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f-harmonic maps which map the boundary of the domain to one point in the target

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Abstract. One considers the class of maps $u : D \to S^2$, which map ∂D to one point in S^2. If u were also harmonic, then it is known that u must be constant. However, if u is instead f-harmonic — a critical point of the energy functional $\frac{1}{2}\int_D f(x)|\nabla u(x)|^2$ — then this need not be true. We shall see that there exist functions $f : D \to (0, \infty)$ and nonconstant f-harmonic maps $u : D \to S^2$ which map the boundary to one point. We will also see that there exist nonconstant f for which, there is no nonconstant f-harmonic map in this class. Finally, we see that there exists a nonconstant f-harmonic map from the torus to the 2-sphere.

Contents

1. Introduction

1.1. Basic definitions. Let $(\mathcal{M}, g)$ be a compact Riemannian surface (with or without boundary). Let $(\mathcal{N}, h)$ be a compact Riemannian manifold without boundary, embedded isometrically in $\mathbb{R}^N$. This embedding is always possible by the Nash Embedding Theorem. Let $f : \mathcal{M} \to (0, \infty)$ be a smooth function. By compactness, f is bounded above, and below by a strictly positive number, say $0 < A \le f(x) \le B$.

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Definition 1.1 (f-harmonic energy)**.** Let $u \in W^{1,2}(\mathcal{M};\mathcal{N})$. The *$f$-harmonic energy functional* is defined to be

$$E_f(u) = \frac{1}{2}\int_{\mathcal{M}} f(x)|\nabla u|^2 d\mathcal{M} \tag{1.1}$$

where $d\mathcal{M} = \sqrt{g}dx^1 \wedge dx^2$ and $\sqrt{g}$ denotes $\big(\det g_{\alpha\beta}\big)^{\frac{1}{2}}$. For consistency of notation, we denote the *harmonic energy* by E_1.

Definition 1.2 (tubular neighbourhood/nearest point projection)**.** For $\rho > 0$, define a tubular neighbourhood of $\mathcal{N}$ by

$$V_\rho\mathcal{N} := \{z \in \mathbb{R}^N : d(z,\mathcal{N}) < \rho\} \subset \mathbb{R}^N. \tag{1.2}$$

Here $d(z,\mathcal{N})$ denotes of course $\inf\{|z-x|_{\mathbb{R}^N} : x \in \mathcal{N}\}$. Choosing $\rho > 0$ sufficiently small, we may let $P : V_\rho\mathcal{N} \to \mathcal{N}$ denote "nearest point" projection. P is well-defined and smooth — see e.g., [9, §2.12.3].

Definition 1.3 (admissible variation)**.** An *admissible variation* of u, is a family of maps $u_s := P \circ (u + s\phi)$, for some $\phi \in C_c^\infty(\mathcal{M},\mathbb{R}^N)$ and for small $|s|$. Notice that $u_0 = u$ and that $u_s \equiv u$ in a neighbourhood of $\partial\mathcal{M}$.

Remark 1.4. If $u \in W^{1,2}(\mathcal{M};\mathcal{N})$ and $\phi \in C_c^\infty(\mathcal{M};\mathbb{R}^N)$, then $P \circ (u + s\phi) \in W^{1,2}(\mathcal{M};\mathcal{N})$ for sufficiently small $|s|$ [9, §2.2], and $\nabla P \circ (u + s\phi)$ is differentiable with respect to s. Hence $E_f(P \circ (u + s\phi))$ is also differentiable. We can then define:

Definition 1.5 (f-harmonic)**.** A map $u \in W^{1,2}(\mathcal{M};\mathcal{N})$ is said to be (weakly) *f-harmonic* if, for any (admissible) variation u_s, of u, we have that

$$\frac{d}{ds}E_f(u_s)\bigg|_{s=0} = 0. \tag{1.3}$$

Remark 1.6. A *harmonic* map satisfies this definition as a 1-*harmonic* map.

Before we proceed, it is worth clearing up any possible confusion with the name "*f-harmonic*". Our *f-harmonic maps* should not be confused with, so-called, *F-harmonic maps* — critical points of $E_F(u) = \int_{\mathcal{M}} F\left(\frac{1}{2}|\nabla u|^2\right) d\mathcal{M}$ for a nonnegative, strictly increasing, C^2 function F on the interval $[0,\infty)$. Neither should an f-harmonic map be confused with a *p-harmonic* map — a critical point of $E_p(u) = \frac{1}{p}\int_{\mathcal{M}} |\nabla u|^p d\mathcal{M}$. Specifically, in the language of p-harmonic maps, an "ordinary" *harmonic* map could be referred to as a 2-harmonic map and the associated energy as E_2. In this work, we may refer to harmonic maps as 1-harmonic and denote the harmonic energy by E_1. Of course a 1-harmonic map (our terminology) is also a λ-harmonic map for any constant $\lambda > 0$.

Remark 1.7. As said previously, we only consider two-dimensional domains. For $\dim\mathcal{M} \neq 2$, any f-harmonic map $(\mathcal{M}, g) \to (\mathcal{N}, h)$ is a harmonic map

$$(\mathcal{M}, f^{\frac{2}{m-2}}g) \to (\mathcal{N}, h)$$

([2, §10.20] or [3, §10.20]). However when $\dim\mathcal{M} = 2$, a conformal change of metric keeps an f-harmonic map, f-harmonic (for the same f). Moreover, when $\dim\mathcal{M} = 2$, we may consider an f-harmonic map $\mathcal{M} \to \mathcal{N}$ as a *harmonic* map on a certain higher-dimensional manifold — on the warped product $\mathcal{M} \times_{f^2} S^1$, perhaps. Thus, an interesting f-harmonic result on a surface *may* yield an interesting 1-harmonic result on some higher-dimensional manifold.

