On the other hand, suppose a gauge Δ for the set E is given. For each $n \in \mathbb{N}$, let

$$E_n = \left\{ x \in E : \Delta(x) \ge \frac{1}{n} \right\}.$$

So $\overline{V}^\varphi_\Delta(E) \ge \overline{V}^\varphi_\Delta(E_n) \ge \overline{V}^\varphi_{1/n}(E_n) = \overline{\mathcal{P}}^\varphi_{1/n}(E_n) \ge \overline{\mathcal{P}}^\varphi_0(E_n) \ge \overline{\mathcal{P}}^\varphi(E_n)$. Now $E_n \nearrow E$ as $n \to \infty$, so by Corollary 5.6, $\lim_{n\to\infty} \overline{\mathcal{P}}^\varphi(E_n) = \overline{\mathcal{P}}^\varphi(E)$. Therefore $\overline{V}^C_\Delta(E) \ge \overline{\mathcal{P}}^\varphi(E)$. This is true for all gauges Δ, so $\overline{V}^\varphi(E) \ge \overline{\mathcal{P}}^\varphi(E)$. □

Example 5.8 (Ultrametric product space: Upper and lower bounds for $\overline{\mathcal{P}}^\varphi$)**.** Let $\Omega = \prod G_n$ be an ultrametric product space with $\rho_n = 1/2^n$ and $k_n \to \infty$. We will prove some elementary bounds for the packing measure $\overline{\mathcal{P}}^\varphi(\Omega)$. There is a gap between the conditions for the upper and lower bounds, however, so this is not a complete analysis.

Lower bound.

$$\text{If } \limsup_{n\to\infty} K_{n-1}\varphi\left(\frac{1}{2^n}-\right) > 0, \text{then } \overline{\mathcal{P}}^{\varphi}(\Omega) = \infty. \tag{11}$$

Let $0 < \alpha < \limsup K_{n-1}\varphi(1/2^n -)$. Let Δ be a gauge on Ω. Then

$$\beta = \left\{ (x,r) : r < \Delta(x), K_{n-1}\varphi(r) > \alpha, \text{ for some } n \text{ with } \frac{1}{2^{n+1}} < r < \frac{1}{2^n} \right\}$$

is a fine cover of Ω. Apply Proposition 3.10 to get a centered closed ball packing $\pi \subseteq \beta$ with $\sum \mu(\overline{B}_{2r}(x)) = \infty$. Now for $(x,r) \in \pi \subseteq \beta$ with $1/2^{n+1} < r < 1/2^n$ we have $\varphi(r) > \alpha\gamma_{n-1} = \alpha\mu(\overline{B}_{2r}(x))$. So $\sum_\pi \varphi(r) = \infty$. Therefore $\overline{V}^{\varphi}_{\Delta}(\Omega) = \infty$. This is true for all Δ, so $\overline{\mathcal{P}}^{\varphi}(\Omega) = \overline{V}^{\varphi}(\Omega) = \infty$.

Upper bound.

$$\text{If } \sum_{n=2}^{\infty} K_{n-1}\varphi\left(\frac{1}{2^n}-\right) < \infty, \text{ then } \overline{\mathcal{P}}^{\varphi}(\Omega) = 0. \tag{12}$$

Let $m \in \mathbb{N}$. Let $\Delta = 1/2^m$ be a constant gauge. Let π be a Δ-fine centered closed ball cover of Ω. For each n, let $\pi_n = \left\{ (x,r) \in \pi : 1/2^{n+1} \le r < 1/2^n \right\}$ so that $\pi = \bigcup_{n=m}^{\infty} \pi_n$. Now for a given n, if $(x,r),(x',r') \in \pi_n$, then $\rho(x,x') > r + r' \ge 2/2^{n+1} = 1/2^n$ so that $\rho(x,x') \ge 1/2^{n-1}$. So there are at most K_{n-1} members of π_n. For $1/2^{n+1} \le r < 1/2^n$ we have $\rho(r) \le \varphi(1/2^n -)$. So

$$\sum_{\pi} \varphi(r) \le \sum_{n=m}^{\infty} K_{n-1}\varphi\left(\frac{1}{2^n}-\right) = \alpha_m.$$

This shows $\overline{V}^{\varphi}(\Omega) \le \overline{V}^{\varphi}_{\Delta}(\Omega) \le \alpha_m$. Take the limit on m to get $\overline{V}^{\varphi}(\Omega) = 0$. Recall that $\overline{V}^{\varphi} = \overline{\mathcal{P}}^{\varphi}$.

Density theorem. Next we see how the lower density may be used for computation of the packing measure.

Theorem 5.9. *Let X be a metric space, let μ be a finite Borel measure on X, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set.*

(a) *Then*

$$\overline{\mathcal{P}}^{\varphi}(E) \inf_{x\in E} \underline{D}^{\varphi}_{\mu}(x) \le \mu(E).$$

(b) *If μ has the Strong Vitali Property, then*

$$\mu(E) \le \overline{\mathcal{P}}^{\varphi}(E) \sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x),$$

provided this product is not 0 times ∞.

(c) *If the Hausdorff function is blanketed, then even if μ fails the Strong Vitali Property,*

$$\mu(E) \le C\overline{\mathcal{P}}^{\varphi}(E) \sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x),$$

provided this product is not 0 times ∞, where $C = \limsup_{t\to 0} \varphi(3t)/\varphi(t)$.

Proof. (a) If $\inf_{x\in E} \underline{D}^{\varphi}_{\mu}(x) = 0$, the claimed inequality is trivial, so we may assume $\inf_{x\in E} \underline{D}^{\varphi}_{\mu}(x) > 0$. Let $h > 0$ be a constant such that $\underline{D}^{\varphi}_{\mu}(x) > h$ for all $x \in E$. We must show that $h\overline{\mathcal{P}}^{\varphi}(E) \leq \mu(E)$. Let $\varepsilon > 0$ be given. Then there is an open set $V \supseteq E$ such that $\mu(V) < \mu(E) + \varepsilon$. For $x \in E$, let $\Delta(x) > 0$ be so small that

$$\frac{\mu\big(\overline{B}_r(x)\big)}{\varphi(r)} > h \text{ for all } r < \Delta(x), \text{ and } \Delta(x) < \operatorname{dist}(x, S \setminus V).$$

Then Δ is a gauge for E. Let π be a Δ-fine packing of E. Then $\bigcup_\pi \overline{B}_r(x)$ is a disjoint union of closed sets contained in V, and

$$\sum_{(x,r)\in\pi} \varphi(r) < \frac{1}{h} \sum_{\pi} \mu\big(\overline{B}_r(x)\big) \leq \frac{1}{h}\mu(V).$$

This shows that

$$\overline{\mathcal{P}}^{\varphi}(E) \leq \overline{V}^{\varphi}_{\Delta}(E) \leq \frac{1}{h}\mu(V) \leq \frac{1}{h}\big(\mu(E) + \varepsilon\big).$$

Let $\varepsilon \to 0$ to obtain $\overline{\mathcal{P}}^{\varphi}(E) \leq (1/h)\mu(E)$ as required.

(b) Assume μ has the Strong Vitali Property. If $\sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x) = \infty$, then either the inequality claimed is trivial, or has the form 0 times ∞. So assume $\sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x) < \infty$. Let $h < \infty$ satisfy $\underline{D}^{\varphi}_{\mu}(x) < h$ for all $x \in E$. We must show that $\mu(E) \leq h\overline{\mathcal{P}}^{\varphi}(E)$. Let Δ be a gauge on E. Then

$$\beta = \left\{ (x,r) : x \in E, r < \Delta(x), \frac{\mu\big(\overline{B}_r(x)\big)}{\varphi(r)} \leq h \right\}$$

is a fine cover of E. By the Strong Vitali Property, there is a packing $\pi \subseteq \beta$ of E with $\mu(E) = \mu\big(E \cap \bigcup_\pi \overline{B}_r(x)\big)$. Thus

$$\mu(E) = \mu\left(E \cap \bigcup_{\pi} \overline{B}_r(x)\right) \leq \sum_{\pi} \mu\big(\overline{B}_r(x)\big) \leq h \sum_{\pi} \varphi(r).$$

So $\mu(E) \leq h\overline{V}^{\varphi}_{\Delta}(E)$. Take the limit on α and Δ to get $\mu(E) \leq h\overline{\mathcal{P}}^{\varphi}(E)$, as required.

