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Schatten class Toeplitz operators on weighted Bergman spaces of the unit ball

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ABSTRACT. For positive Toeplitz operators on Bergman spaces of the unit ball, we determine exactly when membership in the Schatten classes S_p can be characterized in terms of the Berezin transform.

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1. Introduction

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space. For any two points $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$ we write

$$\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n,$$

and

$$|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$$

is the open unit ball in $\mathbb{C}^n$. We denote by dv the usual Lebesgue volume measure on $\mathbb{B}_n$, normalized so that the volume of $\mathbb{B}_n$ is 1.

It is well-known that, for a real parameter α, we have

$$\int_{\mathbb{B}_n} (1 - |z|^2)^\alpha \, dv(z) < \infty$$

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if and only if $\alpha > -1$. Throughout this paper we fix a real parameter α with $\alpha > -1$ and write

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \, dv(z),$$

where c_α is a positive constant chosen so that $v_\alpha(\mathbb{B}_n) = 1$. The precise value of c_α is computable in terms of the gamma function, but it is not important for us.

Let $H(\mathbb{B}_n)$ denote the space of all holomorphic functions in $\mathbb{B}_n$. The space

$$A^2_\alpha = L^2(\mathbb{B}_n, dv_\alpha) \cap H(\mathbb{B}_n)$$

is called a weighted Bergman space of $\mathbb{B}_n$. It is a Hilbert space with the following inner product inherited from $L^2(\mathbb{B}_n, dv_\alpha)$:

$$\langle f, g\rangle = \int_{\mathbb{B}_n} f(z)\,\overline{g(z)}\, dv_\alpha(z).$$

This will be the only inner product we use on A^2_α and the associated norm on A^2_α will be denoted by $\|f\|$.

The orthogonal projection

$$P : L^2(\mathbb{B}_n, dv_\alpha) \to A^2_\alpha$$

is an integral operator. In fact,

$$Pf(z) = \int_{\mathbb{B}_n} \frac{f(w)\, dv_\alpha(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}}, \qquad f \in L^2(\mathbb{B}_n, dv_\alpha).$$

This formula enables us to extend the domain of the operator P, which is usually called a Bergman projection, to the larger space $L^1(\mathbb{B}_n, dv_\alpha)$. Thus P is an integral operator that maps $L^1(\mathbb{B}_n, dv_\alpha)$ into $H(\mathbb{B}_n)$.

More generally, if μ is a complex Borel measure on $\mathbb{B}_n$ with $|\mu|(\mathbb{B}_n) < \infty$, we can densely define an integral operator

$$T_\mu : A^2_\alpha \to H(\mathbb{B}_n)$$

by

$$T_\mu f(z) = \int_{\mathbb{B}_n} \frac{f(w)\, d\mu(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}}.$$

This is usually called the Toeplitz operator on A^2_α with symbol μ. In the case when

$$d\mu(z) = \varphi(z)\, dv_\alpha(z), \qquad \varphi \in L^1(\mathbb{B}_n, dv_\alpha),$$

we write T_φ instead of T_μ. Naturally, T_φ is also called the Toeplitz operator on A^2_α with symbol φ.

A particularly interesting case is when $\varphi \in L^\infty(\mathbb{B}_n)$. In this case we can write

$$T_\varphi(f) = P(\varphi f), \qquad f \in A^2_\alpha.$$

Since P is an orthogonal projection on $L^2(\mathbb{B}_n, dv_\alpha)$, the operator T_φ is bounded on A^2_α with $\|T_\varphi\| \le \|\varphi\|_\infty$.

It is well-known that the operator T_φ, and more generally, the operator T_μ, can turn out to be a bounded linear operator on A^2_α even when φ or μ does not have any boundedness property. More specifically, we say that T_μ is bounded on A^2_α if there exists a positive constant C such that $\|T_\mu f\| \le C\|f\|$ for all $f \in H^\infty(\mathbb{B}_n)$, where $H^\infty(\mathbb{B}_n)$ consists of all bounded holomorphic functions in $\mathbb{B}_n$. It is easy to check that $H^\infty(\mathbb{B}_n)$ is dense in A^2_α.

When μ is nonnegative, the boundedness, and the compactness as well, of the operator T_μ on A^2_α can be characterized in terms of the following two averaging functions.

For $z \in \mathbb{B}_n$ we define

$$\widetilde{\mu}(z) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, w\rangle|^{2(n+1+\alpha)}}\, d\mu(w).$$

The function $\widetilde{\mu}$ is usually called the Berezin transform of μ. For $z \in \mathbb{B}_n$ and $r > 0$ we define

$$\widehat{\mu}_r(z) = \frac{\mu(D(z,r))}{v_\alpha(D(z,r))},$$

where $D(z,r)$ is the Bergman metric ball at z with radius r.

The following result was proved in [5] (for the case of bounded symmetric domains).

Theorem 1. *Suppose $\mu \geq 0$, $p \geq 1$, and $r > 0$. Then:*

(a) *T_μ is bounded on A^2_α if and only if $\widetilde{\mu}$ is bounded on $\mathbb{B}_n$ if and only if $\widehat{\mu}_r$ is bounded on $\mathbb{B}_n$.*
(b) *T_μ is compact on A^2_α if and only if $\widetilde{\mu} \in \mathbb{C}_0(\mathbb{B}_n)$ if and only if $\widehat{\mu}_r \in \mathbb{C}_0(\mathbb{B}_n)$.*
(c) *T_μ belongs to the Schatten class S_p if and only if $\widetilde{\mu} \in L^p(\mathbb{B}_n, d\lambda)$ if and only if $\widehat{\mu}_r \in L^p(\mathbb{B}_n, d\lambda)$.*

Here $\mathbb{C}_0(\mathbb{B}_n)$ is the space of complex-valued continuous functions f on $\mathbb{B}_n$ such that $f(z) \to 0$ as $|z| \to 1^-$, and

$$d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$$

is the Möbius invariant volume measure on $\mathbb{B}_n$.

On the other hand, in the case of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, Daniel Luecking proved the following result in [2].

Theorem 2. *Suppose $\mu \geq 0$, $r > 0$, and $p > 0$. Then T_μ belongs to the Schatten class S_p if and only if $\{\widehat{\mu}_r(a_k)\} \in l^p$, where $\{a_k\}$ is any hyperbolic lattice in the unit disk.*

It was observed in [3] that the condition $\{\widehat{\mu}_r(a_k)\} \in l^p$ is equivalent to $\widehat{\mu}_r \in L^p(\mathbb{D}, d\lambda)$, although no details were given there.

