New York Journal of Mathematics

New York J. Math. **13** (2007) 199–213.

Iterating the Pimsner construction

Valentin Deaconu

Abstract. For A a C^*-algebra, E_1, E_2 two Hilbert bimodules over A, and a fixed isomorphism $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$, we consider the problem of computing the K-theory of the Cuntz–Pimsner algebra $\mathcal{O}_{E_2 \otimes_A \mathcal{O}_{E_1}}$ obtained by extending the scalars and by iterating the Pimsner construction.

The motivating examples are a commutative diagram of Douglas and Howe for the Toeplitz operators on the quarter plane, and the Toeplitz extensions associated by Pimsner and Voiculescu to compute the K-theory of a crossed product. The applications are for Hilbert bimodules arising from rank two graphs and from commuting endomorphisms of abelian C^*-algebras.

Contents

1. Introduction

In his seminal paper [Pi], Pimsner introduced a large class of C^*-algebras generalizing both the Cuntz–Krieger algebras and the crossed products by an automorphism. The central notion is that of a Hilbert bimodule or C^*-correspondence, which appeared also in the theory of subfactors. His construction was modified by Katsura (see [Ka1]) to include bimodules defined from graphs with sinks, or more generally, from topological graphs.

In the same way that a crossed product by $\mathbb{Z}^2$ could be thought as an iterated crossed product by $\mathbb{Z}$, we iterate the Pimsner construction for two "commuting" Hilbert bimodules over a C^*-algebra. A basic example comes from a rank two graph of Kumjian and Pask (see [KP]), where the two bimodules correspond to the horizontal and vertical edges, and the commutation relation is given by the unique

Received January 2, 2007. Revised July 11, 2007.
Mathematics Subject Classification. Primary 46L05; Secondary 46L55, 46L80.
Key words and phrases. C^*-algebra, Hilbert bimodule, K-theory.

ISSN 1076-9803/07

factorization property. We get a particular case of the product systems defined by Fowler (see [F2]), but our approach has the advantage that it generates some exact sequences of K-theory.

After some preliminaries about the algebras $\mathcal{T}_E$ and $\mathcal{O}_E$, we describe how from a Hilbert module E over A and a map $A \to B$ we can get a Hilbert module over B, using a tensor product. Given two Hilbert bimodules E_1, E_2 over A and an isomorphism $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$ we consider $E_2 \otimes_A \mathcal{T}_{E_1}$ and $E_2 \otimes_A \mathcal{O}_{E_1}$ as Hilbert bimodules over $\mathcal{T}_{E_1}$ and $\mathcal{O}_{E_1}$ and repeat the Pimsner construction. The last section deals with a 3×3 diagram involving the iterated Cuntz–Pimsner algebras, inspired from the work of Douglas and Howe (see [DH]) and Pimsner and Voiculescu (see [PV]). As a corollary, we obtain some exact sequences of K-theory, including information about the K-groups of the C^*-algebra $\mathcal{O}_{E_2 \otimes_A \mathcal{O}_{E_1}}$. Several examples are considered, including commuting endomorphisms of abelian C^*-algebras.

Since this paper was completed, we discovered that J. Lindiarni and I. Raeburn ([LR]) already used the diagram which appears in our Lemma 5.2.

2. Preliminaries

Recall that a (right) Hilbert A-module is a Banach space E with a right action of a C^*-algebra A and an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ linear in the second variable such that

$$\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a, \quad \langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*, \quad \langle \xi, \xi \rangle \geq 0, \quad \|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}.$$

A Hilbert module is called full if the closed linear span of the inner products coincides with A. We denote by $\mathcal{L}(E)$ the C^*-algebra of adjointable operators on E, and by $\theta_{\xi,\eta} \in \mathcal{L}(E)$ the rank one operator

$$\theta_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta \rangle.$$

The closed linear span of rank one operators is the ideal $\mathcal{K}(E)$. We have $\mathcal{L}(E) \cong M(\mathcal{K}(E))$, the multiplier algebra. Also, $\mathcal{K}(E)$ can be identified with the balanced tensor product $E \otimes_A E^*$, where E^* is the dual of E, a left Hilbert A-module.

A Hilbert bimodule over A (sometimes called a C^*-correspondence from A to A) is a Hilbert A-module with a left action of A given by a homomorphism $\varphi : A \to \mathcal{L}(E)$. A Hilbert bimodule is called faithful if φ is injective. The left action is nondegenerate if $\overline{\varphi(A)E} = E$. For $n \geq 0$ we denote by $E^{\otimes n}$ the Hilbert bimodule obtained by taking the tensor product of n copies of E, balanced over A (for $n = 0$, $E^{\otimes 0} = A$). Recall that for $n = 2$, the inner product is given by

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \eta, \varphi(\langle \xi, \xi' \rangle)\eta' \rangle,$$

and it is inductively defined for general n.

Definition 2.1. A *Toeplitz representation* of a Hilbert bimodule E over A in a C^*-algebra C is a pair (τ, π) with $\tau : E \to C$ a linear map and $\pi : A \to C$ a *-homomorphism, such that

$$\tau(\xi a) = \tau(\xi)\pi(a), \quad \tau(\xi)^*\tau(\eta) = \pi(\langle \xi, \eta \rangle), \quad \tau(\varphi(a)\xi) = \pi(a)\tau(\xi).$$

Note that the first property actually follows from the second. Indeed,

$$\begin{aligned}
&\|\tau(\xi a) - \tau(\xi)\pi(a)\|^2 \\
&= (\tau(\xi a) - \tau(\xi)\pi(a))^*(\tau(\xi a) - \tau(\xi)\pi(a)) \\
&= \pi(\langle \xi a, \xi a\rangle) - \pi(a)^*\pi(\langle \xi, \xi a\rangle) - \pi(\langle \xi a, \xi\rangle)\pi(a) + \pi(a)^*\pi(\langle \xi, \xi\rangle)\pi(a) = 0.
\end{aligned}$$

The corresponding universal C^*-algebra is called the Toeplitz algebra of E, denoted by $\mathcal{T}_E$. If E is full, then $\mathcal{T}_E$ is generated by elements $\tau^n(\xi)\tau^m(\eta)^*, m, n \geq 0$, where $\tau^0 = \pi$ and for $n \geq 1, \tau^n(\xi_1 \otimes \cdots \otimes \xi_n) = \tau(\xi_1)\ldots\tau(\xi_n)$ is the extension of τ to $E^{\otimes n}$. If E is also faithful, then $A \subset \mathcal{T}_E$.

There is a homomorphism $\psi : \mathcal{K}(E) \to C$ such that $\psi(\theta_{\xi,\eta}) = \tau(\xi)\tau(\eta)^*$. A representation (τ, π) is *Cuntz–Pimsner covariant* if $\pi(a) = \psi(\varphi(a))$ for all a in the ideal

$$I_E = \varphi^{-1}(\mathcal{K}(E)) \cap (\ker\varphi)^\perp.$$

The Cuntz–Pimsner algebra $\mathcal{O}_E$ is universal with respect to the covariant representations, and it is a quotient of $\mathcal{T}_E$.

There is a gauge action of $\mathbb{T}$ on $\mathcal{T}_E$ and $\mathcal{O}_E$ defined by

$$z \cdot (\tau^n(\xi)\tau^m(\eta)^*) = z^{n-m}\tau^n(\xi)\tau^m(\eta)^*, z \in \mathbb{T},$$

and using the universal properties. The *core* $\mathcal{F}_E$ is the fixed point algebra $\mathcal{O}_E^{\mathbb{T}}$, generated by the union of the algebras $\mathcal{K}(E^{\otimes n})$.

The Toeplitz algebra $\mathcal{T}_E$ can be represented by creation operators $T_\xi(\eta) = \xi \otimes \eta$ on the Fock bimodule

$$\ell^2(E) = \bigoplus_{n\geq 0} E^{\otimes n},$$

and there is an ideal $I = \mathcal{K}(\ell^2(E)I_E)$ in $\mathcal{L}(\ell^2(E))$ such that

$$0 \to I \to \mathcal{T}_E \to \mathcal{O}_E \to 0$$

is exact. In particular, if $\mathcal{L}(E) = \mathcal{K}(E)$ or $\varphi(A) \subset \mathcal{K}(E)$, then

$$0 \to \mathcal{K}(\ell^2(E)) \to \mathcal{T}_E \to \mathcal{O}_E \to 0.$$

For more details about the algebras $\mathcal{F}_E, \mathcal{T}_E$, and $\mathcal{O}_E$ we refer to the original paper of Pimsner ([Pi]) and to [Ka1].