(c) Now let $C = \limsup_{t\to 0} \varphi(3t)/\varphi(t)$. Again we may assume $\sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x) < \infty$. Let $h < \infty$ satisfy $\underline{D}^{\varphi}_{\mu}(x) < h$ for all $x \in E$, and let $C_1 > C$. We will show that $\mu(E) \leq hC_1\overline{\mathcal{P}}^{\varphi}(E)$. Let Δ be a gauge on E. We will show that $\mu(E) \leq hC_1\overline{V}^{\varphi}_{\Delta}(E)$. If $\overline{V}^{\varphi}_{\Delta}(E) = \infty$, then this is trivial. So we may assume $\overline{V}^{\varphi}_{\Delta}(E) < \infty$. Now

$$\beta = \left\{ (x,r) : x \in E, r < \Delta(x), \frac{\mu\big(\overline{B}_{3r}(x)\big)}{\varphi(3r)} \leq h, \frac{\varphi(3r)}{\varphi(r)} \leq C_1 \right\}$$

is a fine cover of E. Next we will apply Theorem 3.1 with the fine cover β. Now, for any packing $\{(x_i, r_i)\} \subseteq \beta$ we have $\sum \varphi(r_i) \leq \overline{V}^{\varphi}_{\Delta}(E) < \infty$, and thus $r_i \to 0$. So there is a packing $\{(x_i, r_i)\} \subseteq \beta$ such that $E \subseteq \bigcup_{i=1}^{\infty} \overline{B}_{3r_i}(x_i)$. Thus

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu\big(\overline{B}_{3r_i}(x_i)\big) \leq h \sum_{i=1}^{\infty} \varphi(3r_i) \leq hC_1 \sum_{i=1}^{\infty} \varphi(r_i).$$

So $\mu(E) \leq hC_1\overline{V}^{\varphi}_{\Delta}(E)$ as required. □

Corollary 5.10. *Let X be a metric space, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set.*

(a) *Assume there is a finite Borel measure μ such that* $\sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x) = k < \infty$. *Assume μ has the SVP. Then $\overline{\mathcal{P}}^{\varphi}(E) \geq \mu(E)/k$.*

(a′) *Assume there is a finite Borel measure* μ *such that* $\sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x) = k < \infty$. *Assume* φ *satisfies* $\varphi(3r)/\varphi(r) \le C$. *Then* $\overline{\mathcal{P}}^{\varphi}(E) \ge \mu(E)/(kC)$.

(b) *Assume there is a finite Borel measure* μ *such that* $\inf_{x\in E} \underline{D}^{\varphi}_{\mu}(x) = k > 0$. *Then* $\overline{\mathcal{P}}^{\varphi}(E) \le \mu(E)/k$.

Corollary 5.11. *Let* X *be a metric space, let* φ *be a Hausdorff function, and let* $E \subseteq X$ *be a Borel set such that* $\overline{\mathcal{P}}^{\varphi}(E) < \infty$. *Write* μ *for the restriction of* $\overline{\mathcal{P}}^{\varphi}$ *to* E.

(a) *Then* $\overline{\mathcal{P}}^{\varphi}\left\{\, x \in E : \underline{D}^{\varphi}_{\mu}(x) > 1 \,\right\} = 0$.

(b) *Assume* μ *has the SVP. Then* $\overline{\mathcal{P}}^{\varphi}\left\{\, x \in E : \underline{D}^{\varphi}_{\mu}(x) < 1 \,\right\} = 0$.

(b′) *Assume* φ *satisfies* $\varphi(3r)/\varphi(r) \le C$. *Then* $\overline{\mathcal{P}}^{\varphi}\left\{\, x \in E : \underline{D}^{\varphi}_{\mu}(x) < 1/C \,\right\} = 0$.

Open packing measure. Let X be a metric space, and $A \subseteq X$. A centered *open* ball *packing* of A is a set π of constituents such that $x \in A$ for all $(x, r) \in \pi$, and $\rho(x, x') \ge r + r'$ for all $(x, r), (x', r') \in \pi$ with $(x, r) \ne (x', r')$. If $\delta > 0$, then we say the packing π is δ-*fine* provided $r < \delta$ for all $(x, r) \in \pi$. Define

$$\mathcal{P}^{\varphi}_{\delta}(A) = \sup \left\{ \sum_{(x,r)\in\pi} \varphi(r) : \pi \text{ is a } \delta\text{-fine centered open ball packing of } A \right\},$$

$$\mathcal{P}^{\varphi}_{0}(A) = \inf_{\delta>0} \mathcal{P}^{\varphi}_{\delta}(A) = \lim_{\delta\to 0} \mathcal{P}^{\varphi}_{\delta}(A),$$

$$\mathcal{P}^{\varphi}(A) = \inf \left\{ \sum_{n=1}^{\infty} \overline{\mathcal{P}}^{\varphi}_{0}(E_n) : A \subseteq \bigcup_{n=1}^{\infty} E_n \right\}.$$

This has many of the same properties as the closed version $\overline{\mathcal{P}}^{\varphi}$. But not all. In general, $\overline{\mathcal{P}}^{\varphi}(\Omega) \le \mathcal{P}^{\varphi}(\Omega)$. If φ is left-continuous, then $\overline{\mathcal{P}}^{\varphi} = \mathcal{P}^{\varphi}$, since any centered open ball packing can be approximated from inside by a centered closed ball packing. Example 3.12 is a case with $\overline{\mathcal{P}}^{\varphi}(\Omega) < \mathcal{P}^{\varphi}(\Omega)$. To see this, combine the results of Example 3.12 with Theorems 5.7 and 5.20.

Proposition 5.12. *If* $\mathcal{P}^{\varphi}_{0}(E) < \infty$, *then* E *is totally bounded.*

Proposition 5.13. $\mathcal{P}^{\varphi}$ *is a metric outer measure.*

Corollary 5.14. *All Borel sets are measurable for the outer measures* $\mathcal{P}^{\varphi}$.

However: It is not necessarily true that $\mathcal{P}^{\varphi}_{0}(E) = \mathcal{P}^{\varphi}_{0}(\overline{E})$. This *is* true when φ is left-continuous (since $\overline{\mathcal{P}}^{\varphi}_{0} = \mathcal{P}^{\varphi}_{0}$), or when X is ultrametric (since every point in a ball is a center of the ball). This difficulty is the reason we have taken the closed packing measure as the primary definition, not the open packing measure.

Example 5.15 (Failure of the Closure Theorem)**.** This counterexample cannot be ultrametric, so we describe a different one. Begin with a sequence of positive integers $k_n \ge 2$ increasing to ∞ so that

$$\sum_{j=1}^{\infty} \frac{1}{k_j} \le \frac{1}{2}. \tag{13}$$

For each n, let G_n be a set with k_n elements, including the distinguished element 0. The space Q is the cartesian product $Q = \prod_{n=1}^{\infty} G_n$. For $x \in Q$ we write $x = (x_1, x_2, \dots)$. Cylinders $Q(x_1, \dots, x_n)$ are those elements with the first n coordinates as specified. For a finite initial segment $(x_1, \dots, x_n)$, extend using the distinguished element to get $\tau(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, 0, \dots)$, which will be called the *tag* of the cylinder $Q(x_1, \dots, x_n)$. Let $T \subseteq Q$ be the set of all tags, that is all sequences eventually 0. So T is a countable set.

Define the metric on Q as follows. For $s \in G_n$, if $s = 0$ is the distinguished symbol, then let $\theta(s) = 0$, but for all others $s \neq 0$, then let $\theta(s) = 1$. Now let $x, y \in Q$. If $x = y$, then $\rho(x, y) = 0$. If $x \neq y$, let coordinate n be the first one where they disagree. That is,

$$\begin{aligned} x &= (x_1, \dots, x_{n-1}, x_n, x_{n+1}, \dots), \\ y &= (x_1, \dots, x_{n-1}, y_n, y_{n+1}, \dots), \end{aligned}$$

where $x_n \neq y_n$. Define

$$\rho(x, y) = \frac{1}{2^n} + \sum_{k=n+1}^{\infty} \frac{\theta(x_k)}{2^k} + \sum_{k=n+1}^{\infty} \frac{\theta(y_k)}{2^k}.$$

We may check that this defines a metric on Q. I like to think of the terms in the definition in this way: First, $\sum_{k=n+1}^{\infty} \theta(x_k)/2^k$ is the distance from x to the tag $\tau(x_1, \dots, x_{n-1}, x_n)$, second $1/2^n$ is the distance from the tag $\tau(x_1, \dots, x_{n-1}, x_n)$ to the sibling tag $\tau(x_1, \dots, x_{n-1}, y_n)$, and finally $\sum_{k=n+1}^{\infty} \theta(y_k)/2^k$ is the distance from the tag $\tau(x_1, \dots, x_{n-1}, y_n)$ to the point y.

Note that the diameter of the cylinder $Q(x_1, \dots, x_{n-1})$ is $3/2^n$, and that maximum value is achieved only if $x_n \neq y_n$, $x_k \neq 0$ for all $k \geq n+1$, and $y_k \neq 0$ for all $k \geq n+1$. In particular, the diameter of Q itself is $3/2$.

Write $K_n = k_1 \cdot k_2 \cdots k_n$, $\gamma_n = 1/K_n$, and define the uniform measure μ so that $\mu\big(Q(x_1, \dots, x_n)\big) = \gamma_n$ for all cylinders of generation n. And $\mu(Q) = \gamma_0 = 1$.

Let a discontinuous Hausdorff function φ be defined by:

$$\varphi(r) = \frac{1}{K_n}, \quad \text{for } \frac{3}{2^{n+1}} \leq r < \frac{3}{2^n}.$$

The countable subset $T \subseteq Q$ of tags is dense in Q. But we will show that $\mathcal{P}_0^{\varphi}(T) \leq 1/2$ and $\mathcal{P}_0^{\varphi}(Q) \geq 1$, so that $\mathcal{P}_0^{\varphi}(T) < \mathcal{P}_0^{\varphi}(\overline{T})$.