A natural problem arises now, namely, does part (c) in Theorem 1 remain valid for $0 < p < 1$ in the context of the unit ball? We completely settle the problem in this paper. It turns out that the answer is affirmative in the case of the averaging function $\widehat{\mu}_r$; this is not surprising in view Luecking's theorem above for the unit disk. However, the answer is trickier in the case of the Berezin transform $\widetilde{\mu}$; this part of the solution is even new for the unit disk.

We state the main result of the paper as follows.

Theorem 3. *Suppose $\mu \geq 0$, $r > 0$, and $0 < p < 1$. Then:*

(a) *T_μ belongs to the Schatten class S_p if and only if $\widehat{\mu}_r \in L^p(\mathbb{B}_n, d\lambda)$.*
(b) *If $p > n/(n+1+\alpha)$, then T_μ belongs to the Schatten class S_p if and only if $\widetilde{\mu} \in L^p(\mathbb{B}_n, d\lambda)$.*
(c) *The cut-off point $n/(n+1+\alpha)$ above is best possible.*

In the next secton we gather the necessary technical results that will be needed for the proof of our main result. Section 3 is devoted to the generalization of Luecking's theorem to the unit ball. The characterization of Schatten class Toeplitz operators in terms of the Berezin transform is obtained in Section 4. We conclude the paper in Section 5 with an extension of our results to products of balls, and in particular, to the polydisk in $\mathbb{C}^n$.

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2. Preliminaries

All results in this section are known; we include them here for convenience of reference. The references given here are not necessarily the original ones. We begin with the following integral estimate which has become indispensible in this area of analysis.

Lemma 4. *Suppose $t > -1$ and*

$$I(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^t \, dv(w)}{|1-\langle z, w\rangle|^{n+1+t+s}}.$$

If $s > 0$, then there exists a positive constant C such that

$$I(z) \le \frac{C}{(1-|z|^2)^s}, \quad z \in \mathbb{B}_n.$$

If $s < 0$, then there exists a positive constant C such that $I(z) \le C$ for all $z \in \mathbb{B}_n$.

Proof. See [4] or [7]. □

Let $\mathrm{Aut}(\mathbb{B}_n)$ denote the automorphism group of the unit ball. It is well-known that $\mathrm{Aut}(\mathbb{B}_n)$ is generated by two special classes of automorphisms: the unitary transformations and the involutive automorphisms. Nothing needs to be said about the unitary mappings. For every $z \in \mathbb{B}_n$ there exists a unique automorphism φ_z of $\mathbb{B}_n$ such that $\varphi_z(0) = z$ and $\varphi_z \circ \varphi_z(w) = w$ for all $w \in \mathbb{B}_n$. These maps φ_z are called involutions of $\mathbb{B}_n$, or involutive automorphisms. Explicit formulas are known for them; see [4] or [7]. For example, in the case of the unit disk, we have

$$\varphi_z(w) = \frac{z-w}{1-\bar{z}w}.$$

Recall that the Bergman metric on the unit ball is given by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}.$$

For any $z \in \mathbb{B}_n$ and $r > 0$ we introduce the Bergman metric ball

$$D(z,r) = \{w \in \mathbb{B}_n : \beta(w,z) < r\}.$$

It is well-known that if r is fixed, then the weighted volume $v_\alpha(D(z,r))$ is comparable to $(1-|z|^2)^{n+1+\alpha}$. See [7] for example.

A sequence $\{a_k\}$ in $\mathbb{B}_n$ is called an r-lattice in the Bergman metric if the following conditions are satisfied:

(a) The unit ball is covered by the Bergman metric balls $\{D(a_k,r)\}$.
(b) $\beta(a_i,a_j) \ge r/2$ for all i and j with $i \ne j$.

If $\{a_k\}$ is an r-lattice in $\mathbb{B}_n$, then it also has the following properties:

(c) For any $R > 0$ there exists a positive integer N_1 (depending on r and R) such that every point in $\mathbb{B}_n$ belongs to at most N_1 sets in $\{D(a_k, R)\}$.
(d) For any $R > 0$ there exists a decomposition of $\{a_k\}$ into a finite number of sequences $\{a_{jk}\}$, $1 \le j \le N_2$, such that $\beta(a_{jk}, a_{jm}) \ge R$ for all $k \ne m$.

There are elementary constructions of r-lattices in $\mathbb{B}_n$.

The following result is usually referred to as the atomic decomposition for Bergman spaces. See [1] and [7].

Theorem 5. *For any $b > n + (\alpha+1)/2$ there exists a positive constant r_0 with the following property: if $\{a_k\}$ is any r-lattice in $\mathbb{B}_n$ with $r < r_0$, then the Bergman space A^2_α consists exactly of functions of the form*

$$f(z) = \sum_{k=1}^\infty c_k \frac{(1-|a_k|^2)^{b-(n+1+\alpha)/2}}{(1-\langle z, a_k\rangle)^b}, \tag{1}$$

where $\{c_k\} \in l^2$. Moreover, $\|f\|^2$ is comparable to

$$\|f\|_*^2 = \inf\left\{\sum_{k=1} |c_k|^2 : \{c_k\} \text{ satisfies (1)}\right\}.$$

As a matter of fact, if $\{a_k\}$ is any r-lattice in the Bergman metric, and if $\{c_k\}$ is any sequence in l^2, then the function f defined in (1) belongs to A^2_α and

$$\|f\|^2 \le C \sum_{k=1}^\infty |c_k|^2$$

for some positive constant C that is independent of $\{c_k\}$. This part of Theorem 5 does not require r to be small.

We will also need the following estimate for the Bergman kernel function.

Lemma 6. *Suppose R is a positive radius and b is any real number. Then there exists a positive constant C such that*

$$\left|\frac{(1-\langle z,u\rangle)^b}{(1-\langle z,v\rangle)^b} - 1\right| \le C\beta(u,v)$$

for all z, u, and v in $\mathbb{B}_n$ with $\beta(u,v) \le R$.

