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Classifying higher rank analytic Toeplitz algebras

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Abstract. To a higher rank directed graph (Λ, d), in the sense of Kumjian and Pask, 2000, one can associate natural noncommutative analytic Toeplitz algebras, both weakly closed and norm closed. We introduce methods for the classification of these algebras in the case of single vertex graphs.

Contents

1. Introduction

Let $\mathbb{F}_n^+$ be the free semigroup with n generators. Then the left regular representation of $\mathbb{F}_n^+$ as isometries on the Fock Space $\mathcal{H}_n = \ell^2(\mathbb{F}_n^+)$ generates an operator algebra whose closure in the weak operator topology is known as the free semigroup algebra $\mathcal{L}_n$. This algebra is the weakly closed noncommutative analytic (nonselfadjoint) Toeplitz algebra for the semigroup $\mathbb{F}_n^+$. Together with their norm closed subalgebras $\mathcal{A}_n$, the noncommutative disc algebras, they have been found to have a tractable and interesting analytic structure which extends in many ways the foundational Toeplitz algebra theory for the Hardy space $\mathcal{H}_1 = H^2$ of the unit circle. See, for example, the survey of Davidson [3], and [1], [5], [6], [7], [19], [20], [21].

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Natural generalisations of the algebra $\mathcal{L}_n$ arise on considering the Fock Space $\mathcal{H}_G$ for the discrete semigroupoid formed by the finite paths of a countable directed graph G. These free semigroupoid algebras $\mathcal{L}_G$ were considered in Kribs and Power [13] and in particular it was shown that unitarily equivalent algebras have isomorphic directed graphs. Such uniqueness was subsequently extended to other forms of isomorphism in [12] and [26]. Free semigroupoid algebras and their norm closed counterparts also provide central examples in the more general construction of H^∞-algebras and tensor algebras associated with correspondences, as developed by Muhly and Solel [17], [18]. Current themes in nonselfadjoint graph algebra analysis, embracing generalised interpolation theory, representations into nest algebras, hyper-reflexivity, and ideal structure, can be found in [8], [4], [10], [11], [14], for example.

Generalisations of the algebras $\mathcal{L}_G$ to higher rank were introduced recently in Kribs and Power [15]. Here the discrete path semigroupoid of a directed graph G is replaced by the discrete semigroupoid that is implicit in a higher rank graph (Λ, d) in the sense of Kumjian and Pask [16]. In [15] we extended the basic technique of generalised Fourier series and determined invariant subspaces, reflexivity and the graphs which yield semisimple algebras. The single vertex algebras are generated by the isometric shift operators of the left regular representation and so the associated algebras in this case are, once again, entirely natural generalised analytic Toeplitz algebras. In [26], [27] Solel has recently considered the representation theory of such higher rank analytic Toeplitz algebras and the Toeplitz algebras arising from product systems of correspondences. In particular he obtains a dilation theorem (of Ando type) for contractive representations of certain rank 2 algebras.

In the present article we introduce various methods for the classification of the higher rank analytic Toeplitz algebras $\mathcal{L}_\Lambda$ of higher rank graphs Λ. We confine attention to the fundamental context of single vertex graphs and classification up to isometric isomorphism. Along the way we consider the norm closed subalgebras $\mathcal{A}_\theta$, being higher rank generalisations of Popescu's noncommutative disc algebras $\mathcal{A}_n$, and the function algebras $A_\theta = \mathcal{A}_\theta/\mathrm{com}(\mathcal{A}_\theta)$, being the higher rank variants of Arveson's d-shift algebras. Here θ denotes either a single permutation, sufficient to encode the relations of a 2-graph, or a set of permutations in the case of a k-graph. In fact it is convenient for us to identify a single vertex higher rank graph (Λ, d) with a unital multi-graded semigroup $\mathbb{F}_\theta^+$ as specified in Definition 2.1. In the 2-graph case this is simply the semigroup with generators $e_1, \dots, e_n$ and $f_1, \dots, f_m$ subject only to the relations $e_i f_j = f_{j'} e_{i'}$ where $\theta(i,j) = (i', j')$ for a permutation θ of the nm pairs (i, j).

A useful isomorphism invariant is the Gelfand space of the quotient by the commutator ideal and we show how this is determined in terms of a complex algebraic variety V_θ associated with the set θ of relations for the semigroup $\mathbb{F}_\theta^+$. In contrast to the case of free semigroup algebras the Gelfand space is not a complete invariant and deeper methods are needed to determine the algebraic structure. Nevertheless, the geometric-holomorphic structure of the Gelfand space is useful and we make use of it to show that $\mathbb{Z}_+$-graded isomorphisms are multi-graded with respect to a natural multi-grading. (See Proposition 6.3 and Theorem 7.1) Also the Gelfand space plays a useful role in the differentiation of the 9 algebras $\mathcal{L}_\Lambda$ for the case $(n, m) = (2, 2)$. (Theorem 7.4.)

The relations for the generators can be chosen in a great many essentially different ways, as we see in Section 3. For the 2-graphs with generator multiplicity $(2,3)$ there are 84 inequivalent choices leading to distinct semigroups. Of these we identify explicitly the 14 semigroups which have relations determined by a cyclic permutation. These are the relations which impose the most constraints and so yield the smallest associated algebraic variety $V_{\min}$. In one of the main results, Theorem 7.3, we show that in the minimal variety setting the operator algebras of a single vertex graph can be classified up to isometric isomorphism in terms of product unitary equivalence of the relation set θ. For the case $(n,m) = (2,3)$ we go further and show that product unitary equivalence coincides with product conjugacy and this leads to the fact that there are 14 such algebras.

In the Section 8 we classify algebras for the single vertex 2-graphs with $(n,m) = (n,1)$. These operator algebras are identifiable with natural semicrossed products $\mathcal{L}_n \times_\theta \mathbb{Z}_+$ for a permutation action on the generators of $\mathcal{L}_n$. In this case isometric isomorphisms and automorphisms need not be multi-graded. However we are able to reduce to the graded case. We do so by constructing a counterpart to the unitary Möbius automorphism group of H^∞ and $\mathcal{L}_n$ (see [7]). In our case these automorphisms act transitively on a certain core subset of the Gelfand space.

In a recent article [22] the author and Solel have generalised this automorphism group construction to the general single vertex 2-graph case. In fact we do so for a class of operator algebras associated with more general commutation relations. As a consequence it follows that in the rank 2 case the algebras $\mathcal{A}_\theta$ (and the algebras $\mathcal{L}_\theta$) are classified up to isometric isomorphism by the product unitary equivalence class of their defining permutation.

I would like to thank Martin Cook and Gwion Evans for help in counting graphs.

2. Higher rank analytic Toeplitz algebras

Let $e_1, \dots, e_n$ and $f_1, \dots, f_m$ be sets of generators for the unital free semigroups $\mathbb{F}_n^+$ and $\mathbb{F}_m^+$ and let θ be a permutation of the set of formal products

$$\{e_i f_j : 1 \le i \le n, 1 \le j \le m\}.$$

Write $(ef)^{\mathrm{op}}$ to denote the opposite product fe and define the unital semigroup $\mathbb{F}_n^+ \times_\theta \mathbb{F}_m^+$ to be the universal semigroup with generators $e_1, \dots, e_n,\ f_1, \dots, f_m$ subject to the relations

$$e_i f_j = \big(\theta(e_i f_j)\big)^{\mathrm{op}}$$

for $1 \le i \le n$, $1 \le j \le m$. These equations are commutation relations of the form $e_i f_j = f_k e_l$. In particular, there are natural unital semigroup injections

$$\mathbb{F}_n^+ \to \mathbb{F}_n^+ \times_\theta \mathbb{F}_m^+, \quad \mathbb{F}_m^+ \to \mathbb{F}_n^+ \times_\theta \mathbb{F}_m^+,$$

and any word λ in the generators admits a unique factorisation $\lambda = w_1 w_2$ with w_1 in $\mathbb{F}_n^+$ and w_2 in $\mathbb{F}_m^+$.

This semigroup is in fact the typical semigroup that underlies a finitely generated 2-graph with a single vertex. The additional structure possessed by a 2-graph is a higher rank degree map

$$d : \mathbb{F}_n^+ \times_\theta \mathbb{F}_m^+ \to \mathbb{Z}_+^2$$

given by

$$d(w) = \big(d(w_1), d(w_2)\big)$$

where $\mathbb{Z}_+$ is the unital additive semigroup of nonnegative integers, and $d(w_i)$ is the usual degree, or length, of the word w_i. In particular if e is the unit element then $d(e) = (0, 0)$.

In a similar way we may define a class of multi-graded unital semigroups which contain the graded semigroups of higher rank graphs. Let $n = (n_1, \dots, n_r)$, $|n| = n_1 + \dots + n_r$ and let $\theta = \{\theta_{ij} : 1 \le i < j \le r\}$ be a family of permutations, where θ_{ij}, in the symmetric group $S_{n_i n_j}$, is viewed as a permutation of formal products

$$\{e_{ik}e_{jl} : 1 \le k \le n_i, 1 \le l \le n_j\}.$$

Definition 2.1. The unital semigroup $(\mathbb{F}_\theta^+, d)$ is the semigroup which is universal with respect to the unital semigroup homomorphisms

$$\phi : \mathbb{F}^+_{|\underline{n}|} \to S$$

for which $\phi(ef) = \phi(f'e')$ for all commutation relations $ef = f'e'$ of the relation set θ.

More concretely, $\mathbb{F}_\theta^+$ is simply the semigroup, with unit added, comprised of words in the generators, two words being equal if either can be obtained from the other through a finite number of applications of the commutation relations. Again, each element λ of $\mathbb{F}_\theta^+$ admits a factorisation $\lambda = w_1 w_2 \dots w_r$, with w_i in the subsemigroup $\mathbb{F}^+_{n_i}$ although, for $r \ge 3$, the factorisation need not be unique. In view of the multi-homogeneous nature of the relations it is clear that there is a natural well-defined higher rank degree map $d : \mathbb{F}_\theta^+ \to \mathbb{Z}_+^r$ associated with an ordering of the subsets of freely noncommuting generators. If uniqueness of factorisation $w = w_1 w_2 \dots w_r$ holds, with the factors ordered so that w_i is a word in $\{e_{ik} : 1 \le k \le n_i\}$, then $(\mathbb{F}_\theta^+, d)$ is equivalent to a typical finitely generated single object higher rank graph in the sense of Kumjian and Pask [16]. Although we shall not need k-graph structure theory we note the formal definition from [16] *A k-graph (Λ, d) consists of a countable small category Λ, with range and source maps r and s respectively, together with a functor $d : \Lambda \to \mathbb{Z}_+^k$ satisfying the factorization property: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{Z}_+^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$ and $d(\nu) = n$.*

It is readily seen that for $r \ge 3$ the semigroup $\mathbb{F}_\theta^+$ may fail to be cancelative and therefore may fail to have the unique factorisation property.

For a general unital countable cancelative (left and right) semigroup S we let λ be the isometry representation $\lambda : S \to B(\mathcal{H}_S)$, where each $\lambda(v)$, $v \in S$, is the left shift operator on the Hilbert space $\mathcal{H}_S$, with orthonormal basis $\{\xi_w : w \in S\}$. We write L_v for $\lambda(v)$ and so $L_v \xi_w = \xi_{vw}$ for all $w \in S$. Left cancelation in S ensures that these operators are isometries. Define the operator algebras $\mathcal{L}_S$ and $\mathcal{A}_S$ as the weak operator topology (WOT) closed and norm closed operator algebras on $\mathcal{H}_S$ generated by $\{\lambda(w) : w \in S\}$. We refer to the Hilbert space $\mathcal{H}_S$ as the Fock space of the semigroup and indeed, when $S = \mathbb{F}_n^+$ this Hilbert space is identifiable with the usual Fock space for $\mathbb{C}^n$.