Proposition 5.16. $\mathcal{P}_0^{\varphi}(Q) \geq 1$.

Proof. Let $\delta > 0$. There is n so that $3/2^{n+1} < \delta$. For each $x_1, \dots, x_n$, choose an extension using symbols other than 0, call it $\sigma(x_1, \dots, x_n)$. There are K_n of these. The distance between any two is $\geq 3/2^n$. Using these as centers and $3/2^{n+1}$ as radius, we get a centered open ball weak packing. So

$$\mathcal{P}_\delta^{\varphi}(Q) \geq K_n \varphi\left(\frac{3}{2^{n+1}}\right) = 1.$$

Take the limit as $\delta \to 0$ to get $\mathcal{P}_0^{\varphi}(Q) \geq 1$. □

Proposition 5.17. $\mathcal{P}_0^{\varphi}(T) \leq 1/2$.

Proof. Write

$$\alpha_n = \gamma_{n-1} \sum_{j=n}^{\infty} \frac{1}{k_j}.$$

In particular, since $\gamma_0 = 1$, we have $\alpha_1 \leq 1/2$ by (13). For $n, m \in \mathbb{N}$, we claim that if $\{ (u_i, r_i) : 1 \leq i \leq m \}$ is a centered open ball packing for a cylinder $T \cap$

$Q(x_1, \ldots, x_{n-1})$ with m constituents, with all radii $r_i < 3/2^n$, then $\sum_{i=1}^m \varphi(r_i) < \alpha_n$. We prove this by induction on m. If $m = 1$, then since $r_1 < 3/2^n$ we get

$$\varphi(r_1) \leq \gamma_n = \gamma_{n-1} \frac{1}{k_n} < \alpha_n.$$

Let $m > 1$ and assume the result is known for all smaller values of m. Consider the k_n daughter cylinders $Q(x_1, \ldots, x_{n-1}, x_n)$ obtained by adding one more letter to $x_1, \ldots, x_{n-1}$. If all m of the centers are in the same daughter, then (because the diameter $3/2^{n+1}$ of that daughter is not achieved in T) all of the radii are $< 3/2^{n+1}$; we may keep the same value of $\sum \varphi(r_i)$ by replacing one constituent (u_1, r_1) by another with center in a different daughter, and the same radius r_1; this is possible since there are elements of T in that daughter at least this far from the tag of that daughter. So, in estimating $\sum \varphi(r_i)$ we may assume that not all of the centers are in the same daughter. Say there are m_1 in the first daugher, m_2 in the second, and so on. Of course $m = \sum_{l=1}^{k_n} m_l$ and $m_l < m$.

Consider the case where some $r_i \geq 3/2^{n+1}$. Because this is a packing, and no two points of the cylinder have distance $\geq 3/2^n$, there can be at most one r_i this large. Say it is r_1. Then the ball $B_{r_1}(u_1)$ includes the entire daughter, so all other centers are in other daughters and all other radii are $< (3/2^n) - r_1 < 3/2^{n+1}$. So the portion of the packing in each daughter satisfies the induction hypothesis, and we have

$$\begin{aligned}
\sum_{i=1}^m \varphi(r_i) &< \varphi(r_1) + (k_n - 1)\alpha_{n+1} \leq \gamma_n + k_n \gamma_n \sum_{j=n+1}^{\infty} \frac{1}{k_j} \\
&= \gamma_{n-1} \left(\frac{1}{k_n} + \sum_{j=n+1}^{\infty} \frac{1}{k_j} \right) = \alpha_n.
\end{aligned}$$

Now consider the case where all $r_i < 3/2^{n+1}$. Then we may apply the induction hypothesis to each of the daughters, to get

$$\sum_{i=1}^m \varphi(r_i) < k_n \alpha_{n+1} = k_n \gamma_n \sum_{j=n+1}^{\infty} \frac{1}{k_j} < \gamma_{n-1} \sum_{j=n}^{\infty} \frac{1}{k_j} = \alpha_n.$$

This completes the inductive proof. It follows that for any $3/2$-fine finite packing π of T we have $\sum \varphi(r) \leq \alpha_1 \leq 1/2$. Therefore $\mathcal{P}_0^\varphi(T) \leq 1/2$. $\square$

Example 5.18 (Relative closed ball packing, failure of the Closure Theorem)**.** Write $\widehat{\mathcal{P}}_0^\varphi$ for a packing measure defined using relative centered closed ball packings: that is with condition $\rho(x, x') > r + r'$ replaced by $\overline{B}_r(x) \cap \overline{B}_{r'}(x') = \varnothing$. The Closure Theorem $\widehat{\mathcal{P}}_0^\varphi(A) = \widehat{\mathcal{P}}_0^\varphi(\overline{A})$ may fail. The Closure Theorem is correct if φ is left-continuous or if $\overline{A}$ is complete. So our counterexample is an incomplete metric space. (This example was placed here because it is similar to the metric space of Example 5.15, above.)

Begin with a sequence $k_n \geq 2$ of positive integers increasing to ∞ so that $\sum_n 1/k_n < \infty$. For each n, let G_n be a set with k_n elements, including the distinguished element 0. The space and subset are

$$Q = \left\{ (x_1, x_2, \dots) \in \prod_{n=1}^{\infty} G_n : x_n = 0 \text{ for infinitely many } n \right\}$$
$$T = \left\{ (x_1, x_2, \dots) \in \prod_{n=1}^{\infty} G_n : x_n = 0 \text{ for all but finitely many } n \right\}.$$

Given $x_1 \in G_1, \dots, x_n \in G_n$, the cylinder $Q(x_1, \dots, x_n)$ consists of the elements of Q with the first n coordinates as specified.

Write $K_n = k_1 k_2 \cdots k_n$, so that there are K_n cylinders $Q(x_1, \dots, x_n)$ of generation n. Write $\gamma_n = 1/K_n$, and then define the uniform measure μ so that $\mu(Q(x_1, \dots, x_n)) = \gamma_n$ for cylinders of generation n. In restricting Q to those x with infinitely many zeros, we leave out only a countable set, a set of measure zero for μ, so μ is in fact countably additive even on the subset Q of the product.

Define the metric ρ on Q as follows. For $s \in G_n$, let $\theta(s) = 0$ if s is not the distinguished element 0, and $\theta(0) = 1$. Now let $x, y \in Q$. If $x = y$, then $\rho(x, y) = 0$. If $x \neq y$, let coordinate n be the first one where they disagree. That is,

$$x = (x_1, \dots, x_{n-1}, x_n, x_{n+1}, \dots),$$
$$y = (x_1, \dots, x_{n-1}, y_n, y_{n+1}, \dots),$$

and $x_n \neq y_n$. Define

$$\rho(x, y) = \sum_{j=n+1}^{\infty} \frac{\theta(x_j)}{2^j} + \sum_{j=n+1}^{\infty} \frac{\theta(y_j)}{2^j}.$$

This defines a metric on Q: Because x and y have infinitely many coordinates 0, $\rho(x, y) > 0$. And $\rho(x, y) \leq 1/2^{n-1}$, with strict inequality when $x, y \notin T$. The subset $A = Q \setminus T$ is dense in Q.

Define a Hausdorff function by

$$\varphi(r) = \gamma_n \quad \text{for} \quad \frac{1}{2_n} \leq r < \frac{1}{2^{n-1}}.$$

Proposition 5.19. *Let Q, A, and φ be as defined. Then $\widehat{\mathcal{P}}_0^{\varphi}(A) = 0$ but $\widehat{\mathcal{P}}_0^{\varphi}(\overline{A}) \geq 1$.*

Proof. Of course $\overline{A} = Q$, the whole space. Let n be a natural number. For $x_1 \in G_1, \dots, x_n \in G_n$, let $\tau(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, 0, \dots)$. We have

$$\overline{B}_{1/2^n}\big(\tau(x_1, \dots, x_n)\big) = Q(x_1, \dots, x_n),$$

so there is a (relative) packing made up of K_n pairwise disjoint closed balls of radius $1/2^n$. For $\delta > 1/2^n$ we have $\widehat{\mathcal{P}}_\delta^{\varphi}(Q) \geq K_n \varphi(1/2^n) = 1$. As $\delta \to 0$ we may let $n \to \infty$, and we have $\widehat{\mathcal{P}}_0^{\varphi}(Q) \geq 1$.

Now we consider $A = Q \setminus T$. Let π be a relative packing. That is: for all $(x, r) \in \pi$ we have $x \in A$, and for all $(x, r), (x', r') \in \pi$ with $(x, r) \neq (x', r')$, we have $\overline{B}_r(x) \cap \overline{B}_{r'}(x') = \varnothing$.