Suppose now that u is a "classical" f-harmonic map [6] — that is a C^2 map satisfying Definition 1.5. By combining (1.1) and (1.3) we are able [1] to calculate:

Lemma 1.8 (Euler–Lagrange equation for E_f)**.** *Let $u \in C^2(\mathcal{M};\mathcal{N})$. The following are equivalent:*

(i) *u is f-harmonic.*
(ii) $f\Delta_{\mathcal{M}}u + fA(u)(\nabla u, \nabla u) + \nabla f * \nabla u = 0$.
(iii) *The* harmonic *tension of u is $\tau_1(u) = -\frac{1}{f}\nabla f * \nabla u$.*
(iv) $\operatorname{div}(f\nabla u)$ *is perpendicular to $T\mathcal{N}$.*

Here the notation $\nabla f * \nabla u$ denotes $\left\langle \nabla f, \nabla u^i \right\rangle \frac{\partial}{\partial u^i} \in T_u\mathcal{N}$. Similarly if $u \in W^{1,2}(\mathcal{M};\mathcal{N})$ then: u is weakly f-harmonic if and only if u satisfies part (ii) above, weakly (with test functions $\phi \in C_c^\infty(\mathcal{M};\mathbb{R}^N)$).

Definition 1.9 (domain variation)**.** A *domain variation* is a map $\eta : \mathcal{M}\times(-\varepsilon,\varepsilon) \to \mathcal{M}$ for some small $\varepsilon > 0$, which satisfies

$$\begin{cases} \eta(x,0) = x & \text{on } \mathcal{M} \\ \eta(x,s) = x & \text{on } \partial\mathcal{M}. \end{cases} \tag{1.4}$$

When the boundary of $\mathcal{M}$ is smooth, we are able — by considering convergent sequences of variations — to rely on more general variations. In particular, we shall need the following lemma in the proof of Proposition 2.1.

Lemma 1.10. *Let $u \in C^2(\mathcal{M};\mathcal{N})$ be an f-harmonic map. Let $\{\Omega_j\}$ be a finite partition of $\mathcal{M}$ such that $\partial\Omega_j \in C^\infty$ for each j. Suppose that the domain variation $\eta \in C^0(\mathcal{M}\times(-\varepsilon,\varepsilon);\mathcal{M})$ satisfies:*

(i) *$\eta \in C^2(\overline{\Omega}_j \times (-\varepsilon,\varepsilon);\mathcal{M})$ for each j.*
(ii) $\frac{\partial\eta}{\partial s} \in C^0(\mathcal{M}\times(-\varepsilon,\varepsilon);T\mathcal{M})$.
(iii) *$\frac{\partial\eta}{\partial s} \in C^2(\overline{\Omega}_j \times (-\varepsilon,\varepsilon);T\mathcal{M})$ for each j.*

Then

$$\frac{d}{ds}E_f(u\circ\eta)\bigg|_{s=0} = 0.$$

Again, see [1] for the straightforward proof.

1.2. *f*-harmonic heat flow. Later, we will need to consider the L^2-gradient flow of the functional E_f — namely the following problem which we call the *f-harmonic heat flow*:

$$\begin{cases} u_t - f\Delta_{\mathcal{M}}u = fA(u)(\nabla u,\nabla u) + \nabla f * \nabla u \\ u|_{t=0} = u_0 \\ u(\cdot,t)|_{\partial\mathcal{M}} = u_0|_{\partial\mathcal{M}}. \end{cases} \tag{1.5}$$

As an analogue of Struwe's result for (1-)harmonic maps, we have the following existence and uniqueness result — the proof [1] is very similar to that for the (1-)harmonic case [10]. Indeed, the compactness of the domain means that the extra terms involving f and its derivatives are easily controlled.

Theorem 1.11 (f-harmonic heat flow)**.** *Let $u_0 \in W^{1,2}(\mathcal{M};\mathcal{N})$. If $\partial\mathcal{M}$ is non-empty, suppose further that $u_0|_{\partial\mathcal{M}} \in C^{2,\alpha}(\partial\mathcal{M};\mathcal{N})$. There exists a weak solution*

$$u : \mathcal{M}\times[0,\infty) \to \mathcal{N}$$

of (1.5) *with the following properties:*

(i) *u is smooth on $\mathcal{M}\times(0,\infty)$ away from finitely many points $(\overline{x}_k, \bar{t}_k)$, $1 \le k \le K$, $0 < \bar{t}_k \le \infty$.*
(ii) *$E_f\big(u(t)\big) \le E_f\big(u(s)\big)$ for all $0 \le s \le t$.*
(iii) *u assumes the initial data continuously in $W^{1,2}(\mathcal{M},\mathcal{N})$.*

The solution u is unique in this class.

