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# The Choquet–Deny theorem and distal properties of totally disconnected locally compact groups of polynomial growth

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Abstract. We obtain sufficient and necessary conditions for the Choquet–Deny theorem to hold in the class of compactly generated totally disconnected locally compact groups of polynomial growth, and in a larger class of totally disconnected generalized $\overline{FC}$-groups. The following conditions turn out to be equivalent when $G$ is a metrizable compactly generated totally disconnected locally compact group of polynomial growth:

(1) The Choquet–Deny theorem holds for $G$.
(2) The group of inner automorphisms of $G$ acts distally on $G$.
(3) Every inner automorphism of $G$ is distal.
(4) The contraction subgroup of every inner automorphism of $G$ is trivial.
(5) $G$ is a SIN group.

We also show that for every probability measure $\mu$ on a totally disconnected compactly generated locally compact second countable group of polynomial growth, the Poisson boundary is a homogeneous space of $G$, and that it is a compact homogeneous space when the support of $\mu$ generates $G$.

Contents



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## 1. Introduction

Let $\mu$ be a regular Borel probability measure on a locally compact group $G$. A bounded Borel function $h\colon G \to \mathbb{C}$ is called *$\mu$-harmonic* if it satisfies

$$h(g) = \int_G h(gg')\,\mu(dg'), \quad g \in G. \tag{1.1}$$

The classical Choquet–Deny theorem asserts that when $G$ is abelian then every bounded continuous $\mu$-harmonic function is constant on the (left) cosets of the smallest closed subgroup, $G_\mu$, containing the support of $\mu$.

The Choquet–Deny theorem remains true for many nonabelian locally compact groups, e.g., 2-step nilpotent groups [10], nilpotent [SIN] groups [14], and compact groups. But it does not hold for all groups. If the theorem holds for a probability measure $\mu$ then $G_\mu$ must necessarily be an amenable subgroup [7, 30]. It follows that groups for which the theorem is valid are necessarily amenable. However, the theorem is not true for every amenable group [23].

The stronger condition, that $G$ have polynomial growth, is sufficient for the theorem to hold when $G$ is a finitely generated (discrete) group [23, 21]. When $G$ is finitely generated and solvable then the theorem holds if and only if $G$ has polynomial growth [21]. In general, the theorem fails for discrete groups of polynomial growth that are not finitely generated, in particular, it is not true for locally finite groups [23]. It appears that the largest class of discrete groups known today for which the Choquet–Deny theorem is true is the class of FC-hypercentral groups [15]. This class is a proper subclass of the class of discrete groups of polynomial growth, while finitely generated FC-hypercentral groups are precisely the finitely generated groups of polynomial growth. We do not know of any discrete groups for which the Choquet–Deny theorem is true and which are not FC-hypercentral.

A probability measure $\mu$ on a locally compact group $G$ is called *spread out* if for some $n$ the convolution power $\mu^n$ is nonsingular. With the restriction that $\mu$ be spread out the Choquet–Deny theorem holds for all locally compact nilpotent groups [2, 19] and for compactly generated locally compact groups of polynomial growth [19]. When $G$ is almost connected, then $G$ has polynomial growth if and only if the Choquet–Deny theorem holds for every spread out measure [19]. The same is true when $G$ is a Zariski-connected $p$-adic algebraic group [27, Theorem 4.2]. While it remains an open question whether the spread out assumption can be disposed of when $G$ is nilpotent,[1] it is known that the Choquet–Deny theorem is not true for arbitrary probability measures on compactly generated locally compact groups of polynomial growth [16, Remark 3.15].

The main goal of the present article is to obtain necessary and sufficient conditions for the validity of the Choquet–Deny theorem in the class of compactly generated totally disconnected locally compact groups of polynomial growth, and in a larger class of totally disconnected 'generalized $\overline{FC}$-groups' [3, 24]. It turns out that the key to finding such conditions is a study of distal properties of totally disconnected groups. This motivates our investigations in the next section, which can also be of quite independent interest. The Choquet–Deny theorem for generalized $\overline{FC}$-groups is discussed in Section 3. In Section 4 we remark on the structure

[1] The recently published proof [28] is wrong (the implications (i)⇒(ii) and (i)⇒(iii) in the key Lemma 2.5 are false).

of boundaries of random walks on compactly generated totally disconnected groups of polynomial growth and certain related groups.