Definition 2.2. Let θ be a set of permutations for which $\mathbb{F}_\theta^+$ is a cancelative (left and right) semigroup. Then the associated analytic Toeplitz algebras $\mathcal{A}_\theta$ and $\mathcal{L}_\theta$ are, respectively, the norm closed and WOT closed operator algebras generated by the left regular Fock space representation of $\mathbb{F}_\theta^+$.

In the sequel we shall be mainly concerned with the operator algebras of the single vertex 2-graphs, identified with the bigraded semigroups $(\mathbb{F}_\theta^+, d)$ for a single permutation θ. As we have remarked, these semigroups are cancelative and have the unique factorisation property. In general the multi-graded semigroups $\mathbb{F}_\theta^+$ are naturally $\mathbb{Z}_+$-graded, by total degree ($|w| = |d(w)|$) of elements, and have the further property of being generated by the unit and the elements of total degree 1. We say that a graded semigroup is 1-*generated* in this case. In general, when S is $\mathbb{Z}_+$-graded the Fock space admits an associated grading $\mathcal{H}_S = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$, where $\mathcal{H}_n$ is the closed span of the basis elements ξ_w for which w is of length n. The proof of the following proposition makes use of the block matrix structure induced by this decomposition of $\mathcal{H}$ and is similar to the proofs in [7], [13] for free semigroup and free semigroupoid algebras.

Proposition 2.3. *Let S be a unital countable graded cancelative semigroup which is* 1*-generated. If $A \in \mathcal{L}_S$ then A is the* SOT*-limit of the Cesaro sums*

$$\sum_{|w| \leq n} \Big(1 - \frac{|w|}{n}\Big) a_w L_w,$$

where $a_w = \langle A\xi_e, \xi_w \rangle$ is the coefficient of ξ_w in $A\xi_e$, and where ξ_e is the vacuum vector for the unit of S.

It follows that the nonunital WOT-closed ideal $\mathcal{L}_\theta^0$ generated by the L_w for which $|w| = 1$ is the subspace of operators A whose first coefficient vanishes, that is, $\mathcal{L}_\theta^0 = \{A : \langle A\xi_e, \xi_e \rangle = 0\}$.

One can check that the fact that S is 1-generated implies that for $|w| = 1$ the right shifts R_w, defined in the natural way, satisfy $E_{n+1}R_w = R_w E_n$ where E_n is the projection onto $\mathcal{H}_n$. A consequence of this is that the proofs of the following facts can be obtained using essentially the same proofs as in [7], [15]. We write $\mathcal{R}_S$ for the WOT closed operator algebra generated by the right representation on Fock space.

Proposition 2.4. *Let S be a countable graded cancelative semigroup which is* 1*-generated. Then:*

(i) *The commutant of $\mathfrak{L}_S$ is $\mathfrak{R}_S$.*
(ii) *The commutant of $\mathfrak{R}_S$ is $\mathfrak{L}_S$.*
(iii) *$\mathfrak{R}_S$ is unitarily equivalent to $\mathfrak{L}_{S^{\mathrm{op}}}$ where S^{op} is the opposite semigroup of S.*

Remark. The Fourier series representation of operators in $\mathcal{A}_S$ and $\mathcal{L}_S$ is analogous to similar expansions which are well-known for operators in the free group von Neumann algebra $\mathrm{vN}(\mathbb{F}_n)$ and the reduced free group C*-algebra $C^*_{\mathrm{red}}(\mathbb{F}_n)$. These selfadjoint algebras are the operator algebras generated by the left regular *unitary* representation λ of $\mathbb{F}_n$ on the big Fock space $\ell^2(\mathbb{F}_n)$. We can define the subalgebras $\widetilde{\mathcal{L}}_n$ and $\widetilde{\mathcal{A}}_n$ to be the associated nonselfadjoint operator subalgebras on this Fock space generated by the generators of the semigroup $\mathbb{F}_n^+$ of $\mathbb{F}_n$. Observe however that these algebras are generating subalgebras of the II_1 factor $\mathrm{vN}(\mathbb{F}_n)$ and the finite simple C*-algebra $C^*_{\mathrm{red}}(\mathbb{F}_n)$, while $\mathrm{vN}(\mathcal{L}_n) = \mathcal{L}(\mathcal{H}_n)$ and $C^*(\mathcal{A}_n)$ is an extension of O_n by the compact operators.

3. k-graphs, cycle diagrams and algebraic varieties

A single vertex 2-graph is determined by a pair (n, m), indicating the generator multiplicities, and a single permutation θ in S_{nm}. We shall systematically identify a 2-graph with its unital multi-graded semigroup $\mathbb{F}_\theta^+$. Let us say, if $n \neq m$, that two such permutations θ and τ are *product conjugate* if $\theta = \sigma\tau\sigma^{-1}$ where σ lies in the product subgroup $S_n \times S_m$. In this case the discrete semigroups $\mathbb{F}_n^+ \times_\theta \mathbb{F}_m^+$ and $\mathbb{F}_n^+ \times_\tau \mathbb{F}_m^+$ are isomorphic and it is elementary that there is a unitary equivalence between $\mathcal{L}_\theta$ and $\mathcal{L}_\tau$. Thus, in considering the diversity of isomorphism types we need only consider permutations up to product conjugacy.

The product conjugacy classes can be indicated by a list of representative permutations $\{\theta_1, \dots, \theta_r\}$ each of which may be indicated by an $n \times m$ *directed cycle diagram* which reveals the cycle structure relative to the product structure. For example the permutation $(((11), (12), (21)), ((13), (23)))$ in S_6 is shown in the diagram in Figure 1, where here we have chosen product coordinates (ij) for the cell in the i^{th} row and the j^{th} column. Also, in the next section we obtain cycle diagrams for the 14 product conjugacy classes of the pure cycle permutations.

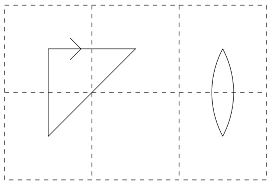

Figure 1. Directed cycle diagram.

For $(n, m) = (2, 2)$ examination reveals that there are nine such classes of permutations which yield distinct semigroups (as ungraded semigroups). In the fourth diagram of Figure 2 the triangular cycle has anticlockwise and clockwise orientations, θ_4^a, θ_4^c say, which, unlike the other 7 permutation, give nonisomorphic semigroups.

For 2-graphs with $n \neq m$ the product conjugacy class of θ gives a complete isomorphism invariant for the isomorphism type of the semigroup. The number of such isomorphism types, $O(n, m)$ say, may be computed using Frobenius' formula for the number of orbits of a group action, as we show below. Note that $O(n, m)$ increases rapidly with n, m; a convenient lower bound, for $n \neq m$, is $\frac{nm!}{(n!m!)}$. For small values of n, m we can calculate (see below) the values summarised in the following proposition.

Proposition 3.1. *Let $O(n, m)$ be the number of 2-graphs (Λ, d) with a single vertex, where $d^{-1}((1, 0)) = n, d^{-1}((0, 1)) = m$. Then*

$$O(2, 2) = 9, \quad O(2, 3) = 84, \quad \textit{and} \quad O(3, 4) = 3,333,212.$$

Let θ be a cancelative permutation set for $\underline{n} = (n_1, \dots, n_r)$. We now associate with $\mathbb{F}_\theta^+$ a complex algebraic variety which will feature in the description of the Gelfand space of $\mathcal{A}_\theta$.

For $1 \leq i \leq r$, let $z_{i,1}, \dots, z_{i,n_i}$ be the coordinate variables for $\mathbb{C}^{n_i}$ so that there is a natural bijective correspondence $e_{i,k} \to z_{i,k}$ between edges and variables. Define

$$V_\theta \subseteq \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r}$$

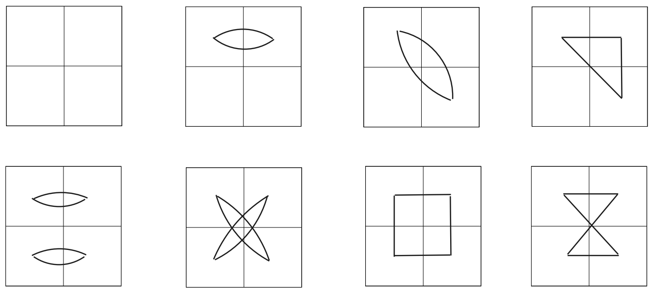

FIGURE 2. Undirected diagrams for $(n,m)=(2,2)$

to be the complex algebraic variety determined by the equation set

$$\hat{\theta}=\left\{z_{i,p}z_{j,q}-\hat{\theta}_{i,j}(z_{i,p}z_{j,q}) : 1\le p\le n_i,\ 1\le q\le n_j,\ 1\le i<j\le r\right\}$$

where $\hat{\theta}_{i,j}$ is the permutation induced by $\theta_{i,j}$ and the bijective correspondence.

Let us identify these varieties in the case of the 2-graphs with $(n,m)=(2,2)$. Let $\theta_1,\theta_2,\theta_3,\theta_4^a,\theta_4^c,\theta_5,\dots,\theta_8$ be the nine associated permutations and let z_1,z_2,w_1,w_2 be the coordinates for $\mathbb{C}^2\times\mathbb{C}^2$. The variety V_{θ_1} for the identity permutation θ_1 is $\mathbb{C}^2\times\mathbb{C}^2$. The 4-cycles θ_7 and θ_8 have the same equation set, namely, $z_1w_1=z_1w_2=z_2w_1=z_2w_2$, and so have the same variety, namely

$$(\mathbb{C}^2\times\{0\})\cup(\{0\}\times\mathbb{C}^2)\cup(E_2\times E_2)$$

where we write $E_n\subseteq\mathbb{C}^n$ for the 1-dimensional "diagonal variety" $z_1=z_2=\cdots=z_n$. In fact, in the general rank 2 setting the variety V_θ for any element θ in S_{nm} contains the subset

$$V_{\min}=(\mathbb{C}^n\times\{0\})\cup(\{0\}\times\mathbb{C}^m)\cup(E_n\times E_m).$$

Also from the irredundancy in each equation set θ it follows that $V_\theta=V_{\min}$ if and only if θ is a pure cycle.

The variety V_{θ_2} for the second cycle diagram is determined by the equations $z_1(w_1-w_2)=0$ and so

$$V_{\theta_2}=(\mathbb{C}^2\times E_2)\cup((\{0\}\times\mathbb{C})\times\mathbb{C}^2),$$

whereas V_{θ_5} is determined by $z_1(w_1-w_2)=0$ and $z_2(w_1-w_2)=0$ and so

$$V_{\theta_5}=(\mathbb{C}^2\times E_2)\cup(\{0\}\times\mathbb{C}^2).$$

The variety $V_{\theta_3}=V(z_1w_1-z_2w_2)$ is irreducible, while θ_4^a and θ_4^c have the same variety

$$V_{\theta_2}\cap V_{\theta_3}=V_{\min}\cup(\mathbb{C}_{z_2}\times\mathbb{C}_{w_1}).$$

Finally,

$$V_{\theta_6}=V(z_1w_1-z_2w_2,z_1w_2-z_2w_1)=V_{\min}\cup(V(z_1+z_2)\times V(w_1+w_2)).$$

There are similar such diagrams and identifications for small higher rank graphs and semigroups $\mathbb{F}_\theta^+$ defined by permutation sets. For example, in the rank 3 case with multiplicities $(n,m,l) = (2,2,2)$ one has generators $e_1, e_2, f_1, f_2, g_1, g_2$ with three 2×2 cycle diagrams for three permutations $\theta_{ef}, \theta_{fg}, \theta_{eg}$ in S_4. Here, $\theta = \{\theta_{ef}, \theta_{fg}, \theta_{eg}\}$. The permutations define equations in the complex variables $z_1, z_2, w_1, w_2, u_1, u_2$ giving in turn a complex algebraic variety in $\mathbb{C}^6$. Once again, in the rank k case a minimal complex algebraic variety $V_{\min}$ arises when the equation set is maximal and this occurs when each of the $k(k-1)/2$ permutations in the set θ is a pure cycle of maximum order;

$$V_{\min} = \left(\cup_{j=1}^{k} (\mathbb{C}^{n_j} \times \{0\}\right) \cup (E_{n_1} \times \cdots \times E_{n_k}).$$

There is a feature of the varieties V_θ that we will find useful in the proof of Proposition 6.3 which follows from the homogeneity of the complex variable equations, namely, the cylindrical property that if $z = (z_1, \ldots, z_k)$ is a point in $\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$ which lies in V_θ then so too does $(\lambda_1 z_1, \ldots, \lambda_k z_k)$ for all λ_i in $\mathbb{C}$.

