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Symplectic geometry on symplectic knot spaces

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Abstract. Symplectic knot spaces are the spaces of symplectic subspaces in a symplectic manifold M. We introduce a symplectic structure and show that the structure can be also obtained by the symplectic quotient method. We explain the correspondence between coisotropic submanifolds in M and Lagrangians in the symplectic knot space. We also define an almost complex structure on the symplectic knot space, and study the correspondence between almost complex submanifolds in M and holomorphic curves in the symplectic knot space.

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1. Introduction

Symplectic geometry on a symplectic manifold M is characterized by a nondegenerate skew symmetric 2-tensor ω which is closed. This gives sharp contrasts between symplectic geometry and Riemannian geometry, which is determined by a nondegenerate symmetric 2-tensor. For example, there is no local invariant in symplectic geometry such as curvatures in Riemannian geometry. Furthermore, unlike Riemannian case, there are submanifolds in M determined by the symplectic structure ω, Lagrangian submanifolds and holomorphic curves. Lagrangian submanifolds are the maximal-dimensional (in fact, a half of the dimension of M) ones among the submanifolds with ω vanishing, and holomorphic curves are the minimal-dimensional (in fact, two) ones whose tangent spaces are preserved by an almost complex structure compatible to ω. These two types of submanifolds are

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playing key roles in the geometrical agenda including Floer homology, Gromov–Witten invariants, and mirror symmetry corresponding to topological A-model in topological string theory.

On the other hand, ω determines another type of submanifolds in M: the so-called coisotropic submanifolds whose each tangent space contains a Lagrangian subspace. It is natural to expect that the geometry of these is closely related to that of Lagrangians. In [7], a correspondence is given between coisotropic subspaces in a symplectic vector space and Lagrangians in a symplectic Grassmannian by N.C. Leung and author. This correspondence is also explained in this paper (Section 4). Note that coisotropic submanifolds are also suggested as the proper objects for topological A-model [4].

In this paper, we consider a space $\mathcal{K}(\Sigma, M) := \mathrm{Map}(\Sigma, M)/\mathrm{Diff}(\Sigma)$ which consists of submanifolds in M given by embeddings from a $2k$-dimensional closed manifold Σ to M. In this paper, M is a $2n$-dimensional symplectic manifold with symplectic structure ω. By applying the transgression method on $\omega^{k+1}/\,(k+1)!$, we obtain a closed 2-form Ω on $\mathcal{K}(\Sigma, M)$, i.e.,

$$\Omega := \int_\Sigma \mathrm{ev}^* \frac{\omega^{k+1}}{(k+1)!}.$$

And it turns out Ω is nondegenerate only on symplectic submanifolds in $\mathcal{K}(\Sigma, M)$ when $k < n-1$. Therefrom we define a symplectic knot space as

$$\mathcal{K}^{\mathrm{Sp}}(\Sigma, M) := \mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)/\mathrm{Diff}(\Sigma),$$

namely the subspace of $\mathcal{K}(\Sigma, M)$ consisting of symplectic submanifolds in M. Note if $k = n-1$, Ω is nondegenerate on $\mathcal{K}(\Sigma, M)$ and this is one of the higher dimensional knot spaces on manifolds with vector cross products (see Remark 3 and [6]).

The symplectic knot space $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ with the symplectic structure Ω can also be constructed as a symplectic quotient. First we consider a space $\mathcal{C}$ with Diff (Σ)-action containing $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$ defined as

$$\mathcal{C} := \mathrm{Map}^{\mathrm{Sp}}(\Sigma, M) \times \Omega^1_{\mathrm{cl}}(\Sigma)$$

where $\Omega^1_{\mathrm{cl}}(\Sigma)$ is the space of closed 1-forms on Σ. And we define a $\mathrm{Diff}(\Sigma)$-invariant symplectic structure 2-form $\hat{\Omega}$ (see Section 3) on $\mathcal{C}$ where Ω is the restriction of $\hat{\Omega}$ on $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$. We show that there is a moment map μ on $\mathcal{C}$ for the $\mathrm{Diff}(\Sigma)$-action defined as

$$\mu(f, A) := f^*\left(\omega^k/k!\right) \otimes A \in \Omega^{2k}\left(\Sigma, T^*_\Sigma\right)$$

where $\Omega^{2k}(\Sigma, T^*_\Sigma)$ is the dual space of $\Gamma(\Sigma, T_\Sigma)$, which is the Lie algebra of $\mathrm{Diff}(\Sigma)$. Since $\mu^{-1}(0)$ is $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$, we conclude that $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ is the symplectic quotient $\mu^{-1}(0)/\mathrm{Diff}(\Sigma)$.

At last, we study the correspondence between coisotropic submanifolds in M and Lagrangian subspaces in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ (see Section 4).

Theorem. *Suppose C is a submanifold in M and the corresponding subknot space $\mathcal{K}^{\mathrm{Sp}}(\Sigma, C)$ is proper. Then C is an $(n+k)$-dimensional coisotropic submanifold in M if and only if $\mathcal{K}^{\mathrm{Sp}}(\Sigma, C)$ is a Lagrangian subknot space in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$.*

We also define an almost complex structure on $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ and study the correspondence between almost complex submanifolds in M and holomorphic curves in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$. We consider a *normal disk* D in $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$, which is an integral two-dimensional disk D in $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$ for the horizontal distribution of a canonical connection on the principal fibration

$$\mathrm{Diff}(\Sigma) \to \mathrm{Map}^{\mathrm{Sp}}(\Sigma, M) \xrightarrow{\pi} \mathcal{K}^{\mathrm{Sp}}(\Sigma, M).$$

And we assume that a $(2k+2)$-dimensional submanifold Z, defined as

$$Z := \bigcup_{f \in D} f(\Sigma),$$

is an embedding in M. Therefrom, we obtain the following theorem.

Theorem. *For a tame normal disk D in $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$, $\hat{D} := \pi(D)$ is a holomorphic disk in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ if and only if Z is a $(2k+2)$-dimensional almost complex submanifold in M and $Z \to D$ is a Riemannian submersion.*

2. Symplectic knot spaces

In this section we define symplectic knot spaces and study symplectic structures on them. The symplectic knot spaces are defined as the spaces of symplectic submanifolds in a symplectic manifold M. To be precise, we consider the spaces of embeddings from an even-dimensional manifold to M whose images are symplectic, and these images form a symplectic knot space. Because the normal bundle of a symplectic submanifold is also symplectic, it is natural to expect the symplectic knot spaces to have symplectic structures and we define one with a symplectic structure on M.

Let M be a $2n$-dimensional symplectic manifold with a symplectic form ω, i.e., a nondegenerate closed 2-form, and Σ be a $2k$-dimensional oriented closed manifold with $k < n-1$. We consider the space of embeddings (resp. symplectic embeddings) from Σ to M,

$$\mathrm{Map}(\Sigma, M) := \{f : \Sigma \to M \mid f \quad \text{embedding}\}$$

(resp. $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$). Because the symplectic condition is open, $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$ and $\mathrm{Map}(\Sigma, M)$ have the same tangent spaces for each $f \in \mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$, namely $T_f\left(\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)\right) = \Gamma(\Sigma, f^*(T_M))$.