## 2. Distal properties of totally disconnected locally compact groups

Let $G$ be a Hausdorff topological group and $\Gamma$ a subgroup of $\mathrm{Aut}(G)$, the group of topological automorphisms of $G$. We will say that $\Gamma$ is *distal* (or acts distally on $G$) if for any $x \in G - \{e\}$, the identity element $e$ is not in the closure of the orbit $\Gamma x = \{\gamma(x)\,;\,\gamma \in \Gamma\}$. A single automorphism $\gamma \in \mathrm{Aut}(G)$ will be called distal if the subgroup, $\langle\gamma\rangle$, it generates acts distally on $G$. An element $g$ of $G$ will be called distal if the corresponding inner automorphism $\gamma(\cdot) = g \cdot g^{-1}$ is distal. We will say that $G$ is distal if the group $\mathrm{Inn}(G)$ of inner automorphisms of $G$ acts distally on $G$.

Trivially, if $G$ is distal then every $g \in G$ is distal. While the converse is not true in general, Rosenblatt [29] proved that when $G$ is an almost connected locally compact group then $G$ is distal if and only if every $g \in G$ is distal; moreover $G$ is distal if and only if it has polynomial growth. According to [26] this remains true also for certain classes of $p$-adic Lie groups. However, there are many locally compact groups of polynomial growth that are not distal. For example, the semidirect product $K \times_\tau \mathbb{Z}$ where $K$ is a nontrivial compact metric group and $\tau$ is an ergodic automorphism of $K$, will never be distal.

Given $\gamma \in \mathrm{Aut}(G)$ the *contraction subgroup* of $\gamma$ is the subgroup $C(\gamma) = \{x \in G\,;\, \lim_{n\to\infty} \gamma^n(x) = e\}$. When $\gamma$ is the inner automorphism $\gamma(\cdot) = g \cdot g^{-1}$, we will write $C(g)$ for $C(\gamma)$. Obviously, if $\tau \in \mathrm{Aut}(G)$ is distal then $C(\tau) = C(\tau^{-1}) = \{e\}$. When $G$ is a Lie group, the three conditions: *$\Gamma$ is distal*; *every $\gamma \in \Gamma$ is distal*; and, *$C(\gamma) = \{e\}$ for every $\gamma \in \Gamma$*, are equivalent for every subgroup $\Gamma$ of $\mathrm{Aut}(G)$ [1].

Recall that a subgroup $\Gamma$ of $\mathrm{Aut}(G)$ is equicontinuous (at $e$) if and only if $G$ admits a neighbourhood base at $e$ consisting of neighbourhoods that are invariant under $\Gamma$. When $G$ is locally compact and totally disconnected then $\Gamma$ is equicontinuous if and only if compact open subgroups invariant under $\Gamma$ form a neighbourhood base at $e$. Equicontinuous automorphism groups are obviously distal. A SIN group is a topological group $G$ for which $\mathrm{Inn}(G)$ is equicontinuous. SIN groups are distal but, in general, distal groups are not SIN groups (e.g., a nilpotent group need not be SIN but every nilpotent group is distal [29]).

Our goal in this section is to prove that for a class of compactly generated totally disconnected locally compact groups, including groups of polynomial growth, the four conditions: *$G$ is distal*; *every $g \in G$ is distal*; *$C(g) = \{e\}$ for every $g \in G$*; and, *$G$ is SIN*, are equivalent. In the following section we will show that for this class of groups the four conditions and the condition that $G$ have polynomial growth, are equivalent to the condition that the Choquet–Deny theorem hold for $G$.

Some of the recent results of Baumgartner and Willis on contractions subgroups [4], based on Willis' theory of tidy subgroups [32], play a key role in our argument. These results are proven for metrizable groups, hence, in many of our results we need to assume metrizability.[2]

**Proposition 2.1.** *If $G$ is a totally disconnected metrizable locally compact group then for every $\tau \in \mathrm{Aut}(G)$ the following conditions are equivalent*:

[2]See note added in proof.