4. Small 2-graphs

For $(n,m) = (2,3)$ there are 84 classes of 2-graph semigroups $\mathbb{F}_\theta^+ = \mathbb{F}_2^+ \times_\theta \mathbb{F}_3^+$. To see this requires computing the number of orbits for the action of $H = S_n \times S_m$ on S_{mn} given by $\alpha_h : g \to hgh^{-1}$. If $\mathrm{Fix}(\alpha_h)$ denotes the fixed point set for α_h then by Frobenius' formula the number of orbits is given by

$$O(n,m) = \frac{1}{|H|} \sum_{h \in H} |\mathrm{Fix}(\alpha_h)| = \frac{1}{|H|} \sum_{h \in H} |C_{S_{mn}}(h)|$$

where $C_{S_{mn}}(h)$ is the centraliser of h in S_{mn}. Suppose that the permutation h has cycles of distinct lengths $a_1, a_2, \ldots, a_t$ and that there are n_i cycles of type a_i. Note that h is conjugate to h' in S_n if and only if they have the same cycle type and so the size of the conjugacy class of h is $n!/(a_1^{n_1} a_2^{n_2} \ldots a_t^{n_t} n_1! n_2! \cdots n_t!)$. To see this consider a fixed partition of positions $1, \ldots, n$ into intervals of the specified cycle lengths. There are $n!$ occupations of these positions and repetitions of a particular permutation occur through permuting equal length intervals (which gives $n_1! n_2! \cdots n_t!$ repetitions) and cycling within intervals (a_i repetitions for each cycle of length a_i). We infer next that the centraliser of h has cardinality

$$|C_{S_{mn}}(h)| = a_1^{n_1} a_2^{n_2} \ldots a_t^{n_t} n_1! n_2! \cdots n_t!$$

In the case of $H = S_2 \times S_3$ an examination of the 12 elements h shows that the cycle types are 1^6, 6^1 (for two elements), 2^3 (for four elements), 3^2 (for two elements) and $2^2 1^2$ (for three). Thus

$$O(2,3) = \frac{1}{2!3!}(6! + 2.6 + 4.8.3! + 2.9.2! + 3.4.2!2!) = 84.$$

In a similar way, with some computer assistance, one can compute that $O(3,4) = 3,333,212$.

We now determine the 2-graphs with $(n,m) = (2,3)$ which have minimal complex variety $V_{\min}$. These are the 2-graphs which have cyclic relations, in the sense that the relations are determined by a permutation θ which is a cycle of order 6. One can use the Frobenius formula or computer checking to determine that there are

14 such classes. However for these small 2-graphs we prefer to determine these classes explicitly through their various properties as this reveals interesting detail of symmetry and antisymmetry.

Proposition 4.1. *There are* 14 2*-graphs of multiplicity type* $(2,3)$ *whose relations are of cyclic type. Representative cycle diagrams for these classes are given in Figures* 3–7.

Proof. Label the cells of the 2×3 rectangle as

1	2	3
4	5	6

Replacing θ by an $S_2 \times S_3$-conjugate we may assume that $\theta(1) = 2$ or $\theta(1) = 5$ or $\theta(1) = 4$. Note that $S_2 \times S_3$ conjugacy preserves the following properties of a cell diagram and that these numerical quantities are useful invariants; the number $h(\theta)$ of horizontal edges, the number $r(\theta)$ of right angles and the number of $v(\theta)$ of vertical edges.

Suppose first that $\theta(1) = 5$ and that $h(\theta) = 0$. Then it is easy to see that there are at most three possible product conjugacy classes; representative cycle diagrams and permutations θ_1, θ_2 and θ_3 are given in Figure 3. We remark that θ_1 and θ_2 have cyclic symmetry and that θ_i and θ_i^{-1} are product conjugate for $i = 1, 2, 3$.

Suppose next that $\theta(1) = 2$ and that there are no diagonal edges (that is, $h(\theta) + v(\theta) = 6$). There are only two possible diagrams, namely the two oriented rectangular cycles, and these are product conjugate, giving a single conjugacy class with representative $\theta_4 = (1\,2\,3\,6\,5\,4)$.

Consider now the remaining classes. Their elements have diagrams which have at least one horizontal and one diagonal edge. We consider first those that do not contain, up to conjugacy, the directed "angular" subgraph, $1 \to 2 \to 4$. Successive examination of the graphs containing $1 \to 2 \to 5, 1 \to 2 \to 6$ and $1 \to 2 \to 3$ shows that, on discarding some obvious conjugates, that there are at most 4 such classes with the representatives $\theta_5, \dots, \theta_8$ given below. Note that θ_7 has horizontal (up-down) symmetry and in fact of the 14 classes it can be seen that only θ_1 and θ_7 have this property.

Finally one can check similarly that there are at most 6 classes with diagrams that do contain the angular subgraph, with representatives $\theta_9, \dots \theta_{14}$.

That these 14 classes really are distinct can be confirmed by considering the invariants for $h(\theta), r(\theta), v(\theta)$ in Table 1.

The table also helps in identifying the possibilities for the class of the inverse permutation. The three permutations $\theta_7, \theta_8, \theta_{12}$ have the same invariants. However θ_7 and θ_8 are not conjugate since the former has its horizontal edges in opposing pairs whilst the latter does not and this property is plainly an $S_2 \times S_3$ conjugacy invariant. Also θ_{12} is conjugate to neither θ_7 or θ_8 by the angular subgraph distinction. We note that θ_7 is self-conjugate while θ_8 is conjugate to θ_{12}^{-1}. Finally, the pair θ_{11} and θ_{13} have the same data but it is an elementary exercise to see that they are not conjugate.

It follows that there are exactly 14 classes, ten of which are conjugate to their inverses, while θ_8 is conjugate to θ_{12}^{-1} and θ_{11} is conjugate to θ_{13}^{-1}. $\square$

TABLE 1. Invariants for $h(\theta), r(\theta), v(\theta)$

	$h(\theta)$	$r(\theta)$	$v(\theta)$
θ_1	0	0	0
θ_2	0	0	3
θ_3	0	0	2
θ_4	4	4	2
θ_5	2	4	3
θ_6	2	2	2
θ_7	4	0	0
θ_8	4	0	0
θ_9	2	0	1
θ_{10}	4	2	1
θ_{11}	2	1	1
θ_{12}	4	0	0
θ_{13}	2	1	1
θ_{14}	2	0	0

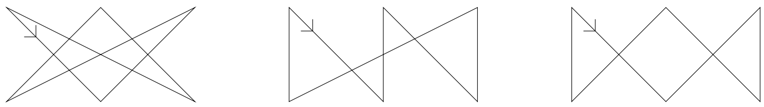

FIGURE 3. $\theta_1, \theta_2, \theta_3$.

FIGURE 4. θ_4 and θ_5 .

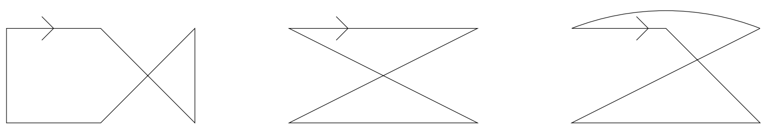

FIGURE 5. θ_6, θ_7 and θ_8.

Product equivalence. We shall meet product unitary equivalence of permutations in Theorem 5.1. Here we show how in a special case product unitary equivalence is the same relation as product conjugacy.

Consider the natural representations $\pi : S_n \to M_n(\mathbb{C})$ for which $\pi(\sigma)(e_i) = e_{\sigma(i)}$ with respect to the standard basis. Identifying $M_{nm}(\mathbb{C})$ with $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ we realise $S_n \times S_m$ as a permutation group of unitaries forming a unitary subgroup of S_{nm}. Here a permutation is viewed as a permutation of the product set

$$\{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m\}$$

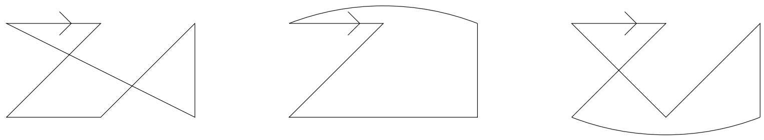

FIGURE 6. θ_9, θ_{10} and θ_{11}.

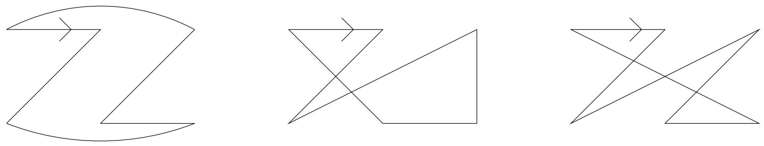

FIGURE 7. θ_{12}, θ_{13} and θ_{14}.

and $\pi(\theta)e_{ij} = e_{\theta(ij)}$. We say that θ_1, θ_2 in S_{nm} are *product similar* (resp. *product equivalent*) if in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ the operators $\pi(\theta_1)$ and $\pi(\theta_2)$ are similar by an invertible (resp. unitary) elementary tensor $A \otimes B$. On the other hand recall that if $n \neq m$ then θ_1 and θ_2 are *product conjugate* if $\sigma\theta_1\sigma^{-1} = \theta_2$ for some element σ in $S_m \times S_n$.

We now show for $(n, m) = (2, 3)$ that two cyclic permutations of order 6 are product unitarily equivalent, relative to $S_2 \times S_3$, if and only if they are product conjugate.

For $\theta \in S_6$ and the 2×3 complex matrix

$$C = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix}$$

define $\theta[C]$ to be the permuted 2×3 matrix

$$\theta[C] = \begin{bmatrix} c_{\theta(1)} & c_{\theta(2)} & c_{\theta(3)} \\ c_{\theta(4)} & c_{\theta(5)} & c_{\theta(6)} \end{bmatrix}.$$

Note that if $\theta \in S_2 \times S_3$ and C has rank 1 then $\theta^k[C]$ has rank 1 for each k.

Lemma 4.2. *Let C be a 2×3 matrix of rank 1 such that at least two of the entries are nonzero and not all entries are equal. Suppose that $\theta \in S_6$ is a cyclic permutation of order 6 such that $\theta^k[C]$ has rank 1 for $k = 1, \dots, 5$. Then one of the following four possibilities holds:*

(i) *θ is product conjugate to θ_1, in which case C can be arbitrary.*
(ii) *θ is product conjugate to one of the (up-down alternating) permutations θ_2, θ_3, in which case C either has a zero row or the rows of C each have 3 equal entries.*
(iii) *θ is product conjugate to the rectangular permutation θ_4, in which case C has exactly two nonzero entries in consecutive locations for the cycle θ.*
(iv) *θ is product conjugate to θ_7, in which case the two rows of C are equal.*

Proof. It is clear that each of the four possibilities can occur. Since we have determined all the conjugacy classes we can complete the proof by checking that if C is any nontrivial rank one matrix, as specified, then each of the permutations θ_5,

$\theta_6, \theta_8, \theta_9, \theta_{10}, \theta_{11}, \theta_{12}, \theta_{13}, \theta_{14}$ fails to create an orbit $\theta^k[C], k = 1, \dots, 5$ consisting of rank 1 matrices.