Let

$$\mathrm{ev} : \Sigma \times \mathrm{Map}(\Sigma, M) \to M$$

be the evaluation map $\mathrm{ev}(x, f) = f(x)$ and let

$$\mathrm{pr} : \Sigma \times \mathrm{Map}(\Sigma, M) \to \mathrm{Map}(\Sigma, M)$$

be the projection map. We define a 2-form Ω_{Map} on $\mathrm{Map}(\Sigma, M)$ by taking the transgression of $\omega^{k+1}/(k+1)!$,

$$\Omega_{\mathrm{Map}} := (\mathrm{pr}_*)(\mathrm{ev})^* \frac{\omega^{k+1}}{(k+1)!} = \int_\Sigma \mathrm{ev}^* \frac{\omega^{k+1}}{(k+1)!}.$$

To be explicit, this is

$$\Omega_{\mathrm{Map}}(a, b) = \int_\Sigma \frac{\iota_{a \wedge b}\omega^{k+1}}{(k+1)!}$$

for tangent vectors a and b to $\mathrm{Map}(\Sigma, M)$ at f, i.e., $a, b \in \Gamma(\Sigma, f^*(T_M))$.

In the following, we consider the degeneracy of Ω_{Map} and define a space from $\mathrm{Map}(\Sigma, M)$ where a 2-form induced from Ω_{Map} achieves nondegeneracy to be a symplectic structure.

First, we observe that Ω_{Map} degenerates along the tangent directions to Σ because $(\iota_{a\wedge b}\omega^{k+1})|_{\Sigma}$ cannot be a volume form on Σ if a or b is tangential, namely given by the natural action of $\mathrm{Diff}(\Sigma)$ on $\mathrm{Map}(\Sigma, M)$. Here $\mathrm{Diff}(\Sigma)$ is the space of orientation preserving diffeomorphisms on Σ. As a matter of fact, this holds true for any form on $\mathrm{Map}(\Sigma, M)$ obtained by transgression. Therefore we consider a quotient space

$$\mathcal{K}(\Sigma, M) := \mathrm{Map}(\Sigma, M)/\mathrm{Diff}(\Sigma),$$

that is the space of $2k$-dimensional submanifolds in M, and Ω_{Map} descends to a 2-form Ω on it. Note that the tangent space of $[f]$ in $\mathcal{K}(\Sigma, M)$ is $\Gamma\left(\Sigma, N_{\Sigma/M}\right)$ where $N_{\Sigma/M}$ is the normal bundle of $f(\Sigma)$ in M.

Second, in the following lemma we see that the 2-form Ω on $\mathcal{K}(\Sigma, M)$ is still degenerate when the submanifolds are not symplectic in M, but it turns out that these are all the possible cases for the degeneracy of Ω.

Lemma 1. *For $k < n - 1$, the 2-form Ω on $\mathcal{K}(\Sigma, M)$ defined as above is nondegenerate only for those $[f]$ in $\mathcal{K}(\Sigma, M)$ whose image in M is symplectic.*

Proof. Consider a fixed $[f]$ in $\mathcal{K}(\Sigma, M)$ and a point x in Σ. Let a be a tangent vector at $[f]$ such that $\Omega_{[f]}(a, b) = 0$ for any b in $\Gamma\left(\Sigma, N_{\Sigma/M}\right)$. We apply the localization lemma (see [6]) to this condition, as follows. For a fixed x in Σ, by multiplying b with a sequence of functions on Σ approaching the delta function at x, we obtain sections $(b)_{\varepsilon}$ which approach $\delta(x)b$ as $\varepsilon \to 0$ where $\delta(x)$ is Dirac delta function. Then

$$\begin{aligned}\frac{\iota_{a(x)\wedge b(x)}\omega^{k+1}\mid_{T_x\Sigma}}{(k+1)!\,\mathrm{vol}_{\Sigma,x}} &= \Omega_{[f],x}\left(a(x), b(x)\right)\\ &= \lim_{\varepsilon\to 0}\left(\int_{\Sigma}\mathrm{ev}^{*}\frac{\omega^{k+1}}{(k+1)!}\right)(a, (b)_{\varepsilon}) = 0.\end{aligned}$$

Therefore, the nondegeneracy of $\Omega_{[f]}$ corresponds to that of

$$\omega^{k+1}\mid_{T_x\Sigma} / \{(k+1)!\,\mathrm{vol}_{\Sigma,x}\}$$

for all x in Σ. Furthermore, one can show that for $k < n-1$, this 2-form on $N_{\Sigma/M,x}$ is nondegenerate iff the tangent space of $f(\Sigma)$ is symplectic at $f(x)$ (or see [7]). Thereby, $a = 0$ iff $f(\Sigma)$ is symplectic in M. This proves the lemma. □

Now, we define the *symplectic knot space* $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ as the space of symplectic submanifolds in M, i.e.,

$$\mathcal{K}^{\mathrm{Sp}}(\Sigma, M) = \mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)/\mathrm{Diff}(\Sigma).$$

Note when $k = 0$, namely Σ is a point, the corresponding symplectic knot space defined for M is M itself.

Besides the nondegeneracy, we need to see that Ω is closed in order to be a symplectic form. This is easily obtained because Ω_{Map} is closed,

$$d\Omega_{\mathrm{Map}} = d\int_{\Sigma}\mathrm{ev}^{*}\frac{\omega^{k+1}}{(k+1)!} = \int_{\Sigma}\mathrm{ev}^{*}\frac{d\omega^{k+1}}{(k+1)!} = 0.$$

As a summary of this section, we have the following theorem.

Theorem 2. *Suppose M is a $2n$-dimensional symplectic manifold and Σ is a $2k$-dimensional closed oriented manifold where $k < n-1$. Then the* symplectic knot space $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ *has a symplectic form Ω defined as above.*

Remark 3. When $k = n-1$, the 2-form Ω is nondegenerate for all $[f]$ in $\mathcal{K}(\Sigma, M)$ since $\omega^{(n-1)+1}/n!$ is the volume form on M. Therefore, symplectic knot space $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ is $\mathcal{K}(\Sigma, M)$ itself. In fact, this space is one of the higher-dimensional knot spaces studied in [6]. In that paper, N.C. Leung and author studied the knot spaces defined for manifolds with vector cross products. The vector cross product is the generalization of the cross product on a 3-dimensional Euclidean space. In our case, the volume form $\omega^n/n!$ is a $(2n-1)$-fold vector cross product on the $2n$-dimensional oriented Riemannian manifold M(see [6, 7]).