(i) *$\tau$ is distal.*
(ii) $C(\tau) = C(\tau^{-1}) = \{e\}$.
(iii) *For every compact open subgroup $U$ there exists $k = 0, 1, \dots$ such that* $\tau\big(\bigcap_{i=0}^{k} \tau^i(U)\big) = \bigcap_{i=0}^{k} \tau^i(U)$.
(iv) *$\langle\tau\rangle$ is equicontinuous.*

**Proof.** The only nonobvious implication in the chain (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(i) is (ii)$\Rightarrow$(iii). Let $U$ be a compact open subgroup. Since $C(\tau)$ is closed, by [4, Theorem 3.32] there exists $k$ such that $V = \bigcap_{i=0}^{k} \tau^i(U)$ is tidy for $\tau$. But as $C(\tau) = C(\tau^{-1}) = \{e\}$, [4, Proposition 3.24] implies that $s(\tau) = s(\tau^{-1}) = 1$ where $s\colon \mathrm{Aut}(G) \to \mathbb{N}$ is the scale function. Since $s(\tau) = [\tau(V) : V \cap \tau(V)]$ and $s(\tau^{-1}) = [\tau^{-1}(V) : V \cap \tau^{-1}(V)]$, so $\tau(V) = V$. $\square$

**Lemma 2.2.** *Let $\Gamma$ be a subgroup of* $\mathrm{Aut}(G)$ *where $G$ is a totally disconnected metrizable locally compact group. If $\tau_1, \tau_2, \dots, \tau_n \in \mathrm{Aut}(G)$ are distal and for every $j = 1, 2, \dots, n$, $[\tau_j, \langle\Gamma \cup \{\tau_1, \dots, \tau_{j-1}\}\rangle] \subseteq \langle\Gamma \cup \{\tau_1, \dots, \tau_{j-1}\}\rangle$, then for every compact open subgroup $U$ invariant under $\Gamma$ there exists a compact open subgroup $V \subseteq U$ invariant under $\langle\Gamma \cup \{\tau_1, \dots, \tau_n\}\rangle$.*

**Proof.** It is clear that the lemma follows by induction once it is verified for $n = 1$. So we suppose that $n = 1$.

By Proposition 2.1 there exists $k$ such that $V = \bigcap_{i=0}^{k} \tau_1^i(U)$ satisfies $\tau_1(V) = V$. It is enough to show that $\gamma(V) = V$ for every $\gamma \in \Gamma$. But our assumption implies that $[\tau_1^i, \Gamma] \subseteq \Gamma$ for every $i = 0, 1, \dots$ . Hence, given $\gamma \in \Gamma$ we obtain $\gamma(V) = \bigcap_{i=0}^{k} (\gamma\tau_1^i)(U) = \bigcap_{i=0}^{k} (\tau_1^i \gamma [\gamma, \tau_1^i])(U) = \bigcap_{i=0}^{k} \tau_1^i(U) = V$. $\square$

**Lemma 2.3.** *Let $\Gamma$ be a subgroup of* $\mathrm{Aut}(G)$ *where $G$ is a totally disconnected metrizable locally compact group. Suppose that every $\gamma \in \Gamma$ is distal and that $\Gamma$ has a normal equicontinuous subgroup $\Gamma_1$ with the quotient $\Gamma/\Gamma_1$ containing a polycyclic subgroup of finite index. Then $\Gamma$ is equicontinuous.*

**Proof.** Let $\Omega$ be a neighbourhood of $e$. Denote by $P$ the polycyclic subgroup of finite index in $\Gamma/\Gamma_1$ and let $P_0 = P$, $P_1 = [P, P]$, $P_2 = [P_1, P_1], \dots, P_m = \{\Gamma_1\}$ be the derived series for $P$. Write $\pi$ for the canonical homomorphism $\pi\colon \Gamma \to \Gamma/\Gamma_1$ and put $\hat{P}_j = \pi^{-1}(P_j)$ for $j = 0, 1, ..., m$.