One can assume that the matrix C has the form

$$\begin{bmatrix} 1 & x & y \\ a & ax & ay \end{bmatrix}.$$

Also, for each of the 9 permutations one can quickly see that there are no solutions for which C has only two nonzero entries, since these entries are put into off-diagonal position by some matrix $\theta^k[C]$. Also there is no solution with $a = 0$ for any such θ. It is then a routine matter to check that for each of the 9 only the excluded case $x = y = a = 1$ is possible, completing the proof. □

Proposition 4.3. *Let $\theta = \theta_i$, $\tau = \theta_j$, with $i \neq j$, $1 \leq i, j \leq 16$. Then θ and τ are not product unitary equivalent.*

Proof. Let $A \in M_2(\mathbb{C})$, $B \in M_3(\mathbb{C})$ be unitary matrices with

$$A \otimes B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} = \left[\begin{array}{ccc|ccc} ar & as & at & br & bs & bt \\ au & av & aw & bu & bv & bw \\ ax & ay & az & bx & by & bz \\ \hline cr & cs & ct & dr & ds & dt \\ cu & cv & cw & du & dv & dw \\ cx & cy & cz & dx & dy & dz \end{array}\right].$$

Suppose that, writing τ for $\pi(\tau)$ etc., we have the intertwining relation, $\tau(A \otimes B) = (A \otimes B)\theta$. We may assume that θ is not conjugate to θ_1. Note that the product $X = \tau(A \otimes B)$, like $A \otimes B$, has the following rank 1 row property, namely, for each row $(x_{i1}, x_{i2}, \dots, x_{i6})$ the associated 2×3 matrix

$$\begin{bmatrix} x_{i1} & x_{i2} & x_{i3} \\ x_{i4} & x_{i5} & x_{i6} \end{bmatrix}$$

is of rank 1. Thus the matrix equation entails that $(A \otimes B)\theta$ has the rank 1 row property, which is to say, in particular, that if C is the rank one matrix

$$C = \begin{bmatrix} ar & as & at \\ br & bs & bt \end{bmatrix}$$

obtained from the first row of $A \otimes B$ then $\theta[C]$ is of rank 1. Similarly, from the intertwining equations $\tau^k(A \otimes B) = (A \otimes B)\theta^k$ we see that $\theta^k[C]$ has rank 1 for $k = 1, \dots, 5$.

Since A and B are unitary we may choose a row of $A \otimes B$, instead of the first row as above, to arrange that $a \neq b$ and that r, s, t are not equal. So we may assume that these conditions hold. If $a \neq 0$ and $b \neq 0$ then the lemma applies and θ is conjugate to θ_1, contrary to our assumption. If $a \neq 0$ and $b = 0$ and two of r, s, t are nonzero then the lemma applies and θ is conjugate to θ_2 or to θ_3. We return to this situation in a moment. First note that the remaining cases not covered are where A and B each have one nonzero unimodular entry in each row, which is to say that apart from a diagonal matrix multiplier, $A \otimes B$ is a permutation matrix in $S_2 \times S_3$. This entails that τ is actually product conjugate to θ_1, contrary to our assumption.

It remains then to show that no two of θ_1, θ_2, θ_3 are unitarily equivalent by an elementary tensor of the form $D \otimes B$ where D, B are unitary and D has two zero entries. Note that $\theta_1 = \sigma_1^{-1}\theta_3\sigma_1$ where $\sigma_1 = (13)$ and $\theta_2 = \sigma_1^{-1}\theta_3\sigma_2$ where $\sigma_2 = (23)$. Suppose that $\theta_1(D\otimes B) = (D\otimes B)\theta_3$. Then $\theta_3\sigma_1(D\otimes B) = \sigma_1(D\otimes B)\theta_3$. However the commutant of θ_3 is the algebra generated by

$$\theta_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

which consists of matrices of the form

$$z = \left[\begin{array}{ccc|ccc} a & b & c & e & f & d \\ c & a & b & f & d & e \\ b & c & a & d & e & f \\ \hline f & e & d & a & c & b \\ e & d & f & b & a & c \\ d & f & e & c & b & a \end{array}\right].$$

On the other hand $\sigma_1(D \otimes B)$ has one of the forms

$$\left[\begin{array}{c|c} \sigma X & 0 \\ \hline 0 & \lambda X \end{array}\right] \qquad \left[\begin{array}{c|c} 0 & \sigma X \\ \hline \lambda X & 0 \end{array}\right]$$

where X is a unitary in $M_3(\mathbb{C})$, $|\lambda| = 1$ and $\sigma \in S_3$ is the unitary permutation matrix for $\sigma = (13)$. The equation $Z = \sigma_1(D \otimes B)$, in the former case, entails

$$\begin{bmatrix} b & c & a \\ c & a & b \\ a & b & c \end{bmatrix} = \lambda \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}.$$

It follows that $\lambda = 1$ and $a = b = c$, which is a contradiction. The other cases are similar. $\square$

5. Graded isomorphisms

We now consider some purely algebraic aspects of graded isomorphisms between higher rank graded semigroup algebras. The equivalences given here play an important role in the classifications of Section 7 and provide a bridge between the operator algebra level and the k-graph level.

Let $\mathbb{C}[\mathbb{F}_n^+ \times_\theta \mathbb{F}_m^+]$ be the complex semigroup algebra for the discrete semigroup $\mathbb{F}_n^+ \times_\theta \mathbb{F}_m^+$ given earlier, where $\theta \in S_{nm}$. We say that an algebra homomorphism $\Phi : \mathbb{C}[\mathbb{F}_n^+ \times_\theta \mathbb{F}_m^+] \to \mathbb{C}[\mathbb{F}_n^+ \times_\tau \mathbb{F}_m^+]$ is *bigraded* if it is determined by linear equations

$$\Phi(e_i) = \sum_{j=1}^{n} a_{ij} e_j, \quad \Phi(f_k) = \sum_{l=1}^{n} b_{kl} f_l,$$

where $\{e_j\}, \{f_k\}$ denote generators, as before, in both the domain and codomain. Furthermore we say that $\Phi = \Phi_{A,B}$ is a *bigraded isomorphism* if $A = (a_{ij})$ and $B = (b_{kl})$ are invertible matrices and that Φ is a *bigraded unitary equivalence* if A and B can be chosen to be unitary matrices. For definiteness we take a strict form of definition in that we assume an order for the two sets of generators is given.

Let us also specify some natural companion algebras which are quotients of the higher rank complex semigroup algebras corresponding to partial abelianisation. Let $\mathbb{C}[z], \mathbb{C}[w]$ be complex multivariable commutative polynomial algebras, where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_m)$, and let θ be a permutation in S_{nm} viewed also as a permutation of the formal products

$$\{z_i w_j : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Thus, if $\theta((i,j)) = (k,l)$ then $\theta(z_i w_j) = z_k w_l$. Define $\mathbb{C}[z, w; \theta]$ to be the complex algebra with these commuting generators $\{z_i\}$, $\{w_k\}$ subject to the relations

$$z_i w_j = \big(\theta(z_i w_j)\big)^{\mathrm{op}}$$

for all i, j. This noncommutative algebra is the quotient of $\mathbb{C}[\mathbb{F}_n^+ \times_\theta \mathbb{F}_m^+]$ by the ideal which is generated by the commutators of the generators of $\mathbb{F}_n^+$ and the commutators of the generators of $\mathbb{F}_m^+$.

It is convenient now to identify $\mathbb{C}[\mathbb{F}_n^+]$ with the tensor algebra for $\mathbb{C}^n$ by means of the identification of words $w_1(e) = e_{i_1} e_{i_2} \ldots e_{i_p}$ in the generators with basis elements $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p}$ of $(\mathbb{C}^n)^{\otimes p}$. Similarly we identify words $w = w_1(e) w_2(f)$ of degree (p,q) in $\mathbb{F}_n^+ \times_\theta \mathbb{F}_m^+$, in their standard factored form, with basis elements

$$(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p}) \otimes (f_{j_1} \otimes f_{j_2} \otimes \cdots \otimes f_{j_q})$$

in $(\mathbb{C}^n)^{\otimes p} \otimes (C^m)^{\otimes q}$. A bigraded isomorphism $\Phi_{A,B}$ now takes the explicit form

$$\Phi_{A,B} = \sum_{(p,q) \in \mathbb{Z}_+^2} (A^{\otimes p}) \otimes (B^{\otimes q}).$$

Likewise, the symmetrised semigroup algebras $\mathbb{C}[z, w; \theta]$ and their bigraded isomorphisms admit symmetric joint tensor algebra presentations.

Theorem 5.1. *The following assertions are equivalent for permutations θ_1, θ_2 in S_{nm}:*

(i) *The complex semigroup algebras $\mathbb{C}[\mathbb{F}_n^+ \times_{\theta_1} \mathbb{F}_m^+]$ and $\mathbb{C}[\mathbb{F}_n^+ \times_{\theta_2} \mathbb{F}_m^+]$ are bigradedly isomorphic* (*resp. bigradedly unitarily equivalent*).

(ii) *The complex algebras $\mathbb{C}[z, w; \theta_1]$ and $\mathbb{C}[z, w; \theta_2]$ are bigradedly isomorphic* (*resp. bigradedly unitarily equivalent*).

(iii) *The permutations θ_1 and θ_2 are product similar* (*resp. product unitarily equivalent*), *that is, there exist matrices A, B such that*

$$\pi(\theta_1)(A \otimes B) = (A \otimes B)\pi(\theta_2)$$

where $A \in M_n(\mathbb{C}), B \in M_m(\mathbb{C})$ are invertible (*resp. unitary*).

Proof. Let us show first that (ii) implies (iii). Let

$$\Phi : \mathbb{C}[z, w; \theta_1] \to \mathbb{C}[z, w, ; \theta_2]$$

be a bigraded isomorphism determined by invertible matrices

$$A = (a_{ij}), B = (b_{kl}).$$

Introduce the notation

$$\theta_1(z_i w_k) = z_\sigma w_\tau, \quad \theta_2(z_i w_k) = z_\lambda w_\mu$$

where

$$\sigma = \sigma(ik), \tau = \tau(ik), \lambda = \lambda(ik), \mu = \mu(ik)$$

are the functions from $\{ik\}$ to $\{i\}$ and to $\{k\}$ which are determined by θ_1 and θ_2. That is

$$\theta_1((i,k)) = (\sigma(ik), \tau(ik)),\ \ \theta_2((i,k) = (\lambda(ik), \mu(ik)).$$

Since Φ is an algebra homomorphism we have

$$\Phi(z_i w_k) = \Phi(z_i)\Phi(w_k) = \left(\sum_{j=1}^{n} a_{ij} z_j\right)\left(\sum_{l=1}^{m} b_{kl}\, w_l\right) = \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ij} b_{kl}\, z_j\, w_l$$

and, similarly,

$$\Phi(w_\tau z_\sigma) = \Phi(w_\tau)\Phi(z_\sigma) = \left(\sum_{j=1}^{m} b_{\tau l} w_l\right)\left(\sum_{j=1}^{n} a_{\sigma,j} z_j\right) = \sum_{j=1}^{n}\sum_{l=1}^{m} a_{\sigma,j} b_{\tau l} w_l\, z_j.$$

Since

$$z_i w_k = (\theta_1(z_i w_k))^{\mathrm{op}} = (z_\sigma w_\tau)^{\mathrm{op}} = w_\tau z_\sigma$$

it follows that the left-hand sides of these expressions are equal. The set $\{z_j w_l\}$ is linearly independent and so $a_{ij}\, b_{kl}$, the coefficient of $z_j w_l$ in the first expression, is equal to the coefficient of $z_j w_l$ in the second expression. Since

$$z_j w_l = (\theta_2(z_j w_l))^{\mathrm{op}} = (z_\lambda w_\mu)^{\mathrm{op}} = w_\mu z_\lambda$$

we have

$$a_{ij}\, b_{kl} = a_{\sigma(ik),\lambda(jl)} b_{\tau(ik),\mu(jl)}$$

for all appropriate i, j, k, l. This set of equations is expressible in matrix terms as

$$A \otimes B = \pi(\theta_1^{-1})(A \otimes B)\pi(\theta_2)$$

and so $A \otimes B$ gives the desired product similarity between $\pi(\theta_1)$ and $\pi(\theta_2)$. The unitary equivalence case is identical.