If E is finitely generated, then it has a basis $\{u_i\}$, in the sense that for all $\xi \in E$,

$$\xi = \sum_{i=1}^n u_i\langle u_i, \xi\rangle.$$

In this case, $\mathcal{L}(E) = \mathcal{K}(E)$, and the Cuntz–Pimsner algebra $\mathcal{O}_E$ is generated by $S_i = S_{u_i}$ with relations

$$\sum_{i=1}^n S_iS_i^* = 1, \quad S_i^*S_j = \langle u_i, u_j\rangle, \quad a \cdot S_j = \sum_{i=1}^n S_i\langle u_i, a \cdot u_j\rangle.$$

The Toeplitz algebra $\mathcal{T}_E$ is generated by $T_i = T_{u_i}$ which satisfy only the last two relations (see [KPW]).

Examples 2.2. (1) For $A = \mathbb{C}$ and $E = H$ a Hilbert space with orthonormal basis $\{\xi_i\}_{i\in I}$, the Toeplitz algebra $\mathcal{T}_H$ is generated by $\{S_i\}_{i\in I}$ satisfying $S_i^* S_j = \delta_{ij}\cdot 1,\ i,j \in I$. In $\mathcal{O}_H$ we also have $\sum_{i\in I} S_i S_i^* = 1$ if the dimension of H is finite. In particular, for $E = \mathbb{C} = A$ we get $\mathcal{T}_E \cong \mathcal{T}$, the classical Toeplitz algebra generated by the unilateral shift, $\mathcal{O}_E \cong C(\mathbb{T})$, the continuous functions on the unit circle, and $\mathcal{F}_E \cong \mathbb{C}$. For $E = \mathbb{C}^n$, we get $\mathcal{T}_E \cong \mathcal{E}_n$, the Cuntz–Toeplitz algebra, $\mathcal{O}_E \cong \mathcal{O}_n$, the Cuntz algebra, and $\mathcal{F}_E \cong UHF(n^\infty)$. For H infinite-dimensional and separable, $\mathcal{T}_H \cong \mathcal{O}_H \cong \mathcal{O}_\infty$.

(2) Let $E = A^n$ with the usual structure:

$$\langle (a_1,\ldots,a_n),(b_1,\ldots,b_n)\rangle = \sum_{i=1}^n a_i^* b_i,$$

$$(a_1,\ldots,a_n)\cdot a = (a_1 a,\ldots,a_n a), \quad a\cdot(a_1,\ldots,a_n) = (aa_1,\ldots,aa_n).$$

We get $\mathcal{T}_E \cong A\otimes\mathcal{E}_n,\ \mathcal{O}_E \cong A\otimes\mathcal{O}_n,\ \mathcal{F}_E \cong A\otimes UHF(n^\infty)$.

(3) Let $\alpha : A \to A$ be an automorphism of a unital C^*-algebra, and let $E = A(\alpha)$ be the Hilbert bimodule obtained from A with the usual inner product and right multiplication, and with left action $\varphi(a)x = \alpha(a)x$. Then $\mathcal{T}_E$ is isomorphic to the Toeplitz extension $\mathcal{T}_\alpha$ used by Pimsner and Voiculescu in [PV], and $\mathcal{O}_E$ is isomorphic to the crossed product $A \rtimes_\alpha \mathbb{Z}$. Indeed, let $\hat{a}$ denote the element in E obtained from $a \in A$. Then $S = \tau(\hat{1})$ is an isometry in any unital Toeplitz representation (τ,π), since $\langle \hat{1},\hat{1}\rangle = 1$, and it is an unitary in any unital covariant representation, since the rank one operator $\theta_{\hat{1},\hat{1}}$ is the identity. We also have

$$\begin{aligned}\pi(\alpha(a)) &= \pi(\langle \hat{1},\widehat{\alpha(a)}\rangle) = \tau(\hat{1})^*\tau(\widehat{\alpha(a)}) = \tau(\hat{1})^*\tau(\varphi(a)\hat{1})\\ &= \tau(\hat{1})^*\pi(a)\tau(\hat{1}) = S^*\pi(a)S.\end{aligned}$$

Therefore $\mathcal{O}_E \cong A \rtimes_\alpha \mathbb{Z}$. In the paper mentioned above, the Toeplitz extension $\mathcal{T}_\alpha$ was defined as the C^*-subalgebra of $(A\rtimes_\alpha\mathbb{Z})\otimes\mathcal{T}$ generated by $A\otimes 1$ and $u\otimes S_+$, where $\alpha(a) = uau^*$ and S_+ is the unilateral shift. It is easy to see that $\mathcal{T}_\alpha \cong \mathcal{T}_E$ by the map which takes $a\otimes 1$ into a and $u\otimes S_+$ into S. Since $\mathcal{K}(\ell^2(E)) \cong A\otimes\mathcal{K}$, we recover the short exact sequence

$$0 \to A\otimes\mathcal{K} \to \mathcal{T}_\alpha \to A\rtimes_\alpha\mathbb{Z} \to 0.$$

(4) Graph C^*-algebras. For an oriented countable graph $G = (G^0, G^1, r, s)$, $C^*(G)$ is defined as the universal C^*-algebra generated by mutually orthogonal projections $\{p_v\}_{v\in G^0}$ and partial isometries $\{s_e\}_{e\in G^1}$ with orthogonal ranges such that $s_e^* s_e = p_{r(e)},\ s_e s_e^* \le p_{s(e)}$ and

$$p_v = \sum_{s(e)=v} s_e s_e^* \text{ if } 0 < |s^{-1}(v)| < \infty. \tag{$*$}$$

We can set $A = C_0(G^0)$, and denote by E the Hilbert module we obtain after we complete $C_c(G^1)$ in the norm given by the inner product

$$\langle \xi,\eta\rangle(v) = \sum_{r(e)=v} \overline{\xi(e)}\eta(e)$$

with the right action defined by $(\xi f)(e) = \xi(e) f(r(e))$. The left action is defined by

$$\varphi : A \to \mathcal{L}(E),\ \varphi(f)\xi(e) = f(s(e))\xi(e),\ \xi \in C_c(G^1) \subset E.$$

We have

$$\begin{aligned}\varphi^{-1}(\mathcal{K}(E)) &= C_0(\{v \in G^0 : |s^{-1}(v)| < \infty\}),\\ \ker(\varphi) &= C_0(\{v \in G^0 : |s^{-1}(v)| = 0\}),\end{aligned}$$

hence

$$I_E = C_0(\{v \in G^0 : 0 < |s^{-1}(v)| < \infty\}).$$

Define

$$\begin{aligned}\pi(f) &= \sum_{v \in G^0} f(v) p_v, \quad f \in C_0(G^0),\\ \tau(\xi) &= \sum_{e \in G^1} \xi(e) s_e, \quad \xi \in C_c(G^1) \subset E.\end{aligned}$$

The pair (τ, π) is a Toeplitz representation into $C^*(G)$ iff $s_e^* s_e = p_{r(e)}$ and $s_e s_e^* \leq p_{s(e)}$, which is covariant iff $(*)$ is satisfied. This proves that $C^*(G) \cong \mathcal{O}_E$ (see [Ka1]). The core $\mathcal{F}_E$ is an AF-algebra. For an irreducible oriented finite graph with no sinks, we obtain the Cuntz–Krieger algebras $\mathcal{O}_A$ as Cuntz–Pimsner algebras.