Consider $(x, r) \in \pi$, $x = (x_1, x_2, \dots)$, $r \geq 1/2^n$. Then not only does $\overline{B}_r(x) \supseteq Q(x_1, \dots, x_n)$, but for every $s \in G_n$ we have $\overline{B}_r(x) \cap Q(x_1, \dots, x_{n-1}, s) \neq \varnothing$. So no other $(x', r') \in \pi$ can have $x' \in Q(x_1, \dots, x_{n-1})$ and $r' \geq 1/2^n$. Thus, in the packing π there are at most K_{n-1} different (x, r) with $r \geq 1/2^n$. In particular, there are at most K_{n-1} different $(x, r) \in \pi$ with $1/2^n \leq r < 1/2^{n-1}$, and for these we have $\varphi(r) \leq \gamma_n$. Now if π is δ-fine and $1/2^m > \delta$, then

$$\sum_{(x,r)\in\pi} \varphi(r) \leq \sum_{n=m+1}^{\infty} K_{n-1}\varphi(\gamma_n) = \sum_{n=m+1}^{\infty} \frac{1}{k_n}.$$

So $\widehat{\mathcal{P}}^{\varphi}_{\delta}(A) \leq \sum_{n=m+1}^{\infty} 1/k_n$, and therefore in the limit we get $\widehat{\mathcal{P}}^{\varphi}_{0}(A) = 0$. □

Properties of $\mathcal{P}^{\varphi}$. I do not know in general whether $\mathcal{P}^{\varphi}$ is regular. Or whether $\mathcal{P}^{\varphi}(E) = V^{\varphi}(E)$. But at least half of the following results remain, with essentially the same proofs:

Theorem 5.20. *Let X be a metric space, let φ be a Hausdorff function. Then $V^{\varphi}(E) \leq \mathcal{P}^{\varphi}(E)$.*

Theorem 5.21. *Let X be a metric space, let μ be a finite Borel measure on X, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set. Then*

$$V^{\varphi}(E) \inf_{x\in E} \underline{\Delta}^{\varphi}_{\mu}(x) \leq \mu(E).$$

Corollary 5.22. *Let X be a metric space, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set. Assume there is a finite Borel measure μ such that $\sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x) = k > 0$. Then $V^{\varphi}(E) \leq \mu(E)/k$.*

Corollary 5.23. *Let X be a metric space, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set such that $V^{\varphi}(E) < \infty$. Write μ for the restriction of V^{φ} to E. Then $V^{\varphi}\left\{ x \in E : \underline{\Delta}^{\varphi}_{\mu}(x) > 1 \right\} = 0$.*

Example 5.24 (Ultrametric product space: Bounds for $\mathcal{P}^{\varphi}(\Omega)$)**.** Let Ω be the ultrametric product space with $\rho_n = 1/2^n$.

Lower bound.

$$\text{If } \limsup_n K_{n-1}\varphi\left(\frac{1}{2^n}-\right) > 0, \text{ then } \mathcal{P}^{\varphi}(\Omega) = \infty.$$

By (11) we have $\overline{\mathcal{P}}^{\varphi}(\Omega) = \infty$, so $\mathcal{P}^{\varphi}(\Omega) = \infty$.

Upper bound.

$$\text{If } \sum_{n=2}^{\infty} K_{n-1}\varphi\left(\frac{1}{2^n}\right) < \infty, \text{ then } \mathcal{P}^{\varphi}(\Omega) = 0.$$

Let $m \in \mathbb{N}$, and $\delta < 1/2^m$. Let π be a δ-fine centered open ball packing of Ω. Let $\pi_n = \left\{ (x, r) \in \pi : 1/2^{n+1} < r \leq 1/2^n \right\}$ so that $\pi = \bigcup_{n=m}^{\infty} \pi_n$. For given n, if $(x, r), (x', r') \in \pi_n$, then $\rho(x, x') \geq r + r' > 2/2^{n+1} = 1/2^n$, so $\rho(x, x') \geq 1/2^{n-1}$. Thus there are at most K_{n-1} elements in π_n. For $1/2^{n+1} < r \leq 1/2^n$ we have $\rho(r) \leq \rho(1/2^n)$. So

$$\sum_{(x,r)\in\pi} \varphi(r) \leq \sum_{n=m}^{\infty} K_{n-1}\varphi\left(\frac{1}{2^n}\right) = \alpha_m.$$

This shows $\mathcal{P}^{\varphi}_0(\Omega) \leq \mathcal{P}^{\varphi}_{\delta}(\Omega) \leq \alpha_m$. Take the limit on m to get $\mathcal{P}^{\varphi}_0(A) = 0$. So $\mathcal{P}^{\varphi}(\Omega) = 0$.

Weak packing. Let X be a metric space, and $A \subseteq X$. A (centered open ball) *weak packing* of A is a set π of constituents such that $x \in A$ for all $(x,r) \in \pi$, and $\rho(x,x') \geq r \vee r'$ for all $(x,r),(x',r') \in \pi$ with $(x,r) \neq (x',r')$. If $\delta > 0$, then we say the weak packing π is δ*-fine* provided $r < \delta$ for all $(x,r) \in \pi$. Define

$$\widetilde{\mathcal{P}}^{\varphi}_{\delta}(A) = \sup\left\{ \sum_{(x,r)\in\pi} \varphi(r) : \pi \text{ is a } \delta\text{-fine weak packing of } A \right\},$$

$$\widetilde{\mathcal{P}}^{\varphi}_{0}(A) = \inf_{\delta>0} \widetilde{\mathcal{P}}^{\varphi}_{\delta}(A) = \lim_{\delta\to 0} \widetilde{\mathcal{P}}^{\varphi}_{\delta}(A),$$

$$\widetilde{\mathcal{P}}^{\varphi}(A) = \inf\left\{ \sum_{n=1}^{\infty} \widetilde{\mathcal{P}}^{\varphi}_{0}(E_n) : A \subseteq \bigcup_{n=1}^{\infty} E_n \right\}.$$

Packing and weak packing measures are within a constant factor of each other when φ is blanketed.

The next few properties of $\widetilde{\mathcal{P}}^{\varphi}$ have the same proofs as $\mathcal{P}^{\varphi}$.

Proposition 5.25. *If $\widetilde{\mathcal{P}}^{\varphi}_{0}(E) < \infty$, then E is totally bounded.*

Proposition 5.26. *$\widetilde{\mathcal{P}}^{\varphi}$ is a metric outer measure.*

Corollary 5.27. *All Borel sets are measurable for the outer measure $\widetilde{\mathcal{P}}^{\varphi}$.*

Theorem 5.28. *$\widetilde{V}^{\varphi}(E) \leq \widetilde{\mathcal{P}}^{\varphi}(E)$.*

A reason to consider the weak packing measure is that this density inequality does not require SVP:

Theorem 5.29. *Let X be a complete separable metric space, let μ be a finite Borel measure on X, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set. Then*

$$\mu(E) \leq \widetilde{\mathcal{P}}^{\varphi}(E) \sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x),$$

provided this product is not 0 times ∞.

Proof. If $\sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x) = \infty$, then either the inequality claimed is trivial, or has the form 0 times ∞. So assume $\sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x) < \infty$. Let $h < \infty$ satisfy $\underline{D}^{\varphi}_{\mu}(x) < h$ for all $x \in E$. We must show that $\mu(E) \leq h\widetilde{\mathcal{P}}^{\varphi}(E)$. Let $\delta > 0$. Then

$$\beta = \left\{ (x,r) : x \in E, r \leq \delta, \frac{\mu\big(B_r(x)\big)}{\varphi(r)} \leq h \right\}$$

is a fine cover of E.

We claim that β is upward closed. Let $(x_n,r_n) \in \beta$ with $x_n \in E$, $x_n \to x \in E$, and $r_n \nearrow r$. First, $r \leq \delta$. Given $\varepsilon > 0$, there is $r' < r$ so that $\mu(B_{r'}(x)) > \mu(B_r(x)) - \varepsilon$; and for n large we have $\rho(x_n,x) < (r-r')/2$ and $r - r_n < (r-r')/2$, so that $B_{r_n}(x_n) \supseteq B_{r'}(x)$ and therefore $\mu(B_{r_n}(x_n)) \geq \mu(B_r(x)) - \varepsilon$. So

$$\mu\big(B_r(x)\big) \leq \mu\big(B_{r_n}(x_n)\big) + \varepsilon \leq h\varphi(r_n) + \varepsilon \leq h\varphi(r) + \varepsilon.$$

This is true for all $\varepsilon > 0$, so $\mu(B_r(x)) \leq h\varphi(r)$. Therefore $(x,r) \in \beta$.