As a consequence, we easily see that for any positive R there exists a positive constant C such that

$$C^{-1} \le \frac{|1-\langle z,u\rangle|}{|1-\langle z,v\rangle|} \le C \tag{2}$$

for all z, u, and v in $\mathbb{B}_n$ with $\beta(u,v) \le R$.

Lemma 7. *Suppose $p > 0$ and $r > 0$. Then there exists a positive constant C such that*

$$|f(z)|^p \le \frac{C}{v(D(z,r))} \int_{D(z,r)} |f(w)|^p \, dv(w)$$

for all $f \in H(\mathbb{B}_n)$ and all $z \in \mathbb{B}_n$.

We will also need a few results concerning Schatten class operators on a separable Hilbert space. Recall that if T is a positive, compact operator on a separable Hilbert space H, then there exists an orthonormal set $\{e_k\}$ in H and a sequence $\{\lambda_k\}$ that decreases to 0 such that

$$Tx = \sum_k \lambda_k \langle x, e_k \rangle e_k$$

for all $x \in H$. This is called the canonical decomposition of T and the numbers λ_k are called the singular values of T. A positive operator T belongs to the Schatten class S_p, where $p > 0$, if the sequence $\{\lambda_k\}$ of its singular values belongs to the sequence space l^p. In this case, we write

$$\|T\|_p = \left[\sum_k \lambda_k^p\right]^{1/p}.$$

More generally, a compact (not necessarily positive) operator T on H belongs to the Schatten class S_p if the positive operator $|T| = (T^*T)^{1/2}$ belongs to S_p. In this case, we define $\|T\|_p = \||T|\|_p$.

Lemma 8. *Suppose A is a bounded surjective operator on H and T is any bounded linear operator on H. Then $T \in S_p$ if and only if $A^*TA \in S_p$.*

Proof. Each Schatten class S_p is an ideal in the full algebra of bounded linear operators on H. Therefore, if $T \in S_p$, then the operator $S = A^*TA$ is also in S_p. On the other hand, since A is surjective, it has bounded right inverse. Thus there exists a bounded linear operator B on H such that $AB = I$, the identity operator on H. It follows that $T = B^*SB$, so $S \in S_p$ implies $T \in S_p$. □

Lemma 9. *Suppose T is a bounded linear operator on H and $\{e_k\}$ is any orthonormal basis of H. Then for any $0 < p \le 2$ we have*

$$\|T\|_p^p \le \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle Te_i, e_j \rangle|^p.$$

Proof. See [2] for example. □

Lemma 10. *Suppose T is a positive operator on H and $\{e_k\}$ is an orthonormal basis for H. If $0 < p < 1$ and*

$$\sum_{k=1}^{\infty} \langle Te_k, e_k \rangle^p < \infty,$$

then T belongs to the Schatten class S_p.

Proof. See [6] for example. □

In the rest of the paper we use C to denote a positive constant whose value may change from one occurence to another.

3. Characterization by $\widehat{\mu}_r$

In this section we characterize Schatten class Toeplitz operators T_μ on A^2_α with positive symbols μ based on the averaging function $\widehat{\mu}_r$. This was already done in [5] for the unit ball when $p \geq 1$, and in [2] for the unit disk when $p > 0$. So the important case here is when $0 < p < 1$. The ideas of this section are clearly from [2].

Lemma 11. *Suppose $\mu \geq 0$, $r > 0$, $0 < p < 1$, and $\widehat{\mu}_{2r} \in L^p(\mathbb{B}_n, d\lambda)$. Then $T_\mu \in S_p$.*

Proof. Fix an r-lattice $\{a_k\}$ in the Bergman metric of $\mathbb{B}_n$. If $z \in D(a_k, r)$, then $D(a_k, r) \subset D(z, 2r)$ by the triangle inequality. Since $v_\alpha(D(a_k, r))$ is comparable to $v_\alpha(D(z, 2r))$ whenever $z \in D(a_k, r)$, we can find a positive constant C, independent of k, such that

$$\widehat{\mu}_r(a_k) = \frac{\mu(D(a_k, r))}{v_\alpha(D(a_k, r))} \leq C \frac{\mu(D(z, 2r))}{v_\alpha(D(z, 2r))} = C\widehat{\mu}_{2r}(z)$$

for all $z \in D(a_k, r)$. Also, for any fixed $r > 0$, there exists a constant $C > 0$ such that

$$C^{-1} \leq \lambda(D(z, r)) \leq C$$

for all $z \in \mathbb{B}_n$. It follows that there exists a constant $C > 0$ such that

$$\widehat{\mu}_r(a_k)^p \leq C \int_{D(a_k,r)} \widehat{\mu}_{2r}(z)^p \, d\lambda(z)$$

for all k. Recall that every point of $\mathbb{B}_n$ belongs to at most N of the sets $D(a_k, r)$. So

$$\begin{aligned} \sum_{k=1}^{\infty} \widehat{\mu}_r(a_k)^p &\leq C \sum_{k=1}^{\infty} \int_{D(a_k,r)} \widehat{\mu}_{2r}(z)^p \, d\lambda(z) \\ &\leq CN \int_{\mathbb{B}_n} \widehat{\mu}_{2r}(z)^p \, d\lambda(z) < \infty. \end{aligned}$$

Fix a positive constant $b > n+(n+1+\alpha)/2$. If $0 < r_1 < r_2$ and $\widehat{\mu}_{r_2} \in L^p(\mathbb{B}_n, d\lambda)$, then clearly $\widehat{\mu}_{r_1} \in L^p(\mathbb{B}_n, d\lambda)$. So by shrinking r if necessary, we may assume that atomic decomposition for A^2_α (Theorem 5) already holds on the lattice $\{a_k\}$.