Suppose that for some $j = 1, 2, ..., m$, $V \subseteq \Omega$ is a compact open subgroup invariant under $\hat{P}_j$. We will show that there is then a compact open subgroup $W \subseteq V$ invariant under $\hat{P}_{j-1}$. Now, since $P$ is polycyclic, $P_{j-1}$ is generated by a finite set $\{p_1, \dots, p_n\}$. For every $i = 1, 2, \dots, n$ find $\tau_i \in \hat{P}_{j-1}$ with $p_i = \pi(\tau_i)$. Applying Lemma 2.2 to $\hat{P}_j$ and $\tau_1, \dots, \tau_n$ we conclude that there is a compact open subgroup $W \subseteq V$ invariant under $\langle\hat{P}_j \cup \{\tau_1, \dots, \tau_n\}\rangle = \hat{P}_{j-1}$.

Our assumption is that there is a compact open subgroup $V \subseteq \Omega$ such that $\gamma(V) = V$ for every $\gamma \in \hat{P}_m = \Gamma_1$. With the aid of the preceding paragraph it then follows that there is a compact open subgroup $W \subseteq \Omega$ invariant under $\hat{P}_0$. Next, since $P = P_0$ has finite index in $\Gamma/\Gamma_1$, $\hat{P}_0$ has finite index in $\Gamma$. Hence, the intersection $U = \bigcap_{\gamma \in \Gamma} \gamma(W)$ is a compact open subgroup invariant under $\Gamma$ and contained in $\Omega$. $\square$

**Corollary 2.4.** *Let $G$ be a totally disconnected metrizable locally compact group. If a subgroup $\varGamma$ of* $\mathrm{Aut}(G)$ *contains a polycyclic subgroup of finite index then the following conditions are equivalent*:

(i) *$\varGamma$ is distal.*
(ii) *Every $\gamma \in \varGamma$ is distal.*
(iii) *$\varGamma$ is equicontinuous.*

As the following examples show, 'polycyclic' in Corollary 2.4 cannot be replaced by 'solvable'. In fact, the three conditions can be different for countable abelian groups of automorphisms. We do not know if 'polycyclic' can be replaced by 'finitely generated solvable'.

**Example 2.5.** Let $\varphi\colon \mathbb{R} \to \mathbb{T}$ denote the function $\varphi(t) = e^{2\pi i t}$ and let $H$ be any infinite subgroup of $\varphi(\mathbb{Q})$. Note that every $h \in H$ has finite order. Let $G$ be the totally disconnected compact abelian group $G = \mathbb{Z}_2^H$. $H$ acts on $G$ by left translations: $(hf)(x) = f(h^{-1}x)$ $(h \in H,\ f \in G,\ x \in H)$. Let $\varGamma$ be the resulting subgroup of $\mathrm{Aut}(G)$. Then every element of $\varGamma$ is distal because it has finite order. However, $\varGamma$ is not distal. Indeed, let $f \in G$ be the function $f(x) = \delta_{1x}$ and let $U$ be any neighbourhood of $e$ in $G$. Then for some finite subset $F \subseteq H$, $U$ contains the set $\{g \in G\,;\, g(x) = 0 \text{ for every } x \in F\}$. Hence, if $h \in H - F$ then $(hf)(x) = 0$ for every $x \in F$, i.e., $hf \in U$.

Thus for a countable abelian group of automorphisms (ii) does not imply (i) (nor (iii)).

**Example 2.6.** Let for $j \in \mathbb{Z}$, $G_j = \{x \in \mathbb{Z}_2^{\mathbb{Z}}\,;\, x_i = 0 \text{ for every } i \le j\}$ and let $G = \bigcup_{j\in\mathbb{Z}} G_j$. There is a locally compact totally disconnected group topology on $G$ in which the subgroups $G_j$, $j \in \mathbb{Z}$, form a neighbourhood base at $e$ (and are compact open). Given $j \in \mathbb{N}$ define $\tau_j \in \mathrm{Aut}(G)$ by $\tau_j(x) = y$ where $y_i = x_i$ for $i \in \mathbb{Z} - \{\pm j\}$, $y_j = x_{-j}$, and $y_{-j} = x_j$. The subgroup $\varGamma$ of $\mathrm{Aut}(G)$ generated by $\tau_j$, $j \in \mathbb{N}$, is then abelian and distal: if $x \ne e$ and $j$ is the smallest integer with $x_j \ne 0$, then $\tau(x) \notin G_{|j|}$ for any $\tau \in \varGamma$. However, $\varGamma$ is not equicontinuous because if it were, there would exist $k \ge 0$ with $\tau(x) \in G_0$ for every $x \in G_k$ and $\tau \in \varGamma$. However, if $x = (\delta_{k+1\,n})_{n\in\mathbb{Z}}$ then $x \in G_k$, but $(\tau_{k+1}(x))_{-k-1} = x_{k+1} = 1$, so that $\tau_{k+1}(x) \notin G_0$.