By Proposition 3.5, there is a weak packing $\{(x_n,r_n)\} \subseteq \beta_1$ of E with $\mu\big(E \setminus \bigcup_n B_{r_n}(x_n)\big) = 0$. Thus

$$\mu(E) \leq \sum_n \mu\big(B_{r_n}(x_n)\big) \leq h \sum_n \varphi(r_n).$$

So $\mu(E) \leq h\widetilde{\mathcal{P}}^\varphi_\delta(E)$. But δ was arbitrary, so $\mu(E) \leq h\widetilde{\mathcal{P}}^\varphi_0(E)$. Finally, μ is countably subadditive, so we get $\mu(E) \leq h\widetilde{\mathcal{P}}^\varphi(E)$, as required. □

Corollary 5.30. *Let X be a metric space, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set. Assume there is a finite Borel measure μ such that $\sup_{x\in E} \underline{D}^\varphi_\mu(x) = k < \infty$. Then $\widetilde{\mathcal{P}}^\varphi(E) \geq \mu(E)/k$.*

Corollary 5.31. *Let X be a metric space, let φ be a Hausdorff function, and let $E \subseteq X$ be a Borel set such that $\widetilde{\mathcal{P}}^\varphi(E) < \infty$. Write μ for the restriction of $\widetilde{\mathcal{P}}^\varphi$ to E. Then $\widetilde{\mathcal{P}}^\varphi \left\{ x \in E : \underline{D}^\varphi_\mu(x) < 1 \right\} = 0$.*

Summary of the density inequalities. In the summary table, we list the inequalities with the conditions used in the proofs. And of course we can imagine many other combinations that have not been considered. And there are many more inequalities with constant c.

[1 means "φ is blanketed", 2 means "μ has the Strong Vitali Property"]

$\mu(E) \leq \mathcal{C}^\varphi(E)\ \sup \overline{D}^\varphi_\mu(x)$	
$\mu(E) \leq \overline{\mathcal{C}}^\varphi(E)\ \sup \overline{\Delta}^\varphi_\mu(x)$	
$\mu(E) \leq v^\varphi(E)\ \sup \overline{D}^\varphi_\mu(x)$	1 or 2
$\mu(E) \leq \overline{v}^\varphi(E)\ \sup \overline{\Delta}^\varphi_\mu(x)$	1 or 2
$\mathcal{C}^\varphi(E)\ \inf \overline{D}^\varphi_\mu(x) \leq \mu(E)$	1
$\overline{\mathcal{C}}^\varphi(E)\ \inf \overline{\Delta}^\varphi_\mu(x) \leq \mu(E)$	1
$v^\varphi(E)\ \inf \overline{D}^\varphi_\mu(x) \leq \mu(E)$	
$\overline{v}^\varphi(E)\ \inf \overline{\Delta}^\varphi_\mu(x) \leq \mu(E)$	
$\mu(E) \leq \overline{\mathcal{P}}^\varphi(E)\ \sup \underline{D}^\varphi_\mu(x)$	2
$\mu(E) \leq c\,\overline{\mathcal{P}}^\varphi(E)\ \sup \underline{D}^\varphi_\mu(x)$	1
$\mu(E) \leq \widetilde{\mathcal{P}}^\varphi(E)\ \sup \underline{D}^\varphi_\mu(x)$	
$\overline{\mathcal{P}}^\varphi(E)\ \inf \underline{D}^\varphi_\mu(x) \leq \mu(E)$	
$V^\varphi(E)\ \inf \underline{\Delta}^\varphi_\mu(x) \leq \mu(E)$.	

6. Product inequalities

In this section we intend to discuss the product inequalities 0.5:

$$\mathcal{C}^s(E)\mathcal{C}^t(F) \leq c\,\mathcal{C}^{s+t}(E\times F),$$
$$\mathcal{C}^{s+t}(E\times F) \leq c\,\mathcal{C}^s(E)\mathcal{P}^t(F),$$
$$\mathcal{C}^s(E)\mathcal{P}^t(F) \leq c\,\overline{\mathcal{P}}^{s+t}(E\times F),$$
$$\mathcal{P}^{s+t}(E\times F) \leq c\,\mathcal{P}^s(E)\mathcal{P}^t(F).$$

To what extent do these generalize to metric spaces, and to general (possibly discontinuous) Hausdorff functions? Howroyd [20] has a fairly complete discussion of this. Here we will try to see to what extent the density inequalities are relevant. However, my attempt to use densities here has been a disappointment.

We can think of certain density inequalities as "local" versions of the product inequalities. For example, the inequality $\overline{\mathcal{P}}^{s+t}(E \times F) \leq c\, \overline{\mathcal{P}}^{s}(E)\overline{\mathcal{P}}^{t}(F)$ is a consequence of a density inequality $c\,\underline{D}_{\mu\times\nu}^{\varphi\psi}(x,y) \geq \underline{D}_{\mu}^{\varphi}(x)\ \underline{D}_{\nu}^{\psi}(y)$. The advantages of the density approach over the traditional one include the local nature of the inequality, and the use of a single unified method for all four of the inequalities. The disadvantages include the inability to handle sets of measure 0 or ∞. (In fact, this disadvantage goes back to Marstrand [26] in the first study of Hausdorff dimension of products.)

To overcome the problem of measure zero, we will have to add extra nullset lemmas in certain cases. To overcome the problem of infinite measure, we will require semifiniteness.

Let X be a metric space, and let μ be a Borel measure on X. Then μ is *semifinite* iff for every Borel set $E \subseteq X$ with $\mu(E) = \infty$, there exists a Borel set $F \subseteq E$ with $0 < \mu(F) < \infty$. We say μ is semifinite on A iff the restriction of μ to A is semifinite. That is, the above holds for subsets $E \subseteq A$.

Once μ is semifinite, the conclusion can be improved: if X is a complete separable metric space and μ is semifinite, then for every Borel set $E \subseteq X$ we have

$$\mu(E) = \sup\{\, \mu(F) : F \subseteq E, F \text{ compact}\,\}.$$

Here are the relevant semifiniteness results from the literature:

Theorem 6.1 (Howroyd [19, Corollary 7])**.** *Let X be a complete separable metric space, and let φ be a blanketed Hausdorff function. Then $\mathcal{C}^\varphi$ is semifinite.*

Theorem 6.2 (Joyce & Preiss [22, Theorem 1, Corollary 1])**.** *Let X be a complete separable metric space, and let φ be an arbitrary Hausdorff function. Then $\widetilde{\mathcal{P}}^\varphi$ is semifinite. If φ is blanketed, then $\overline{\mathcal{P}}^\varphi$ is semifinite.*

There are examples of Hausdorff functions φ and spaces X where the Hausdorff measure $\mathcal{H}^\varphi$ is not semifinite. I have not checked whether these examples will also provide $\mathcal{C}^\varphi$ not semifinite. And how about v^φ?

Wen & Wen [34] show: for every Hausdorff function φ that is not blanketed, there is a compact metric space X where $\overline{\mathcal{P}}^\varphi$ is not semifinite. They use relative packings, what about absolute packings?

Let X and Y be two metric spaces. The Cartesian product space $X \times Y$ may be metrized in more than one way. We will use the *maximum metric* defined by

$$\rho\big((x_1,y_1),(x_2,y_2)\big) = \max\{\rho(x_1,x_2), \rho(y_1,y_2)\}.$$

Although this is not the metric to make the product $\mathbb{R}^k \times \mathbb{R}^l$ of two Euclidean spaces into Euclidean space $\mathbb{R}^{k+l}$, it is within a constant factor. To use the Euclidean metric in any of our product results 0.5, the constant c on the right-hand side will need to be adjusted. In particular, we usually do not have $c = 1$ when the Euclidean metric is used for $\mathbb{R}^{k+l}$. But (when the Hausdorff function is blanketed), changing the metric by at most a constant factor will change the measures $\mathcal{C}^\varphi, \overline{\mathcal{P}}^\varphi$, etc. also only by at most a constant factor.

When we use the maximum metric, we have an easy description for balls in the product:

$$B_r\big((x,y)\big) = B_r(x) \times B_r(y),$$
$$\overline{B}_r\big((x,y)\big) = \overline{B}_r(x) \times \overline{B}_r(y).$$

6a. Generalizing $\mathcal{C}^s(E)\mathcal{C}^t(F) \le \mathcal{C}^{s+t}(E \times F)$.

If φ and ψ are two Hausdorff functions, define $\varphi\psi$ by $(\varphi\psi)(r) = \varphi(r)\psi(r)$. Of course $\varphi\psi$ is also a Hausdorff function. In particular, if $\varphi(r) = (2r)^s$ and $\psi(r) = (2r)^t$, then $\varphi\psi(r) = (2r)^{s+t}$.