Fix an orthonormal basis $\{e_k\}$ for A^2_α and define an operator

$$A : A^2_\alpha \to A^2_\alpha$$

by

$$A\left(\sum_{k=1}^{\infty} c_k e_k\right) = \sum_{k=1}^{\infty} c_k h_k,$$

where

$$h_k(z) = \frac{(1 - |a_k|^2)^{b-(n+1+\alpha)/2}}{(1 - \langle z, a_k \rangle)^b}.$$

By Theorem 5, A is a bounded and surjective operator on A^2_α. According to Lemma 8, the Toeplitz operator T_μ will be in S_p if we can show that the operator $T = A^* T_\mu A$ belongs to S_p. By Lemma 10, we just need to verify the following condition:

$$S=\sum_{k=1}^\infty \langle Te_k,e_k\rangle^p<\infty. \tag{3}$$

It is clear that

$$\langle Te_k,e_k\rangle=\langle T_\mu h_k,h_k\rangle=\int_{\mathbb{B}_n}|h_k(z)|^2\,d\mu(z).$$

Since $\{D(a_j,r)\}$ is an open cover of $\mathbb{B}_n$, we have

$$\langle Te_k,e_k\rangle\le\sum_{j=1}^\infty\int_{D(a_j,r)}|h_k(z)|^2\,d\mu(z).$$

By Lemma 6, there exists a positive constant C such that

$$\langle Te_k,e_k\rangle\le C\sum_{j=1}^\infty |h_k(a_j)|^2\mu(D(a_j,r)).$$

Since $0<p<1$, an application of Hölder's inequality gives

$$\langle Te_k,e_k\rangle^p\le C\sum_{j=1}^\infty |h_k(a_j)|^{2p}\mu(D(a_j,r))^p.$$

Recall that $v_\alpha(D(a_j,r))$ is comparable to $(1-|a_j|^2)^{n+1+\alpha}$. So there is another constant $C>0$ such that

$$\langle Te_k,e_k\rangle^p\le C\sum_{j=1}^\infty (1-|a_j|^2)^{p(n+1+\alpha)}|h_k(a_j)|^{2p}\,\widehat{\mu}_r(a_j)^p.$$

By Fubini's theorem, we have

$$S\le C\sum_{j=1}^\infty (1-|a_j|^2)^{p(n+1+\alpha)}\,\widehat{\mu}_r(a_j)^p\sum_{k=1}^\infty |h_k(a_j)|^{2p}.$$

For each $j\ge 1$ we consider the sum

$$S_j=\sum_{k=1}^\infty |h_k(a_j)|^{2p}=\sum_{k=1}^\infty\frac{(1-|a_k|^2)^{p(2b-n-1-\alpha)}}{|1-\langle a_j,a_k\rangle|^{2pb}}.$$

By Lemma 7, there is a constant $C>0$ such that

$$\frac{1}{|1-\langle a_j,a_k\rangle|^{2pb}}\le\frac{C}{v(D(a_k,r))}\int_{D(a_k,r)}\frac{dv(z)}{|1-\langle a_j,z\rangle|^{2pb}}$$

for all j and k. Since $v(D(a_k,r))$ is comparable to $(1-|a_k|^2)^{n+1}$, and $1-|z|^2$ is comparable to $1-|a_k|^2$ for $z\in D(a_k,r)$, we obtain

$$S_j\le C\sum_{k=1}^\infty\int_{D(a_k,r)}\frac{(1-|z|^2)^{p(2b-n-1-\alpha)-(n+1)}}{|1-\langle a_j,z\rangle|^{2pb}}\,dv(z).$$

Since every point of $\mathbb{B}_n$ belongs to at most N of the sets $D(a_k,r)$, we obtain

$$S_j\le CN\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{p(2b-n-1-\alpha)-(n+1)}}{|1-\langle a_j,z\rangle|^{2pb}}\,dv(z).$$

By Lemma 4, there is a positive constant C such that

$$S_j \leq \frac{C}{(1-|a_j|^2)^{p(n+1+\alpha)}}$$

for all $j \geq 1$. It follows that

$$S \leq C \sum_{j=1}^{\infty} \widehat{\mu}_r(a_j)^p < \infty.$$

This completes the proof of Lemma 11. □

Lemma 12. *Suppose $T_\mu \in S_p$, $0 < p < 1$, and $\{a_k\}$ is an r-lattice in the Bergman metric of $\mathbb{B}_n$. Then the sequence $\{\widehat{\mu}_r(a_k)\}$ is in l^p.*

Proof. Fix a sufficiently large positive radius R and partition the lattice $\{a_k\}$ into N subsequences such that the Bergman metric between any two points in each subsequence is at least R. Let $\{\zeta_j\}$ be such a subsequence and define a measure ν on $\mathbb{B}_n$ as follows:

$$d\nu(z) = \sum_{k=1}^{\infty} \chi_k(z)\, d\mu(z),$$

where χ_k is the characteristic function of $D(\zeta_k, r)$. We assume that $R > 2r$, so that the Bergman metric balls $\{D(\zeta_k, r)\}$ are disjoint.

Since $T_\mu \in S_p$ and $0 \leq \nu \leq \mu$, we must also have $T_\nu \in S_p$. In fact, $0 \leq T_\nu \leq T_\mu$ implies $0 \leq T_\nu^p \leq T_\mu^p$, which in turn implies that $\|T_\nu\|_p \leq \|T_\mu\|_p$.

Fix an orthonormal basis $\{e_k\}$ for A^2_α and define an operator A on A^2_α by

$$A\left[\sum_{k=1}^{\infty} c_k e_k\right] = \sum_{k=1}^{\infty} c_k h_k,$$

where b is a sufficiently large positive constant and

$$h_k(z) = \frac{(1-|\zeta_k|^2)^{(2b-n-1-\alpha)/2}}{(1-\langle z, \zeta_k\rangle)^b}.$$

Since $T_\nu \in S_p$, we also have $T = A^* T_\nu A \in S_p$ with

$$\|T\|_p \leq \|A\|^2 \|T_\nu\|_p.$$

Here the boundedness of A on A^2_α follows from the remarks after Theorem 5; all that is needed is the fact that $\{\zeta_k\}$ is separated in the Bergman metric. In fact, we can find a positive constant C (that only depends on r, b, and α, but not on the particular subsequence $\{\zeta_k\}$ of $\{a_k\}$ being used) such that

$$\|T\|_p^p \leq C \|T_\mu\|_p^p.$$

We split the operator T as $T = D + E$, where D is the diagonal operator on A^2_α defined by

$$Df = \sum_{k=1}^{\infty} \langle Te_k, e_k\rangle \langle f, e_k\rangle e_k, \qquad f \in A^2_\alpha,$$

and $E = T - D$. By the triangle inequality, we have

$$\|T\|_p^p \geq \|D\|_p^p - \|E\|_p^p. \tag{4}$$

Since D is a positive diagonal operator, we have

$$\begin{aligned}\|D\|_p^p &= \sum_{k=1}^\infty \langle Te_k, e_k\rangle^p = \sum_{k=1}^\infty \langle T_\nu h_k, h_k\rangle^p \\ &= \sum_{k=1}^\infty \left[\int_{\mathbb{D}} |h_k(z)|^2\, d\nu(z)\right]^p \\ &\ge \sum_{k=1}^\infty \left[\int_{D(\zeta_k,r)} |h_k(z)|^2\, d\nu(z)\right]^p \\ &\ge C\sum_{k=1}^\infty \widehat{\nu}_r(\zeta_k)^p.\end{aligned}$$

