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Knotted Hamiltonian cycles in spatial embeddings of complete graphs

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Abstract. We show the complete graph on n vertices contains a knotted Hamiltonian cycle in every spatial embedding, for $n > 7$. Moreover, we show that for $n > 8$, the minimum number of knotted Hamiltonian cycles in every embedding of K_n is at least $(n-1)(n-2)\dots(9)(8)$. We also prove K_8 contains at least 3 knotted Hamiltonian cycles in every spatial embedding.

Contents

1. Introduction

Recall that a *Hamiltonian cycle* in a graph is a cycle that passes through every vertex of the graph. Generalizing Conway and Gordon's result [1] that K_7, the complete graph on 7 vertices, contains a knotted Hamiltonian cycle in every embedding, we show that K_n, for $n \geq 7$, contains a knotted Hamiltonian cycle in every spatial embedding. Furthermore, we show that for $n > 8$, the minimum number of knotted Hamiltonian cycles in every embedding of K_n is at least $(n-1)(n-2)\dots(9)(8)$. We strongly suspect that this lower bound can be improved. Finally, using techniques inspired by Shimabara [3], we show that K_8 must have at least 3 knotted Hamiltonian cycles in every embedding.

The results of Section 2 were obtained by the third author. The remaining results were obtained by all of the authors in an NSF and NSA-sponsored research experience for undergraduates program during the summer of 2003.

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2. Hamiltonian knotted cycles in embeddings of K_n, for $n \geq 7$

For background on arf invariant, see [2].

Lemma 2.1. *In every spatial embedding of K_7, there exists an edge of K_7 that is contained in an odd number of Hamiltonian cycles with nonzero arf invariant.*

Proof. Consider an arbitrary embedding of K_7. By Conway–Gordon's result [1], the sum of the arf invariants of all Hamiltonian cycles in an arbitrary embedding of K_7 must be odd. Thus, in the given embedding there must be an odd number of Hamiltonian cycles with nonzero arf invariant. Let's say the number of such cycles is $2n+1$. Now, if we count up the edges of such cycles, we get that a grand total of $7(2n+1)$ edges (counting multiplicities) are in a cycle with nonzero arf invariant. On the other hand, if we number the edges of K_7 as $e_1, \dots e_{21}$, and let n_i, $i = 1, 2, \dots, 21$ stand for the number of Hamiltonian cycles that contain e_i, then we must have that $\sum_{i=1}^{21} n_i = 7(2n+1)$, thus $\sum_{i=1}^{21} n_i$ must be odd. It follows that at least one of the n_i must be odd, and our lemma is proven. □

Theorem 2.2. *Every K_n, for $n \geq 7$ contains a knotted Hamiltonian cycle in every spatial embedding.*

Proof. We will prove the theorem for K_8. The proof for general n is similar. Embed K_8. Consider the embedding of the subgraph induced by seven vertices of K_8, and let v denote the eigth vertex, and let G_7 denote the subgraph on 7 vertices. By the previous lemma, the embedded G_7 contains an edge that is contained in an odd number of Hamiltonian cycles with nonzero arf invariant; we denote this edge e, and let w_1 and w_2 denote the vertices of e. Now, we ignore the edge e, and consider the subdivided K_7 that results from replacing e with the edges (v, w_1) and (v, w_2). We denote this subdivided K_7 by G'_7. Ignoring the degree 2 vertex v, the embedded G'_7 must have an odd number of Hamiltonian cycles with nonzero arf invariant. Since there was an odd number of Hamiltonian cycles of G_7 through the edge e with nonzero arf invariant, there is an even number of Hamiltonian cycles in G_7 that do not contain e and with nonzero arf invariant. The Hamiltonian cycles of G_7 not containing e are exactly the same as the Hamiltonian cycles in G'_7 not containing the edges (v, w_1) and (v, w_2). Thus, in the embedding of G'_7, there must be an odd number of Hamiltonian cycles through the edges (v, w_1) and (v, w_2) with nonzero arf invariant. Such a cycle is a Hamiltonian cycle in K_8. Thus, in the original embedded K_8, there must be a knotted Hamiltonian cycle. □

We note here that the above proof can be used to show that every edge of K_9 is contained in at least two knotted Hamiltonian cycles in every spatial embedding of K_9. This can be seen by removing an edge, call it e, from K_9. The vertices disjoint from e induce a K_7 subgraph. In an arbitrary embedding of K_9, consider the embedded sub-K_7. One of its edges must lie in an odd number of Hamiltonian cycles with nonzero arf invariant. We denote this edge f. The edges e and f are connected by 4 different edges, which we shall denote e_1, e_2, e_3, e_4. Without loss of generality, e_1 and e_2 share no vertex, and neither do e_3 and e_4. If we replace the edge f with the 4- (vertex) path (e_1, e, e_2), then there is a knotted Hamiltonian cycle through the 4-path. Similarly, there is a knotted Hamiltonian cycle through the 4-path (e_3, e, e_4). Thus, there are at least two different knotted Hamiltonian cycles through the edge e. One can use an analogous argument to show that every 3-path in K_{10} is contained in at least two knotted Hamiltonian cycles in every spatial embedding, and in general, every $(n-7)$-path in K_n is contained in at least two knotted Hamiltonian cycles in every spatial embedding, for $n \geq 9$.