Furthermore, at a singular (*or bubble*) *point $(\overline{x}, \bar{t}) \in \mathcal{M} \times (0,\infty]$, there exist sequences $x_m \to \overline{x}$, $t_m \nearrow \bar{t}$, $R_m \searrow 0$ and a nonconstant harmonic map $\widetilde{u} : \mathbb{R}^2 \to \mathcal{N}$ with finite* (*harmonic*) *energy, such that as $m \to \infty$,*

$$u_m(x) := u\big(\exp_{x_m}(R_m x), t_m\big) \to \widetilde{u} \tag{1.6}$$

in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^2;\mathcal{N})$. Moreover $\widetilde{u}$ extends to a smooth harmonic map $\overline{u} : \mathbb{R}^2 \cup \{\infty\} = S^2 \to \mathcal{N}$ which we call a 'bubble'.

There exists a further sequence of times $t_m \to \infty$ such that the sequence of maps $u(\cdot, t_m)$ converges weakly in $W^{1,2}(\mathcal{M};\mathcal{N})$ to a smooth f-harmonic map $u_\infty : \mathcal{M} \to \mathcal{N}$, and smoothly away from finitely many points $\overline{x}_k$.

2. Analogue of a theorem by Lemaire

The reader is asked now to recall a theorem of Lemaire [5, Theorem 3.2], regarding harmonic maps, which states: "*Let $\mathcal{M}$ be a compact contractible surface with boundary, and let p be a point in $\mathcal{N}$. Every harmonic map $\mathcal{M} \to \mathcal{N}$ which maps $\partial\mathcal{M}$ onto p is constant, and takes value p.*" As we will see in Section 3, the direct analogue involving f-harmonic maps is not true; it is for example, possible to find a nontrivial f-harmonic map $D \to S^2$ which maps ∂D to a point.

We do however find a partial analogue of the quoted result — if we place suitable restrictions on f. Presented here is a simple demonstration of a restriction applied to f that denies the existence of any nonconstant f-harmonic maps $D \to S^2$.

Proposition 2.1. *Suppose that $f : D \to (0,\infty)$ satisfies $\nabla f(x) \cdot x \ge 0$ for all $x \in D$. Then every smooth f-harmonic map $u \in C^\infty(D;\mathcal{N})$ which maps ∂D to a point p, is constant and takes the value p.*

The strategy for the proof is as follows: Assuming that there is a nonconstant f-harmonic map, for such an f, we precompose u with a particular rotationally symmetric variation $D \to D$ (based on the $[0,1] \to [0,1]$ map shown in Figure 1 on page 429). The purpose of this is to "squash" the energy away from the boundary (high f) towards the origin (low f). This "should" decrease the overall energy, hence proving that u cannot be f-harmonic. However, the distortion on an annulus close to the boundary (i.e., $D_b \setminus D_a$ for $a < b$ both close to 1) "may" add enough to the f-energy to cancel out the decrease elsewhere. Fortunately, we are able to rule this possibility out by studying the Hopf differential. The following technical lemma gives this calculation.

Lemma 2.2. *Let $u \in C^\infty(D,\mathcal{N})$ be an f-harmonic map which maps ∂D to a point. Then, for $0<a<b<1$ we have that*

$$-\left(\frac{b}{b-a}\right)\int_{D_b\setminus D_a}\frac{f}{r}\left(\left|u_r\right|^2-\frac{1}{r^2}\left|u_\phi\right|^2\right)\,dxdy \le \frac{1}{a}\left\{-\int_0^{2\pi} f|u_r|^2\,d\phi\Big|_{|z|=1}+\|\nabla f\|_{L^\infty}\int_{D\setminus D_a}|\nabla u|^2\,dxdy\right\}. \tag{2.1}$$

Proof. Consider the Hopf differential ψdz^2, where

$$\psi(u)=|u_x|^2-|u_y|^2-2i\,\langle u_x,u_y\rangle$$

in Cartesian coordinates and $\psi(u) = \frac{\bar z^2}{r^2}\left(|u_r|^2-\frac{1}{r^2}|u_\phi|^2-\frac{2i}{r}\langle u_r,u_\phi\rangle\right)$ in polar coordinates. For this proof, we use the notation $z=x+iy$ to denote coordinates in the two-disc D:

$$\begin{pmatrix} z=x+iy & r_x=x/r & x_r=x/r \\ x=r\cos\phi & r_y=y/r & x_\phi=-y \\ y=r\sin\phi & \phi_x=-y/r^2 & y_r=y/r \\ & \phi_y=x/r^2 & y_\phi=x \end{pmatrix}.$$

It is well-known that if u is (1-)harmonic then $\bar\partial\psi=0$. For u f-harmonic, we calculate that

$$\begin{aligned}\bar\partial\psi(u) := \frac{1}{2}\left(\psi_x+i\psi_y\right) &= \langle u_x-iu_y,\tau_1(u)\rangle \\ &= \frac{-1}{f}\left\langle \bar z\left(\frac{u_r}{r}-i\frac{u_\phi}{r^2}\right), f_ru_r+\frac{1}{r^2}f_\phi u_\phi\right\rangle \\ &= \left(\frac{-\bar z}{rf}\right)\left(\left[f_r|u_r|^2+\tfrac{1}{r^2}f_\phi\langle u_r,u_\phi\rangle\right]-\frac{i}{r}\left[f_r\langle u_r,u_\phi\rangle+\tfrac{1}{r^2}f_\phi|u_\phi|^2\right]\right)\end{aligned} \tag{2.2}$$

and that

$$\begin{aligned}&\operatorname{Re}\left[(f_x+if_y)\psi z\right] \\ &\quad= \operatorname{Re}\left(\frac{1}{r}f_r+\frac{i}{r^2}f_\phi\right)\frac{z^2\bar z^2}{r^2}\left(|u_r|^2-\frac{1}{r^2}|u_\phi|^2-\frac{2i}{r}\langle u_r,u_\phi\rangle\right) \\ &\quad= rf_r\left(|u_r|^2-\frac{1}{r^2}|u_\phi|^2\right)+2\frac{1}{r}f_\phi\langle u_r,u_\phi\rangle.\end{aligned} \tag{2.3}$$