3. Symplectic knot spaces as symplectic quotients

In the previous section we have seen that the symplectic knot space $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ has a symplectic structure induced from a symplectic structure on M. In this section we show $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ and its symplectic structure Ω can be obtained as a symplectic quotient. The key lines of this section are as follows. At first, we consider a space with $\mathrm{Diff}(\Sigma)$-action that contains $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$, and define a $\mathrm{Diff}(\Sigma)$-invariant symplectic structure on it. Then we identify a moment map on it given by the action of $\mathrm{Diff}(\Sigma)$. At last, $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ is constructed as a symplectic quotient.

Let $\mathcal{C}$ be a space containing $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$ defined as

$$\mathcal{C} := \mathrm{Map}^{\mathrm{Sp}}(\Sigma, M) \times \Omega^1_{\mathrm{cl}}(\Sigma)$$

where $\Omega^1_{\mathrm{cl}}(\Sigma)$ is the space of closed 1-forms on Σ. The group $\mathrm{Diff}(\Sigma)$ acts on $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$ by composition and on $\Omega^1_{\mathrm{cl}}(\Sigma)$ by pullback. Note the Lie algebra of $\mathrm{Diff}(\Sigma)$ is isomorphic to $\Omega^1(\Sigma)$, but we define $\mathcal{C}$ with $\Omega^1_{\mathrm{cl}}(\Sigma)$ so that the following 2-form is symplectic on $\mathcal{C}$.

Suppose (a_1, B_1) and (a_2, B_2) are tangent vectors of $\mathcal{C}$ at (f, A) i.e., $(a_i, B_i) \in \Gamma(\Sigma, f^*(T_M)) \times \Omega^1_{\mathrm{cl}}(\Sigma)$, then we define a 2-form $\hat{\Omega}$ on $\mathcal{C}$ as

$$\begin{aligned}\hat{\Omega}_{(f,A)}\left((a_1, B_1), (a_2, B_2)\right) := & \int_\Sigma \frac{\iota_{a_1 \wedge a_2}\omega^{k+1}}{(k+1)!} + \int_\Sigma \frac{\iota_{a_1}\omega^k \wedge B_2}{k!} \\ & - \int_\Sigma \frac{\iota_{a_2}\omega^k \wedge B_1}{k!} + \int_\Sigma \frac{\omega^{k-1} \wedge B_1 \wedge B_2}{(k-1)!}.\end{aligned}$$

Note that ω stands for $f^*(\omega)$ and we suppress f afterwards unless there is confusion.

It is easy to see $\hat{\Omega}$ is preserved by the action of $\mathrm{Diff}\,(\Sigma)$, and we show that it is a closed and nondegenerate 2-form on $\mathcal{C}$ as follows.

Lemma 4. *$\hat{\Omega}$ is a symplectic form on $\mathcal{C}$.*

Proof. First, we prove that $\hat{\Omega}$ is *closed.* Note that the first term of $\hat{\Omega}$ can be obtained by pulling back the closed 2-form Ω_{Map} on $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$ to $\mathcal{C}$. Therefore, we only need to show that $\hat{\Omega} - \Omega_{\mathrm{Map}}$, denoted as Φ, is closed. And it suffices to check that

$$\begin{aligned}3d\Phi\left(X_1, X_2, X_3\right) = & X_1\Phi\left(X_2, X_3\right) - X_2\Phi\left(X_1, X_3\right) + X_3\Phi\left(X_1, X_2\right) \\ & - \Phi\left(\left[X_1, X_2\right], X_3\right) + \Phi\left(\left[X_1, X_3\right], X_2\right) - \Phi\left(\left[X_2, X_3\right], X_1\right)\end{aligned}$$

is zero for any tangent vector fields X_1, X_2 and X_3 on $\mathcal{C}$.

Consider a fixed point $([f], A)$ in $\mathcal{C}$ and denote tangent vector of X_1, X_2 and X_3 at $([f], A)$ as (a_1, A_1), (a_2, A_2) and (a_3, A_3) respectively, where $a_i \in \Gamma(\Sigma, f^*T_M)$ and $A_i \in \Omega^1_{\mathrm{cl}}(\Sigma)$. Since $\mathcal{C}$ is the product space, $[X_i, X_j]$ at $([f], A)$ is

$$\left([a_i, a_j], \mathcal{L}_{a_i} A_j - \mathcal{L}_{a_j} A_i\right)$$

where $[a_i, a_j]$ is the usual Lie bracket of vector fields, and $[A_i, A_j] = 0$ on $\Omega^1_{\mathrm{cl}}(\Sigma)$. To get $[A_i, A_j] = 0$, we need to choose a vector field extended from the tangent vector A_i to have constant coefficients, and this is possible because $\Omega^1_{\mathrm{cl}}(\Sigma)$ is a vector space.

The followings are typical calculation in each terms of $d\Phi(X_1, X_2, X_3)$. At the point $([f], A)$ and for distinct i, j and m,

$$\begin{aligned} d\left(\int_\Sigma \iota_{a_i}\omega^k \wedge A_j\right)((a_m, A_m)) &= \int_\Sigma \mathcal{L}_{a_m}\left(\iota_{a_i}\omega^k \wedge A_j\right) \\ &= \int_\Sigma \left(\iota_{a_m} d\iota_{a_i}\omega^k \wedge A_j + \iota_{a_i}\omega^k \wedge \mathcal{L}_{a_m} A_j\right), \end{aligned}$$

$$\begin{aligned} d\left(\int_\Sigma \omega^{k-1} \wedge A_i \wedge A_j\right)((a_m, A_m)) &= \int_\Sigma \mathcal{L}_{a_m}\left(\omega^{k-1} \wedge A_i \wedge A_j\right) \\ &= \int_\Sigma d\iota_{a_m}\left(\omega^{k-1} \wedge A_i \wedge A_j\right) = 0, \end{aligned}$$

and

$$\begin{aligned} &\Phi_{([f],A)}\left(\left([a_i, a_j], \mathcal{L}_{a_i} A_j - \mathcal{L}_{a_j} A_i\right), (a_m, A_m)\right) \\ &= \frac{1}{k!}\int_\Sigma \iota_{[a_i,a_j]}\omega^k \wedge A_m - \frac{1}{k!}\int_\Sigma \iota_{a_m}\omega^k \wedge \left(\mathcal{L}_{a_i} A_j - \mathcal{L}_{a_j} A_i\right). \end{aligned}$$

Note these are obtained by using Stokes' theorem and the fact that $\mathcal{L}_{a_i} A_j - \mathcal{L}_{a_j} A_i$ is exact.