The last inequality follows from (2). In particular, the constant C only depends on r, b, and α. Since $\nu = \mu$ on each $D(\zeta_k, r)$, we obtain

$$\|D\|_p^p \ge C_1 \sum_{k=1}^\infty \widehat{\mu}_r(\zeta_k)^p. \tag{5}$$

On the other hand, according to Lemma 9, we have

$$\begin{aligned}\|E\|_p^p &\le \sum_{k=1}^\infty \sum_{j=1}^\infty |\langle Ee_j, e_k\rangle|^p = \sum_{j\ne k} |\langle T_\nu h_j, h_k\rangle|^p \\ &= \sum_{j\ne k} \left|\int_{\mathbb{B}_n} h_j(z)\overline{h_k(z)}\, d\nu(z)\right|^p \\ &\le \sum_{j\ne k} \left[\int_{\mathbb{B}_n} |h_j(z)h_k(z)|\, d\nu(z)\right]^p.\end{aligned}$$

For any j and k we write

$$I_{jk} = \int_{\mathbb{B}_n} |h_j(z)h_k(z)|\, d\nu(z).$$

Then

$$I_{jk} = \sum_{i=1}^\infty \int_{D(\zeta_i,r)} |h_j(z)h_k(z)|\, d\mu(z),$$

and by Lemma 6, there exists a positive constant C (depending on r, b, and α) such that

$$\begin{aligned}I_{jk} &\le C\sum_{i=1}^\infty |h_j(\zeta_i)h_k(\zeta_i)|\mu(D(\zeta_i,r)) \\ &\le C\sum_{i=1}^\infty (1-|\zeta_i|^2)^{n+1+\alpha}|h_j(\zeta_i)h_k(\zeta_i)|\widehat{\mu}_r(\zeta_i).\end{aligned}$$

Here in the last inequality we used the fact that $v_\alpha(D(\zeta_i, r))$ is comparable to $(1-|\zeta_i|^2)^{n+1+\alpha}$.

Since $0 < p < 1$, we apply Hölder's inequality to get

$$|I_{jk}|^p \le C \sum_{i=1}^{\infty} (1-|\zeta_i|^2)^{p(n+1+\alpha)} |h_j(\zeta_i)h_k(\zeta_i)|^p \widehat{\mu}_r(\zeta_i)^p.$$

Combining this with Fubini's theorem, we obtain

$$\|E\|_p^p \le C \sum_{i=1}^{\infty} (1-|\zeta_i|^2)^{p(n+1+\alpha)} \widehat{\mu}_r(\zeta_i)^p I_i,$$

where

$$\begin{aligned} I_i &= \sum_{j\ne k} |h_j(\zeta_i)h_k(\zeta_i)|^p \\ &= \sum_{j\ne k} \frac{\big[(1-|\zeta_j|^2)(1-|\zeta_k|^2)\big]^{pb-p(n+1+\alpha)/2}}{\big[|1-\langle \zeta_j,\zeta_i\rangle||1-\langle \zeta_k,\zeta_i\rangle|\big]^{pb}} \end{aligned}$$

for all $i \ge 1$. Let

$$\Omega = \bigcup_{j\ne k} D(\zeta_n, r) \times D(\zeta_k, r) \subset \mathbb{B}_n \times \mathbb{B}_n.$$

Since the union above is a disjoint one, we can find a positive constant C (depending on r, b, and α) such that

$$I_i \le C \iint_\Omega \frac{\big[(1-|z|^2)(1-|w|^2)\big]^{pb-n-1-p(n+1+\alpha)/2}}{\big[|1-\langle z,\zeta_i\rangle||1-\langle w,\zeta_i\rangle|\big]^{pb}} \, dv(z)\, dv(w).$$

By assumption, we have

$$\Omega \subset G_R = \{(z,w) \in \mathbb{B}_n \times \mathbb{B}_n : \beta(z,w) \ge R - 2r\}.$$

Therefore,

$$I_i \le C \iint_{G_R} \frac{\big[(1-|z|^2)(1-|w|^2)\big]^{pb-p(n+1+\alpha)/2}}{\big[|1-\langle z,\zeta_i\rangle||1-\langle w,\zeta_i\rangle|\big]^{pb}} \, d\lambda(z)\, d\lambda(w).$$

Making the change of variables $z = \varphi_{\zeta_i}(u)$ and $w = \varphi_{\zeta_i}(v)$, we obtain

$$I_i \le C(1-|\zeta_i|^2)^{-p(n+1+\alpha)} \iint_{G_R} F(u,v)\, dv(u)\, dv(v),$$

where

$$F(u,v) = \frac{\big[(1-|u|^2)(1-|v|^2)\big]^{pb-n-1-p(n+1+\alpha)/2}}{\big[|1-\langle u,\zeta_i\rangle||1-\langle v,\zeta_i\rangle|\big]^{pb-p(n+1+\alpha)}}.$$

We can assume b is large enough so that

$$pb \ge n+1+p(n+1+\alpha)/2.$$

Since

$$(1-|u|^2)(1-|v|^2) \le 4|1-\langle u,\zeta_i\rangle||1-\langle v,\zeta_i\rangle|,$$

there exists another positive constant C (independent of R) such that

$$I_i \le C(1-|\zeta_i|^2)^{-p(n+1+\alpha)} \iint_{G_R} \frac{dv(u)\, dv(v)}{\big[|1-\langle u,\zeta_i\rangle||1-\langle v,\zeta_i\rangle|\big]^{n+1-p(n+1+\alpha)/2}}.$$

Choose $t \in (1,\infty)$ and $s \in (1,\infty)$ (independent of R) such that

$$A = t\left[n+1-\frac{p(n+1+\alpha)}{2}\right] < n+1, \quad \frac{1}{t}+\frac{1}{s}=1.$$