This reasoning allows us to estimate a minimum number of knotted Hamiltonian cycles in every spatial embedding of K_n for $n > 8$. One need only compute the number of paths of length $(n-7)$, then multiply by 2 and divide by n (because every Hamiltonian cycle in K_n contains exactly n paths of length $(n-7)$). To get double the number of paths of length $(n-7)$ in K_n, one merely computes $n(n-1)(n-2)\dots(8)$. Dividing by n gives our lower bound:

Theorem 2.3. *For $n > 8$, the minimum number of knotted Hamiltonian cycles in every embedding of K_n is at least $(n-1)(n-2)\dots(9)(8)$.*

3. A lower bound for K_8

Here we adapt Shimabara's techniques to show that K_8 contains at least 3 knotted Hamiltonian cycles in every spatial embedding. First, we need to recall some definitions. Let a_2 represent the coefficient of the degree 2 term of the Conway polynomial. For background on the Conway polynomial, see [2].

Definition 3.1. Let G be a graph with Γ a set of cycles in G. Given an embedding f of G, let $\mu_f(G, \Gamma; n)$ be given by

$$\mu_f(G, \Gamma; n) \equiv \sum_{\gamma \in \Gamma} a_2(f(\gamma)) \pmod{n},$$

where $\sum_{\gamma \in \Gamma}$ is the summation over all cycles γ in Γ.

Given a directed graph with edges E_1 and E_2 lying in a cycle ϕ, E_1 and E_2 are said to be *coherent* if they induce the same orientation on ϕ. Given adjacent edges A and B we have the following definition.

Definition 3.2. Let $\nu_1(\Gamma; A, B, E) = |n_1 - n_2|$,where n_1 is the number of cycles in Γ containing A, B and E such that A and E are coherent, and n_2 is the number of cycles in Γ containing A, B and E such that A and E are not coherent.

Given pairs of nonadjacent edges $\{A, B\}$ and $\{E, F\}$ we have the following definition. Note that we refer to the cycles in Γ such that the edges A, E, B, F lie in this order as Γ_1.

Definition 3.3. Let $\nu_2(\Gamma; A, B; E, F) = |n_3 - n_4|$,where n_3 is the number of cycles in Γ_1 such that an even number of pairs in A, B, E, F are coherent, and n_4 is the number of cycles in Γ_1 such that an odd number of pairs in A, B, E, F are coherent.

The following lemmas are results of [3].

Lemma 3.1. *The number*

$$\nu_2(\Gamma; A, B; E, F) = \nu_2(\Gamma; A, B; F, E) = \nu_2(\Gamma; B, A; E, F) = \nu_2(\Gamma; B, A; F, E).$$

Moreover, the numbers $\nu_1(\Gamma; A, B, E)$ and $\nu_2(\Gamma; A, B; E, F)$ are independent of the direction of a graph G.

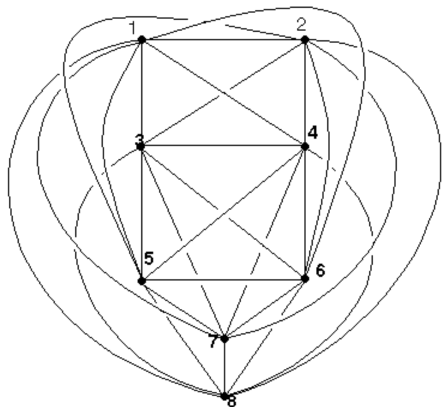

FIGURE 1. An embedding of K_8 containing exactly 21 knotted Hamiltonian cycles.

Lemma 3.2. *Let* Γ *be a set of cycles in an undirected graph* G*. The invariant* $\mu_f(G,\Gamma;n)$ *does not depend on the spatial embedding* f *of* G *if the following two conditions hold:*

(1) *For any edges* A*,* B*,* E *such that* A *is adjacent to* B*,*

$$\nu_1(\Gamma;A,B,E) \equiv 0 \pmod{n}.$$

(2) *For any pairs of nonadjacent edges* (A,B) *and* (E,F)*,*

$$\nu_2(\Gamma;A,B;E,F) \equiv 0 \pmod{n}.$$

Finally, we are ready for our main result of this section:

Theorem 3.3. *Given an embedding of* K_8*, there exists at least* 3 *knotted Hamiltonian cycles.*