(5) For a C^*-algebra A and an injective unital endomorphism $\alpha \in \mathrm{End}(A)$ such that there is a conditional expectation P onto the range $\alpha(A)$, one can define a Hilbert bimodule $E = A(\alpha, P)$, using the transfer operator $L = \alpha^{-1} \circ P : A \to A$ as in [EV]. We complete the vector space A with respect to the inner product $\langle \xi, \eta \rangle = L(\xi^* \eta)$, and define the right and left multiplications by $\xi \cdot a = \xi \alpha(a),\ a \cdot \xi = a\xi$. We have

$$\begin{aligned}\langle \xi, \eta \cdot a \rangle &= \langle \xi, \eta\alpha(a) \rangle = L(\xi^* \eta \alpha(a)) = \alpha^{-1}(P(\xi^* \eta \alpha(a)))\\ &= \alpha^{-1}(P(\xi^*\eta)\alpha(a)) = \alpha^{-1}(P(\xi^*\eta))a = \langle \xi, \eta \rangle a.\end{aligned}$$

For α an automorphism and $P = \mathrm{id}$, the corresponding C^*-algebra $\mathcal{O}_E$ is isomorphic to the crossed product $A \rtimes_{\alpha^{-1}} \mathbb{Z}$. Indeed, let F be the Hilbert bimodule $A(\alpha^{-1})$ with the structure as in Example 2.2.3. The map $\alpha^{-1} : A \to A$ induces an isomorphism of Hilbert bimodules $h : E \to F$. For $A = C(X)$ with X compact, and α induced by a surjective local homeomorphism $\sigma : X \to X$, we take

$$(Pf)(x) = \frac{1}{\nu(x)} \sum_{\sigma(y) = \sigma(x)} f(y),$$

where $\nu(x)$ is the number of elements in the fiber $\sigma^{-1}(x)$. It was proved in [D] that the corresponding algebra $\mathcal{O}_{A(\alpha,P)}$ is isomorphic to $C^*(\Gamma(\sigma))$, where $\Gamma(\sigma)$ is the Renault groupoid

$$\Gamma(\sigma) = \{(x, p-q, y) \in X \times \mathbb{Z} \times X \ \mid \ \sigma^p(x) = \sigma^q(y)\}.$$

3. Extending the scalars

Let E be a Hilbert module over A and let $\rho : A \to B$ be a C^*-algebra homomorphism (we will be interested mostly in the case when ρ is an inclusion). Then B is a left A-module with multiplication $a \cdot b = \rho(a)b$, and $E \otimes_A B$ becomes a Hilbert module over B, with the inner product given by

$$\langle \xi_1 \otimes b_1, \xi_2 \otimes b_2 \rangle = b_1^* \rho(\langle \xi_1, \xi_2 \rangle) b_2$$

and right multiplication $(\xi \otimes b_1) \cdot b_2 = \xi \otimes b_1 b_2$. We have

$$\mathcal{K}(E \otimes_A B) \cong (E \otimes_A B) \otimes_B (E \otimes_A B)^* \cong E \otimes_A B \otimes_A E^*,$$

which is strongly Morita equivalent to B in the case E is full. Also, we have an inclusion $\mathcal{K}(E) \subset \mathcal{K}(E \otimes_A B)$ for B unital, given by $\xi \otimes \eta^* \mapsto \xi \otimes 1 \otimes \eta^*$. If E is a Hilbert bimodule over A, and if there is a *-morphism $B \to \mathcal{L}(E)$ which extends the left multiplication of A on E, then $E \otimes_A B$ becomes a Hilbert bimodule over B, and one can form the tensor powers $(E \otimes_A B)^{\otimes n}$. Assuming that the left action of B on $E \otimes_A B$ is nondegenerate, we get

$$(E \otimes_A B)^{\otimes n} \cong E^{\otimes n} \otimes_A B.$$

In particular, the Toeplitz algebra $\mathcal{T}_{E \otimes_A B}$ is represented on

$$\ell^2(E \otimes_A B) \cong \ell^2(E) \otimes_A B,$$

and depends on the left multiplication of B. An interesting question is to relate the C^*-algebras $\mathcal{T}_{E \otimes_A B}$ and $\mathcal{O}_{E \otimes_A B}$ to $\mathcal{T}_E$, $\mathcal{O}_E$, and B.

Example 3.1. Let $A = \mathbb{C}$, let $E = H$ be a separable infinite-dimensional Hilbert space, and let B be a separable unital C^*-algebra. Then $H \otimes B$ is a Hilbert module over B, and $\mathcal{K}(H \otimes B) \cong \mathcal{K}(H) \otimes B$. If B is faithfully represented on H, then $H \otimes B$ becomes a Hilbert bimodule over B. Assuming, in addition, that the intersection of B with $\mathcal{K}(H)$ is trivial, Kumjian (see [K]) showed that $\mathcal{T}_{H \otimes B} \cong \mathcal{O}_{H \otimes B}$ is simple and purely infinite, with the same K-theory as B.

Example 3.2. Let $A = C_0(X)$ and let E be a Hilbert module given by a continuous field of elementary C^*-algebras over X. Then for an abelian C^*-algebra B containing $C_0(X)$, the tensor product $E \otimes_A B$ is obtained by a pullback. In particular, let (G^0, G^1, r, s) be a (topological) graph, and consider a covering map $p : \widetilde{G^0} \to G^0$ which gives an inclusion $A = C_0(G^0) \subset C_0(\widetilde{G^0}) = B$. The Hilbert module $E \otimes_A B$, where E is obtained from $C_c(G^1)$ as in Example 2.2.4, is associated to a"pullback graph" in which the set of vertices is $\widetilde{G^0}$ and the set of edges is $\widetilde{G^1} := \{(x, e, y) \in \widetilde{G^0} \times G^1 \times \widetilde{G^0} \mid s(e) = p(x), r(e) = p(y)\}$. The new range and source maps are $s(x, e, y) = x, r(x, e, y) = y$. It is known that $\mathcal{K}(E)$ is isomorphic to $C^*(R)$, where R is the equivalence relation

$$R = \{(e_1, e_2) \in G^1 \times G^1 \mid r(e_1) = r(e_2)\}.$$

Then $\mathcal{K}(E \otimes_A B)$ is isomorphic to $C^*(p^*(R))$, where

$$p^*(R) = \{((x_1, e_1, y_1), (x_2, e_2, y_2)) \in \widetilde{G^1} \times \widetilde{G^1} \mid p(y_1) = p(y_2)\}.$$

Example 3.3. Let $\alpha : A \to A$ be an automorphism of a C^*-algebra A, which extends to $\widetilde{\alpha} : B \to B$, where $A \subset B$. Consider $E = A(\alpha)$ as in Example 2.2.3.

Then $E \otimes_A B \cong B(\widetilde{\alpha})$, $\mathcal{T}_{E\otimes_A B} \cong \mathcal{T}_{\widetilde{\alpha}}$ and $\mathcal{O}_{E\otimes_A B} \cong B \rtimes_{\widetilde{\alpha}} \mathbb{Z}$, which contains $\mathcal{O}_E \cong A \rtimes_\alpha \mathbb{Z}$.

Example 3.4. For a Hilbert bimodule E over A, Pimsner used the Hilbert module $E_\infty = E \otimes_A \mathcal{F}_E$ (see [Pi], Section 2) in order to get an inclusion $\mathcal{T}_E \subset \mathcal{T}_{E_\infty}$, an isomorphism $\mathcal{O}_E \cong \mathcal{O}_{E_\infty}$, and a completely positive map $\phi : \mathcal{O}_E \to \mathcal{T}_{E_\infty}$ which is a cross-section to the quotient map $\mathcal{T}_{E_\infty} \to \mathcal{O}_E$.

4. Iterating the Pimsner construction

Consider now two full finitely generated Hilbert A-bimodules E_1 and E_2 such that A is unital and the left actions $\varphi_i : A \to \mathcal{L}(E_i)$ are injective and nondegenerate. We assume that there is an isomorphism of Hilbert A-bimodules $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$. This isomorphism should be understood as a kind of commutation relation. The most interesting cases are when E_1 and E_2 are independent, in the sense that no tensor power of one is isomorphic to the other.

Note that the isomorphism χ induces and isomorphism $E_1^* \otimes_A E_2^* \to E_2^* \otimes_A E_1^*$ because $(E_1 \otimes_A E_2)^* \cong E_2^* \otimes_A E_1^*$.