By Hölder's inequality, the integral

$$\iint_{G_R} \frac{dv(u)\,dv(v)}{\left[|1-\langle u,\zeta_i\rangle||1-\langle v,\zeta_i\rangle|\right]^{n+1-p(n+1+\alpha)/2}}$$

is less than or equal to

$$\left[\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{dv(u)\,dv(v)}{|1-\langle u,\zeta_i\rangle|^A|1-\langle v,\zeta_i\rangle|^A}\right]^{\frac{1}{t}} \left[v^2(G_R)\right]^{\frac{1}{s}},$$

where

$$v^2(G_R) = \iint_{G_R} dv(u)\,dv(v).$$

Since $A < n+1$, we apply Lemma 4 to find another positive constant C (independent of R) such that

$$I_i \le C(1-|\zeta_i|^2)^{-p(n+1+\alpha)}\left[v^2(G_R)\right]^{\frac{1}{s}}.$$

We conclude that there exists a constant $C_2 > 0$ (independent of R) such that

$$\|E\|_p^p \le C_2\left[v^2(G_R)\right]^{\frac{1}{s}}\sum_{i=1}^{\infty}\widehat{\mu}_r(\zeta_i)^p.$$

Combining this with (4) and (5), we obtain

$$\|T\|_p^p \ge \left(C_1 - C_2\left[v^2(G_R)\right]^{\frac{1}{s}}\right)\sum_{i=1}^{\infty}\widehat{\mu}_r(\zeta_i)^p,$$

where C_1 and C_2 are positive constant independent of R. If we chose R large enough so that

$$C_1 - C_2\left[v^2(G_R)\right]^{\frac{1}{s}} > 0,$$

then we could find a positive constant C (independent of μ) such that

$$\sum_{i=1}^{\infty}\widehat{\mu}_r(\zeta_i)^p \le C\|T_\mu\|_p^p.$$

Since this holds for each one of the N subsequences of $\{a_n\}$, we obtain

$$\sum_{n=1}^{\infty}\widehat{\mu}_r(a_n)^p \le CN\|T_\mu\|_p^p \tag{6}$$

for all positive Borel measures μ such that

$$\sum_{n=1}^{\infty}\widehat{\mu}_r(a_n)^p < \infty.$$

An easy approximation argument then shows that (6) actually holds for all positive Borel measures μ. This completes the proof of Lemma 12. $\square$

Lemma 13. *Suppose μ is a positive Borel measure on $\mathbb{B}_n$ and $r > 0$. If for every $2r$-lattice $\{a_k\}$ in the Bergman metric we have*

$$\sum_{k=1} \widehat{\mu}_{2r}(a_k)^p < \infty,$$

then $\widehat{\mu}_r \in L^p(\mathbb{B}_n, d\lambda)$.

Proof. Fix an r-lattice $\{a_k\}$ in the Bergman metric and partition it into N subsequences $\{a_{jk}\}$, $1 \le j \le N$, such that each subsequence is a $2r$-lattice in the Bergman metric. We have

$$\sum_{k=1}^{\infty} \widehat{\mu}_{2r}(a_k)^p = \sum_{j=1}^{N} \sum_{k=1}^{\infty} \widehat{\mu}_{2r}(a_{jk})^p < \infty,$$

because each subsequence $\{a_{jk}\}$ is a $2r$-lattice. We now show that the integral

$$I = \int_{\mathbb{B}_n} \widehat{\mu}_r(z)^p \, d\lambda(z)$$

is finite.

Since $\{D(a_k, r)\}$ is an open cover of $\mathbb{B}_n$, we have

$$\begin{aligned} I &\le \sum_{k=1}^{\infty} \int_{D(a_k,r)} \widehat{\mu}_r(z)^p \, d\lambda(z) \\ &\le \sum_{k=1}^{\infty} \lambda(D(a_k, r)) \sup\{\widehat{\mu}_r(z)^p : z \in D(a_k, r)\}. \end{aligned}$$

There exists a positive constant C such that $\lambda(D(a_k, r)) \le C$ for all $k \ge 1$. Also, if $z \in D(a_k, r)$, then by the triangle inequality, $D(z, r) \subset D(a_k, 2r)$. Combining this with the fact that $v_\alpha(D(a_k, 2r))$ and $v_\alpha(D(z, r))$ are comparable whenever $z \in D(a_k, r)$, we can find another positive constant C such that

$$I \le C \sum_{k=1}^{\infty} \widehat{\mu}_{2r}(a_k)^p < \infty.$$

This completes the proof of Lemma 13. □

As a consequence of the three lemmas above, we have proved the main result of the section.

Theorem 14. *Suppose μ is a positive Borel measure on $\mathbb{B}_n$, $0 < p < 1$, and $r > 0$. Then the following conditions are equivalent.*

(a) $T_\mu \in S_p$.
(b) $\widehat{\mu}_r \in L^p(\mathbb{B}_n, d\lambda)$.
(c) $\{\widehat{\mu}_r(a_k)\} \in l^p$ *for some r-lattice $\{a_k\}$.*
(d) $\{\widehat{\mu}_r(a_k)\} \in l^p$ *for every r-lattice $\{a_k\}$.*

Since the condition $T_\mu \in S_p$ does not involve r, we see that $\widehat{\mu}_r \in L^p(\mathbb{B}_n, d\lambda)$ for some $r > 0$ if and only if $\widehat{\mu}_r \in L^p(\mathbb{B}_n, d\lambda)$ for every $r > 0$. The same remark applies to the characterizations in (c) and (d) above.

4. Characterization by $\widetilde{\mu}$

In this section we consider the problem of characterizing membership of T_μ in the Schatten classes by integral properties of the Berezin transform $\widetilde{\mu}$. It turns out that this cannot be done for the full range $0 < p < 1$. We will exhibit an obvious obstruction, and we will then show that this obstruction is the only one.

First observe that if μ is a positive Borel measure with compact support in $\mathbb{B}_n$, then the function $\widehat{\mu}_r(z)$ is also compactly supported in $\mathbb{B}_n$, so the condition

$$\int_{\mathbb{B}_n} \widehat{\mu}_r(z)^p \, d\lambda(z) < \infty$$

is satisfied for all $p > 0$. By Theorem 14, the operator T_μ belongs to the Schatten class S_p for every $p > 0$.