For this generalization, we use the inequality

$$\operatorname{u}\limsup_{r\to 0}\left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\;\frac{\mu\big(B_r(y)\big)}{\psi(r)}\right] \le \operatorname{u}\limsup_{r\to 0}\left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\right]\;\operatorname{u}\limsup_{r\to 0}\left[\frac{\mu\big(B_r(y)\big)}{\psi(r)}\right]$$

so that

$$\overline{D}^{\varphi\psi}_{\mu\times\nu}(x,y) \le \overline{D}^{\varphi}_{\mu}(x)\;\overline{D}^{\psi}_{\nu}(y). \tag{14}$$

Theorem 6.3. *Let X and Y be complete separable metric, let $X \times Y$ have the maximum metric, let $E \subseteq X$ and $F \subseteq Y$ be Borel sets, and let φ, ψ be two Hausdorff functions. Assume $v^\varphi(E) < \infty$ and $v^\psi(F) < \infty$. Then*

$$v^\varphi(E)\; v^\psi(F) \le \mathcal{C}^{\varphi\psi}(E \times F).$$

Proof. Let μ be the restriction of v^φ to E. That is, $\mu(A) = v^\varphi(E \cap A)$ for all A. Then μ is a finite Borel measure on X. Similarly, let ν be the restriction of v^φ to F, a finite Borel measure on Y. Now by Corollary 4.20(a) we have $\overline{D}^{\varphi}_{\mu}(x) \le 1$ for almost all $x \in E$. Write

$$E_1 = \left\{ x \in E : \overline{D}^{\varphi}_{\mu}(x) \le 1 \right\}, \qquad \mu(E_1) = \mu(E).$$

Similarly, we have $\overline{D}^{\psi}_{\nu}(y) \le 1$ for almost all $y \in F$. Write

$$F_1 = \left\{ y \in F : \overline{D}^{\psi}_{\nu}(y) \le 1 \right\}, \qquad \nu(F_1) = \nu(F).$$

Now the product measure $\mu \times \nu$ is a finite Borel measure on $X \times Y$. For $(x,y) \in E_1 \times F_1$, we have from (14) that $\overline{D}^{\varphi\psi}_{\mu\times\nu}\big((x,y)\big) \le 1$. Therefore, by Corollary 4.16(a), we have

$$\mathcal{C}^{\varphi\psi}(E_1 \times F_1) \ge (\mu \times \nu)(E_1 \times F_1) = \mu(E_1)\,\nu(F_1) = \mu(E)\,\nu(F),$$

so that $\mathcal{C}^{\varphi\psi}(E \times F) \ge \mathcal{C}^{\varphi\psi}(E_1 \times F_1) \ge \mu(E)\,\nu(F) = v^\varphi(E)\,v^\psi(F)$. □

Corollary 6.4. *Let X and Y be complete separable metric, let $X \times Y$ have the maximum metric, let $E \subseteq X$ and $F \subseteq Y$ be Borel sets, and let φ, ψ be two Hausdorff functions. Assume v^φ is semifinite on E and v^ψ is semifinite on F. Then*

$$v^\varphi(E)\; v^\psi(F) \le \mathcal{C}^{\varphi\psi}(E \times F).$$

Proof. Let $E_1 \subseteq E$ be compact with $v^\varphi(F_1) < \infty$ and let $F_1 \subseteq F$ be compact with $v^\psi(F_1) < \infty$. Then by the theorem, we have

$$v^\varphi(E_1)\; v^\psi(F_1) \le \mathcal{C}^{\varphi\psi}(E_1 \times F_1) \le \mathcal{C}^{\varphi\psi}(E \times F).$$

Taking supremum over E_1 and F_1 we get $v^\varphi(E)\; v^\psi(F) \le \mathcal{C}^{\varphi\psi}(E \times F)$ using the semifiniteness. □

The previous inequalities have v^φ on one side and $\mathcal{C}^\varphi$ on the other. With added hypotheses, we may prove inequalities with the same type of measure on both sides.

Proposition 6.5. *Let X and Y be complete separable metric spaces, let $X \times Y$ have the maximum metric, let $E \subseteq X$ and $F \subseteq Y$ be Borel sets, and let φ, ψ be two Hausdorff functions. Assume that φ and ψ are blanketed. Assume $\mathcal{C}^\varphi(E) < \infty$ and $\mathcal{C}^\psi(F) < \infty$. Then*

$$\mathcal{C}^\varphi(E)\, \mathcal{C}^\psi(F) \leq \mathcal{C}^{\varphi\psi}(E \times F).$$

Proposition 6.6. *Let X and Y be complete separable metric, let $X \times Y$ have the maximum metric, let $E \subseteq X$ and $F \subseteq Y$ be Borel sets, and let φ, ψ be two Hausdorff functions. Assume either that $X \times Y$ has the SVP or that $\varphi\psi$ is blanketed. Assume $v^\varphi(E) < \infty$ and $v^\psi(F) < \infty$. Then*

$$v^\varphi(E)\, v^\psi(F) \leq v^{\varphi\psi}(E \times F).$$

Varying the hypotheses appropriately, we can get other variants, such as

$$\begin{aligned} v^\varphi(E)\, \mathcal{C}^\psi(F) &\leq \mathcal{C}^{\varphi\psi}(E \times F), \\ v^\varphi(E)\, \mathcal{C}^\psi(F) &\leq v^{\varphi\psi}(E \times F), \\ \mathcal{C}^\varphi(E)\, \mathcal{C}^\psi(F) &\leq v^{\varphi\psi}(E \times F). \end{aligned}$$

Remark. The results are also true with $\overline{\mathcal{C}}^\varphi$, $\overline{v}^\varphi$, etc. For this, we use the inequality

$$\limsup_{r\to 0} \left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\, \frac{\mu\big(B_r(y)\big)}{\psi(r)}\right] \leq \limsup_{r\to 0} \left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\right]\, \limsup_{r\to 0} \left[\frac{\mu\big(B_r(y)\big)}{\psi(r)}\right]$$

so that

$$\overline{\Delta}^{\varphi\psi}_{\mu\times\nu}(x,y) \leq \overline{\Delta}^{\varphi}_{\mu}(x)\, \overline{\Delta}^{\psi}_{\nu}(y). \tag{15}$$

6b. Generalizing $\mathcal{C}^{s+t}(E \times F) \leq \mathcal{C}^s(E)\mathcal{P}^t(F)$.

For this generalization, we use the inequality

$$\operatorname{u}\limsup_{r\to 0} \left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\, \frac{\mu\big(B_r(y)\big)}{\psi(r)}\right] \geq \operatorname{u}\limsup_{r\to 0} \left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\right]\, \operatorname{u}\liminf_{r\to 0} \left[\frac{\mu\big(B_r(y)\big)}{\psi(r)}\right]$$

so that

$$\overline{D}^{\varphi\psi}_{\mu\times\nu}(x,y) \geq \overline{D}^{\varphi}_{\mu}(x)\, \underline{D}^{\psi}_{\nu}(y). \tag{16}$$

The proofs then proceed much as before.

Theorem 6.7. *Let X and Y be complete separable metric, let $X \times Y$ have the maximum metric, and let φ, ψ be two Hausdorff functions.*

$$v^{\varphi\psi}(E \times F) \leq \mathcal{C}^\varphi(E)\, \widetilde{\mathcal{P}}^\psi(F).$$

is true for all Borel sets $E \subseteq X$ and $F \subseteq Y$ provided it is true in the "null" cases when one of the factors on the right is zero.

Proof. If either $\mathcal{C}^\varphi(E) = \infty$ or $\widetilde{\mathcal{P}}^\psi(F) = \infty$, then there is nothing to prove. So assume both are finite. Write μ for the restriction of $\mathcal{C}^\varphi$ to E and ν for the restriction of $\widetilde{\mathcal{P}}^\psi$ to F. By Corollaries 4.17(a) and 5.31 we have $\mu(E_1) = \mu(E)$ and $\nu(F_1) = \nu(F)$, where

$$E_1 = \left\{ x \in E : \overline{D}^\varphi_\mu(x) \geq 1 \right\}, \qquad F_1 = \left\{ y \in F : \underline{D}^\psi_\nu(y) \geq 1 \right\}.$$

So on the set $E_1 \times F_1$ we have $\overline{D}^{\varphi\psi}_{\mu\times\nu}(x,y) \geq 1$ by (16). Therefore, by Corollary 4.19(b) we conclude $v^{\varphi\psi}(E_1 \times F_1) \leq \mathcal{C}^\varphi(E)\widetilde{\mathcal{P}}^\psi(F)$. Finally, by our assumption for the "null" case, we get the result with $E \times F$. □

As before, with proper hypotheses we can prove variants, such as the ones below.

Proposition 6.8. *Let X and Y be complete separable metric, let $X \times Y$ have the maximum metric, and let φ, ψ be two Hausdorff functions. Assume that Y has the SVP. Assume that $\varphi\psi$ is blanketed. Then*

$$\mathcal{C}^{\varphi\psi}(E \times F) \leq \mathcal{C}^\varphi(E)\, \overline{\mathcal{P}}^\psi(F).$$

is true for all Borel sets $E \subseteq X$ and $F \subseteq Y$ provided it is true in the "null" cases when one of the factors on the right is zero.

A variant will use the inequality

$$\limsup_{r\to 0} \left[\frac{\mu\big(\overline{B}_r(x)\big)}{\varphi(r)}\, \frac{\mu\big(\overline{B}_r(y)\big)}{\psi(r)} \right] \geq \limsup_{r\to 0} \left[\frac{\mu\big(\overline{B}_r(x)\big)}{\varphi(r)} \right] \liminf_{r\to 0} \left[\frac{\mu\big(\overline{B}_r(y)\big)}{\psi(r)} \right]$$

so that

$$\overline{\Delta}^{\varphi\psi}_{\mu\times\nu}(x,y) \geq \overline{\Delta}^\varphi_\mu(x)\, \underline{D}^\psi_\nu(y). \tag{17}$$

Proposition 6.9. *Let X and Y be complete separable metric, let $X \times Y$ have the maximum metric, and let φ, ψ be two Hausdorff functions. Then*

$$\overline{v}^{\varphi\psi}(E \times F) \leq \overline{\mathcal{C}}^\varphi(E)\, \widetilde{\mathcal{P}}^\psi(F).$$

is true for all Borel sets $E \subseteq X$ and $F \subseteq Y$ provided it is true in the "null" cases when one of the factors on the right is zero.