Notice that by Cauchy–Stokes

$$\begin{aligned}\int_{|z|=1} f\psi(z)z\,dz-\int_{|z|=r} f\psi(z)z\,dz &= \int_{D\setminus D_r} f\left(\bar\partial\psi\right)z\,d\bar z\wedge dz \\ &\quad+\frac{1}{2}\int_{D\setminus D_r}(f_x+if_y)\psi(z)z\,d\bar z\wedge dz.\end{aligned} \tag{2.4}$$

By (2.3), we see that

$$\operatorname{Re}\psi z^2 = |z|^2\left|u_r\right|^2-\left|u_\phi\right|^2. \tag{2.5}$$

Therefore,

$$\int_0^{2\pi} f\Big[r^2\big|u_r\big|^2 - \big|u_\phi\big|^2\Big]\ d\phi\Big|_{|z|=r} = \operatorname{Re}\int_0^{2\pi} f\psi(z)z^2\ d\phi\Big|_{|z|=r} = \operatorname{Im}\int_{|z|=r} f\psi(z)z\ dz. \tag{2.6}$$

In the case of u harmonic (i.e., $f \equiv 1$), we could use $\operatorname{Im}\int_{|z|=r}\psi(z)z\ dz = 0$ to obtain a stronger result [7]. For u f-harmonic, we instead calculate

$$\begin{aligned} Q :&= -\int_{D_b\setminus D_a} \frac{f}{r}\left[\big|u_r\big|^2 - \frac{1}{r^2}\big|u_\phi\big|^2\right]\ dxdy \\ &= -\int_a^b \frac{1}{r^2}\int_0^{2\pi} f\left[r^2\big|u_r\big|^2 - \big|u_\phi\big|^2\right]\ d\phi\ dr \\ &= -\int_a^b \frac{1}{r^2}\left[\operatorname{Im}\int_{|z|=r} f\psi(z)z\ dz\right]\ dr \end{aligned}$$

by (2.6). It follows that

$$\begin{aligned} Q = &-\int_a^b \frac{1}{r^2}\operatorname{Im}\Big[\int_{|z|=1} f\psi(z)z\ dz - \int_{D\setminus D_r} f\big(\overline{\partial}\psi\big)z\ d\overline{z}\wedge dz \\ &\quad - \int_{D\setminus D_r}\frac{1}{2}(f_x + if_y)\psi z\ d\overline{z}\wedge dz\Big]\ dr \\ = &-\int_a^b \frac{1}{r^2}\ dr\,\operatorname{Re}\left[\int_0^{2\pi} f\psi(z)z^2\ d\phi\right]_{|z|=1} \\ &+\int_a^b \frac{1}{r^2}\operatorname{Im}\left[\int_{D\setminus D_r} f\big(\overline{\partial}\psi\big)z2i\ dx\wedge dy\right]\ dr \\ &+\int_a^b \frac{1}{r^2}\operatorname{Im}\left[\int_{D\setminus D_r}\frac{1}{2}(f_x + if_y)\psi z2i\ dx\wedge dy\right]\ dr \end{aligned}$$

by (2.4) and then (2.6) again. Here we have used $d\overline{z}\wedge dz = 2idx\wedge dy$. Then by (2.2), (2.3) and (2.5), and since $u_\phi\big|_{|z|=1} = 0$, we see that

$$\begin{aligned} Q = &-\int_a^b \frac{1}{r^2}\ dr\left[\int_0^{2\pi} f|u_r|^2\ d\phi\right]_{|z|=1} \\ &-\int_a^b \frac{2}{r^2}\int_{D\setminus D_r}|z|\left[f_r|u_r|^2 + \frac{1}{|z|^2}f_\phi\left\langle u_r, u_\phi\right\rangle\right]\ dx\wedge dy\ dr \\ &+\int_a^b \frac{1}{r^2}\int_{D\setminus D_r}\left[|z|f_r\left(|u_r|^2 - \frac{1}{|z|^2}|u_\phi|^2\right) + \frac{2}{|z|}f_\phi\left\langle u_r, u_\phi\right\rangle\right]\ dx\wedge dy\ dr \\ = &-\int_a^b \frac{1}{r^2}\ dr\left[\int_0^{2\pi} f|u_r|^2\ d\phi\right]_{|z|=1} - \int_a^b \frac{1}{r^2}\int_{D\setminus D_r}|z|f_r|\nabla u|^2\ dx\wedge dy\ dr. \end{aligned}$$