We show next that the single tensor condition of (iii) is enough to ensure that the linear map $\Phi = \Phi_{A,B}$, when *defined* by the multiple tensor formula is indeed an algebra homomorphism.

Note first that the equality $\Phi(w_1(e)aw_2(f)) = \Phi(w_1(e))\Phi(a)\Phi(w_2(f))$ is elementary. It will suffice therefore to show that $\Phi(w_1(f)w_2(e)) = \Phi(w_1(f))\Phi(w_2(e))$. However the calculation above shows that the equality follows from the single tensor condition when w_1 and w_2 are single letter words. Combining these two principles we obtain the equality in general. Thus $\Phi(f_i e_j e_k) = \Phi(e_p f_q e_k) = \Phi(e_p)\Phi(f_q e_k) = \Phi(e_p)\Phi(f_q)\Phi(e_k) = \Phi(e_p f_q)\Phi(e_k) = \Phi(f_i e_j)\Phi(e_t)$ and in this manner we obtain the equality when the total word length is three, and simple induction completes the proof. □

The arguments above apply to the higher rank setting, with only notational accommodation, to yield the following.

Theorem 5.2. *Let* $\theta = \{\theta_{i,j}; 1 \le i < j \le r\}$, $\tau = \{\tau_{i,j}; 1 \le i < j \le r\}$ *be cancelative permutation sets for the* r*-tuple* $\underline{n} = (n_1, \dots, n_r)$. *Then the following statements are equivalent:*

(i) *There are unitary matrices* $A_i = \left(a_{pq}^{(i)}\right)$ *in* $M_{n_i}(\mathbb{C}), 1 \leq i \leq r$, *and a graded algebra isomorphism* $\Phi : \mathbb{C}[\mathbb{F}_\theta^+] \to \mathbb{C}[\mathbb{F}_\tau^+]$ *for which, for each* i,

$$\Phi(e_{ip}) = \sum_{q=1}^{n_i} a_{pq}^{(i)} e_{iq}.$$

(ii) *There are unitary matrices as in* (i) *that implement the product unitary equivalences*

$$\pi(\theta_{ij}) = (A_i \otimes A_j)\pi(\tau_{ij})(A_i \otimes A_j)^{-1}.$$

6. Gelfand spaces

Let θ be a permutation set for which $\mathbb{F}_\theta^+$ is cancelative. In the rank one free semigroup case the noncommutative polynomial ring $\mathbb{C}[\mathbb{F}_n^+]$ has abelian quotient equal to the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$. Similarly the semigroup ring $\mathbb{C}[\mathbb{F}_\theta^+]$ has abelianisation

$$\mathbb{C}[z_{1,1}, \ldots, z_{1,n_1}, z_{2,1}, \ldots\ldots, z_{k,n_k}]/I_\theta$$

where I_θ is the ideal determined by the associated equation set $\hat{\theta}$. It follows that each point α of V_θ gives rise to a complex algebra homomorphism $\hat{\alpha} : \mathbb{C}[\mathbb{F}_\theta^+] \to \mathbb{C}$ and all such homomorphisms arise this way. In particular, for each word w in $\mathbb{F}_\theta^+$ with arbitrary factorisation $w_1 \ldots w_r$ the product $\hat{\alpha}(w_1) \ldots \hat{\alpha}(w_s)$ agrees with $\hat{\alpha}(w)$.

We now identify the set of complex homomorphisms for the nonselfadjoint Toeplitz algebra $\mathcal{A}_\theta$ and hence the Gelfand spaces of the abelian quotients.

Let us first recall the function algebra implicit in Arveson's analysis of row contractions and the d-shift [2]. This is a function algebra on the unit ball $\overline{\mathbb{B}}_d$ obtained by completing the algebra of polynomials $p(z)$ with respect to the large norm

$$\|p(z)\|_a = \|p(S_1, \ldots, S_d)\|$$

where $[S_1, \ldots, S_d]$ is the d-shift, the row contraction arising from the coordinate shift operators on the symmetric Fock space of $\mathbb{C}^d$. These coordinate shifts are weighted shifts for which $S_1S_1^* + \cdots + S_dS_d^*$ is the projection onto the constant functions. Let us simply write A_d for this algebra which we refer to as the d-shift algebra. It can be shown readily that A_d is naturally isometrically isomorphic to the quotient algebra $\mathcal{A}_d/\mathrm{com}(\mathcal{A}_d)$ where $\mathcal{A}_d$ is the noncommutative disc algebra for $\mathbb{F}_d^+$ and for our present purposes we take this perspective.

Definition 6.1. Let θ be a cancelative permutation set for $\underline{n} = (n_1, \ldots, n_k)$ with norm closed analytic Toeplitz algebra $\mathcal{A}_\theta$. Then the higher rank d-shift algebra, or Arveson algebra, for θ is the commutative Banach algebra $A_\theta = \mathcal{A}_\theta/\mathrm{com}(\mathcal{A}_\theta)$, viewed as a function algebra on Ω_θ.

Let $S = \mathbb{F}_\theta^+$ and let $\alpha \in V_\theta \cap \mathbb{B}_{\underline{n}}$ where $\mathbb{B}_{\underline{n}} = \mathbb{B}_{n_1} \times \cdots \times \mathbb{B}_{n_k}$ is the product of open unit balls in $\mathbb{C}^{n_i}$, $1 \leq i \leq k$. If $w \in S$ then $w(\alpha)$ denotes the well-defined evaluation of w at α as indicated above. Define the vectors

$$\omega_\alpha = \sum_{w \in S} w(\alpha)\xi_w, \quad \nu_\alpha = \omega_\alpha/\|\omega_\alpha\|_2$$

in the Fock space $\mathcal{H}_S$, noting that $\|\omega_\alpha\|$ is finite, since with $\alpha = (\alpha^{(1)}, \dots, \alpha^{(k)})$ we have

$$\begin{aligned} ||\omega_\alpha||_2^2 &= \sum_{w \in S} |w(\alpha)|^2 \\ &= \sum_{w_1 \in \mathbb{F}^+_{n_1}} \cdots \sum_{w_k \in \mathbb{F}^+_{n_k}} |w_1(\alpha^{(1)})|^2 \dots |w_k(\alpha^{(k)})|^2 \\ &= \prod_{i=1}^{k} \left(1 - ||\alpha^{(i)}||_2^2\right)^{-1}. \end{aligned}$$

Note that $(e_{ij}w)(\alpha) = \hat{\alpha}(e_{ij}w) = \hat{\alpha}(e_{ij})\hat{\alpha}(w) = \alpha_j^{(i)} w(\alpha)$. From this we see that $L^*_{e_{ij}}\omega_\alpha = \alpha_j^{(i)}\omega_\alpha$. Indeed, write e for e_{ij} and note that for all w,

$$\begin{aligned} \langle L_e^*\omega_\alpha, \xi_w \rangle &= \langle \omega_\alpha, \xi_{ew} \rangle = (ew)(\alpha) \\ &= \alpha_j^{(i)} w(\alpha) = \alpha_j^{(i)} \langle \omega_\alpha, \xi_w \rangle = \langle \alpha_j^{(i)} \omega_\alpha, \xi_w \rangle. \end{aligned}$$

It follows that the unit vector $\nu_{\overline{\alpha}}$ defines a vector functional

$$\rho(A) = \langle A\nu_{\overline{\alpha}}, \nu_{\overline{\alpha}} \rangle$$

which in turn gives a character ρ in $\mathcal{M}(\mathcal{A}_\theta)$ for which $\rho(L_{e_{ij}}) = \alpha_j^{(i)}$. These characters and their boundary limits in $V_\theta \cap \overline{\mathbb{B}}_{\underline{n}}$ in fact determine the Gelfand space, as in the following characterisation from [15]. Here we write Ω_θ for the closed set $V_\theta \cap \overline{\mathbb{B}}_{\underline{n}}$, carrying the relative topology from $\mathbb{C}^{|\underline{n}|}$.

Theorem 6.2. *Let $\mathfrak{L}_\theta$ and $\mathcal{A}_\theta$ be the operator algebras associated with a cancelative unital semigroup $\mathbb{F}_\theta^+$. Then:*

(i) *Each invariant subspace of $\mathfrak{L}_\theta$ of codimension one has the form $\{\omega_\alpha\}^\perp$ for some α in $\mathbb{B}_{\underline{n}} \cap V_\theta$.*

(ii) *The character space $\mathcal{M}(\mathcal{A}_\theta)$ is homeomorphic to Ω_θ under the map φ given by*

$$\varphi(\rho) = \left(\rho\left(L_{e_1^{(1)}}\right), \dots, \rho\left(L_{e_{n_k}^{(k)}}\right)\right), \quad \textit{for} \quad \rho \in \Omega_\theta.$$

The identification of the Gelfand spaces for the 2-graphs with $(n,m) = (2,2)$ now follows from our earlier descriptions in Section 3. In particular there are two algebras with Gelfand space of minimal type corresponding to the two permutations of order 4 indicated in Figure 2. Likewise, algebras for the fourteen 2-graphs with $(n,m) = (2,3)$ and relations of cyclic type have the "minimal" Gelfand space

$$\Omega_\theta = (\overline{\mathbb{B}}_2 \times \{0\}) \cup (\{0\} \times \overline{\mathbb{B}}_3) \cup ((\overline{\mathbb{B}}_2 \times \overline{\mathbb{B}}_3) \cap (E_2 \times E_3)).$$

The 2-graphs with $(n,m) = (n,1)$ are readily seen to be in bijective correspondence with the conjugacy classes in S_n and so $O(n,1)$ coincides with the number of possible cycle types for permutations τ in S_n. In this case the variety V_τ for τ in S_n is simply given; write $\tau(z)$ for the permuted vector $(z_{\tau(1)}, z_{\tau(2)}, \dots, z_{\tau(n)})$ and we have

$$V_\tau = (\mathbb{C}^n \times \{0\}) \cup (U_\tau \times \mathbb{C})$$

where $U_\tau = \{z \in \mathbb{C}^n : z = \tau(z)\}$. This variety does not determine the cycle type of τ but we see below that the geometric structure of $(\overline{\mathbb{B}}_n \times \overline{\mathbb{B}}_1) \cap V_\tau$ determines τ up to conjugacy, as does biholomorphic type of $(\mathbb{B}_n \times \mathbb{B}_1) \cap V_\tau$. In particular for each n there is one 2-graph algebra with minimal Gelfand space

$$V_{\min} = (\overline{\mathbb{B}}_n \times \{0\}) \cup ((\overline{\mathbb{B}}_n \cap E_n) \times \overline{\mathbb{B}}_1).$$

The Gelfand space $\Omega_\theta = V_\theta \cap \overline{\mathbb{B}}_{\underline{n}}$ of the generalised Arveson algebra A_θ splits naturally into (overlapping) parts determined by the algebraic components of V_θ. In particular the "interior" $V_\theta \cap \mathbb{B}_{\underline{n}}$ is generally a union of domains of various dimensions and A_θ is realised as an algebra of holomorphic functions in the sense that restrictions to these domains are holomorphic. In view of the homogeneous nature of the relations θ it follows that if $z \in \Omega_\theta$ then $\xi z \in \Omega_\theta$ for all complex numbers ξ with $|\xi| < 1$. Moreover $\xi \to f(\xi z)$ is holomorphic for each $f \in A_\theta$. Using this we can obtain a generalised Schwarz principal for maps between these spaces sufficient for the proof of the following proposition. The proposition will be useful in determining the multi-graded nature of graded isometric isomorphisms between higher rank analytic Toeplitz algebras.

Proposition 6.3. *Let θ, τ be permutation sets determining the spaces*

$$\Omega_\theta \subseteq \mathbb{C}^n = \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_s}, \quad \Omega_\tau \subseteq \mathbb{C}^m = \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_t}$$

and let γ be a biholomorphic automorphism from Ω_θ to Ω_τ with $\gamma(0) = 0$. Then $n = m$ and there is a unitary matrix X such that $\gamma(z) = Xz$. Moreover, up to a permutation, $(n_1, \ldots, n_s) = (m_1, \ldots, m_t)$ and with respect to this identification X is a block diagonal unitary matrix.