Nullset lemmas. Of course the assumption of the "null" case is a blemish on this proof. Under the right conditions there are "nullset lemmas" to prove these cases. But I do not know proofs in terms of densities. As a sample, the nullset lemmas for the inequality $\overline{\mathcal{C}}^{\varphi\psi}(E \times F) \leq \overline{\mathcal{C}}^\varphi(E)\widetilde{\mathcal{P}}^\psi(F)$ are given next. But (as far as I know, in general) nullset lemmas are unfortunately no simpler to prove than the general theorems.

Lemma 6.10. *Let X and Y be complete separable metric spaces, let φ, ψ be Hausdorff functions, let $E \subseteq X$, $F \subseteq Y$ be Borel sets. Assume $\overline{\mathcal{C}}^\varphi(E) = 0$ and $\widetilde{\mathcal{P}}^\psi(F) < \infty$. Then $\overline{\mathcal{C}}^{\varphi\psi}_0(E \times F) = 0$. If $\varphi\psi$ is blanketed we may conclude $\overline{\mathcal{C}}^{\varphi\psi}(E \times F) = 0$.*

Proof. First, $\widetilde{\mathcal{P}}^\psi(F) < \infty$, so F is the union of countably many sets F_n with $\widetilde{\mathcal{P}}^\psi_0(F_n) < \infty$. So we may assume $\widetilde{\mathcal{P}}^\psi_0(F) < \infty$. By Proposition 5.25, F is totally bounded. There is $\delta > 0$ so that $\widetilde{\mathcal{P}}^\psi_\delta(F) < \infty$.

Let $\varepsilon > 0$ be given. Write $\varepsilon_1 = \varepsilon/\widetilde{\mathcal{P}}^\psi_\delta(F)$.

Now $\overline{\mathcal{C}}^{\varphi}(E) = 0$, so $\overline{\mathcal{C}}_0^{\varphi}(E) = 0$. There is a centered closed ball cover $\{(x_i, r_i)\}$ of E with $r_i < \delta$ such that $\sum_i \varphi(r_i) < \varepsilon_1$. For each i, by the total boundedness of F there is a maximal finite set $\{\, y_{ij} : 1 \leq j \leq K_i \,\}$ subject to $\rho(y_{ij}, y_{ij'}) > r_i$ for all $j \neq j'$. So $\{\, (y_{ij}, r_i) : 1 \leq j \leq K_i \,\}$ is a centered closed ball weak packing with equal radii, and $F = \bigcup_j \overline{B}_{r_i}(y_{ij})$. So we have $K_i \psi(r_i) \leq \widetilde{\mathcal{P}}_\delta^{\psi}(F)$. Now we may form a cover for the product $E \times F$:

$$E \times F \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{K_i} \overline{B}_{r_i}(x_i, y_{ij}).$$

But

$$\sum_{i=1}^{\infty} \sum_{j=1}^{K_i} \varphi\psi(r_i) \leq M \widetilde{\mathcal{P}}_\delta^{\psi}(F) \sum_{i=1}^{\infty} \varphi(r_i) < \varepsilon.$$

This is true for every ε, so $\overline{\mathcal{C}}_\delta^{\varphi\psi}(E \times F) = 0$. Thus $\overline{\mathcal{C}}_0^{\varphi\psi}(E \times F) = 0$. If $\varphi\psi$ is blanketed, then by Corollary 4.3 we get $\mathcal{C}^{\varphi\psi}(E \times F) = 0$. $\square$

Lemma 6.11. *Let X and Y be complete separable metric spaces, let φ, ψ be Hausdorff functions, let $E \subseteq X$, $F \subseteq Y$ be Borel sets. Assume $\overline{\mathcal{C}}^{\varphi}(E) < \infty$ and $\widetilde{\mathcal{P}}^{\psi}(F) = 0$. Then $\overline{\mathcal{C}}_0^{\varphi\psi}(E \times F) = 0$. If $\varphi\psi$ is blanketed, we may conclude $\overline{\mathcal{C}}^{\varphi\psi}(E \times F) = 0$.*

Proof. Let $\varepsilon > 0$ be given. First, $\overline{\mathcal{C}}_0^{\varphi}(E) \leq \overline{\mathcal{C}}^{\varphi}(E) < \infty$. Fix a number $c > \overline{\mathcal{C}}_0^{\varphi}(E)$. Let $\varepsilon_1 = \varepsilon / c$. Now $\widetilde{\mathcal{P}}^{\psi}(F) = 0$, so there exist sets F_n with $F \subseteq \bigcup F_n$ such that $\sum \widetilde{\mathcal{P}}_0^{\psi}(F_n) < \varepsilon_1$. Choose numbers $p_n > \widetilde{\mathcal{P}}_0^{\psi}(F_n)$ with $\sum p_n < \varepsilon_1$. Fix n. There exists $\delta > 0$ (depending on n) so that $\widetilde{\mathcal{P}}_\delta^{\psi}(F_n) < p_n$. Now $\overline{\mathcal{C}}_\delta^{\varphi}(E) \leq \overline{\mathcal{C}}_0^{\varphi}(E) < c$, so there is a centered closed ball cover $\{(x_i, r_i)\}$ of E with $r_i < \delta$ and $\sum \varphi(r_i) < c$. Now by Proposition 5.25 F_n is totally bounded, so for each i there is a maximal finite r_i-separated set $\{\, y_{ij} : 1 \leq j \leq K_i \,\} \subseteq F_n$. (That is: with $\rho(y_{ij}, y_{ij'}) > r_i$ for $j \neq j'$ and the set is maximal subject to that restriction.) So $\{\, (y_{ij}, r_i) : 1 \leq j \leq K_i \,\}$ is a centered closed ball weak packing of F_n, so $K_i \psi(r_i) \leq \widetilde{\mathcal{P}}_\delta^{\psi}(F_n) < p_n$. By the maximality, $F_n \subseteq \bigcup_{j=1}^{K_i} \overline{B}_{r_i}(y_{ij})$. So we have a δ-fine cover of $E \times F_n$:

$$E \times F_n \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{K_i} \overline{B}_{r_i}\big((x_i, y_{ij})\big).$$

But $\sum_{i=1}^{\infty} K_i \varphi\psi(r_i) \leq c p_n$. So $\overline{\mathcal{C}}_\delta^{\varphi\psi}(E \times F_n) \leq c p_n$. This is true for all small enough δ, so $\overline{\mathcal{C}}_0^{\varphi\psi}(E \times F_n) \leq c p_n$. This is true for all n and $\overline{\mathcal{C}}_0^{\varphi\psi}$ is countably subadditive, so $\overline{\mathcal{C}}_0^{\varphi\psi}(E \times F) \leq c \sum p_n < \varepsilon$. This holds for all $\varepsilon > 0$, so $\overline{\mathcal{C}}_0^{\varphi\psi}(E \times F) = 0$. If $\varphi\psi$ is blanketed, then by Corollary 4.3 we get $\mathcal{C}^{\varphi\psi}(E \times F) = 0$. $\square$

Example 6.12 (Ultrametric product space: Counterexample for a nullset lemma: $\mathcal{C}^{\varphi\psi}(X \times Y) \leq \mathcal{C}^{\varphi}(X) \overline{\mathcal{P}}^{\varphi}(Y)$ fails in general)**.** Let X be the ultrametric product space with $k_n = 2$, $K_n = 2^n$. Let Y be the ultrametric product space with $k'_n = n^3$, $K'_n = (n!)^3$. These are compact metric spaces. Let φ be the Hausdorff function with $\varphi(r) = 1/(n 2^n)$ for all r with $1/2^{n+1} < r \leq 1/2^n$. Let ψ be the Hausdorff function with $\psi(r) = n/(n!)^3$ for all r with $1/2^{n+1} < r \leq 1/2^n$. Let $E = X$ and $F = Y$. These are Borel sets. The Cartesian product $X \times Y$ is another ultrametric product space with $k''_n = 2n^3$, $K''_n = 2^n(n!)^3$. The product $\varphi\psi$ is the Hausdorff function with $\varphi\psi(r) = 1/(2^n(n!)^3)$ for all r with $1/2^{n+1} < r \le 1/2^n$. But we claim $\mathcal{C}^\varphi(E) = \overline{\mathcal{P}}^\psi(F) = 0$ and $\mathcal{C}^{\varphi\psi}(E \times F) > 0$. Indeed:

(a) $K_n\varphi(1/2^n) = 2^n/(n2^n) \to 0$, so by (9) we have $\mathcal{C}^\varphi(X) = 0$.

(b) $K'_{n-1}\psi(1/2^n -) = ((n-1)!)^3 n/(n!)^3 = 1/n^2$ and $\sum 1/n^2 < \infty$, so by (12) we have $\overline{\mathcal{P}}^\varphi(Y) = \overline{V}^\varphi(Y) = 0$.