By applying above three identities to $3d\Phi(X_1, X_2, X_3)$ at $([f], A)$,

$$\begin{aligned} &k!\, 3d\Phi_{([f],A)}\left((a_1, A_1), (a_2, A_2), (a_3, A_3)\right) \\ &= \int_\Sigma \left(\iota_{a_1} d\iota_{a_2}\omega^k \wedge A_3 + \iota_{a_2}\omega^k \wedge \mathcal{L}_{a_1} A_3 - \iota_{a_1} d\iota_{a_3}\omega^k \wedge A_2 - \iota_{a_3}\omega^k \wedge \mathcal{L}_{a_1} A_2\right) \\ &\quad - \int_\Sigma \left(\iota_{a_2} d\iota_{a_1}\omega^k \wedge A_3 + \iota_{a_1}\omega^k \wedge \mathcal{L}_{a_2} A_3 - \iota_{a_2} d\iota_{a_3}\omega^k \wedge A_1 - \iota_{a_3}\omega^k \wedge \mathcal{L}_{a_2} A_1\right) \\ &\quad + \int_\Sigma \left(\iota_{a_3} d\iota_{a_1}\omega^k \wedge A_2 + \iota_{a_1}\omega^k \wedge \mathcal{L}_{a_3} A_2 - \iota_{a_3} d\iota_{a_2}\omega^k \wedge A_1 - \iota_{a_2}\omega^k \wedge \mathcal{L}_{a_3} A_1\right) \\ &\quad - \int_\Sigma \left\{\iota_{[a_1,a_2]}\omega^k \wedge A_3 - \iota_{a_3}\omega^k \wedge \left(\mathcal{L}_{a_1} A_2 - \mathcal{L}_{a_2} A_1\right)\right\} \\ &\quad + \int_\Sigma \left\{\iota_{[a_1,a_3]}\omega^k \wedge A_2 - \iota_{a_2}\omega^k \wedge \left(\mathcal{L}_{a_1} A_3 - \mathcal{L}_{a_3} A_1\right)\right\} \\ &\quad - \int_\Sigma \left\{\iota_{[a_2,a_3]}\omega^k \wedge A_1 - \iota_{a_1}\omega^k \wedge \left(\mathcal{L}_{a_2} A_3 - \mathcal{L}_{a_3} A_2\right)\right\} \\ &= 0. \end{aligned}$$

The last equality is obtained from

$$\int_\Sigma \iota_{a_i} d\iota_{a_j}\omega^k \wedge A_m - \iota_{a_j} d\iota_{a_i}\omega^k \wedge A_m = \int_\Sigma \iota_{[a_i,a_j]}\omega^k \wedge A_m.$$

Hence this shows that $\hat{\Omega}$ is closed.

Second, we show that $\hat{\Omega}$ is *nondegenerate* at $\mathcal{C}$. Let (b_1, B_1) be a nonzero vector at $([f], A)$. We separate cases with respect to the vector field b_1.

Case 1. b_1 is a nonzero vector field on Σ and B_1 is any closed 1-form on Σ.

Since Σ is symplectic for $f^*\omega$, there is a pair of vector fields

$$(a_T, a_S) \in \Gamma(\Sigma, T_\Sigma) \times \Gamma\left(\Sigma, f^*T_M \cap T_\Sigma^{f^*\omega}\right)$$

with $b_1 = a_T + a_S$. Here, $T_\Sigma^{f^*\omega}$is the $f^*\omega$-orthogonal complement of T_Σ in f^*T_M.

(i) If a_T is zero, i.e., b_1 is in $\Gamma\left(\Sigma, f^*T_M \cap T_\Sigma^{f^*\omega}\right)$, then there is a vector $(b_2, 0)$ at $([f], A)$ such that b_2 is in $\Gamma\left(\Sigma, f^*T_M \cap T_\Sigma^{f^*\omega}\right)$ with $\omega(b_1, b_2) > 0$. And we have

$$\begin{aligned}\hat{\Omega}_{(f,A)}((b_1, B_1), (b_2, 0)) &= \frac{1}{(k+1)!}\int_\Sigma \iota_{b_1\wedge b_2}\omega^{k+1} - \frac{1}{k!}\int_\Sigma \iota_{b_2}\omega^k \wedge B_1 \\ &= \int_\Sigma \omega(b_1, b_2)\frac{\omega^k}{k!} > 0 .\end{aligned}$$

Note the second equality uses the vanishing of $\iota_{b_2}\omega$ on T_Σ.

(ii) If a_T is nonzero, there is an exact form B_2 on Σ such that $B_2(a_T) \le 0$ but $B_2(a_T)$ is not a zero function on Σ. As a local question, such an exact form always exists. Therefrom we have

$$\begin{aligned}k!\hat{\Omega}_{(f,A)}((b_1, B_1), (0, B_2)) &= \int_\Sigma \iota_{b_1}\omega^k \wedge B_2 + k\int_\Sigma \omega^{k-1}\wedge B_1 \wedge B_2 \\ &= \int_\Sigma \iota_{a_T}\omega^k \wedge B_2 = -\int_\Sigma B_2(a_T)\,\omega^k > 0 .\end{aligned}$$

Note the second equality uses the vanishing of $\iota_{a_S}\omega$ on T_Σ and Stokes' theorem.

Case 2. $b_1 = 0$.

Therefore B_1 must be nonzero and there is a tangent vector field v_1 on Σ where $\iota_{v_1} f^*\omega = B_1$ because $\Omega^1_{\mathrm{cl}}(\Sigma)$ be identified with the space of tangent vector fields preserving $f^*\omega$. Moreover one can find a tangent vector field v_2 on Σ such that $\omega(v_1, v_2) > 0$, since Σ is symplectic for $f^*\omega$. By using $(v_2, 0)$ at $([f], A)$, we obtain

$$\begin{aligned}k!\hat{\Omega}_{(f,A)}((0, B_1), (v_2, 0)) &= -\int_\Sigma \iota_{v_2}\omega^k \wedge B_1 = -\int_\Sigma \iota_{v_2}\omega^k \wedge \iota_{v_1}\omega \\ &= \int_\Sigma \omega^k \iota_{v_2}\iota_{v_1}\omega > 0 .\end{aligned}$$

From above two cases, we have the nondegeneracy of $\hat{\Omega}$ on $\mathcal{C}$. $\square$

Remark. Since Σ is a symplectic subspace in M, the space $\Omega^1_{\mathrm{cl}}(\Sigma)$ can be identified with the space of vector fields preserving the induced symplectic form; called symplectic vector fields; and there is another Lie algebra structure on $\Omega^1_{\mathrm{cl}}(\Sigma)$ defined by identifying the Lie bracket on the vector fields on Σ. Furthermore, it is easy to show that Lie bracket of two symplectic vector fields induces an exact 1-form in $\Omega^1_{\mathrm{cl}}(\Sigma)$. But with this Lie algebra structure, the 2-form $\hat{\Omega}$ is not closed.

Recall that the infinite-dimensional Lie group $\mathrm{Diff}(\Sigma)$ acts naturally on $\mathcal{C}$ and preserves its symplectic structure $\hat{\Omega}$. Therefrom we define a $\mathrm{Diff}(\Sigma)$-equivariant map

$$\mu : \mathcal{C} \to \Omega^{2k}\left(\Sigma, T^*_\Sigma\right)$$
$$\mu\left(f, A\right) := f^*\left(\omega^k/k!\right) \otimes A,$$

where $\Omega^{2k}\left(\Sigma, T^*_\Sigma\right)$ is equivalent to the dual space of $\Gamma\left(\Sigma, T_\Sigma\right)$ which is the Lie algebra of $\mathrm{Diff}(\Sigma)$. By the following lemma, μ is actually a moment map.