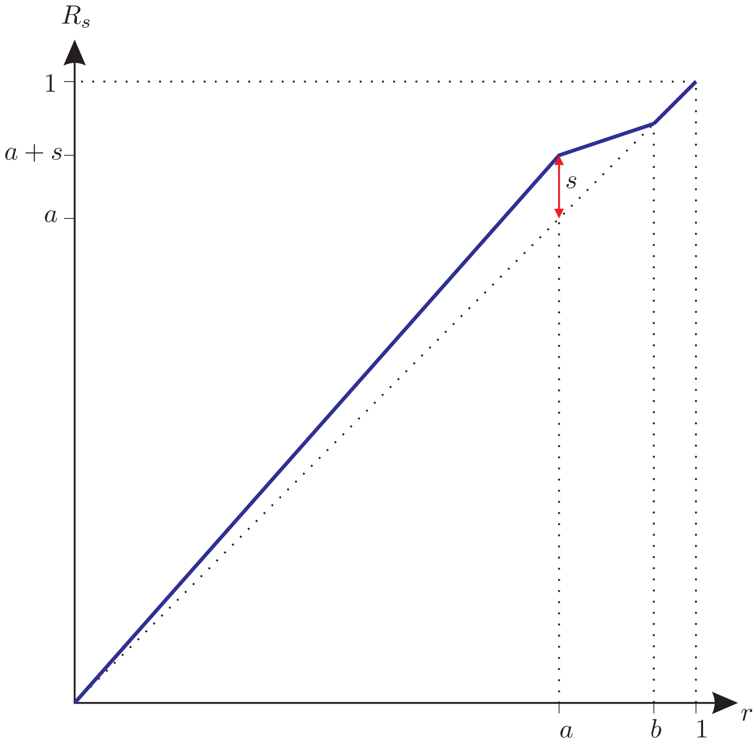

FIGURE 1. The map $R_s : [0,1] \to [0,1]$.

Finally, we estimate the second integral to see

$$Q \le -\left(\frac{b-a}{ab}\right)\left[\int_0^{2\pi} f|u_r|^2 \, d\phi\right]_{|z|=1} + \|\nabla f\|_{L^\infty}\left(\frac{b-a}{ab}\right)\int_{D\setminus D_a} |\nabla u|^2 \, dx \wedge dy. \qquad \square$$

To ensure that an f-harmonic map cannot be constant in some regions and nonconstant elsewhere, we need the result:

Lemma 2.3. *Let u and v be two f-harmonic maps $\mathcal{M} \to \mathcal{N}$. If they agree on an open subset, then they are identical. In particular if an f-harmonic map is constant on an open subset, then it is a constant map.*

This lemma is the f-harmonic analogue of [8, Theorem 2], and the proof is almost identical — the extra term in the f-harmonic equation $(\nabla f * \nabla u)$ can easily be absorbed into the estimate $|\Delta u^i| \le C[\sum_{\alpha,j} |u^j_\alpha| + \sum_j |u^j|]$ in Sampson's proof. Note that we are implicitly assuming that $\mathcal{M}$ is connected.

We are now ready to prove Proposition 2.1:

Proof of Proposition 2.1. If f is a constant function, then u is harmonic and we are done. Suppose instead that f is nonconstant. We must then have $\nabla f(x)\cdot x > 0$ on at least one point in D, and hence on some open set Ω. Suppose that $u : D \to \mathcal{N}$ is a f-harmonic map, mapping ∂D to a point and suppose that u is nonconstant on Ω.

Define $R_s : [0,1] \to [0,1]$ by (see Figure 1)

$$R_s(r) = \begin{cases} (1+\frac{s}{a})r & r \in [0,a), \\ (r-a)(1-\frac{s}{b-a}) + a + s & r \in [a,b), \\ r & r \in [b,1], \end{cases} \tag{2.7}$$

for $0 < a < b < 1$, where $(1-a)$ is small. Consider the variation $u_s(x) := u(y_s)$ where in polar coordinates $x = (r,\phi)$ and $y_s = (R_s,\phi)$. We will omit the "$_s$" notation on y_s and R_s.

Recall that while this variation is not admissible: we may, by Lemma 1.10 on page 425, use it for our purposes.

Suppose now (until after (2.8)) that $r \in [a,b)$. Then

$$r = \frac{R - \frac{sb}{b-a}}{1 - \frac{s}{b-a}}.$$

So

$$\frac{dr}{dR} = \frac{1}{1-\frac{s}{b-a}}$$

and the volume elements dx and dy satisfy

$$dx = r dr d\phi = r\frac{dr}{dR} dR d\phi = \frac{R - \frac{sb}{b-a}}{(1-\frac{s}{b-a})^2} dR d\phi = \frac{1}{R}\frac{R-\frac{sb}{b-a}}{(1-\frac{s}{b-a})^2} dy.$$

Moreover

$$\begin{aligned}
\nabla u_s(x) &= \hat{\boldsymbol{r}}\frac{\partial u_s}{\partial r}(x) + \frac{1}{r}\hat{\boldsymbol{\phi}}\frac{\partial u_s}{\partial \phi}(x) \\
&= \hat{\boldsymbol{r}}\frac{\partial u}{\partial R}(y)\frac{\partial R}{\partial r}(x) + \frac{1}{r}\hat{\boldsymbol{\phi}}\frac{\partial u}{\partial \phi}(y) \\
&= \hat{\boldsymbol{r}}\frac{\partial u}{\partial R}(y)\left(1 - \frac{s}{b-a}\right) + \frac{1-\frac{s}{b-a}}{R-\frac{sb}{b-a}}\hat{\boldsymbol{\phi}}\frac{\partial u}{\partial \phi}(y) \\
&= R\frac{1-\frac{s}{b-a}}{R-\frac{sb}{b-a}}\left[\frac{R-\frac{sb}{b-a}}{R}\hat{\boldsymbol{R}}\frac{\partial u}{\partial R} + \frac{1}{R}\hat{\boldsymbol{\phi}}\frac{\partial u}{\partial \phi}\right](y) \\
&= R\frac{1-\frac{s}{b-a}}{R-\frac{sb}{b-a}}\left[\nabla u - \frac{sb}{R(b-a)}\hat{\boldsymbol{R}}\frac{\partial u}{\partial R}\right](y)
\end{aligned}$$