Proof. Let $\gamma(z) = (\gamma_1(z), \ldots, \gamma_t(z))$ with $z = (z_1, \ldots, z_s)$ and $z_i = (z_{i,1}, \ldots, z_{i,n_i})$, $1 \leq i \leq s$, and where $\gamma_j : \Omega_\theta \longrightarrow V_\tau \cap \mathbb{B}_{m_j}$, $1 \leq j \leq t$. Fix j and let $\gamma_j(z) = (\gamma_{j,1}(z), \ldots, \gamma_{j,m_j}(z))$ where $\gamma_{j,q} : \Omega_\theta \longrightarrow \overline{\mathbb{D}}$ are coordinate functions. Our hypotheses imply $\gamma_{j,q}(0) = 0$. Let β be a vector in Ω_θ. Also let $\xi \in \overline{\mathbb{D}}$ and note that $\xi\beta$ is in Ω_θ. Let $\alpha \in \mathbb{C}^{m_j}$ and consider the scalar holomorphic function $h(\xi)$ given by

$$h(\xi) = \alpha_1 \gamma_{j,1}(\xi\beta) + \cdots + \alpha_{m_j} \gamma_{j,m_j}(\xi\beta).$$

If α is a unit vector then by the Cauchy–Schwarz inequality we have $|h(z)| \leq 1$ since $\gamma_j(\xi\beta) \in \overline{\mathbb{B}}_{m_j}$. It follows now from Schwarz' inequality that $|h(\xi)| \leq |\xi|$. This is true for all α and so $\|\gamma_j(\xi\beta)\|_2 \leq |\xi|$.

Let $\|z\|_m = \max\{\|z_1\|_2, \ldots, \|z_s\|_2\}$ be the usual polyball norm. We have shown that $\|\gamma(\xi\beta)\|_m \leq |\xi|$ if $\beta \in \Omega_\theta$. If $w \in \Omega_\theta$ then $w = \xi\beta$ with $\|\beta\|_m = 1$, $|\xi| \leq 1$, $\|w\|_m = |\xi|$ and so it follows that $\|\gamma(w)\|_m \leq \|w\|_m$ for $w \in \Omega_\theta$. In view of the hypothesis γ is isometric with respect to polyball norms.

For notational convenience we assume that in the remainder of the proof that $s = t = 2$. Changing notation we have, for $(z, w) \in \Omega_\theta \subseteq \mathbb{B}_{n_1} \times \mathbb{B}_{n_2}$,

$$\gamma(z, w) = (\gamma_1(z, w), \gamma_2(z, w)),$$

where, for $l = 1, 2$,

$$\gamma_l(z, w) = (\gamma_{l,1}(z, w), \ldots, \gamma_{l,m_l}(z, w)).$$

Since $\gamma(0, 0) = (0, 0)$ the Taylor expansion takes the form

$$\gamma_{l,i}(z, w) = \sum_p a^l_{ip} z_p + \sum_q b^l_{iq} w_q + \delta_{l,i}(z, w)$$

where $\delta_{l,i}(tz, tw) = O(t^2)$. The isometric nature of γ with respect to $\| \|_m$ to now implies that for all z in $\mathbb{B}_{n_1}$ we have

$$\begin{aligned}\|z\|_2^2 &= \max\left(\|\gamma_1(z,0)\|_2^2, \|\gamma_2(z,0)\|_2^2\right)\\ &= \max_{l=1,2}\left(\sum_i\left|\sum_p a^l_{ip} z_p\right|^2\right)\\ &= \max\left(\left\|A^{(1)}z\right\|_2^2, \left\|A^{(2)}z\right\|_2^2\right)\end{aligned}$$

where $A^{(l)}$ is the $n_1 \times m_l$ matrix $\left(a^l_{ip}\right)$. It follows readily that one of these matrices is isometric and hence unitary while the other matrix is zero. Thus $N_1 = m_1$ or m_2 and, considering $\|w\|_2^2$ in a similar way the block unitary nature of γ follows. □

7. Isomorphism

The canonical generators for the analytic Toeplitz algebra of a single vertex k-graph, or semigroup $\mathbb{F}_\theta^+$, gives an associated $\mathbb{Z}_+$-grading and multi-grading. Let us say that an algebra homomorphism between such algebras is *graded* if it maps each generating isometry L_e, of total degree one, to a linear combination of such generators. Also, let us say that a graded homomorphism is *multi-graded* if it respects the given multi-gradings, up to reorderings of the k sets of generators, so that the image of each generator of total degree one and multi-degree δ_i is a linear combination of generators of a fixed multi-degree δ_j.

We now characterise isometric graded isomorphisms and see that they are unitarily implemented. In particular graded isometric automorphisms take a natural unitary form extending the notion of gauge automorphisms familiar in the free semigroup case.

First we make explicit the nature of bigraded unitary isomorphisms. Let $\mathbb{F}_n \times_{\theta_1} \mathbb{F}_m$, $\mathbb{F}_n \times_{\theta_2} \mathbb{F}_m$ be as in the last section. Then we have natural identifications for the Fock spaces for θ_1 and θ_2, namely,

$$\mathcal{H}_{\theta_i} = \ell^2(\mathbb{F}_n \times_{\theta_i} \mathbb{F}_m) = \sum_{(p,q)\in\mathbb{Z}_+^2} \oplus \mathcal{H}_{p,q}$$

where $\mathcal{H}_{p,q} = (\mathbb{C}^n)^{\otimes p} \otimes (C^m)^{\otimes q}$. Let $A \in M_n(\mathbb{C}), B \in M_n(\mathbb{C})$ be unitary matrices. Define $U : \mathcal{H}_{\theta_1} \to \mathcal{H}_{\theta_2}$ by the same formula as given in Section 5 for the map $\Phi_{A,B}$, that is,

$$U = U_{A,B} = \sum_{(p,q)\in\mathbb{Z}_+^2} (A^{\otimes p}) \otimes (B^{\otimes q}).$$

Assume now that we have the product unitary equivalence

$$\pi(\theta_1) = (A \otimes B)\pi(\theta_2)(A \otimes B)^*.$$

By Theorem 5.1 and its proof we have the commuting diagram

$$\begin{CD} \mathbb{C}[\mathbb{F}_n^+ \times_{\theta_1} \mathbb{F}_m^+] @>>> \mathcal{H}_{\theta_1} \\ @V{\Phi_{A,B}}VV @VV{U}V \\ \mathbb{C}[\mathbb{F}_n^+ \times_{\theta_2} \mathbb{F}_m^+] @>>> \mathcal{H}_{\theta_2} \end{CD}$$

where the horizontal maps are the natural linear space inclusions. It follows that the map $X \to UXU^*$ defines a unitarily implemented isomorphism $\mathcal{L}_{\theta_1} \to \mathcal{L}_{\theta_2}$. The higher rank multi-graded unitary isomorphisms are described in the same way, via Theorem 5.2, and are implemented by unitary operators of the form

$$U = U_{A_1,\dots,A_r} = \sum_{p \in \mathbb{Z}_+^r} (A_1^{\otimes p_1}) \otimes \cdots \otimes (A_r^{\otimes p_r}).$$

Theorem 7.1. *Let $\mathcal{L}_\theta$, $\mathcal{L}_\tau$, $\mathcal{A}_\theta$, $\mathcal{A}_\tau$ be the weakly closed and norm closed analytic Toeplitz algebras associated with the semigroups of cancelative permutation sets θ, τ. Then the following assertions are equivalent:*

(i) *The algebras $\mathcal{A}_\theta$ and $\mathcal{A}_\tau$ are gradedly isometrically isomorphic.*
(i′) *The algebras $\mathcal{A}_\theta$ and $\mathcal{A}_\tau$ are multi-gradedly isometrically isomorphic.*
(ii) *The algebras $\mathcal{L}_\theta$ and $\mathcal{L}_\tau$ are gradedly isometrically isomorphic.*
(ii′) *The algebras $\mathcal{L}_\theta$ and $\mathcal{L}_\tau$ are multi-gradedly gradedly isometrically isomorphic.*
(iii) *The permutation sets are product unitarily equivalent (after a possible relabeling) and the algebras $\mathcal{L}_\theta$ and $\mathcal{L}_\tau$ are unitarily equivalent by an isomorphism of the form $X \to UXU^*$ where $U = U_{A_1,\dots,A_r}$.*

Proof. To see that (iii) implies (i) and (ii) recall that the weakly closed subalgebra $\mathcal{L}_\theta^0$ generated by $\{L_w : |w| = 1\}$ is equal to the set of operators A with $\langle A\xi, \xi\rangle = 0$. Since $U\xi' = \xi$ it follows that $U\mathcal{L}_\theta^0 U^* = \mathcal{L}_\tau^0$. Let $\mathcal{M} = \{\xi\}^\perp$ and let $\mathcal{W}$ be the (wandering) subspace $\mathcal{M} \ominus (\mathcal{L}_\theta^0\mathcal{M})^-$ with $\mathcal{M}'$, $\mathcal{W}'$ similarly defined for $\mathcal{L}_\tau$. Then $U\mathcal{W}' = \mathcal{W}$. However, $\mathcal{W}$ is the linear span of ξ_w for $|w| = 1$ and so U gives a linear bijection $\mathcal{W}' \to \mathcal{W}$ effected by a unitary matrix, V say. Since ξ is a separating vector for $\mathcal{L}_\theta$ it follows that for $|w'| = 1$, we have $U^*L_{w'}U \in \operatorname{span}\{L_w : |w| = 1\}$. Hence the map $A \to UAU^*$ gives a graded isomorphism $\mathcal{L}_\theta \to \mathcal{L}_\tau$ which restricts to a graded isomorphism $\mathcal{A}_\theta \to \mathcal{A}_\tau$.

Plainly (ii) implies (i). Suppose that (i) holds. We show that (iii) holds, which will complete the proof. The given isomorphism, Φ say, induces an isometric algebra isomorphism $\mathcal{A}_\theta \to \mathcal{A}_\tau$ and hence a homeomorphism $\gamma : \Omega_\theta \to \Omega_\tau$ of their Gelfand spaces. These spaces have canonical realisations in $\mathbb{C}^N$ arising from the generators, as given in the last section, and it follows from elementary Banach algebra that γ is biholomorphic in the sense given in Section 6. Furthermore, since Φ is graded it follows that γ maps the origin to the origin. Proposition 6.3 applies and it follows that γ is implemented by a unitary matrix, X say, and that, after a permutation of coordinates, we may assume that θ and τ are permutation sets associated with $\underline{n} = \underline{m} = (n_1, \dots, n_r)$ and that X has the form $A_1 \oplus \cdots \oplus A_r$. We now see that γ is multi-graded and since Φ is graded, by assumption, it follows that Φ is multi-graded. In particular, with the usual notational convention, for each generator e_{ip} we have $\Phi(L_{e_{ip}}) = L_{A_ie_{ip}}$. Since Φ is an algebra isomorphism it follows readily that $\Phi = \Phi_{A_1,\dots,A_r}$ and that Φ is implemented by the unitary $U_{A_1,\dots,A_r}$. $\square$

Theorem 7.2. *Let $\mathcal{A}_\theta$, $\mathcal{A}_\tau$ be as in the statement of the last theorem. Let $\mathcal{I}_\theta$, $\mathcal{I}_\tau$ be the ideals of operators with vanishing constant term (so that, $\mathcal{I}_\theta = \mathcal{A}_\theta \cap \mathcal{L}_\theta^0$) and let $\Phi : \mathcal{A}_\theta \to \mathcal{A}_\tau$ be an isometric isomorphism with $\Phi(\mathcal{I}_\theta) = \mathcal{I}_\tau$. Then Φ is a multi-graded unitarily implemented isomorphism and θ, τ are product unitarily equivalent.*

Proof. As in the last proof the isomorphism induces a homeomorphism $\gamma : \Omega_\theta \to \Omega_\tau$ and in view of the stated ideal preservation γ preserves the origin ; $\gamma(0) = 0$. By Proposition 6.3 γ is given by a unitary matrix X which we may assume is in block diagonal form. Suppose that a generator e for $\mathbb{F}_\theta^+$ corresponds to basis element in $\mathbb{C}^{n_i}$, also denoted e. Write L_{Xe} for the linear combination of generators arising from the sum Xe. We now want to show that Φ is multi-graded and we have $\Phi(L_e) = L_{Xe} + c$ where $c = \sum_w \beta_w L_w$ where the summation extends over elements w of total degree at least 2. Since Φ is an isometry $\|L_{Xe} + c\| = 1$. Since X is a block diagonal unitary it follows that L_{Xe} is an isometry. Recall that the Fock space admits a graded decomposition $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots$. The isometry L_{Xe} has a subdiagonal block matrix structure which is disjoint from the block matrix support of c. It follows readily that $c = 0$. Thus Φ is a multi-graded isomorphism and the previous theorem completes the proof. $\square$

Up to this point we have not examined the local structure of the Gelfand spaces but it is clear that this information as well as general decomposition theory for algebraic varieties provides useful invariants, particularly for the analysis of automorphisms. We now appeal to the local structure of the minimal varieties $V_{\min}$ to see that in this case biholomorphic maps between the Gelfand spaces necessarily map 0 to 0.