Thus for a countable abelian group of automorphisms (i) does not imply (iii).

**Theorem 2.7.** *Suppose that a totally disconnected metrizable locally compact group $G$ admits an open normal SIN subgroup $N$ such that the quotient $G/N$ contains a polycyclic subgroup of finite index. Then the following conditions are equivalent*:

(i) *$G$ is distal.*
(ii) *Every $g \in G$ is distal.*
(iii) *$G$ is SIN.*

**Proof.** (ii)⇒(iii): Let $\varGamma = \mathrm{Inn}(G)$ and $\alpha : G \to \varGamma$ be the canonical homomorphism. Put $\varGamma_1 = \alpha(N)$. $\varGamma_1$ is a normal subgroup of $\varGamma$ and $\varGamma/\varGamma_1$ is a homomorphic image of $G/N$. Therefore $\varGamma/\varGamma_1$ contains a polycyclic subgroup of finite index. Since $N$ is open and SIN, it follows that $\varGamma_1$ is equicontinuous. Thus Lemma 2.3 applies. □

Following [3] and [24] we call a locally compact group a *generalized $\overline{FC}$-group* if $G$ has a series $G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_n = \{e\}$ of closed normal subgroups such that for every $i = 0, 1, \ldots, n-1$, $G_i/G_{i+1}$ is a compactly generated group with precompact conjugacy classes. Every compactly generated locally compact group of polynomial growth is a generalized $\overline{FC}$ group [24, Theorem 2]. Every closed subgroup of a generalized $\overline{FC}$ group is compactly generated [24, Proposition 2]. A locally compact solvable group $G$ is a generalized $\overline{FC}$ group if and only if each closed subgroup of $G$ is compactly generated [10, Théorème III.1]. Using Propositions 1 and 7(ii) in [24] it is straighforward to give the following characterization of totally disconnected generalized $\overline{FC}$ groups:

**Proposition 2.8.** *A totally disconnected locally compact group $G$ is a generalized $\overline{FC}$ group if and only if it admits a compact open normal subgroup $N$ with the quotient $G/N$ containing a polycyclic subgroup of finite index.*

**Theorem 2.9.** *Conditions* (i), (ii)*, and* (iii) *of Theorem* 2.7 *are equivalent when $G$ is a totally disconnected generalized $\overline{FC}$-group.*

**Proof.** When $G$ is metrizable, this is a special case of Theorem 2.7. We need to show that the implication (ii)⇒(iii) is also true when $G$ is not metrizable.

Note that $G$ is necessarily $\sigma$-compact (as it is compactly generated). Let $U$ be a neighbourhood of $e$ contained in the subgroup $N$ of Proposition 2.8. Find a neighbourhood $V$ of $e$ with $V^2 \subseteq U$. By [11, Theorem 8.7] $V$ contains a compact normal subgroup $K$ such that $G/K$ is metrizable. Let $\pi \colon G \to G/K$ denote the canonical homomorphism. Since $N$ is compact, we can use the theorem stating that a factor of a distal flow is distal [5, Corollary 6.10, p. 52] to conclude that the restriction of every inner automorhism of $G/K$ to $N/K$ is distal. As $N/K$ is open, every $g \in G/K$ is then distal. Hence, by Theorem 2.7, $G/K$ is SIN. Thus $\pi(V)$ contains a compact open normal subgroup $W$. Then $\hat{W} = \pi^{-1}(W) \subseteq VK \subseteq U$ and $\hat{W}$ is a compact open normal subgroup of $G$. □

Since nilpotent groups are distal, Theorem 2.9 implies that a totally disconnected compactly generated locally compact nilpotent group is a SIN group, a result due to Hofmann, Liukkonen, and Mislove [12].