Since $A \subset \mathcal{T}_{E_1}$, the tensor product $E_2 \otimes_A \mathcal{T}_{E_1}$ becomes a Hilbert module over $\mathcal{T}_{E_1}$ as in Section 3, with the inner product given by $\langle \xi \otimes x, \eta \otimes y\rangle = x^*\langle \xi, \eta\rangle y$, and the right multiplication by $(\xi \otimes x)y = \xi \otimes xy$. Since $\mathcal{T}_{E_1}$ is generated by E_1, in order to define a left multiplication of $\mathcal{T}_{E_1}$ on $E_2 \otimes_A \mathcal{T}_{E_1}$, it is sufficient to define the left multiplication by elements in E_1. This is done via the composition

$$E_1 \otimes_A E_2 \otimes_A \mathcal{T}_{E_1} \to E_2 \otimes_A E_1 \otimes_A \mathcal{T}_{E_1} \to E_2 \otimes_A \mathcal{T}_{E_1},$$

where the first map is $\chi \otimes \mathrm{id}$, and the second is given by absorbing E_1 into $\mathcal{T}_{E_1}$. For the adjoint, note that there is a map

$$E_1^* \otimes_A E_2 \otimes_A E_1 \to E_1^* \otimes_A E_1 \otimes_A E_2 \to E_2,$$

where the first map is $\mathrm{id} \otimes \chi^{-1}$, and the second is given by left multiplication with the inner product in E_1. We will denote by $E_2 \otimes_A^\chi \mathcal{T}_{E_1}$ the bimodule obtained using this left multiplication, when χ is not understood. In the same way, we may consider the bimodule $E_1 \otimes_A \mathcal{T}_{E_2}$ with the left multiplication induced by χ^{-1}.

Lemma 4.1. *With the above structure, $E_2 \otimes_A \mathcal{T}_{E_1}$ is a Hilbert bimodule over $\mathcal{T}_{E_1}$, and we can consider the Toeplitz algebra $\mathcal{T}_{E_2\otimes_A\mathcal{T}_{E_1}}$. We have*

$$\mathcal{T}_{E_2\otimes_A\mathcal{T}_{E_1}} \cong \mathcal{T}_{E_1\otimes_A\mathcal{T}_{E_2}}.$$

Proof. Both algebras $\mathcal{T}_{E_2\otimes_A\mathcal{T}_{E_1}}, \mathcal{T}_{E_1\otimes_A\mathcal{T}_{E_2}}$ are represented on the Fock space

$$\ell^2(E_2) \otimes_A \ell^2(E_1) \cong \ell^2(E_1) \otimes_A \ell^2(E_2),$$

and are generated by $\mathcal{T}_{E_1}$ and $\mathcal{T}_{E_2}$ with the commutation relation given by the isomorphism χ. □

Similarly, we can construct the Hilbert bimodules $E_2 \otimes_A \mathcal{O}_{E_1}$ and $E_1 \otimes_A \mathcal{O}_{E_2}$. We get

Lemma 4.2. *With the above notation,*

$$\mathcal{T}_{E_2\otimes_A\mathcal{O}_{E_1}} \cong \mathcal{O}_{E_1\otimes_A\mathcal{T}_{E_2}}, \quad \mathcal{O}_{E_2\otimes_A\mathcal{O}_{E_1}} \cong \mathcal{O}_{E_1\otimes_A\mathcal{O}_{E_2}}.$$

Proof. The first two algebras are quotients of $\mathcal{T}_{E_2\otimes_A\mathcal{T}_{E_1}} \cong \mathcal{T}_{E_1\otimes_A\mathcal{T}_{E_2}}$ by the ideal generated by $\mathcal{K}(\ell^2(E_1))$, and the last two are quotients by the ideal generated by $\mathcal{K}(\ell^2(E_1))$ and $\mathcal{K}(\ell^2(E_2))$. $\square$

Note that there is a gauge action of $\mathbb{T}^2$ on $\mathcal{O}_{E_2\otimes_A\mathcal{O}_{E_1}} \simeq \mathcal{O}_{E_1\otimes_A\mathcal{O}_{E_2}}$.

Remark 4.3. Given E_1, E_2 as above, we can define a product system $E = E^\chi$ of Hilbert bimodules over the semigroup $(\mathbb{N}^2, +)$ (see [F2]), as follows. Define the fibers by $E_{(m,n)} = E_1^{\otimes m} \otimes_A E_2^{\otimes n}$ for $(m, n) \in \mathbb{N}^2$, and the multiplication induced by the isomorphism χ. It is easy to see that we get associativity, and therefore we may consider the C^*-algebras $\mathcal{T}_E$, $\mathcal{O}_E$ as defined by Fowler. Note that if we change χ, the product system also changes. In particular, a path of isomorphisms χ_t will determine a family of product systems $E^t = E^{\chi_t}$.

Recall that a Toeplitz representation of a product system E in a C^*-algebra is obtained from a family of Toeplitz representations of the fibers, compatible with the product. In this particular case, it is sufficient to consider two Toeplitz representations $(\tau_1, \pi), (\tau_2, \pi)$ of the generators E_1 and E_2, respectively, and to define $\tau(\xi_1 \otimes \cdots \otimes \xi_m \otimes \eta_1 \otimes \cdots \otimes \eta_n) = \tau_1(\xi_1)\ldots\tau_1(\xi_m)\tau_2(\eta_1)\ldots\tau_2(\eta_n)$ for $\xi_i \in E_1, i = 1, \ldots, m$ and $\eta_j \in E_2, j = 1, \ldots, n$. The Toeplitz algebra $\mathcal{T}_E$ is represented on the Fock space $\ell^2(E) \cong \ell^2(E_1) \otimes_A \ell^2(E_2)$ by creation operators. The covariance condition requires that each (τ_i, π) is covariant, $i = 1, 2$. This means that $\psi_i(\varphi_i(a)) = \pi(a)$ for $a \in \varphi_i^{-1}(\mathcal{K}(E_i)), i = 1, 2$, where $\psi_i(\theta_{\xi,\eta}) = \tau_i(\xi)\tau_i(\eta)^*$. We get that

$$\mathcal{T}_{E_2\otimes_A\mathcal{T}_{E_1}} \cong \mathcal{T}_{E_1\otimes_A\mathcal{T}_{E_2}} \cong \mathcal{T}_E \text{ and } \mathcal{O}_{E_2\otimes_A\mathcal{O}_{E_1}} \cong \mathcal{O}_{E_1\otimes_A\mathcal{O}_{E_2}} \cong \mathcal{O}_E.$$

Example 4.4. Let A be a unital C^*-algebra, and α, β two commuting automorphisms of A. We assume that α and β generate $\mathbb{Z}^2$ as a subgroup of $\mathrm{Aut}(A)$. Denote by $A(\alpha), A(\beta)$ the Hilbert bimodules as in Example 2.2.3, which are independent. We have $A(\alpha) \otimes_A A(\beta) \cong A(\beta) \otimes_A A(\alpha)$ by the map $\chi(\hat{a} \otimes \hat{b}) = \beta(\widehat{\alpha^{-1}(a)}) \otimes \hat{b}$. Indeed,

$$\begin{aligned}
\langle \chi(\hat{a}_1 \otimes \hat{b}_1), \chi(\hat{a}_2 \otimes \hat{b}_2)\rangle &= \langle \beta(\widehat{\alpha^{-1}(a_1)}) \otimes \hat{b}_1, \beta(\widehat{\alpha^{-1}(a_2)}) \otimes \hat{b}_2\rangle \\
&= \langle \hat{b}_1, \langle \beta(\widehat{\alpha^{-1}(a_1)}), \beta(\widehat{\alpha^{-1}(a_2)})\rangle \cdot \hat{b}_2\rangle \\
&= b_1^*\alpha(\beta(\alpha^{-1}(a_1))^*\beta(\alpha^{-1}(a_2)))b_2 \\
&= b_1^*\beta(a_1^*a_2)b_2 = \langle \hat{b}_1, \langle \hat{a}_1, \hat{a}_2\rangle \cdot \hat{b}_2\rangle \\
&= \langle \hat{a}_1 \otimes \hat{b}_1, \hat{a}_2 \otimes \hat{b}_2\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\chi(a' \cdot (\hat{a} \otimes \hat{b}) \cdot b') &= \chi(\widehat{\alpha(a')a} \otimes \widehat{bb'}) = \beta(\widehat{\alpha^{-1}(\alpha(a')a)}) \otimes \widehat{bb'} \\
&= \beta(\widehat{a'\alpha^{-1}(a)}) \otimes \widehat{bb'} = \beta(a')\widehat{\beta(\alpha^{-1}(a))} \otimes \widehat{bb'} \\
&= a' \cdot \chi(\hat{a} \otimes \hat{b}) \cdot b'.
\end{aligned}$$