where $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are unit vectors in the directions of increasing r and ϕ respectively. So

$$\begin{aligned}
&|\nabla u_s(x)|^2 \\
&= R^2\left(\frac{1-\frac{s}{b-a}}{R-\frac{sb}{b-a}}\right)^2\left[|\nabla u|^2 - \frac{2sb}{R(b-a)}\left|\frac{\partial u}{\partial R}\right|^2 + \left(\frac{sb}{R(b-a)}\right)^2\left|\frac{\partial u}{\partial R}\right|^2\right](y).
\end{aligned} \tag{2.8}$$

We may then calculate that

$$
\begin{aligned}
E_f(u_s) &= \frac{1}{2}\int_{D_a} f(x)\,|\nabla u_s(x)|^2\,dx \\
&\quad + \frac{1}{2}\int_{D_b\setminus D_a} f(x)\,|\nabla u_s(x)|^2\,dx \\
&\quad + E_f(u_s; D\setminus D_b) \\
&= \frac{1}{2}\int_{D_{a+s}} f\left(y\left(\frac{1}{1+\frac{s}{a}}\right)\right)|\nabla u|^2\,dy \\
&\quad + \frac{1}{2}\int_{D_b\setminus D_{a+s}} f\left(y\frac{1}{|y|}\left(\frac{|y|-\frac{sb}{b-a}}{1-\frac{s}{b-a}}\right)\right)\frac{R}{R-\frac{sb}{b-a}}\Big[\cdots\Big]\,dy \\
&\quad + E_f(u; D\setminus D_b)
\end{aligned}
$$

where the notation "$\big[\cdots\big]$" refers to the contents of the square parentheses in (2.8). We may then calculate

$$
\begin{aligned}
\frac{d}{ds}E_f(u_s)\bigg|_{s=0} &= \frac{1}{2}\int_{D_a} \nabla f\cdot y\left(\frac{-1}{a}\right)|\nabla u|^2\,dy \\
&\quad + \frac{1}{2}\int_{D_b\setminus D_a} \nabla f\cdot\frac{y}{|y|}\left(\frac{|y|-b}{b-a}\right)|\nabla u|^2\,dy \\
&\quad + \frac{1}{2}\int_{D_b\setminus D_a} f(y)\frac{1}{R}\left(\frac{b}{b-a}\right)|\nabla u|^2\,dy \\
&\quad + \frac{1}{2}\int_{D_b\setminus D_a} f(y)\left(\frac{-2b}{R(b-a)}\right)\left|\frac{\partial u}{\partial R}\right|^2\,dy \\
&= -\frac{1}{2a}\int_{D_a} \nabla f\cdot y|\nabla u|^2\,dy \\
&\quad - \frac{1}{2}\int_{D_b\setminus D_a} \nabla f\cdot\frac{y}{|y|}\left(\frac{b-|y|}{b-a}\right)|\nabla u|^2\,dy \\
&\quad - \frac{1}{2}\int_{D_b\setminus D_a} f(y)\left(\frac{b}{b-a}\right)\frac{1}{R}\left[\left|\frac{\partial u}{\partial R}\right|^2-\frac{1}{R^2}\left|\frac{\partial u}{\partial \phi}\right|^2\right]\,dy.
\end{aligned}
$$

The reader should notice that the two boundary derivatives — i.e., the two integrals over $|z| = a+s$ — cancel when $s = 0$.

It follows by Lemma 2.2 that

$$
\begin{aligned}
\frac{d}{ds}E_f(u_s)\bigg|_{s=0} &\le -\frac{1}{2a}\int_{D_a} \nabla f\cdot y|\nabla u|^2\,dy \\
&\quad - \frac{1}{2}\int_{D_b\setminus D_a} \nabla f\cdot\frac{y}{|y|}\left(\frac{b-|y|}{b-a}\right)|\nabla u|^2\,dy \\
&\quad - \frac{1}{2a}\left[\int_0^{2\pi} f|u_r|^2\,d\phi\right]_{|z|=1} \\
&\quad + \frac{1}{2a}\|\nabla f\|_{L^\infty}\int_{D\setminus D_a}|\nabla u|^2\,dy \\
&< 0
\end{aligned}
\tag{2.9}
$$

for a sufficiently close to 1, contradicting u being f-harmonic. Therefore u must be constant on the open set Ω and thus, by Lemma 2.3, be a constant map. $\square$

Remark 2.4. The previous lemma has the hypothesis $\nabla f(x) \cdot x \geq 0$. One would expect this result to also hold with an alternate hypothesis of f being a convex function — that is if for all $x, y \in D$, $x \neq y$, and all $\lambda \in (0,1)$ there holds

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y).$$

Certainly if f has minimum at x_0 in the interior of D then $\nabla f(x) \cdot (x - x_0) \geq 0$ and so such a proof should be possible by distorting about x_0, instead of about 0 as we did above.