We note that for totally disconnected groups of polynomial growth which are not compactly generated, conditions (i), (ii), (iii) are different. In fact, the equivalence fails already for metabelian groups of polynomial growth. Examples of totally disconnected 2-step nilpotent groups which are not SIN groups can be found in [12] and [31]. An example of a metabelian group of polynomial growth which satisfies (ii) but not (i) (nor (iii)) is also readily available:

**Example 2.10.** Let $H$ be as in Example 2.5 and let $W$ be the complete wreath product $W = \mathbb{Z}_2 \wr H$. Evidently, $W$ is not distal but every $w \in W$ has finite order, so is distal.

In the remainder of this section we prove that conditions (i), (ii), (iii) of Theorem 2.7 are equivalent for every metrizable compactly generated totally disconnected locally compact metabelian group.

**Lemma 2.11.** *If a locally compact group $G$ contains a normal finitely generated subgroup $N$ and a compact set $K$ such that $KN = G$, then $G$ is a SIN group.*

**Proof.** Recall that the group of automorphisms of a finitely generated group is countable. Since the centralizer $C_G(N)$ of $N$ in $G$ is the kernel of the homomorphism which maps $g \in G$ to the restriction of the inner automorphism $g \cdot g^{-1}$ to $N$, it follows that $G/C_G(N)$ is countable. As $C_G(N)$ is a closed subgroup and $G$ is of the second category, we conclude that $C_G(N)$ is open.

Let $U$ be a neighbourhood of $e$. Put $V = U \cap C_G(N)$ and let $V'$ be a neighbourhood of $e$ with $g^{-1}V'g \subseteq V$ for every $g \in K$. Then $W = \bigcap_{g\in G} gVg^{-1} = \bigcap_{g\in K} gVg^{-1} \supseteq V'$. Thus $W$ is a neighbourhood of $e$, invariant under $\mathrm{Inn}(G)$ and contained in $U$. $\square$

**Proposition 2.12.** *If a compactly generated totally disconnected locally compact group $G$ contains a closed cocompact normal SIN subgroup, then $G$ is a SIN group.*

**Proof.** Let $N$ denote the closed cocompact normal SIN subgroup and let a compact open subgroup $U$ of $G$ be given. A routine argument shows that $\mathrm{Inn}(G)$ acts equicontinuously on $N$. Hence, $U$ contains a compact subgroup $V$ of $N$ which is open in $N$ and normal in $G$.

Let $\pi\colon G \to G/V$ denote the canonical homomorphism. Since $N$ is cocompact, it is compactly generated [25]. Since $V$ is open in $N$, $\pi(N)$ is then finitely generated. It is also normal and there is a compact $K \subseteq G/V$ with $K\pi(N) = G/V$. Hence, by Lemma 2.11 $G/V$ is a (totally disconnected) SIN group. Thus $\pi(U)$ contains a compact open normal subgroup $W$. Then $\pi^{-1}(W)$ is a compact open normal subgroup contained in $UV = U$. $\square$

It is well-known that Proposition 2.12 is false for locally compact groups in general (e.g., the motion groups). The following example shows that it can also fail for totally disconnected groups which are not compactly generated:

**Example 2.13.** Let $\mathbb{Z}^{*\mathbb{N}} = \{x \in \mathbb{Z}^{\mathbb{N}}\,;\, x_i \neq 0 \text{ for finitely many } i\}$ and give $\mathbb{Z}^{*\mathbb{N}}$ the discrete topology. Give the multiplicative group $\{-1,1\}^{\mathbb{N}}$ the product topology. Let $\varphi\colon \{-1,1\}^{\mathbb{N}} \to \mathrm{Aut}(\mathbb{Z}^{*\mathbb{N}})$ be given by $\varphi\big((\omega_i)_{i=1}^{\infty}\big)\big((x_i)_{i=1}^{\infty}\big) = (\omega_i x_i)_{i=1}^{\infty}$ and let $G$ be the semidirect product $G = \mathbb{Z}^{*\mathbb{N}} \times_{\varphi} \{-1,1\}^{\mathbb{N}}$.