Denote by $\mathcal{T}_\alpha, \mathcal{T}_\beta$ the corresponding Toeplitz algebras, each generated by A and an isometry as in Example 2.2.3. Then $A(\beta) \otimes_A \mathcal{T}_\alpha$ becomes a Hilbert bimodule over $\mathcal{T}_\alpha$, isomorphic to $\mathcal{T}_\alpha(\beta)$, where β is extended to $\mathcal{T}_\alpha$ in the natural way, fixing the isometry. Similarly, $A(\alpha) \otimes_A \mathcal{T}_\beta \simeq \mathcal{T}_\beta(\alpha)$. We have

$$\mathcal{T}_{\mathcal{T}_\alpha(\beta)} \cong \mathcal{T}_{\mathcal{T}_\beta(\alpha)} \cong \mathcal{T}_E,$$

where E is the product system over $\mathbb{N}^2$ constructed from $A(\alpha)$, $A(\beta)$, and the isomorphism χ. Also, we may consider the Hilbert bimodules $A(\beta) \otimes_A (A \rtimes_\alpha \mathbb{Z}) \cong (A \rtimes_\alpha \mathbb{Z})(\beta)$, and $A(\alpha) \otimes_A (A \rtimes_\beta \mathbb{Z}) \cong (A \rtimes_\beta \mathbb{Z})(\alpha)$, where again α and β are extended to $A \rtimes_\beta \mathbb{Z}$ and $A \rtimes_\alpha \mathbb{Z}$, respectively, by fixing the unitaries implementing the actions of $\mathbb{Z}$. We have

$$\begin{aligned}\mathcal{T}_{(A\rtimes_\alpha\mathbb{Z})(\beta)} &\cong \mathcal{T}_\beta \rtimes_\alpha \mathbb{Z},\\ \mathcal{T}_{(A\rtimes_\beta\mathbb{Z})(\alpha)} &\cong \mathcal{T}_\alpha \rtimes_\beta \mathbb{Z},\\ \mathcal{O}_{(A\rtimes_\alpha\mathbb{Z})(\beta)} \cong \mathcal{O}_{(A\rtimes_\beta\mathbb{Z})(\alpha)} &\cong \mathcal{O}_E \cong A \rtimes_{\alpha,\beta} \mathbb{Z}^2.\end{aligned}$$

Example 4.5. Let $E = F = \mathbb{C}^2$ with the usual structures of Hilbert bimodules over $A = \mathbb{C}$. Then $\mathcal{O}_E \cong \mathcal{O}_2$, and $F \otimes \mathcal{O}_2$ becomes a Hilbert module over $\mathcal{O}_2$ with the usual operations. The left action will depend on a fixed isomorphism $\chi : E \otimes F \to F \otimes E$. If $\{e_1, e_2\}$ and $\{f_1, f_2\}$ are the canonical bases in E and F respectively, and $\chi_1(e_i \otimes f_j) = f_j \otimes e_i,\ i, j = 1, 2$, then the left multiplication is given by $S_i(f_j \otimes S_k) = f_j \otimes S_i S_k$, where $\mathcal{O}_2$ has generators $\{S_1, S_2\}$, and $\mathcal{O}_{F\otimes^{\chi_1}\mathcal{O}_2} \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. On the other hand, if $\chi_2(e_i \otimes f_j) = f_i \otimes e_j,\ i, j = 1, 2$, then the left action is given by $S_i(f_j \otimes S_k) = f_i \otimes S_j S_k$, and the $\mathcal{O}_2$-bimodule $F \otimes^{\chi_2} \mathcal{O}_2$ is degenerate. The corresponding Cuntz–Pimsner algebra $\mathcal{O}_{F\otimes^{\chi_2}\mathcal{O}_2}$ is isomorphic to $C(\mathbb{T}) \otimes \mathcal{O}_2$. Indeed, if we interpret E and F as being each associated to the 1-graph Γ defining $\mathcal{O}_2$ (Γ has two edges and one vertex), then the isomorphism χ_2 is defining the rank 2 graph $\phi^*(\Gamma)$ where $\phi : \mathbb{N}^2 \to \mathbb{N}, \phi(m, n) = m + n$, and the last assertion follows from Example 6.1 and Proposition 2.10 in [KP].

Note that an arbitrary isomorphism χ does not necessarily define a rank 2 graph. For example, χ could be given by the unitary matrix $U = (u_{kl})$, where

$$\begin{aligned}&u_{11} = \cos\alpha, \quad u_{12} = -\sin\alpha, \quad u_{21} = \sin\alpha, \quad u_{22} = \cos\alpha,\\ &u_{33} = \cos\beta, \quad u_{34} = -\sin\beta, \quad u_{43} = \sin\beta, \quad u_{44} = \cos\beta,\end{aligned}$$

for some angles α and β, and the rest of the entries equal to zero. For other C^*-algebras defined by a product system of finite-dimensional Hilbert spaces over the semigroup $\mathbb{N}^2$, see [F1].

Example 4.6. Consider two star-commuting onto local homeomorphisms σ_1 and σ_2 of a compact space X. By definition, that means that for every $x, y \in X$ such that $\sigma_1(x) = \sigma_2(y)$, there exists a unique $z \in X$ such that $\sigma_2(z) = x$ and $\sigma_1(z) = y$. This condition ensures that the associated conditional expectations P_1, P_2 commute (see [ER]). Unfortunately, this condition was omitted in the Proposition on page 8 in [D2]. I would like to thank Ruy and Jean for pointing this to me.

We can define the Hilbert bimodules $E_i = A(\alpha_i, P_i),\ i = 1, 2$ over $A = C(X)$ as in Example 2.2.5. Then there is an isomorphism

$$h : A(\alpha_1, P_1) \otimes_A A(\alpha_2, P_2) \to A(\alpha_1 \circ \alpha_2, P_1 \circ P_2),$$

given by $h(\hat{a} \otimes \hat{b}) = \widehat{a\alpha_1(b)}$, with inverse $h^{-1}(\hat{x}) = \hat{x} \otimes \hat{1}$, which induces an isomorphism $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$. The resulting C^*-algebra $\mathcal{O}_{E_2\otimes_A\mathcal{O}_{E_1}} \cong \mathcal{O}_{E_1\otimes_A\mathcal{O}_{E_2}}$ can be understood as a crossed product of $C(X)$ by the semigroup $\mathbb{N}^2$. For other examples of semigroups of local homeomorphisms, see [ER].