Remark 2.5. Proposition 2.1 appeared in [1] with the alternate hypothesis "$\nabla f \cdot \nabla x > 0$ *almost everywhere*".

3. The existence of nontrivial f-harmonic maps from the disc to the two sphere which map the boundary of the domain to one point

We show now that the quoted result of Lemaire does not extend directly to f-harmonic maps:

Lemma 3.1. *There exist an f and a smooth nonconstant, f-harmonic map from the disc to the 2-sphere which maps ∂D to a point.*

To prove this lemma, we will construct such an f-harmonic map. A first guess, given Proposition 2.1, is that a convex f is of no use to use here. Instead, one would perhaps guess that we would require an f with a maximum at the origin.

In order to simplify our calculations we introduce so-called "longitudinally symmetric maps". We say that the function $f : D \to (0, \infty)$ is *rotationally symmetric* if we can write $f(x) = f_1(|x|)$ for some $f_1 : [0,1] \to (0,\infty)$. For $\alpha \in C^\infty([0,1], \mathbb{R})$ such that $\alpha(0) = 0$, define $U_\alpha : D \to S^2 \subset \mathbb{R}^3$ by

$$U_\alpha(x) = \left(\frac{x}{|x|} \sin \alpha(|x|), \cos \alpha(|x|) \right).$$

The map $u : D \to S^2$ is said to be *longitudinally symmetric* if $u(x) = U_\theta(x)$ for some $\theta : [0,1] \to \mathbb{R}$.

Throughout the rest of this section, we use polar coordinates (r, ϕ) and (ϕ, θ) on D and S^2 respectively. In these coordinates, we may easily calculate the Euler–Lagrange equation for longitudinally symmetric maps:

Lemma 3.2. *Let $f : D \to (0,\infty)$ depend only on r. Suppose that the map $u : D \to S^2$ is of the form $(r, \phi) \mapsto (\phi, \theta(r))$ (i.e., u is longitudinally symmetric), that $\theta(0) = 0$ and that $\theta(1) = \pi$. Then u is f-harmonic if and only if*

$$r^2 \theta_{rr} + \theta_r \left(r + r^2 \frac{f_r}{f} \right) = \sin\theta \cos\theta. \tag{3.1}$$

Notice that, for a fixed odd function $\theta : [0,1] \to [0,\pi]$ satisfying $\theta(0) = 0$, $\theta(1) = \pi$ and $\theta_r(0) \neq 0$, we could rearrange (3.1) to obtain a formula for $f : [0,1] \to (0,\infty)$.

Indeed this is what we do: Define $f : D \to (0, \infty)$ and $\theta : [0, 1] \to [0, \pi]$ by

$$\begin{aligned} f(r) &:= \exp\Big(\int_0^r \frac{-\pi s + \cos(\pi s)\sin(\pi s)}{\pi s^2}\, ds\Big), \\ \theta(r) &:= \pi r. \end{aligned} \tag{3.2}$$

Then θ and f satisfy (3.1); so $u : (r, \phi) \mapsto (\phi, \pi r)$ is a nontrivial f-harmonic map, which maps the boundary of the domain to a point in the target. We remark that f is smooth and that for small r, $f(r) \approx e^{-\frac{1}{2}\pi^2 r^2}$.

Remark 3.3. The reader should not assume here that, given a general odd function θ, the f constructed in this way, that is defined by

$$f(r) := \exp\Big(\int_0^r \Big[\frac{\sin\theta(s)\cos\theta(s)}{s^2\theta_r(s)} - \frac{1}{r} - \frac{\theta_{rr}(s)}{\theta_r(s)}\Big]\, ds\Big), \tag{3.3}$$

will always have a maximum at the origin — our earlier intuition was false. Indeed, consider the odd function $\theta(r) := ar + br^3 + (\pi - a - b)r^5$. One may check that if $(a, b) = (1.5, -1.5)$, then f has a local minimum at 0. Moreover, if $(a, b) = (1.5, -4.5)$, then f achieves it's minimum over the whole disc D, at the origin.

4. The existance of a nontrivial f-harmonic map from the square torus to S^2

There is result by Eells–Wood [4] which states that: *There does not exist a harmonic map of degree* 1*, from the torus to the* 2*-sphere.* However:

Lemma 4.1. *There exist $f : T^2 \to (0, \infty)$ and a smooth degree* 1*, f-harmonic map from the square torus to the* 2*-sphere.*

Our strategy is as follows: We first construct an f with maxima at two particular points, and a certain symmetry. By careful choice of initial map u_0, we find that; if the f-harmonic heat flow (with $u(0) = u_0$) bubbles, then it *must* bubble at a maximum of f. We argue this bubbling would "use up" too much energy and is hence impossible. Thus the flow converges smoothly to the asserted map.