Lemma 5. *μ is a moment map.*

Proof. For each tangent vector field X on Σ, $(X, \mathcal{L}_X A)$ is the corresponding fundamental vector on $\mathcal{C}$ at (f, A) induced from the action of $\mathrm{Diff}(\Sigma)$ and a map $\mu_X : \mathcal{C} \to \mathbb{R}$ is defined as

$$\mu_X\left(f, A\right) = \int_\Sigma \left(\iota_X A\right) f^*\left(\omega^k/k!\right).$$

For any tangent vector (a, B) and a fixed fundamental vector $(X, d\iota_X A)$ at (f, A), we have

$$\begin{aligned}
&hat\Omega_{(f,A)}\left(\left(X, d\iota_X A\right), \left(a, B\right)\right)\\
&= \frac{1}{(k+1)!}\int_\Sigma \iota_{X\wedge a}\omega^{k+1} + \frac{1}{(k-1)!}\int_\Sigma \omega^{k-1}\wedge d\iota_X A\wedge B\\
&\quad + \frac{1}{k!}\int_\Sigma \left(\iota_X\omega^k\wedge B - \iota_a\omega^k\wedge d\iota_X A\right)\\
&= 0 - \frac{1}{k!}\int_\Sigma \left\{\left(\iota_X B\right)\omega^k + \left(\iota_X A\right)\mathcal{L}_a\omega^k\right\} + 0\\
&= -d\mu_X\left(a, B\right).
\end{aligned}$$

In the second equality, we use the fact that $\iota_{X\wedge a}\omega^{k+1}$ can not be a volume form on Σ and Stokes' theorem. $\square$

Since $f^*\left(\omega^k/k!\right) \neq 0$ for each f in $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)$, $\mu^{-1}(0)$ is $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M)\times\{0\}$, and the symplectic quotient $\mu^{-1}(0)/\mathrm{Diff}(\Sigma)$ is $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$. Therefore $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ is a symplectic space with a induced symplectic structure Ω. This section is summarized in the following theorem.

Theorem 6. *Let M be a symplectic manifold and let Σ be an even-dimensional closed oriented manifold with $\dim\Sigma + 2 < \dim M$. Then the symplectic space $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ can be constructed as a symplectic quotient.*

Remark 7. The assumption $\dim\Sigma + 2 < \dim M$ is necessary since the above argument does not work when $\dim\Sigma + 2 = \dim M = 2n$. For this case, $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ is the same with $\mathcal{K}(\Sigma, M)$, therefore $\mathcal{C}$ is defined as $\mathrm{Map}(\Sigma, M)\times\Omega^1_{\mathrm{cl}}(\Sigma)$. But $\hat{\Omega}$ is degenerate on $\mathrm{Map}(\Sigma, M) \times \Omega^1_{\mathrm{cl}}(\Sigma)$. For example, consider $(f, 0)$ in $\mathrm{Map}(\Sigma, M) \times \Omega^1_{\mathrm{cl}}(\Sigma)$ where the image of f in M is coisotropic. Let $(v, 0)$ be a tangent vector on $\mathcal{C}$ at $(f, 0)$ where v is tangent vector field on Σ. Then for any tangent vector (a, A) on $\mathcal{C}$ at $(f, 0)$, we have

$$\hat{\Omega}_{(f,0)}\left((v,0),(a,A)\right) = \int_\Sigma \frac{\iota_{v\wedge a}\omega^n}{n!} + \int_\Sigma \frac{\iota_v\omega^{n-1}\wedge A}{(n-1)!} = 0 - \int_\Sigma \frac{\omega^{n-1}\iota_v A}{(n-1)!} = 0.$$

The last equality is given by $f(\Sigma)$ being coisotropic in M.

4. Symplectic geometry of symplectic knot spaces

In a $2n$-dimensional manifold M with a symplectic structure ω, the most important features characterized by ω are Lagrangian submanifolds and holomorphic curves. These are submanifolds calibrated by ω; equivalently preserved by a corresponding almost complex structure, and those are n-dimensional submanifolds with vanishing ω. These two types of submanifolds are playing key roles in the study of Mirror symmetry, Gromov–Witten invariants, and Floer homology. In particular, sigma A-model in the mirror symmetry is modeled with Lagrangian submanifolds equipped with flat unitary bundles. On the other hand, there are suggestions (see [4]) that A-model is rather properly modeled with coisotropic submanifolds. Since the tangent space at each point in a coisotropic submanifold contains Lagrangian subspaces, it is natural that the geometry of coisotropic submanifolds is consistent with Lagrangian geometry. In the paper [7], a correspondence is given between Lagrangian subgrassmannians in a symplectic Grassmannian space and coisotropic subspaces in a symplectic vector space. In this section we show that this correspondence holds true for the symplectic knot spaces. We also define an almost complex structure and study the correspondence between the almost complex submanifolds and the holomorphic curves in the symplectic knot spaces.

Suppose X is a submanifold in M and Σ is a $2k$-dimensional oriented closed manifold with $k < n-1$ as before, the symplectic subknot space corresponding to X is defined as

$$\mathcal{K}^{\mathrm{Sp}}(\Sigma, X) := \{\mathrm{Map}^{\mathrm{Sp}}(\Sigma, M) \cap \mathrm{Map}(\Sigma, X)\}/\mathrm{Diff}(\Sigma).$$

The Ω-orthogonal space of $\mathcal{K}^{\mathrm{Sp}}(\Sigma, X)$ at $[f]$ is defined as

$$\mathcal{N}^{\Omega}_{[f]}(\Sigma, X) := \left\{a \in \Gamma\left(\Sigma, N_{\Sigma/M}\right) : \Omega_{[f]}\left(a, b\right) = 0 \text{ for all } b \in \Gamma\left(\Sigma, N_{\Sigma/X}\right)\right\},$$

and a symplectic subknot space is *Lagrangian* if the symplectic structure Ω vanishes on it and its Ω-orthogonal space is the same as its tangent space.

A submanifold C in M is called *coisotropic* if for each point x in C, the tangent space T_xC is contained in its ω-orthogonal space, i.e., $(T_xC)^\omega \subset T_xC$. In fact, one can show that a $(n+k)$-dimensional submanifold C is coisotropic iff ω^k never vanishes but ω^{k+1} does on C (see [7, 8] for details).

In the following theorem, we need an assumption that for each point x in X there is an element in $\mathrm{Map}^{\mathrm{Sp}}(\Sigma, X)$ whose image contains x, and the corresponding symplectic subknot space $\mathcal{K}^{\mathrm{Sp}}(\Sigma, X)$ is called *proper*. The author suspects that the properness condition on $\mathcal{K}^{\mathrm{Sp}}(\Sigma, X)$ is unnecessary if X is a $(n+k)$-dimensional coisotropic submanifold in M. But it is not clear whether the following theorem holds true without this condition.