5. Exact sequences in K-theory

To study the C^*-algebra of Toeplitz operators on the quarter plane, Douglas and Howe (see [DH]) considered the commutative diagram with exact rows and columns, where j is the inclusion map, and π is the quotient map:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{K}\otimes\mathcal{K} \arrow[r, "j\otimes 1"] \arrow[d, "1\otimes j"] & \mathcal{T}\otimes\mathcal{K} \arrow[r, "\pi\otimes 1"] \arrow[d, "1\otimes j"] & C(\mathbb{T})\otimes\mathcal{K} \arrow[r] \arrow[d, "1\otimes j"] & 0 \\
0 \arrow[r] & \mathcal{K}\otimes\mathcal{T} \arrow[r, "j\otimes 1"] \arrow[d, "1\otimes\pi"] & \mathcal{T}\otimes\mathcal{T} \arrow[r, "\pi\otimes 1"] \arrow[d, "1\otimes\pi"] & C(\mathbb{T})\otimes\mathcal{T} \arrow[r] \arrow[d, "1\otimes\pi"] & 0 \\
0 \arrow[r] & \mathcal{K}\otimes C(\mathbb{T}) \arrow[r, "j\otimes 1"] \arrow[d] & \mathcal{T}\otimes C(\mathbb{T}) \arrow[r, "\pi\otimes 1"] \arrow[d] & C(\mathbb{T})\otimes C(\mathbb{T}) \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Corollary 5.1. *We have the short exact sequences*

$$0 \to \mathcal{T}\otimes\mathcal{K}+\mathcal{K}\otimes\mathcal{T} \stackrel{1\otimes j+j\otimes 1}{\longrightarrow} \mathcal{T}\otimes\mathcal{T} \stackrel{\pi\otimes\pi}{\longrightarrow} C(\mathbb{T})\otimes C(\mathbb{T}) \to 0,$$

$$0 \to \mathcal{K}\otimes\mathcal{K} \stackrel{j\otimes 1+1\otimes j}{\longrightarrow} \mathcal{T}\otimes\mathcal{K}+\mathcal{K}\otimes\mathcal{T} \stackrel{\pi\otimes 1+1\otimes\pi}{\longrightarrow} C(\mathbb{T})\otimes\mathcal{K}\oplus\mathcal{K}\otimes C(\mathbb{T}) \to 0.$$

The above 3×3 diagram and the exact sequences in the corollary are particular cases of a more general situation, for which we provide a proof.

Lemma 5.2. *Let A be a C^*-algebra and I, J two closed two-sided ideals of A. Then we have the commutative diagram with exact rows and columns, where the maps are the canonical ones:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I\cap J \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"] & I \arrow[r, "\omega_1"] \arrow[d, "\iota_1"] & I/(I\cap J) \arrow[r] \arrow[d, "\sigma_1"] & 0 \\
0 \arrow[r] & J \arrow[r, "\iota_2"] \arrow[d, "\omega_2"] & A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & A/J \arrow[r] \arrow[d, "\rho_2"] & 0 \\
0 \arrow[r] & J/(I\cap J) \arrow[r, "\sigma_2"] \arrow[d] & A/I \arrow[r, "\rho_1"] \arrow[d] & A/(I+J) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

From this we get the exact sequences

$$0 \to I+J \stackrel{\iota_1+\iota_2}{\longrightarrow} A \stackrel{\pi}{\longrightarrow} A/(I+J) \to 0,$$

where $\pi = \rho_1\circ\pi_1 = \rho_2\circ\pi_2$, and

$$0 \to I\cap J \stackrel{\lambda_1+\lambda_2}{\longrightarrow} I+J \stackrel{\omega_1+\omega_2}{\longrightarrow} I/(I\cap J)\oplus J/(I\cap J) \to 0.$$

Applying the K-theory functor, we get

$$\begin{array}{ccccc}
K_0(I+J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/(I+J)) \\
\uparrow & & & & \downarrow \\
K_1(A/(I+J)) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I+J),
\end{array}$$

$$\begin{array}{ccccc}
K_0(I\cap J) & \longrightarrow & K_0(I+J) & \longrightarrow & K_0(I/(I\cap J)) \oplus K_0(J/(I\cap J)) \\
\uparrow & & & & \downarrow \\
K_1(I/(I\cap J)) \oplus K_1(J/(I\cap J)) & \longleftarrow & K_1(I+J) & \longleftarrow & K_1(I\cap J).
\end{array}$$

Proof. Note that the first two rows and the first two columns are obviously exact. For the third row, the map $\pi_1 \circ \iota_2$ has kernel $I \cap J$. This defines a map $\sigma_2 : J/(I\cap J) \to A/I$ such that $\sigma_2 \circ \omega_2 = \pi_1 \circ \iota_2$. By the second isomorphism theorem, $(I+J)/I \cong J/(I\cap J)$, and by the third isomorphism theorem, $(A/I)/((I+J)/I) \cong A/(I+J)$, hence the third row is exact. The exactness of the third column is proved similarly. Consider now the diagonal morphism $\pi = \rho_1 \circ \pi_1 = \rho_2 \circ \pi_2 : A \to A/(I+J)$. If $a \in \ker \pi$, then $\pi_2(a) \in \ker \rho_2 = \sigma_1(I/(I\cap J)) = \sigma_1(\omega_1(I))$, hence there is $b \in I$ with $\pi_2(a) = \sigma_1(\omega_1(b)) = \pi_2(\iota_1(b))$. It follows that $a - \iota_1(b) \in \ker \pi_2 = \iota_2(J)$, and there is $c \in J$ with $a = \iota_1(b) + \iota_2(c)$. This gives the first exact sequence. For the second, we use the map $\omega_1 + \omega_2 : I + J \to I/(I\cap J) \oplus J/(I\cap J)$ which has kernel $I \cap J$. $\square$

We generalize the above diagram of Douglas and Howe to certain iterated Toeplitz and Cuntz–Pimsner algebras. By the lemma, we get some exact sequences which we hope will help to do K-theory computations in some particular cases.

Theorem 5.3. *Consider a C^*-algebra A and two finitely generated Hilbert bimodules E_1, E_2 with a fixed isomorphism $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$. We assume that $E_2 \otimes_A \mathcal{T}_{E_1}, E_2 \otimes_A \mathcal{O}_{E_1}, E_1 \otimes_A \mathcal{T}_{E_2}, E_1 \otimes_A \mathcal{O}_{E_2}$ are nondegenerate as Hilbert bimodules, with the structure described in Section* 3*. Then we have the following commuting diagram* (*depending on χ*)*, with exact rows and columns, where the maps are canonical:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{K}(\ell^2(E_1 \otimes_A E_2)) & \longrightarrow & \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{T}_{E_1})) & \longrightarrow & \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{O}_{E_1})) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{T}_{E_2})) & \to & \mathcal{T}_{E_1 \otimes_A \mathcal{T}_{E_2}} \cong \mathcal{T}_{E_2 \otimes_A \mathcal{T}_{E_1}} & \to & \mathcal{O}_{E_1 \otimes_A \mathcal{T}_{E_2}} \cong \mathcal{T}_{E_2 \otimes_A \mathcal{O}_{E_1}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{O}_{E_2})) & \to & \mathcal{T}_{E_1 \otimes_A \mathcal{O}_{E_2}} \cong \mathcal{O}_{E_2 \otimes_A \mathcal{T}_{E_1}} & \to & \mathcal{O}_{E_1 \otimes_A \mathcal{O}_{E_2}} \cong \mathcal{O}_{E_2 \otimes_A \mathcal{O}_{E_1}} & \to & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. Recall that the algebra $\mathcal{T}_{E_i}$ is represented on the Fock space $\ell^2(E_i)$ and we have a short exact sequence

$$0 \to \mathcal{K}(\ell^2(E_i)) \to \mathcal{T}_{E_i} \to \mathcal{O}_{E_i} \to 0,$$

for $i = 1, 2$. We apply the above lemma for the C^*-algebra $\mathcal{T}_{E_1 \otimes_A \mathcal{T}_{E_2}} \cong \mathcal{T}_{E_2 \otimes_A \mathcal{T}_{E_1}}$ with ideals $I = \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{T}_{E_1}))$ and $J = \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{T}_{E_2}))$. The nondegeneracy assumption implies that $\ell^2(E_2 \otimes_A \mathcal{T}_{E_1}) \cong \ell^2(E_2) \otimes_A \mathcal{T}_{E_1}$ and $\ell^2(E_2 \otimes_A \mathcal{O}_{E_1}) \cong \ell^2(E_2) \otimes_A \mathcal{O}_{E_1}$. Note that the map $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$ induces isomorphisms $\mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{K}(\ell^2(E_1)))) \cong \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{K}(\ell^2(E_2)))) \cong \mathcal{K}(\ell^2(E_1 \otimes E_2))$, therefore $I \cap J = \mathcal{K}(\ell^2(E_1 \otimes_A E_2))$. $\square$

Corollary 5.4. *Under the same assumptions as in the theorem, we get the short exact sequences*

$$0 \to \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{T}_{E_1})) + \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{T}_{E_2})) \to \mathcal{T}_{E_1 \otimes_A \mathcal{T}_{E_2}} \to \mathcal{O}_{E_1 \otimes_A \mathcal{O}_{E_2}} \to 0$$

and

$$0 \to \mathcal{K}(\ell^2(E_1 \otimes_A E_2)) \to \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{T}_{E_1})) + \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{T}_{E_2})) \to$$

$$\to \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{O}_{E_1})) \oplus \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{O}_{E_2})) \to 0.$$

The corresponding six-term exact sequences of K-theory give us information about the K-theory of $\mathcal{O}_{E_1 \otimes_A \mathcal{O}_{E_2}}$, once we identify the maps between the various K-groups. Note that the Toeplitz algebras are KK-equivalent to the C^*-algebra A.