On the other hand, if μ is any positive Borel measure on $\mathbb{B}_n$ with $\mu(\mathbb{B}_n) > 0$, then an elementary estimate shows that

$$\begin{aligned}\widetilde{\mu}(z) &= (1-|z|^2)^{n+1+\alpha} \int_{\mathbb{B}_n} \frac{d\mu(w)}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} \\ &\geq \frac{\mu(\mathbb{B}_n)}{4^{n+1+\alpha}}(1-|z|^2)^{n+1+\alpha}.\end{aligned}$$

It follows that

$$\int_{\mathbb{B}_n} \widetilde{\mu}(z)^p \, d\lambda(z) = \infty$$

whenever $p(n+1+\alpha) \leq n$. Therefore, in the range $0 < p \leq n/(n+1+\alpha)$, it is not possible to characterize the membership of T_μ in S_p in terms of the Berezin transform $\widetilde{\mu}$. Our next result shows that this is the only obstruction.

Theorem 15. *Suppose μ is a positive Borel measure on $\mathbb{B}_n$ and*

$$n/(n+1+\alpha) < p < 1.$$

Then $T_\mu \in S_p$ if and only if $\widetilde{\mu} \in L^p(\mathbb{B}_n, d\lambda)$.

Proof. Fix any positive radius r. For any $z \in \mathbb{B}_n$ we have

$$\begin{aligned}\widetilde{\mu}(z) &= \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} \, d\mu(w) \\ &\geq \int_{D(z,r)} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} \, d\mu(w).\end{aligned}$$

By Lemma 6, there exists a positive constant C such that

$$\frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} \geq \frac{C}{v_\alpha(D(z,r))}$$

for all $z \in \mathbb{B}_n$ and all $w \in D(z,r)$. It follows that $C\widehat{\mu}_r(z) \leq \widetilde{\mu}(z)$ for all $z \in \mathbb{B}_n$. So the condition $\widetilde{\mu} \in L^p(\mathbb{B}_n, d\lambda)$ implies $\widehat{\mu}_r \in L^p(\mathbb{B}_n, d\lambda)$, which, in view of Theorem 14, implies that $T_\mu \in S_p$. This argument works regardless of the range of p.

Next we suppose that $T_\mu \in S_p$. Fix an r-lattice $\{a_k\}$ in the Bergman metric and estimate the Berezin transform $\widetilde{\mu}$ as follows.

$$\begin{aligned}\widetilde{\mu}(z) &= \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}\,d\mu(w)\\ &\le \sum_{k=1}^{\infty}\int_{D(a_k,r)} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}\,d\mu(w)\\ &\le C\sum_{k=1}^{\infty} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,a_k\rangle|^{2(n+1+\alpha)}}\,\mu(D(a_k,r)).\end{aligned}$$

The last step above follows from Lemma 6. Since $v_\alpha(D(a_k,r))$ is comparable to $(1-|a_k|^2)^{n+1+\alpha}$, we obtain another positive constant C such that

$$\begin{aligned}\widetilde{\mu}(z) &\le C\sum_{k=1}^{\infty} \frac{(1-|z|^2)^{n+1+\alpha}(1-|a_k|^2)^{n+1+\alpha}}{|1-\langle z,a_k\rangle|^{2(n+1+\alpha)}}\widehat{\mu}_r(a_k)\\ &= C\sum_{k=1}^{\infty}\left(1-|\varphi_{a_k}(z)|^2\right)^{n+1+\alpha}\widehat{\mu}_r(a_k).\end{aligned}$$

Here we have used the well-known identity

$$1-|\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2}, \qquad a,w\in\mathbb{B}_n,$$

which can be found in [4] and [7] for example.

When $0<p<1$, an application of Hölder's inequality leads to

$$\widetilde{\mu}(z)^p \le C\sum_{k=1}^{\infty}\left(1-|\varphi_{a_k}(z)|^2\right)^{p(n+1+\alpha)}\widehat{\mu}_r(a_k)^p.$$

It follows that

$$\int_{\mathbb{B}_n}\widetilde{\mu}(z)^p\,d\lambda(z) \le C\sum_{k=1}^{\infty}\widehat{\mu}_r(a_k)^p\int_{\mathbb{B}_n}(1-|\varphi_{a_k}(z)|^2)^{p(n+1+\alpha)}\,d\lambda(z).$$

Since λ is invariant under the action of automorphisms, we have

$$\int_{\mathbb{B}_n}(1-|\varphi_{a_k}(z)|^2)^{p(n+1+\alpha)}\,d\lambda(z) = \int_{\mathbb{B}_n}(1-|z|^2)^{p(n+1+\alpha)}\,d\lambda(z)$$

for all k. The last integral above can be written as

$$\int_{\mathbb{B}_n}(1-|z|^2)^{p(n+1+\alpha)-(n+1)}\,dv(z),$$

which is finite because of the assumption that $p(n+1+\alpha)>n$.

Therefore, there exists a positive constant C such that

$$\int_{\mathbb{B}_n}\widetilde{\mu}(z)^p\,d\lambda(z) \le C\sum_{k=1}^{\infty}\widehat{\mu}_r(a_k)^p.$$

This combined with Lemma 12 shows that the condition $T_\mu \in S_p$ implies the condition $\widetilde{\mu}\in L^p(\mathbb{B}_n,d\lambda)$. The proof of Theorem 15 is now complete. $\square$

Once again, we recall that Theorem 15 above was shown in [5] to hold for all $p \geq 1$ as well. We have now completed the proof of our main result which was stated as Theorem 3 in the introduction.

5. Further generalizations

We can combine the main results of [5], [2], and the previous sections as follows.

Theorem 16. *Suppose μ is a positive Borel measure on $\mathbb{B}_n$, $0 < p < \infty$, and $0 < r < \infty$. Then the following conditions are equivalent.*

(a) $T_\mu \in S_p$.
(b) $\widehat{\mu}_r \in L^p(\mathbb{B}_n, d\lambda)$.
(c) $\{\widehat{\mu}_r(a_k)\} \in l^p$ *for every r-lattice* $\{a_k\}$.
(d) $\{\widehat{\mu}_r(a_k)\} \in l^p$ *for some r-lattice* $\{a_k\}$.

Moreover, if $p > n/(n+1+\alpha)$, then the above conditions are also equivalent to

(e) $\widetilde{\mu} \in L^p(\mathbb{B}_n, d\lambda)$.