Theorem 8. *Let M be a $2n$-dimensional symplectic submanifold and Σ be a $2k$-dimensional closed oriented manifold where $k < n-1$. Suppose C is a submanifold in M and the corresponding subknot space $\mathcal{K}^{\mathrm{Sp}}(\Sigma, C)$ is proper, then the following statements are equivalent:*

(1) *C is a $(n+k)$-dimensional coisotropic submanifold in M.*
(2) *The subknot space $\mathcal{K}^{\mathrm{Sp}}(\Sigma, C)$ is Lagrangian in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$.*

Proof. ((1) $\Leftarrow$ (2)) Assume $\mathcal{K}^{\mathrm{Sp}}(\Sigma, C)$ is Lagrangian in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$. As $\mathcal{K}^{\mathrm{Sp}}(\Sigma, C)$ is proper, ω^k never vanishes on T_xC for each x in C. But ω^{k+1} vanishes on C because of the vanishing condition of Ω on $\mathcal{K}^{\mathrm{Sp}}(\Sigma, C)$ and the localization lemma in [6] (or see Lemma 1 in Section 2). Therefore we have $\dim C \le n+k$. If $\dim C < n+k$, one can show that $\mathcal{K}^{\mathrm{Sp}}(\Sigma, C)$ is isotropic in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ but can not be Lagrangian since $\mathcal{N}^{\Omega}_{[f]}(\Sigma, X)$ is bigger than $T_{[f]}\mathcal{K}^{\mathrm{Sp}}(\Sigma, X)$. Therefore C is a $(n+k)$-dimensional submanifold with $\omega^k \neq 0$ but $\omega^{k+1} = 0$ on T_xC for each x in C, namely a coisotropic submanifold.

((1) $\Rightarrow$ (2)) Assume C is a $(n+k)$-dimensional coisotropic submanifold in M. Then it is obvious that Ω vanishes on $\mathcal{K}^{\mathrm{Sp}}(\Sigma, C)$ since ω^{k+1} vanishes on C. This implies $\mathcal{K}^{\mathrm{Sp}}(\Sigma, C)$ is isotropic in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$. Suppose $\mathcal{N}^{\Omega}_{[f]}(\Sigma, X)$ is greater than $T_{[f]}\mathcal{K}^{\mathrm{Sp}}(\Sigma, X)$ for some $[f]$ in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, X)$, there is v contained in the complement of $T_{[f]}\mathcal{K}^{\mathrm{Sp}}(\Sigma, X)$ in $\mathcal{N}^{\Omega}_{[f]}(\Sigma, X)$. By choosing a proper point x in Σ and using the localization lemma, we have

$$\Omega_{[f],x}(v, b) = 0 \text{ for all } b \in \Gamma\left(\Sigma, N_{\Sigma/X}\right),$$

and this implies ω^{k+1} vanishes on $T_{f(x)}C + \langle v_{f(x)} \rangle$. But this is a contradiction because $\omega^{k+1} \neq 0$ on $T_{f(x)}C + \langle v_{f(x)} \rangle$, which is a $(n+k+1)$-dimensional coisotropic subspace in $T_{f(x)}M$. Therefore $\mathcal{K}^{\mathrm{Sp}}(\Sigma, C)$ is Lagrangian in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$. $\square$

In the remaining section, we discuss the holomorphic curves in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ and their correspondence to almost complex submanifolds in M. At first we define an almost complex structure on $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ compatible to the symplectic structure Ω. Recall that there is an L^2-metric $g^{\mathcal{K}}$ on $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ defined as

$$g^{\mathcal{K}}_{[f]}(a, b) := \int_{f(\Sigma)} g(a, b)\,\mathrm{vol}_{\Sigma},$$

where a and b in $\Gamma\left(\Sigma, N_{\Sigma/M}\right)$ for each $[f]$ in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ and g is a Riemannian metric on M. Since Ω is a symplectic structure, there is an endomorphism compatible to $g^{\mathcal{K}}$ and Ω, but the endomorphism may not be an almost complex structure. To get an almost complex structure corresponding to Ω, it is necessary to modify the metric on $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$. For each $[f]$ in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ and x in Σ, the symplectic structure Ω can be localized as

$$\frac{\iota_{(a_x \wedge b_x)}\omega^{k+1}\mid_{T_x\Sigma}}{(k+1)!\,\mathrm{vol}_{\Sigma,x}},$$

where vol_{Σ} is given by induced metric on $f(\Sigma)$. Therefore by performing the following linear algebra method to the normal bundle of $f(\Sigma)$ for each $[f]$ in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$, one can obtain new metric $\widetilde{g}^{\mathcal{K}}_{[f]}$ on M where the corresponding endomorphism $\mathcal{J}_{[f]}$ is a complex structure.

Let S be a $2k$-dimensional symplectic subspace in a $2n$-dimensional vector space V with a symplectic structure ω compatible to a metric g and a complex structure J. We consider a 2-form ϕ defined on $S^{\perp}$ as

$$\phi\left(a,b\right):=\frac{\iota_{(a\wedge b)}\omega^{k+1}\mid_{S}}{(k+1)!\mathrm{vol}_{S}},$$

for any vectors a and b in $S^{\perp}$. Note that ϕ is nondegenerate on $S^{\perp}$ since S is symplectic. Therefore there is an endomorphism K on $S^{\perp}$ satisfying

$$\phi\left(a,b\right)=g\mid_{S^{\perp}}\left(Ka,b\right),$$

but K may not be a complex structure on $S^{\perp}$. Notice that K^2 is self adjoint and negative definite, and there is unique self-adjoint positive operator B with $B^2=-K^2$. Note that B and K share the same eigenspaces. Now, we define a metric $\widetilde{g}$ on $S^{\perp}$ as

$$\widetilde{g}\left(a,b\right):=g\mid_{S^{\perp}}\left(\sqrt{B}a,\sqrt{B}b\right),$$

and an endomorphism $\widetilde{J}$ as $\widetilde{J}:=K\ B^{-1}$. One can show $\widetilde{J}$ is compatible with ϕ and $\widetilde{g}$, furthermore it satisfies $\widetilde{J}^2=-\mathrm{id}$ on $S^{\perp}$. Note that $g\mid_{S^{\perp}}$and $\widetilde{g}$ are equal on the 1-eigenspace of $-K^2$. We call a subspace W in $S^{\perp}$ *ϕ-tame* if it is a K-invariant subspace in the 1-eigenspace of $-K^2$. One can obtain the following characterizations of ϕ-tame.