(c) $K''_n\varphi\psi(1/2^{n+1}+) = (2^n(n!)^3)/(2^n(n!)^3) = 1$, so by (9) we have

$$\mathcal{C}^{\varphi\psi}(X \times Y) \ge 1.$$

6c. Generalizing $\mathcal{C}^s(E)\mathcal{P}^t(F) \le \mathcal{P}^{s+t}(E \times F)$.

For this generalization, we use the inequality

$$\operatorname{u\,lim\,inf}_{r\to 0}\left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\;\frac{\mu\big(B_r(y)\big)}{\psi(r)}\right] \le \operatorname{u\,lim\,sup}_{r\to 0}\left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\right]\;\operatorname{u\,lim\,inf}_{r\to 0}\left[\frac{\mu\big(B_r(y)\big)}{\psi(r)}\right]$$

so that

$$\underline{D}^{\varphi\psi}_{\mu\times\nu}(x,y) \le \overline{D}^\varphi_\mu(x)\,\underline{D}^\psi_\nu(y). \tag{18}$$

Theorem 6.13. *Let X and Y be complete separable metric, let $X \times Y$ have the maximum metric, let $E \subseteq X$ and $F \subseteq Y$ be Borel sets, and let φ, ψ be two Hausdorff functions. Assume $v^\varphi(E) < \infty$ and $\overline{\mathcal{P}}^\psi(F) < \infty$. Then*

$$v^\varphi(E)\,\overline{\mathcal{P}}^\psi(F) \le \widetilde{\mathcal{P}}^{\varphi\psi}(E \times F).$$

Proof. Let μ be the restriction of v^φ to E and ν the restriction of $\overline{\mathcal{P}}^\psi$ to F. By Corollary 4.20(a), we have $\overline{D}^\varphi_\mu(x) \le 1$ almost everywhere on E. By Corollary 5.11(a), we have $\underline{D}^\psi_\nu(y) \le 1$ almost everywhere on F. Write

$$E_1 = \left\{\, x \in E : \overline{D}^\varphi_\mu(x) \le 1 \,\right\}, \qquad F_1 = \left\{\, y \in F : \underline{D}^\psi_\nu(y) \le 1 \,\right\},$$

so $\mu(E_1) = \mu(E)$ and $\nu(F_1) = \nu(F)$. Therefore, by (18), for all $(x,y) \in E_1 \times F_1$ we have $\underline{D}^{\varphi\psi}_{\mu\times\nu}(x,y) \le 1$. So by Corollary 5.30 we have

$$\widetilde{\mathcal{P}}^{\varphi\psi}(E_1 \times F_1) \ge (\mu\times\nu)(E_1 \times F_1) = \mu(E_1)\nu(F_1) = \mu(E)\nu(F),$$

so that $\widetilde{\mathcal{P}}^{\varphi\psi}(E \times F) \ge \widetilde{\mathcal{P}}^{\varphi\psi}(E_1 \times F_1) \ge \mu(E)\nu(F) = v^\varphi(E)\overline{\mathcal{P}}^\psi(F)$. □

And of course with hypotheses we can prove variants, such as the following.

Theorem 6.14. *Let X and Y be complete separable metric, let $X \times Y$ have the maximum metric, let $E \subseteq X$ and $F \subseteq Y$ be Borel sets, and let φ, ψ be two Hausdorff functions. Assume that φ is blanketed. Assume $X \times Y$ has the SVP. Assume $\mathcal{C}^\varphi(E) < \infty$ and $\overline{\mathcal{P}}^\psi(F) < \infty$. Then*

$$\mathcal{C}^\varphi(E)\,\overline{\mathcal{P}}^\psi(F) \le \overline{\mathcal{P}}^{\varphi\psi}(E \times F).$$

For the alternate densities, we get a variant

$$\liminf_{r\to 0}\left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\;\frac{\mu\big(B_r(y)\big)}{\psi(r)}\right] \le \limsup_{r\to 0}\left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\right]\;\liminf_{r\to 0}\left[\frac{\mu\big(B_r(y)\big)}{\psi(r)}\right]$$

so that

$$\underline{\Delta}^{\varphi\psi}_{\mu\times\nu}(x,y) \le \overline{D}^\varphi_\mu(x)\,\underline{\Delta}^\psi_\nu(y).$$

But we have not done density inequalities for the density $\underline{\Delta}_\mu^\varphi(x)$, so we do not yet have a way to apply it.

Proposition 6.15. *Let X and Y be complete separable metric, let $X \times Y$ have the maximum metric, let $E \subseteq X$ and $F \subseteq Y$ be Borel sets, and let φ, ψ be two Hausdorff functions. Assume $X \times Y$ has the SVP. Assume $v^\varphi(E) < \infty$ and $\overline{\mathcal{P}}^\psi(F) < \infty$. Then*

$$v^\varphi(E)\,\overline{\mathcal{P}}^\psi(F) \le \overline{\mathcal{P}}^{\varphi\psi}(E \times F).$$

As before, we may avoid the restriction to finite measure by adding hypotheses to make the semifiniteness theorems applicable.

6d. Generalizing $\mathcal{P}^{s+t}(E \times F) \le \mathcal{P}^s(E)\mathcal{P}^t(F)$.

For this generalization, we use the inequality

$$\liminf_{r\to 0}\left[\frac{\mu\big(\overline{B}_r(x)\big)}{\varphi(r)}\,\frac{\mu\big(\overline{B}_r(y)\big)}{\psi(r)}\right] \ge \liminf_{r\to 0}\left[\frac{\mu\big(\overline{B}_r(x)\big)}{\varphi(r)}\right]\;\liminf_{r\to 0}\left[\frac{\mu\big(\overline{B}_r(y)\big)}{\psi(r)}\right]$$

so that

$$\underline{D}_{\mu\times\nu}^{\varphi\psi}(x,y) \ge \underline{D}_\mu^\varphi(x)\,\underline{D}_\nu^\psi(y). \tag{19}$$

Theorem 6.16. *Let X and Y be complete separable metric, let $X \times Y$ have the maximum metric, and let φ, ψ be two Hausdorff functions. Then*

$$\overline{\mathcal{P}}^{\varphi\psi}(E \times F) \le \widetilde{\mathcal{P}}^\varphi(E)\,\widetilde{\mathcal{P}}^\psi(F)$$

for all Borel sets $E \subseteq X$ and $F \subseteq Y$ provided it is true in the "null" cases when one of the factors on the right is zero.

Proof. If $\widetilde{\mathcal{P}}^\varphi(E) = \infty$ or $\widetilde{\mathcal{P}}^\psi(F) = \infty$, there is nothing to prove, so assume they are both finite. Let μ be the restriction of $\widetilde{\mathcal{P}}^\varphi$ to E and ν the restriction of $\widetilde{\mathcal{P}}^\psi$ to F. By Corollary 5.31 we have $\underline{D}_\mu^\varphi(x) \ge 1$ almost everywhere on E and $\underline{D}_\nu^\psi(y) \ge 1$ almost everywhere on F. So if

$$E_1 = \left\{\, x \in E : \underline{D}_\mu^\varphi(x) \ge 1 \,\right\}, \qquad F_1 = \left\{\, y \in F : \underline{D}_\nu^\psi(y) \ge 1 \,\right\},$$

then $\mu(E) = \mu(E_1)$ and $\nu(F) = \nu(F_1)$. Then by (19), for every $(x,y) \in E_1 \times F_1$ we have $\underline{D}_{\mu\times\nu}^{\varphi\psi}(x,y) \ge 1$. So by Corollary 5.10(b) we have $\overline{\mathcal{P}}^{\varphi\psi}(E_1 \times F_1) \le \mu(E_1)\nu(F_1) = \widetilde{\mathcal{P}}^\varphi(E)\widetilde{\mathcal{P}}^\psi(F)$. By the assumption for the "null" cases, we get the result with $E \times F$. □

Variants are possible, for example:

Theorem 6.17. *Let X and Y be complete separable metric, let $X \times Y$ have the maximum metric, and let φ, ψ be two Hausdorff functions. Assume that X and Y both have the SVP. Then*

$$\overline{\mathcal{P}}^{\varphi\psi}(E \times F) \le \overline{\mathcal{P}}^\varphi(E)\,\overline{\mathcal{P}}^\psi(F)$$

for all Borel sets $E \subseteq X$ and $F \subseteq Y$ provided it is true in the "null" cases when one of the factors on the right is zero.

As usual, there is a variant

$$\liminf_{r\to 0}\left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\;\frac{\mu\big(B_r(y)\big)}{\psi(r)}\right] \geq \liminf_{r\to 0}\left[\frac{\mu\big(B_r(x)\big)}{\varphi(r)}\right]\;\liminf_{r\to 0}\left[\frac{\mu\big(B_r(y)\big)}{\psi(r)}\right]$$

so that

$$\underline{\Delta}^{\varphi\psi}_{\mu\times\nu}(x,y) \geq \underline{\Delta}^{\varphi}_{\mu}(x)\;\underline{\Delta}^{\psi}_{\nu}(y). \tag{20}$$

But we have not proved density inequalities that can be applied to this case.

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