Let Ω be the minimal Gelfand space associated with the multiplicities $(n_1, \ldots, n_k)$ and realised as the subset of $\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$ given by

$$\Omega = \left(\cup_{j=1}^k (\overline{\mathbb{B}}_{n_j} \times \{0\})\right) \cup ((\overline{\mathbb{B}}_{n_1} \cap E_{n_1}) \times \cdots \times (\overline{\mathbb{B}}_{n_k} \cap E_{n_k})),$$

with relative Euclidean topology. Let $z = (z_1, \ldots, z_k)$ be a point of Ω with $z_i \neq 0$. If $z_i \notin E_{n_i}$ then necessarily $z_j = 0$ for all $j \neq i$ and every open neighbourhood of z contains a basic open neighbourhood of the form

$$U_1(z, r) = (B(z_i, r)) \times \{0\}$$

where $B(z_i, r)$ is the intersection of $\overline{\mathbb{B}}_{n_i}$ with the open ball in $\mathbb{C}^{n_i}$ centred at z_i with radius r. Let us say that such a point is of type 1.

On the other hand, suppose that $z_i \neq 0$ and $z_i \in E_{n_i}$. If $z_j \neq 0$ for some $j \neq i$ then $z \in E_{n_1} \times \cdots \times E_{n_k}$ and z has a basic open neighbourhood of the form

$$U_2(z, r) = (B(z_1, r) \cap E_{n_1}) \times \cdots \times (B(z_k, r) \cap E_{n_k})),$$

whereas if $z_j = 0$ for all $j \neq i$ then z has the larger basic neighbourhood of the form

$$U_3(z, r) = U_1(z, r) \cup U_2(z, r).$$

Let us say that the points in these two cases are of types 2 and 3 respectively. Finally, if $z = 0$ then z has basic neighbourhoods of the form

$$r\Omega^o := \left(\cup_{j=1}^k (r\mathbb{B}_{n_j} \times \{0\})\right) \cup ((r\mathbb{B}_{n_1} \cap E_{n_1}) \times \cdots \times (r\mathbb{B}_{n_k} \cap E_{n_k})).$$

We shall show that in fact any homeomorphism $\gamma : \Omega \to \Omega$ maps the origin to the origin. There is a prima facie suggestion of this in the detail above, although basic open neighbourhoods and coordinates are not topologically determined. However we have the following connectivity argument.

Let

$$C = \cup_{j=1}^{k} (\overline{\mathbb{B}}_{n_j} \times \{0\}) \cap (E_{n_j} \times \{0\})$$

and note that C is the union of k closed discs, where, by a disc we mean a homeomorphic image of the set $\{(x, y) : x^2 + y^2 \leq 0\}$ in $\mathbb{R}^2$. These discs become disjoint on removal of the origin. Furthermore, the set $\Omega\backslash C$ is the disjoint union

$$(\cup_{j=1}^{k} (\overline{\mathbb{B}}_{n_j} \times \{0\}) \backslash (E_{n_j} \times \{0\})) \cup ((\mathbb{B}(z_1, r) \cap E_{n_1}) \times \cdots \times (\mathbb{B}(z_k) \cap E_{n_k}) \backslash C).$$

Suppose first that $n_i \geq 2$ for all i. Then this set has $k + 1$ pathwise connected components. Moreover, *every* open neighbourhood U of 0 has the property that $U\backslash C$ has $k + 1$ pathwise connected components. It remains to check that for each of the points of type 1, 2 and 3 the basic open neighbourhoods fail to have such a degree of disconnection on the removal of a homeomorph of C. In general, if $n_i = 1$ for some or several i, there are fewer disconnected components but the distinction of the origin persists.

In view of Theorem 7.2 we may now deduce the following result which applies in particular to the analytic Toeplitz algebras of k-graphs with cyclic relations.

Theorem 7.3. *Let $\mathcal{A}_\theta$ and $\mathcal{A}_\tau$ be the analytic Toeplitz algebras associated with the cancelative rank k semigroups $\mathbb{F}_\theta^+, \mathbb{F}_\tau^+$ with generator multiplicities $(n_1, \ldots, n_k)$, and assume that the Gelfand spaces are of minimal type. Then the following statements are equivalent:*

(i) *$\mathcal{A}_\theta$ and $\mathcal{A}_\tau$ are isometrically isomorphic.*
(ii) *$\mathcal{L}_\theta$ and $\mathcal{L}_\tau$ are isometrically isomorphic.*
(iii) *θ and τ are product unitarily equivalent.*

Furthermore the unitary automorphisms of $\mathcal{A}_\theta$ are implemented by the unitaries $U_{A_1,\ldots,A_k}$ where

$$\pi(\theta_{ij})(A_i \otimes A_j) = (A_i \otimes A_j)\pi(\theta_{ij})$$

for all $1 \leq i < j \leq k$.

We now focus attention on the rank 2 case. The next theorem shows that there are nine algebras $\mathcal{A}_G$ arising from single vertex 2-graphs with 1-skeleton (corresponding to the generators) consisting of two blue edges and two red edges.

Theorem 7.4. *Let Λ_1 and Λ_2 be single vertex 2-graphs with generating edge multiplicities 2, 2. Then the norm closed Toeplitz algebras $\mathcal{A}_{\Lambda_1}, \mathcal{A}_{\Lambda_2}$ are isometrically isomorphic if and only if their 2-graphs are isomorphic.*

Proof. By Theorem 6.1 and the descriptions of the varieties in Section 3 the Gelfand spaces of the quotient function algebras are all distinct up to homeomorphism except for the pair $\theta_4^a = (142), \theta_4^c = (124)$ and the pair $\theta_7 = (1243)$, $\theta_8 = (1234)$.

Suppose by way of contradiction that $\mathcal{A}_{\theta_7}$ and $\mathcal{A}_{\theta_8}$ are isometrically isomorphic and let $\gamma : \Omega_{\theta_7} \to \Omega_{\theta_8}$ be the induced biholomorphic homeomorphism. These Gelfand spaces are of minimal type and from the local structure it follows as before that $\gamma(0) = 0$. By Theorem 7.2, θ_7 and θ_8 are product unitarily equivalent. However, this is not the case as can be seen in a similar but simpler way to our earlier arguments for $(n, m) = (2, 3)$. Suppose, by way of contradiction, that $X \otimes Y$ is a tensor product of unitary matrices in $M_2 \otimes M_2$ and $(X \otimes Y)\theta_7 = \theta_8(X \otimes Y)$. We have $\theta_7 = \sigma\theta_8\sigma$ where $\sigma = \sigma^{-1} = (34)$ and so $[(X \otimes Y)\sigma]\theta_8 = \theta_8[(X \otimes Y)\sigma]$. In view of the matrix form of matrices that commute with the shift θ_8 this entails

$$(X \otimes Y)\sigma = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix}$$

and hence

$$X \otimes Y = \left(\begin{array}{cc|cc} a & d & b & c \\ b & a & c & d \\ \hline c & b & d & a \\ d & c & a & b \end{array}\right) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

On the other hand the matrix form of an elementary tensor entails that the 2×2 submatrices A, B, C, D are scalar multiples of each other. In our case these must be nonzero scalar multiples or else all but one of a, b, c, d is nonzero and the matrix fails then fails to have the form $X \otimes Y$. Similarly it follows now that a, b, c, d are nonzero. With $c = \lambda$ we have $d = \lambda b = \lambda^2 d$ and λ is $+1$ or -1. If $+1$ then $d = b$ and $b = +a$ or $-a$. However, in all cases all solutions $X \otimes Y$ fail to be invertible. The same is true when $\lambda = -1$, completing the contradiction.

The argument for the pair $\theta_4^a = (142), \theta_4^c = (124)$ is similar; the Gelfand space has four components and the origin is distinguished, as before. So it suffices to show that there is no unitary tensor with $X \otimes Y(142) = (124)X \otimes Y$.

To this end let

$$X = \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \quad Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The matrix equation implies

$$\begin{array}{cccc} wa = yc & wb = ya & xa = zc & xb = zd \\ wc = wa & wd = wb & xc = xa & xd = xb \\ yc = wc & ya = wd & zc = xc & zd = xd \end{array}$$

Now if $w \neq 0$ then $a = c, d = b$ and Y is not unitary. However, if $w = 0$ then x must be nonzero, since X is unitary, and we see once again that $a = c, d = b$ and Y is not unitary. Thus θ_4^c and θ_4^a are not product unitary equivalent, as required. □

Theorem 7.5. *Let Λ_1 and Λ_2 be single vertex 2-graphs with generating edge multiplicites 2, 3 and suppose that the relations for these 2-graphs are of cyclic type (or, equivalently, that each $\mathcal{A}_{\Lambda_i}$ has Gelfand space of minimal type). Then the norm closed Toeplitz algebras $\mathcal{A}_{\Lambda_1}, \mathcal{A}_{\Lambda_2}$ are isometrically isomorphic if and only if the 2-graphs Λ_1 and Λ_2 are isomorphic. Moreover there are exactly 14 isomorphism classes and these are in correspondence with the permutations of Figures 3–7.*

Proof. The proof has the same structure as the previous proof and so the relations θ, τ underlying Λ_1, Λ_2 are product unitarily equivalent. By Proposition 4.3, θ and τ are product conjugate and so Λ_1 and Λ_2 are isomorphic 2-graphs. □

The following corollary shows that in the higher rank case the commutant algebra need not be isomorphic to the original algebra. Theorem 7.4 shows that this also occurs when $(n, m) = (2, 2)$ for the algebra $\mathcal{A}_{\theta_4^a}$ whose commutant is isomorphic to $\mathcal{A}_{\theta_4^c}$.

Corollary 7.6. *Let $(n, m) = (2, 3)$ and let $\theta \in S_6$ be the permutation* (124653) *defining the* 2*-graph* Λ*. Then the algebra* $\mathcal{L}_\Lambda$ *is not isometrically isomorphic to its commutant.*

Proof. The permutation is θ_{12} in the list given in Section 4 and we have seen in Proposition 4.1 that this permutation is not product conjugate to its inverse. The associated 2-graphs are therefore not isomorphic. By the previous theorem the algebras $\mathcal{L}_\Lambda$ and $\mathcal{L}_{\Lambda^{\text{op}}}$ are not isometrically isomorphic, and so the corollary follows from Proposition 2.3. □

We expect that algebra isomorphism corresponds to graph isomorphism, or generator exchange graph isomorphism. There are two main issues to resolve in order to establish this.

Firstly it seems plausible that in general product unitary equivalence gives the same equivalence relation as product conjugacy. If this is true then, for example, we obtain from the last theorem a more definitive classification, akin to the (2,3) case, of the single vertex k-graph algebras with character space of minimal type.