Example 5.5. For α_1, α_2 two commuting independent automorphisms of a C^*-algebra A, the diagram is

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{K} \otimes A \otimes \mathcal{K} \arrow[r] \arrow[d] & \mathcal{T}_{\alpha_1} \otimes \mathcal{K} \arrow[r] \arrow[d] & A \rtimes_{\alpha_1} \mathbb{Z} \otimes \mathcal{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} \otimes \mathcal{T}_{\alpha_2} \arrow[r] \arrow[d] & \mathcal{T}_{\mathcal{T}_{\alpha_2}(\alpha_1)} \cong \mathcal{T}_{\mathcal{T}_{\alpha_1}(\alpha_2)} \arrow[r] \arrow[d] & \mathcal{T}_{\alpha_2} \rtimes_{\alpha_1} \mathbb{Z} \cong \mathcal{T}_{(A \rtimes_{\alpha_1} \mathbb{Z})(\alpha_2)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} \otimes A \rtimes_{\alpha_2} \mathbb{Z} \arrow[r] \arrow[d] & \mathcal{T}_{\alpha_1} \rtimes_{\alpha_2} \mathbb{Z} \cong \mathcal{T}_{(A \rtimes_{\alpha_2} \mathbb{Z})(\alpha_1)} \arrow[r] \arrow[d] & A \rtimes_{\alpha_1, \alpha_2} \mathbb{Z}^2 \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We get the exact sequences

$$\begin{tikzcd}
K_0(A) \arrow[r] & K_0(\mathcal{T}_{\alpha_1} \otimes \mathcal{K} + \mathcal{K} \otimes \mathcal{T}_{\alpha_2}) \arrow[d] \\
K_1(A \rtimes_{\alpha_1} \mathbb{Z}) \oplus K_1(A \rtimes_{\alpha_2} \mathbb{Z}) \arrow[u, "\partial^1 + \partial^2"] & K_0(A \rtimes_{\alpha_1} \mathbb{Z}) \oplus K_0(A \rtimes_{\alpha_2} \mathbb{Z}) \arrow[d, "\partial^1 + \partial^2"'] \\
K_1(\mathcal{T}_{\alpha_1} \otimes \mathcal{K} + \mathcal{K} \otimes \mathcal{T}_{\alpha_2}) \arrow[u] & K_1(A) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
K_0(\mathcal{T}_{\alpha_1}\otimes\mathcal{K}+\mathcal{K}\otimes\mathcal{T}_{\alpha_2}) \arrow[r] & K_0(A) \arrow[d] \\
K_1(A\rtimes_{\alpha_1,\alpha_2}\mathbb{Z}^2) \arrow[u] & K_0(A\rtimes_{\alpha_1,\alpha_2}\mathbb{Z}^2) \arrow[d] \\
K_1(A) \arrow[u] & K_1(\mathcal{T}_{\alpha_1}\otimes\mathcal{K}+\mathcal{K}\otimes\mathcal{T}_{\alpha_2}). \arrow[l]
\end{tikzcd}$$

Let now $A = C(X)$ with X a Cantor set, and let $\sigma,\tau : X \to X$ be two commuting local homeomorphisms. Denote by α,β the induced endomorphisms of A, and assume that the associated conditional expectations commute. Applying the theorem for the Hilbert bimodules $E = A(\alpha, P)$, $F = A(\beta, Q)$ and the canonical isomorphism $E \otimes_A F \cong F \otimes_A E$, we get

$$\begin{tikzcd}
C(X,\mathbb{Z}) \arrow[r] & K_0(\mathcal{T}_E\otimes\mathcal{K}+\mathcal{K}\otimes\mathcal{T}_F) \arrow[d] \\
\ker(1-\alpha_*)\oplus\ker(1-\beta_*) \arrow[u] & C(X,\mathbb{Z})/\mathrm{im}(1-\alpha_*)\oplus C(X,\mathbb{Z})/\mathrm{im}(1-\beta_*) \arrow[d] \\
K_1(\mathcal{T}_E\otimes\mathcal{K}+\mathcal{K}\otimes\mathcal{T}_F) \arrow[u] & 0 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
K_0(\mathcal{T}_E\otimes\mathcal{K}+\mathcal{K}\otimes\mathcal{T}_F) \arrow[r] & C(X,\mathbb{Z}) \arrow[d] \\
K_1(\mathcal{O}_{F\otimes_A\mathcal{O}_E}) \arrow[u] & K_0(\mathcal{O}_{F\otimes_A\mathcal{O}_E}) \arrow[d] \\
0 \arrow[u] & K_1(\mathcal{T}_E\otimes\mathcal{K}+\mathcal{K}\otimes\mathcal{T}_F). \arrow[l]
\end{tikzcd}$$

Here we used the fact that $K_0(A) = C(X,\mathbb{Z})$ and that $K_1(A) = 0$.

Example 5.6. Let $E = \mathbb{C}^m, F = \mathbb{C}^n$ and fix $\chi : E \otimes F \to F \otimes E$ an isomorphism. The corresponding diagram (depending on χ) is

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{K}\otimes\mathcal{K} \arrow[r] \arrow[d] & \mathcal{E}_m\otimes\mathcal{K} \arrow[r] \arrow[d] & \mathcal{O}_m\otimes\mathcal{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K}\otimes\mathcal{E}_n \arrow[r] \arrow[d] & \mathcal{T}_{\mathbb{C}^m\otimes\mathcal{E}_n}\cong\mathcal{T}_{\mathbb{C}^n\otimes\mathcal{E}_m} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{C}^m\otimes\mathcal{E}_n}\cong\mathcal{T}_{\mathbb{C}^n\otimes\mathcal{O}_m} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K}\otimes\mathcal{O}_n \arrow[r] \arrow[d] & \mathcal{T}_{\mathbb{C}^m\otimes\mathcal{O}_n}\cong\mathcal{O}_{\mathbb{C}^n\otimes\mathcal{E}_m} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{C}^m\otimes\mathcal{O}_n}\cong\mathcal{O}_{\mathbb{C}^n\otimes\mathcal{O}_m} \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The exact sequences (depending on χ) are

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & K_0(\mathcal{E}_m \otimes \mathcal{K} + \mathcal{K} \otimes \mathcal{E}_n) \arrow[r] & \mathbb{Z}_{m-1} \oplus \mathbb{Z}_{n-1} \arrow[d] \\
0 \arrow[u] & K_1(\mathcal{E}_m \otimes \mathcal{K} + \mathcal{K} \otimes \mathcal{E}_n) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd}
K_0(\mathcal{E}_m \otimes \mathcal{K} + \mathcal{K} \otimes \mathcal{E}_n) \arrow[r] & \mathbb{Z} \arrow[r] & K_0(\mathcal{O}_{\mathbb{C}^m \otimes \mathcal{O}_n}) \arrow[d] \\
K_1(\mathcal{O}_{\mathbb{C}^m \otimes \mathcal{O}_n}) \arrow[u] & 0 \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

In particular, if χ is just the flip, we have

$$\mathcal{T}_{\mathbb{C}^m \otimes \mathcal{E}_n} \cong \mathcal{T}_{\mathbb{C}^n \otimes \mathcal{E}_m} \cong \mathcal{E}_n \otimes \mathcal{E}_m, \quad \mathcal{O}_{\mathbb{C}^m \otimes \mathcal{E}_n} \cong \mathcal{T}_{\mathbb{C}^n \otimes \mathcal{O}_m} \cong \mathcal{O}_m \otimes \mathcal{E}_n,$$

$$\mathcal{T}_{\mathbb{C}^m \otimes \mathcal{O}_n} \cong \mathcal{O}_{\mathbb{C}^n \otimes \mathcal{E}_m} \cong \mathcal{E}_m \otimes \mathcal{O}_n, \quad \mathcal{O}_{\mathbb{C}^m \otimes \mathcal{O}_n} \cong \mathcal{O}_{\mathbb{C}^n \otimes \mathcal{O}_m} \cong \mathcal{O}_m \otimes \mathcal{O}_n,$$

and we recover the K-theory of $\mathcal{O}_m \otimes \mathcal{O}_n$.