Lemma 9. *With above setup, the followings are equivalent.*

(1) *W is ϕ-tame.*
(2) *W is a subspace where K serves as a complex structure and $g\mid_{S^{\perp}}=\widetilde{g}$.*
(3) *For any a and b in W, we have $\phi\left(a,b\right)=\left|a\wedge b\right|$ where $|..|$ is induced from $g\mid_{S^{\perp}}$.*

Remark. The equivalence of (3) to (1) is essentially Wirtinger's inequality, and the remaining equivalences are straight-forward from the definition of ϕ-tame.

The following lemma will be used when we explain the correspondence between holomorphic curves in $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$ and almost complex submanifolds in M.

Lemma 10. *Let S be a $2k$-dimensional symplectic subspace in a Hermitian vector space (V,g,ω,J) and ϕ be a symplectic structure on $S^{\perp}$ as above. If*

$$\dim\left((S+JS)\cap S^{\perp}\right)=2,$$

then $(S+JS)\cap S^{\perp}$ is ϕ-tame. And if S is complex subspace, then any complex subspace in $S^{\perp}$ is ϕ-tame, in particular $S^{\perp}$ is ϕ-tame.

Proof. Let $e_1,...,e_{2k}$ be an oriented orthonormal basis of S. Suppose

$$\dim\left((S+JS)\cap S^{\perp}\right)=2,$$

then $S+JS$ is a $(2k+2)$-dimensional complex subspace in V. Therefore $\mathrm{vol}_{(S+JS)}=\omega^{k+1}/\left(k+1\right)!\mid_{(S+JS)}$, and we have

$$\begin{aligned}\phi\left(a,b\right)=\frac{\iota_{(a\wedge b)}\omega^{k+1}\mid_{S}}{(k+1)!\mathrm{vol}_{S}}&=\iota_{(a\wedge b\wedge e_1\wedge\ldots\wedge e_{2k})}\frac{\omega^{k+1}}{(k+1)!}\\&=\iota_{(a\wedge b..e_1,\ldots,e_{2k})}\mathrm{vol}_{(S+JS)}=\left|a\wedge b\right|\end{aligned}$$

for any a and b in $(S+JS)\cap S^{\perp}$. This implies $(S+JS)\cap S^{\perp}$ is ϕ-tame.

Suppose S is complex and W is any complex subspace in $S^{\perp}$, then W is in the symplectic orthogonal space of S. Therefore, for any a and b in W, we have

$$\phi\left(a,b\right)=\frac{\iota_{(a\wedge b)}\omega^{k+1}|_S}{(k+1)!\,\mathrm{vol}_S}=\frac{\omega\left(a,b\right)\omega^k|_S}{k!\,\mathrm{vol}_S}=\omega\left(a,b\right)=\left|a\wedge b\right|.$$

Note that the last two equalities are obtained from S and W being complex subspaces. This shows W is ϕ-tame. □

From above process, we obtain a new metric $\widetilde{g}^{\mathcal{K}}$ and compatible complex structure $\mathcal{J}$. With the metric $\widetilde{g}^{\mathcal{K}}$, the symplectic 2-form Ω serves as a calibration and the corresponding calibrated submanifolds in $\mathcal{K}^{\mathrm{Sp}}\left(\Sigma,M\right)$ are called *$\mathcal{J}$-holomorphic curves.*

We consider a *normal disk* D in $\mathrm{Map}^{\mathrm{Sp}}(\Sigma,M)$, which is a two-dimensional disk D in $\mathrm{Map}^{\mathrm{Sp}}(\Sigma,M)$ such that for each tangent vector $v\in T_fD$, the corresponding vector field in $\Gamma\left(\Sigma,f^*T_M\right)$ is normal to Σ. Note that D being a normal disk is equivalent to it being an integral submanifold for the horizontal distribution of a canonical connection (see [1]) on the principal fibration

$$\mathrm{Diff}(\Sigma)\to\mathrm{Map}^{\mathrm{Sp}}(\Sigma,M)\xrightarrow{\pi}\mathcal{K}^{\mathrm{Sp}}(\Sigma,M).$$

We denote the corresponding disk in $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$ as $\hat{D}:=\pi\left(D\right)$. Furthermore the normal disk D is called *tame* if the tangent vector space $T_{[f]}\hat{D}$ is $\Omega_{[f]}$-tame in Σ. Note that normal disks in other types of knot spaces are introduced and studied in [6].

For simplicity we assume that the $(2k+2)$-dimensional submanifold Z defined as

$$Z:=\bigcup_{f\in D}f(\Sigma),$$

is an embedding in M. For the small enough D, this is always the case. Note that Z is diffeomorphic to $D\times\Sigma$.

In the following theorem, we describe the relationship between a disk $\hat{D}$ in $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$ defined above and the corresponding $(2k+2)$-dimensional submanifold Z in M. The proof of the following theorem is adapted from [6].

Theorem 11. *Suppose that M is a $2n$-dimensional Hermitian manifold with the compatible symplectic structure ω and $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$ is its symplectic knot space for a $2k$-dimensional oriented closed manifold Σ as before. For a tame normal disk D in $\mathrm{Map}^{\mathrm{Sp}}(\Sigma,M)$, $\hat{D}:=\pi\left(D\right)$ is a $\mathcal{J}$-holomorphic disk in $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$, i.e., calibrated by Ω, if and only if Z is a $(2k+2)$-dimensional almost complex submanifold in M and $Z\to D$ is a Riemannian submersion.*

Proof. For a fixed $[f]\in\hat{D}$, we consider $\nu,\mu\in T_{[f]}(\hat{D})\subset\Gamma(\Sigma,N_{\Sigma/Z})$. Since $\omega^{k+1}/\left(k+1\right)!$ is a calibrating form, we have,

$$\omega^{k+1}/\left(k+1\right)!(\nu,\mu,e_1,...,e_{2k})\leq\mathrm{Vol}_Z(\nu,\mu,e_1,...,e_{2k})=|\nu\wedge\mu|$$

where $e_1,...,e_{2k}$ is any orthonormal frame on $f(\Sigma)$. In particular we have

$$\int_{f(\Sigma)}\frac{\iota_{\nu\wedge\mu}\mathrm{ev}^*\omega^{k+1}}{(k+1)!}\leq\int_{f(\Sigma)}|\nu\wedge\mu|\,\mathrm{vol}_\Sigma,$$

and the equality sign holds for every $[f] \in \hat{D}$ if and only if Z is an almost complex submanifold in M. $\int_{f(\Sigma)}$ will be simply denoted by $\int_{\Sigma}$. Notice that the symplectic form Ω on $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$ is defined as,

$$\Omega_{[f]}(\nu, \mu) = \int_{\Sigma} \frac{\iota_{\nu\wedge\mu}\mathrm{ev}^* \omega^{k+1}}{(k+1)!}.$$

Since Ω is a calibration on $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$, we have

$$\Omega_{[f]}(\nu, \mu) \leq \left(|\nu|_{\mathcal{K}}^2 \, |\mu|_{\mathcal{K}}^2 - \langle \nu, \mu \rangle_{\mathcal{K}}^2 \right)^{1/2}$$

where $\langle \nu, \mu \rangle_{\mathcal{K}} := \widetilde{g}^{\mathcal{K}}(\nu, \mu)$ and $|\nu|_{\mathcal{K}}^2 := \widetilde{g}^{\mathcal{K}}(\nu, \nu)$. Furthermore the equality sign holds when $\hat{D}$ is a $\mathcal{J}$-holomorphic disk in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$.