Proof. Consider $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\delta \in (0, \frac{1}{100})$ be chosen later. Take an f with $f \equiv 1$ on $T^2 \setminus [B_\delta(0) \cup B_\delta(\frac{1}{2}, \frac{1}{2})]$, $f(0) = f(\frac{1}{2}, \frac{1}{2}) = 2$, and $1 < f(x) < 2$ otherwise. Suppose further that f is invariant under isometries $T^2 \to T^2$ which fix 0. Such isometries form a subset Φ, of the set of all isometries $\mathbb{R}^2 \to \mathbb{R}^2$. So Φ must be $\{e, R, R^2, R^3, r_{x_1=0}, r_{x_2=0}, r_{x_1=x_2}, r_{x_1=-x_2}\}$, where e, R and $r_{x_1=x_2}$ denote respectively; the identity, anticlockwise rotation by $\frac{1}{2}\pi$ about $(\frac{1}{2}, \frac{1}{2})$ (or equivalently about $(0, 0)$), and reflection in the line $x_1 = x_2$. Here we use the notaion $x = (x_1, x_2) \in \mathbb{R}^2$. Note that the only points fixed by every element of Φ are 0 and $(\frac{1}{2}, \frac{1}{2})$.

Now consider an initial map $u_0 : T^2 \to S^2$ which, for small ε, maps $T^2 \setminus B_\varepsilon(0)$ onto a small neighbourhood of the "south pole" and maps $B_\varepsilon(0)$ once around the remainder of S^2. So u_0 is of degree 1. We suppose further that u_0 has the "same" symmetry as f. Precisely, we suppose that for each $\phi \in \Phi$, the initial map u_0 satisfies $u_0 \circ \phi = \phi \circ u_0$, where the action of the group Φ on the target S^2 is defined in the obvious way. It is known that we may define such a u_0, with the following

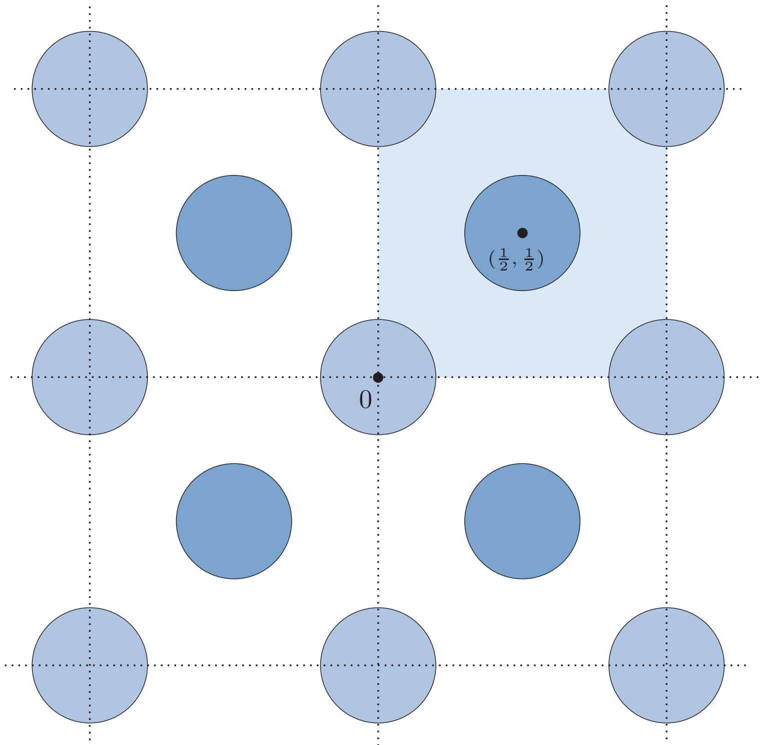

FIGURE 2. The flat-square torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The small balls $B_\delta(0)$ and $B_\delta(\frac{1}{2}, \frac{1}{2})$ are indicated.

additional property: the (1-)harmonic energy satisfies $E_1(u_0) < 6\pi$. Now choose δ sufficiently small so that we have $E_f(u_0) < 7\pi$.

We now study the heat flow $u : T^2 \times [0, \infty) \to S^2$,

$$\begin{cases} u_t - f\Delta u = fu|\nabla u|^2 + \nabla f * \nabla u \\ u|_{t=0} = u_0. \end{cases} \tag{4.1}$$

We know from the f-harmonic heat flow theorem (Theorem 1.11), that away from bubble points $(z_0, t_0) \in T^2 \times (0, \infty]$, the map u is smooth and $u(\cdot, t_n)$ converges smoothly to an f-harmonic map, u_∞ say, as $t_n \to \infty$ (for some suitable sequence t_n). If we can show that there can be no bubbling in this flow, then we would have the existence of a degree 1, f-harmonic map $T^2 \to S^2$.

Because $E_f(u_0) < 7\pi$, and because the "energy lost in a bubble" at z_0 is $4\pi f(z_0) \geq 4\pi$ (if $f \equiv 1$, it is well-known that the energy lost due to bubbling would be a multiple of 4π), there can be at most one bubble in this heat flow. Now suppose that a bubble does indeed form. Suppose a bubble forms at the point and time (z_0, t_0). By z_0, we mean $z_0 + \mathbb{Z}^2$ where $z_0 \in [0, 1)^2$. Due to the symmetry that we started with, if we have a bubble at z_0 then we must also have one at $\phi(z_0)$ for any isometry $\phi \in \Phi$. As noted previously, the only points fixed under such isometries are 0 and $(\frac{1}{2}, \frac{1}{2})$, and because we can have at most one bubble, these are the only places where a bubble could possibly form.

However, the "energy lost" if a bubble formed at either one of these points would be $4\pi f(z_0) = 8\pi$ — greater than the amount of energy that we started with. Therefore there can be no bubbling. So there exists a smooth degree 1, f-harmonic map $u_\infty : T^2 \to S^2$. $\square$

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