Example 5.7. Let $A = C(\mathbb{T})$ and $\sigma_i(x) = x^{p_i}, i = 1, 2$ with p_1, p_2 relatively prime. Then the associated conditional expectations commute, and using the K-theory computations in [D1], we get

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & K_0(\mathcal{T}_{\alpha_1} \otimes \mathcal{K} + \mathcal{K} \otimes \mathcal{T}_{\alpha_2}) \arrow[r] & \mathbb{Z}^2 \oplus \mathbb{Z}_{p_1-1} \oplus \mathbb{Z}_{p_2-1} \arrow[d] \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[u] & K_1(\mathcal{T}_{\alpha_1} \otimes \mathcal{K} + \mathcal{K} \otimes \mathcal{T}_{\alpha_2}) \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
K_0(\mathcal{T}_{\alpha_1} \otimes \mathcal{K} + \mathcal{K} \otimes \mathcal{T}_{\alpha_2}) \arrow[r] & \mathbb{Z} \arrow[r] & K_0(\mathcal{O}_{E_2 \otimes_A \mathcal{O}_{E_1}}) \arrow[d] \\
K_1(\mathcal{O}_{E_2 \otimes_A \mathcal{O}_{E_1}}) \arrow[u] & \mathbb{Z} \arrow[l] & K_1(\mathcal{T}_{\alpha_1} \otimes \mathcal{K} + \mathcal{K} \otimes \mathcal{T}_{\alpha_2}). \arrow[l]
\end{tikzcd}$$

More generally, we may consider coverings of the n-torus $\mathbb{T}^n$.

References

[D] DEACONU, VALENTIN. Generalized solenoids and C^*-algebras. *Pacific J. Math.* **190** (1999), no. 2, 247–260. MR1722892 (2000j:46103).

[D1] DEACONU, VALENTIN. Generalized Cuntz–Krieger algebras. *Proc. Amer. Math. Soc.* **124** (1996) 3427–3435. MR1343688 (97a:46081), Zbl 0864.46033.

[D2] DEACONU, VALENTIN. C^*-algebras of commuting endomorphisms. *Advances in operator algebras and mathematical physics*, Theta Ser. Adv. Math., 5, *Theta, Bucharest*, 2005, 47–55. MR2238282.

[DH] DOUGLAS, RONALD G.; HOWE, ROGER. On the C^*-algebra of Toeplitz operators on the quarter-plane. *Trans. Amer. Math. Soc.* **158** (1971), no.1, 203–217. MR0288591 (44 #5787), Zbl 0224.47015.

[ER] EXEL, RUY; RENAULT, JEAN. Semigroups of local homeomorphisms and interaction groups. Preprint arXiv:math.OA/0608589.

[EV] EXEL, RUY; VERSHIK, ANATOLY M. C^*-algebras of irreversible dynamical systems. *Canad. J. Math.* **58** (2006), no. 1, 39–63. MR2195591 (2007d:46058).

[F1] FOWLER, NEAL J. Discrete product systems of finite-dimensional Hilbert spaces, and generalized Cuntz algebras. Preprint arXiv:math.OA/9904116.

[F2] FOWLER, NEAL J. Discrete product systems of Hilbert bimodules. *Pacific J. Math.* **204** (2002), no. 2, 335–375. MR1907896 (2003g:46070), Zbl 1059.46034.

[FMR] FOWLER, NEAL J.; MUHLY, PAUL S.; RAEBURN, IAIN. Representations of Cuntz–Pimsner algebras. *Indiana Univ. Math. J.* **52** (2003), no. 3, 569–605. MR1986889 (2005d:46114), Zbl 1034.46054.

[KPW] KAJIWARA, TSUYOSHI; PINZARI, CLAUDIA; WATATANI, YASUO. Ideal structure and simplicity of the C^*-algebras generated by Hilbert bimodules. *J. Funct. Anal.* **159**(1998), no. 2, 295–323. MR1658088 (2000a:46094), Zbl 0942.46035.

[Ka1] KATSURA, TAKESHI. On C^*-algebras associated with C^*-correspondences. *J. Funct. Anal.* **217** (2004), no. 2, 366–401. MR2102572 (2005e:46099), Zbl 1067.46054.

[Ka2] KATSURA, TAKESHI. A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras II, examples. *International J. Math.* **17** (2006), no. 7, 791–833. MR2253144 (2007e:46051).

[K] KUMJIAN, ALEX. On certain Cuntz–Pimsner algebras. *Pacific J. Math.* **217** (2004), no. 2, 275–289. MR2109936 (2005g:46107), Zbl 1075.46048.

[KP] KUMJIAN, ALEX; PASK, DAVID. Higher rank graph C^*-algebras. *New York J. Math.* **6** (2000) 1–20. MR1745529 (2001b:46102), Zbl 0946.46044.

[LR] LINDIARNI, JANNY; RAEBURN, IAIN. Partial-isometric crossed products by semigroups of endomorphisms. *J. Operator Theory* **52** (2004), no. 1, 61–87. MR2091460 (2005g:46125), Zbl 1078.46047.

[MS] MUHLY, PAUL S.; SOLEL, BARUCH. Tensor algebras over C^*-correspondences: representations, dilations, and C^*-envelopes. *J. Funct. Anal.* **158** (1998) 389–457. MR1648483 (99j:46066), Zbl 0912.46070.

[Pi] PIMSNER, MICHAEL V. A class of C^*-algebras generalizing both Cuntz–Krieger algebras and crossed products by $\mathbb{Z}$. *Free Probability Theory* (D. Voiculescu, Ed.), 189–212, Fields Institute Communications, 12, *Amer. Math. Soc., Providence*, 1997. MR1426840 (97k:46069), Zbl 0871.46028.

[PV] PIMSNER, MIHAI V.; VOICULESCU, DAN-VIRGIL. Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras. *J. Operator Theory* **4** (1980), no. 1, 93–118. MR0587369 (82c:46074), Zbl 0474.46059.

[RS1] ROBERTSON, GUYAN; STEGER, TIM. Affine buildings, tiling systems and higher rank Cuntz–Krieger algebras. *J. Reine Angew. Math.* **513** (1999) 115–144. MR1713322 (2000j:46109), Zbl 1064.46504.

[RS2] ROBERTSON, GUYAN; STEGER, TIM. K-theory for rank two Cuntz–Krieger algebras. Preprint.

[S] SCHWEIZER, JÜRGEN. Crossed products by C^*-correspondences and Cuntz–Pimsner algebras. *C^*-algebras* (Münster, 1999), 203–226, *Springer, Berlin*, 2000. MR1798598 (2002f:46133), Zbl 0995.46045.

[VE] VAN ELST, ANDREAS. Gap-labeling theorems for Schrödinger operators on the square and cubic lattice. *Rev. Math. Phys.* **6** (1994), no. 2, 319–342. MR1269302 (95f:46122), Zbl 0804.47053.

[Z] ZACHARIAS, JOACHIM. Quasi-free automorphisms of Cuntz–Krieger–Pimsner algebras. *C^*-algebras* (Münster, 1999), 262–272, *Springer, Berlin*, 2000. MR1798602 (2002a:46079), Zbl 0976.46046.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEVADA, RENO NV 89557-0084, USA
vdeaconu@unr.edu

This paper is available via http://nyjm.albany.edu/j/2007/13-12.html.