Proof. The embedding of K_8 in Figure 1 contains 21 Hamiltonian knots, all of which have arf invariant 1. The Hamiltonian knots are the following:

$$18452376, 18457236, 18632745, 14723568, 14752368, 13724586,$$

$$17342586, 17432586, 17425386, 17845236, 17452836, 18745236,$$

$$13685472, 17425863, 17458263, 17458632, 17325846, 17325864,$$

$$14723685, 14856237, 14852367.$$

Let Γ denote the set of all Hamiltonian cycles in K_8. We wish to compute $\nu_1(\Gamma;A,B,E)$ and $\nu_2(\Gamma;A,B;E,F)$, in all possible cases. First, we consider the possible values for $\nu_1(\Gamma;A,B,E)$, when A and B are adjacent. In the case that edge E is adjacent to neither A nor B, then there are an equal number of cycles in Γ with A and E coherent and not coherent. To see this, denote the vertices of E as v_1 and v_2. Assign an arbitrary orientation on edge A. For every cycle in Γ through E that passes in the direction of A to v_1 first, there is a corresponding cycle that starts at A and passes through v_2 first. For one of these cycles, A and E

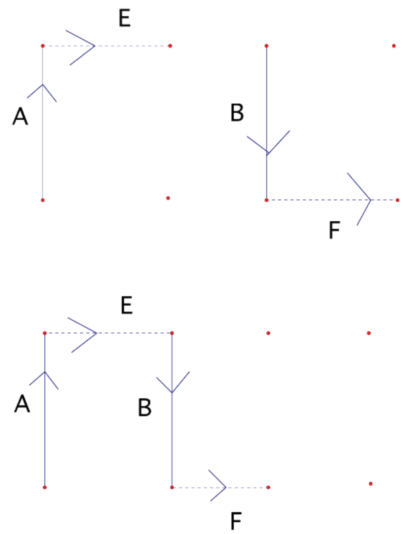

FIGURE 2. Up to symmetry, the above illustration demonstrates the two cases that can occur when all of the edges in $\{A, B, E, F\}$ are adjacent to at least one other edge in the set.

are coherent, for the other, they are not. Thus in this case, $n_1 = n_2$, and therefore $\nu_1(\Gamma; A, B, E) = 0$.

Next, we consider the case where E is adjacent to exactly one of A and B and A and B are adjacent. Without loss of generality, suppose E is adjacent to B and not A. In this case, there are 4! cycles in Γ that contain A, B and E; and A and E are either coherent for all of these cycles or not coherent for all of these cycles. In any case, $\nu_1(\Gamma; A, B, E) = 24$ and thus is a multiple of 6.

Finally, it is impossible for E to be adjacent to both A and B and for the edges A, B and E to belong to a Hamiltonian cycle.

Now, we consider the possible values taken on by $\nu_2(\Gamma; A, B; E, F)$, when A and B are nonadjacent, and E and F are nonadjacent. First, we consider the case where one of the edges in the set $\{A, B, E, F\}$ is nonadjacent to all other edges in the set. Without loss of generality, suppose this edge is edge F. Assign an arbitrary orientation to F. In this case, for every cycle in Γ_1, there is a corresponding cycle in Γ_1 for which F has the opposite orientation. Such a change in F changes the coherentness (or lack thereof) of 3 pairs of edges in the cycle. Thus, if the first cycle has an odd number of pairs of edges from $\{A, B, E, F\}$ that are coherent, then the second cycle will have an even number of coherent pairs of edges, and vice versa. Therefore in this case, $n_3 = n_4$, and thus $\nu_2(\Gamma; A, B; E, F) = 0$.

Now we consider what happens when every edge in the set $\{A, B, E, F\}$ is adjacent to at least one other edge in the set. In order for the edges A, B, E and F to be part of a Hamiltonian cycle, they must form either two disjoint paths of length 2, or a path of length 4. If they form two disjoint paths of length 2, then either A is adjacent to E and B is adjacent to F, or A is adjacent to F and B is adjacent to E. Without loss of generality, we may assume we have the top case depicted in Figure 2. We may pick the directions, since by Lemma 3.1, $\nu_2(\Gamma, A, B; E, F)$ is independent of the direction of a graph G. If the edges A, B, E and F form a path of length 4, then there are four possible ways they can do so. The path of length 4 can be (A, E, B, F), (A, F, B, E), (B, E, A, F), or (B, F, A, E). Without loss of generality, we may take the path to be (A, E, B, F), and by Lemma 3.1, we may choose directions for each edge as shown in the bottom case in Figure 2. In each case depicted in Figure 2, all cycles in Γ_1 have all six possible pairs of edges coherent. It then follows that in each case, $n_3 = 6$ and $n_4 = 0$, and thus $\nu_2(\Gamma; A, B; E, F) = 6$.

All possible values of ν_1 and ν_2 are congruent to 0 (mod 6). Thus, by Lemma 3.2, together with the embedding given in Figure 1, $\mu_f(G, \Gamma; 6) = 3$ is independent of embedding. Hence, K_8 contains at least 3 knotted Hamiltonian cycles in every embedding. □

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