Secondly, it seems likely that for general finitely generated single vertex k-graph one can reduce to graded isomorphisms by means of composition with a unitary automorphism. For general (multi-vertex) 1-graphs this was shown in [13]. In the next section we show how this may be done for a special class of 2-graphs. As we have remarked in the introduction, in [22] it has now been proven for general 2-graphs.

8. The 2-graph algebras $\mathcal{A}_n \times_\theta \mathbb{Z}_+$

Consider the algebras associated with single vertex 2-graphs with $(n, m) = (n, 1)$. Such a 2-graph is specified by a permutation τ in S_n and we may consider the relations to be $e_i f = f e_{\tau(i)}$, $i = 1, \ldots, n$. As usual we write $\mathcal{A}_\tau$, $\mathcal{L}_\tau$ for the corresponding nonselfadjoint Toeplitz algebras. We remark that $\mathcal{A}_\tau$ is identifiable with a crossed product algebra $\mathcal{A}_n \times_\theta \mathbb{Z}_+$ which in turn may be identified with a subalgebra of the full crossed product $O_n \times_\theta \mathbb{Z}$ of the Cuntz algebra O_n.

Isometric isomorphisms $\mathcal{A}_\tau \to \mathcal{A}_\sigma$ need not be graded. However we shall identify explicit unitary automorphisms of $\mathcal{A}_\tau$ (and also $\mathcal{L}_\tau$) which allow us to reduce to the graded case.

Suppose that τ has cycle type $r_1 r_2 \ldots r_t$, that is, t distinct cycles of length $r_i, i = 1, \ldots, t$. Then the Gelfand space Ω_τ is identifiable with the subset

$$(\overline{\mathbb{B}}_n \times \{0\}) \cup \big((U \cap \overline{\mathbb{B}}_n) \times \overline{\mathbb{B}}_1\big) \subseteq \mathbb{C}^n \times \mathbb{C}$$

where U is the variety of points z in $\mathbb{C}^n$ with $\tau(z) = z$. Functions in the Arveson algebra $A_\tau = \mathcal{A}_\tau / \operatorname{com} \mathcal{A}_\tau$ have holomorphic restrictions to $(\overline{\mathbb{B}}_n \times \{0\})$ and to $(U \cap \overline{\mathbb{B}}_n) \times \overline{\mathbb{B}}_1$ and we shall simply say that A_τ is an algebra of holomorphic functions with this sense understood. Likewise, a holomorphic function $\phi : \Omega_\tau \to \Omega_\sigma$ is biholomorphic if both ϕ and ϕ^{-1} have coordinate functions which are holomorphic in this sense.

Define the subset $(U \cap \mathbb{B}_n) \times \{0\}$ to be the *open core* of Ω_τ. If $\varphi : \Omega_\tau \to \Omega_\sigma$ is a biholomorphic map then it is clear that such a map respects the open core. We show that the biholomorphic automorphisms of Ω_τ act transitively on the open core. Furthermore the automorphisms of Ω_τ that derive from unitary automorphisms of

$\mathcal{A}_\tau$ also act transitively on the open core. To construct these automorphisms we make use of the explicit automorphisms of Cuntz algebras obtained by Voiculescu [28]. Our account below relies on Voiculescu's automorphisms but is otherwise self-contained, and uses notation similar to that of the discussion in Davidson and Pitts [6]. For an alternative discussion of Voiculescu's construction see also [22].

Proposition 8.1. *Let α be a real vector in $\mathbb{B}_n$. Then there is biholomorphic automorphism $\theta : \mathbb{B}_n \to \mathbb{B}_n$ with $\theta(0) = \alpha$. Furthermore θ may be defined by*

$$\theta(\lambda) = \frac{X_1\lambda + \eta}{x_0 + \langle \lambda, \eta\rangle},$$

where $x_0 = (1 - |\alpha|^2)^{-\frac{1}{2}}, \eta = x_0\alpha,$ and X_1 is the positive square root of $I_n + \eta\eta*$.

Proof. We have $X_1\eta = X_1^*\eta = x_0\eta$. Using this and the equation $X_1^*X_1 = I_n + \eta\eta*$ we obtain

$$\begin{aligned}
&|x_0{+}\langle \lambda, \eta\rangle|^2 - \|(X_1\lambda + \eta)\|^2 \\
&\quad = |x_0|^2 + 2\mathrm{Re}\langle x_0\eta, \lambda\rangle + |\langle \lambda, \eta\rangle|^2 - \|X_1\lambda\|^2 - 2\mathrm{Re}\langle \lambda, X_1^*\eta\rangle - \|\eta\|^2 \\
&\quad = |x_0|^2 - \|\eta\|^2 + |\langle \lambda, \eta\rangle|^2 - |\lambda|^2 - \langle \eta\eta^*\lambda, \lambda\rangle \\
&\quad = 1 - |\lambda|^2.
\end{aligned}$$

Thus θ_X maps $\mathbb{B}_n$ into $\mathbb{B}_n$ and maps 0 to $\eta/x_0 = \alpha$. Let $\theta' : \mathbb{B}_n \to \mathbb{B}_n$ be defined by

$$\theta'(\lambda) = \frac{X_1\lambda - \eta}{x_0 - \langle \lambda, \eta\rangle}.$$

Then $\theta \circ \theta'(x) = \lambda$ for all λ in $\mathbb{B}_n$. Indeed

$$\begin{aligned}
\theta(\theta'(\lambda)) &= \frac{X_1\left(\frac{X_1\lambda - \eta}{x_0 - \langle \lambda, \eta\rangle}\right) + \eta}{x_0 + \left\langle \frac{X_1\lambda - \eta}{x_0 - \langle \lambda, \eta\rangle}, \eta\right\rangle} \\
&= \frac{X_1^2\lambda - X_1\eta + \eta(x_0 - \langle \lambda, \eta\rangle)}{x_0(x_0 - \langle \lambda, \eta\rangle) + \langle X_1\lambda, \eta\rangle - \langle \eta, \eta\rangle} \\
&= \frac{\lambda + \eta\eta^*(\lambda) - x_0\eta + \eta x_0 - \eta\langle \lambda, \eta\rangle}{|x_0|^2 - x_0\langle \lambda, \eta\rangle + x_0\langle \lambda, \eta\rangle - |\eta|^2} \\
&= \lambda.
\end{aligned}$$

It follows that θ', and similarly θ, is injective on $\mathbb{B}_n$, and that θ, and similarly θ', is onto $\mathbb{B}_n$, as required. □

Proposition 8.2. *Let $\mathcal{A}_\tau$ be the 2-graph algebra for the permutation $\tau \in S_n$ and let $\alpha \in E$ where $E \times \{0\}$ is the open core of the Gelfand space Ω_τ of A_τ.*

(i) *There is a biholomorphic automorphism $\theta : \mathbb{B}_n \to \mathbb{B}_n$ with $\theta(E) = E$, $\theta(0) = \alpha$ and with $\theta = \tau^{-1}\theta\tau$, where τ also denotes the coordinate shift automorphism.*

(ii) *There is an isometric operator algebra automorphism $\Theta : \mathcal{A}_n \to \mathcal{A}_n$ which extends the ball automorphism θ in* (i) *and which satisfies $\Theta = T^{-1}\Theta T$ where $T : \mathcal{A}_n \to \mathcal{A}_n$ is the coordinate shift automorphism such that $T(L_{e_i}) = L_{e_{\tau(i)}}$.*

Proof. (i) Since τ commutes with diagonal gauge automorphisms

$$\gamma : z \to (d_1 z_1, d_2 z_2, \ldots, d_n z_n)$$

when the coefficient sequence d satisfies $d = \tau(d)$ it is clear that we may assume that α is a real vector in E. Consider now the automorphism θ of Proposition 8.1 associated with α. We claim that θ satisfies the desired requirements. Indeed, η is a scalar multiple of α and so $\tau(\eta) = \eta$. Since η is a fixed vector for τ the matrix $I_n + \eta\eta^*$ is diagonalised by a complete set of eigenvectors for τ, where τ is considered as a unitary permutation matrix as before. It follows that the square root matrix X_1 is similarly diagonalised and so commutes with τ. It follows now from the formula for θ that $\theta(\tau(x)) = \tau(\theta(x))$ for λ in $\mathbb{B}_n$.

(ii) Following Voiculescu [28], for $\xi \in \mathbb{C}^n$ define

$$\Theta(L_\xi) = (x_0 I - L_\eta)^{-1}(L_{X_1\xi} - \langle \xi, \eta\rangle I)$$

where x_0, η, X_1 are as in Proposition 8.1. That Θ determines an automorphism of $\mathcal{A}_n$ follows from Theorem 2.10 of [28].

In the semigroup ring generated by the e_i and f we have, writing $X_1 = (x_{ij})$,

$$\begin{aligned}(X_1 e_i) f &= \left(\sum_t x_{ti} e_t\right) f \\ &= f \sum_t x_{ti} e_{\tau(t)} \\ &= f \sum_s x_{\tau^{-1}(s),i} e_s \\ &= f(\pi(\tau) X_1 e_i) \\ &= f \pi(\tau) X_1 \pi(\tau^{-1}) e_{\tau(i)} \\ &= f X_1 e_{\tau(i)},\end{aligned}$$

since X_1 commutes with $\pi(\tau)$. It follows that $L_{X_1 e_i} L_f = L_f L_{X_1 e_{\tau(i)}}$ for each i. Since $\tau(\eta) = \eta$ it now follows that

$$\begin{aligned}\Theta(L_{e_i}) L_f &= L_f \Theta(L_{e_{\tau(i)}}) \\ L_f^* \Theta(L_{e_i}) L_f &= \Theta(L_f^* L_{e_i} L_f).\end{aligned}$$

Since $A \to L_f^* A L_f$, is an implementation of the automorphism T we have $T \circ \Theta = \Theta \circ T$ and the proof of (ii) is complete. □

Theorem 8.3. *Let Λ_1, Λ_2 be single vertex 2-graphs with generating graphs having a single red edge and finitely many blue edges. Then the following statements are equivalent:*

(i) *Λ_1 and Λ_2 are isomorphic 2-graphs.*
(ii) *$\mathcal{A}_{\Lambda_1}$ and $\mathcal{A}_{\Lambda_2}$ are isometrically isomorphic.*
(iii) *$\mathcal{L}_{\Lambda_1}$ and $\mathcal{L}_{\Lambda_2}$ are unitarily equivalent.*

Proof. Let $\Phi : \mathcal{L}_{\Lambda_1} \to \mathcal{L}_{\Lambda_2}$ be a unitary equivalence. Let $M^*(\mathcal{L}_{\Lambda_i})$ be the space of weak star continuous multiplicative linear functionals on $\mathcal{L}_{\Lambda_i}$, $i = 1, 2$ with the weak star topology. These spaces are identifiable with the Euclidean space $\Omega^o_{\Lambda_i} = \Omega_{\Lambda_i} \cap (\mathbb{B}_{n_i} \times \mathbb{B}_1)$. The map Φ induces a weak star continuous map $\gamma : M^*(\mathcal{L}_{\Lambda_1}) \to M^*(\mathcal{L}_{\Lambda_2})$ and hence a homeomorphism $\gamma : \Omega^o_{\Lambda_1} \to \Omega^0_{\Lambda_2}$. This map respects the open core and so $\gamma(0) = \alpha$ lies in $\{(z, 0) : \theta_2(z) = z\}$ where, for $i = 1, 2$, θ_i is the permutation in S_{n_i} determining Λ_i. Composing Φ with a unitary automorphism of $\mathcal{L}_{\Lambda_2}$ mapping α to 0 we may assume, without loss of generality, that $\gamma(0) = 0$. Theorem 7.2 now applies and it follow that $n_1 = n_2$ and θ_1 and θ_2 are unitarily equivalent permutation matrices in $\pi(S_n)$. It follows from spectral theory that θ_1 and θ_2 are conjugate in S_n from which (i) follows.

The direction (i) $\Rightarrow$ (iii) is elementary while the equivalence of (i) and (ii) follows as above. $\square$

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