Proof. Everything has been proved except conditions (c) and (d) for $p \geq 1$ in the case of higher dimensions. But this follows from exactly the same arguments used in [2] together with the high-dimensional preliminaries included in Section 2. We leave the details to the interested reader. □

Our main result remains valid for certain other domains in $\mathbb{C}^n$. In particular, our result holds for the polydisk in $\mathbb{C}^n$. We will now make this precise.

Suppose $n = n_1 + \cdots + n_m$, where each n_k is a positive integer. Let

$$\Omega = \prod_{k=1}^{m} \mathbb{B}_{n_k}$$

be the product of m unit balls. When each $n_k = 1$, the resulting domain is the polydisk in $\mathbb{C}^n$.

Suppose $\alpha_k > -1$ for each $1 \leq k \leq m$. Consider the measure

$$dv_\alpha(Z) = dv_{\alpha_1}(Z_1) \cdots dv_{\alpha_m}(Z_m),$$

where a point $Z \in \mathbb{C}^n$ is written as $Z = (Z_1, \cdots, Z_m)$, with each $Z_k \in \mathbb{C}^{n_k}$, and dv_{α_k} is the normalized volume measure on $\mathbb{B}_{n_k}$ defined in the introduction.

We define a weighted Bergman space on Ω by

$$A^2_\alpha(\Omega) = H(\Omega) \cap L^2(\Omega, dv_\alpha),$$

where $H(\Omega)$ is the space of holomorphic functions in Ω.

If μ is a finite Borel measure on Ω, then the Toeplitz operator

$$T_\mu : A^2_\alpha(\Omega) \to H(\Omega)$$

is densely defined by

$$T_\mu f(Z) = \int_\Omega K_\alpha(Z, W) f(W)\, d\mu(W),$$

where

$$K_\alpha(Z, W) = \prod_{k=1}^{m} K_{\alpha_k}(Z_k, W_k),$$

and each $K_{\alpha_k}(Z_k, W_k)$ is the reproducing kernel of $A^2_{\alpha_k}(\mathbb{B}_{n_k})$.

The averaging function $\widehat{\mu}_r$ with respect to the Bergman metric is defined exactly as before, and the Berezin transform of μ is simply defined by

$$\widetilde{\mu}(Z) = \int_\Omega \frac{|K_\alpha(Z,W)|^2}{K_\alpha(Z,Z)}\, d\mu(W), \qquad Z \in \Omega.$$

We can now state the corresponding theorems for Ω.

Theorem 17. *Suppose μ is a positive Borel measure on Ω, $p > 0$, and $r > 0$. Then:*

(a) *T_μ is bounded on $A^2_\alpha(\Omega)$ if and only if $\widehat{\mu}_r \in L^\infty(\Omega)$.*
(b) *T_μ is compact on $A^2_\alpha(\Omega)$ if and only if $\widehat{\mu}_r(Z) \to 0$ as Z approaches the full boundary of Ω.*
(c) *T_μ is in the Schatten class S_p if and only if $\widehat{\mu}_r \in L^p(\Omega, d\lambda)$, where*

$$d\lambda(Z) = \prod_{k=1}^m \frac{dv(Z_k)}{(1-|Z_k|^2)^{n_k+1}}$$

is the Möbius invariant volume measure on Ω.

Theorem 18. *Suppose μ is a positive Borel measure on Ω, $p > 0$, and $\{a_k\}$ is an r-lattice in the Bergman metric of Ω. Then:*

(a) *T_μ is bounded on $A^2_\alpha(\Omega)$ if and only if $\{\widehat{\mu}_r(a_k)\} \in l^\infty$.*
(b) *T_μ is compact on $A^2_\alpha(\Omega)$ if and only if $\{\widehat{\mu}_r(a_k)\} \in c_0$, where c_0 is the space of sequences that tend 0.*
(c) *T_μ is in the Schatten class S_p if and only if $\{\widehat{\mu}_r(a_k)\} \in l^p$.*

Conditions (a) and (b) here warrant some comments, because I believe they have not appeared explicitly before. First, if T_μ is bounded (or compact), then by [5], $\widehat{\mu}_r$ is in $L^\infty(\Omega)$ (or $\mathbb{C}_0(\Omega)$). So $\{\widehat{\mu}_r(a_k)\} \in l^\infty$ (or c_0).

On the other hand, if $\{\widehat{\mu}_r(a_k)\} \in l^\infty$ (or c_0) for some r-lattice $\{a_k\}$, then $\{\widehat{\mu}_{2r}(a_k)\} \in l^\infty$ (or c_0) as well. This is because we can find a positive integer N, indepedent of j, such that every set $D(a_j, 2r)$ can be covered by at most N of the sets in $\{D(a_k, r)\}$. If $z \in D(a_k, r)$, then $D(z, r) \subset D(a_k, 2r)$. Since $v_\alpha(D(z, r))$ is comparable to $v_\alpha(D(a_k, 2r))$ whenever $z \in D(a_k, r)$, we can find a constant $C > 0$, independent of k, such that $\widehat{\mu}_r(z) \le C\widehat{\mu}_r(a_k, 2r)$ for all k and all $z \in D(a_k, r)$. This shows that $\widehat{\mu}_r \in L^\infty(\Omega)$ (or $\mathbb{C}_0(\Omega)$).

Theorem 19. *Suppose μ is a positive Borel measure on Ω and*

$$p > \max\left(\frac{n_k}{n_k + 1 + \alpha_k} : 1 \le k \le m\right). \tag{7}$$

Then:

(a) *T_μ is bounded on $A^2_\alpha(\Omega)$ if and only if $\widetilde{\mu} \in L^\infty(\Omega)$.*
(b) *T_μ is compact on $A^2_\alpha(\Omega)$ if and only if $\widetilde{\mu}(Z) \to 0$ as Z approaches the full boundary of Ω.*
(c) *$T_\mu \in S_p$ if and only if $\widetilde{\mu} \in L^p(\Omega, d\lambda)$.*

Moreover, part (c) *above becomes false if p does not satisfy condition* (7).

The main obstacle to a generalization of our results to arbitrary bounded symmetric domains is the validity of Lemma 4, which is known to be false for general bounded symmetric domains. When $p \ge 1$, the approach taken in [5] does not involve the usage of Lemma 4. In conclusion, it is still an open problem whether part (c) of Theorems 17, 18, and 19, in the case $0 < p < 1$, can be extended to domains more general than products of balls.

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