($\Rightarrow$) We suppose that $\hat{D}$ is a $\mathcal{J}$-holomorphic disk in $\mathcal{K}^{\mathrm{Sp}}(\Sigma, M)$. From above discussions, we have

$$\begin{aligned} \int_{\Sigma} |\nu \wedge \mu| &\geq \int_{\Sigma} \iota_{\nu\wedge\mu}\mathrm{ev}^* \omega^{k+1} / (k+1)! = \left(|\nu|_{\mathcal{K}}^2 \, |\mu|_{\mathcal{K}}^2 - \langle \nu, \mu \rangle_{\mathcal{K}}^2 \right)^{1/2} \\ &= \left(\int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2 - \left(\int_{\Sigma} \langle \nu, \mu \rangle \right)^2 \right)^{1/2}, \end{aligned}$$

where $\langle \nu, \mu \rangle := g^{\mathcal{K}}(\nu, \mu)$ and $|\nu|^2 := g^{\mathcal{K}}(\nu, \nu)$. Note that the second equality is obtained from tame condition. We also have the Hölder inequality,

$$\begin{aligned} \left(\int_{\Sigma} |\nu \wedge \mu| \right)^2 + \left(\int_{\Sigma} \langle \nu, \mu \rangle \right)^2 &\leq \left(\int_{\Sigma} |\sin \theta_x \nu| \, |\mu| \right)^2 + \left(\int_{\Sigma} |\cos \theta_x \nu| \, |\mu| \right)^2 \\ &\leq \int_{\Sigma} |\sin \theta_x \nu|^2 \int_{\Sigma} |\mu|^2 + \int_{\Sigma} |\cos \theta_x \nu|^2 \int_{\Sigma} |\mu|^2 \\ &= \int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2, \end{aligned}$$

where θ_x is $\angle(\nu(x), \mu(x))$ for each x in Σ. By combining these two inequalities, we obtain

(i) $$\int_{\Sigma} \iota_{\nu\wedge\mu}\mathrm{ev}^* \omega^{k+1} / (k+1)! = \int_{\Sigma} |\nu \wedge \mu| \, \mathrm{vol}_{\Sigma} \quad \text{and}$$

(ii) $$\left(\int_{\Sigma} |\nu \wedge \mu| \right)^2 + \left(\int_{\Sigma} \langle \nu, \mu \rangle \right)^2 = \int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2.$$

Condition (i) says that Z is a $\omega^{k+1}/(k+1)!$-calibrated submanifold, i.e., an almost complex submanifold in M. Condition (ii) implies that given any $[f]$, there exists constant C_1 and C_2 such that for any $x \in \Sigma$,

$$|\nu(x)| = C_1 \, |\mu(x)| \, , \; \angle(\nu(x), \mu(x)) = C_2.$$

Therefore, once we fix a tangent vector $\mu \in T_{[f]}(\hat{D})$, $\nu(x)$ for any $x \in \Sigma$ is completely determined by $\langle \nu, \mu \rangle$ and $|\nu| / |\mu|$. This implies that $Z \to D$ is a Riemannian submersion.

($\Leftarrow$) We notice that Z being an almost complex submanifold in M implies that D is a tame normal disk by Lemma 10. Therefrom, we obtain

$$\begin{aligned}\int_\Sigma |\nu\wedge\mu| &= \int_\Sigma \frac{\iota_{\nu\wedge\mu}\mathrm{ev}^*\omega^{k+1}}{(k+1)!} = \Omega^{\mathcal{K}}_{[f]}(\nu,\mu) \le \left(|\nu|^2_{\mathcal{K}}\,|\mu|^2_{\mathcal{K}} - \langle\nu,\mu\rangle^2_{\mathcal{K}}\right)^{1/2}\\ &= \left(\int_\Sigma |\nu|^2\int_\Sigma|\mu|^2 - \left(\int_\Sigma\langle\nu,\mu\rangle\right)^2\right)^{1/2}.\end{aligned}$$

Note that the last equality is obtained from tame condition. Recall that the Riemannian submersion condition implies that $|\nu|$, $|\mu|$ and $\langle\nu,\mu\rangle$ determine the norms and inner product of $\nu(x)$ and $\mu(x)$ for any $x\in\Sigma$. Therefrom, we have an equality,

$$\left(\int_\Sigma |\nu\wedge\mu|\right)^2 + \left(\int_\Sigma\langle\nu,\mu\rangle\right)^2 = \int_\Sigma|\nu|^2\int_\Sigma|\mu|^2$$

so the above inequality is turned into an equality so that it gives

$$\Omega_{[f]}(\nu,\mu) = \int_\Sigma \frac{\iota_{\nu\wedge\mu}\mathrm{ev}^*\omega^{k+1}}{(k+1)!} = \left(|\nu|^2_{\mathcal{K}}\,|\mu|^2_{\mathcal{K}} - \langle\nu,\mu\rangle^2_{\mathcal{K}}\right)^{1/2}$$

i.e., $\hat{D}$ is $\mathcal{J}$-holomorphic in $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$. □

Remark 12. Recall the symplectic knot space of the case $k=n-1$ is a higher-dimensional knot space (see Remark 3). For this case, this section was explained in [6] as the correspondence between branes (resp. instantons) in the symplectic manifold M and Lagrangians (resp. holomorphic normal disks) in the knot space. Here, branes and instantons in M are hypersurfaces (therefore coisotropic) and open subsets, respectively.

5. Further remarks

In this paper, symplectic knot space $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$ is introduced as a space of symplectically embedded submanifolds from an even-dimensional oriented closed manifold Σ to a symplectic manifold M. The space $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$ has a symplectic structure induced from a symplectic structure on M, and we study the symplectic geometry on $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$. When Σ is a Riemann surface, one can consider a subspace of $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$ consists of pseudo-holomorphic maps. The geometry and topology of this subspace has been studied corresponding to Gromov–Witten invariants (see [2]). It is interesting to explore the geometry and topology of $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$ along the development of the pseudo-holomorphic curves.

In Section 4, we explain the correspondence between coisotropic submanifolds in M and Lagrangians in $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$. The deformation theory of coisotropic submanifolds is studied in [9] and [10]. And it is natural to study the relationships between the deformation theory of coisotropic submanifolds and that of the corresponding Lagrangians in $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$. Furthermore, we also ask similar questions for the intersection theories on coisotropic submanifolds in M and those on Lagrangians in $\mathcal{K}^{\mathrm{Sp}}(\Sigma,M)$. Note that another approach to the coisotropic intersections with respect to Lagrangian geometry is introduced in [3].

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