$\mathbb{Z}^{*\mathbb{N}} \times \{e\}$ is trivially a closed cocompact normal SIN subgroup of $G$ but $G$ is not a SIN group because for every nonidentity element $g = (e,w) \in \{e\} \times \{-1,1\}^{\mathbb{N}}$ there is $a \in \mathbb{Z}^{*\mathbb{N}}$ with $(a,e)(e,w)(a,e)^{-1} \notin \{e\} \times \{-1,1\}^{\mathbb{N}}$. Indeed, if $w_j = -1$ and $a = (\delta_{ji})_{i=1}^{\infty}$ then $(a,e)(e,w)(a,e)^{-1} = (v,w)$ where $v_j = 2$.

**Lemma 2.14.** *Let $G$ be a locally compact compactly generated totally disconnected solvable group. Then there exists a closed normal cocompact subgroup $N$ such that $\overline{[G,G]} \subseteq N$ and $N/\overline{[G,G]}$ is topologically isomorphic to $\mathbb{Z}^d$ for some $d \geq 0$.*

**Proof.** $G/\overline{[G,G]}$ is a compactly generated totally disconnected abelian group. Hence, it is the direct product $AB$ where $A \cong \mathbb{Z}^d$ and $B$ is a totally disconnected compact abelian group. Put $N = \pi^{-1}(A)$ where $\pi\colon G \to G/\overline{[G,G]}$ is the canonical homomorphism. $\square$

**Theorem 2.15.** *Conditions* (i), (ii), *and* (iii) *of Theorem* 2.7 *are equivalent when $G$ is a metrizable compactly generated totally disconnected locally compact metabelian group.*

**Proof.** Let $N$ be as in Lemma 2.14. To prove the nontrivial implication (ii)$\Rightarrow$(iii) observe that as $\overline{[G,G]}$ is abelian, Theorem 2.7 applies to $N$. Thus if (ii) holds then $N$ is a SIN group. But then $G$ is a SIN group by Proposition 2.12. $\square$

## 3. The Choquet–Deny theorem

Let $\mu$ be a regular Borel probability measure on a locally compact group $G$. Recall that $G_\mu$ denotes the smallest closed subgroup containing the support of $\mu$. $\mu$ is called *adapted* if $G_\mu = G$. We will say that $\mu$ is a *Choquet–Deny measure* if every bounded continuous $\mu$-harmonic function is constant on the left cosets of $G_\mu$.

We note that in the literature the Choquet–Deny theorem is often understood as the statement that every adapted $\mu \in M_1(G)$ is a Choquet–Deny measure (i.e., all bounded continuous $\mu$-harmonic functions are constant). We emphasize that in this paper the Choquet–Deny theorem is understood as the (formally) stronger statement that *every* $\mu \in M_1(G)$ is a Choquet–Deny measure. It is not known if the two versions of the Choquet–Deny theorem are equivalent. However, we know of examples of almost connected Lie groups with the property that every adapted spread out probability measure is Choquet–Deny but some nonadapted spread out measures are not. It can be shown (see Lemma 4.1) that the strong version of the theorem is true about $G$ if and only if the weak version holds for every closed subgroup of $G$.

Throughout the sequel by the weak topology on the set $M_1(\mathcal{X})$ of probability measures on a locally compact space $\mathcal{X}$ we mean the $\sigma(M_1(\mathcal{X}), C_b(\mathcal{X}))$-topology where $C_b(\mathcal{X})$ is the algebra of bounded continuous functions on $\mathcal{X}$.

**Lemma 3.1.** (a) *If $\mu$ is a Choquet–Deny measure on $G$ and $N \subseteq G$ is a closed normal subgroup, then the projection of $\mu$ onto $G/N$ is a Choquet–Deny measure on $G/N$.*

(b) *If every neighbourhood of $e$ contains a compact normal subgroup $N$ such that the projection of $\mu$ onto $G/N$ is a Choquet–Deny measure, then $\mu$ is a Choquet–Deny measure.*

**Proof.** We omit a straightforward proof of (a). To prove (b) let us choose, for every neighbourhood $\Omega$ of $e$, a compact normal subgroup $N_\Omega \subseteq \Omega$ such that the projection of $\mu$ onto $G/N_\Omega$ is a Choquet–Deny measure. Denote by $\pi_\Omega \colon G \to G/N_\Omega$ the canonical homomorphism and by $\omega_\Omega$ the normalized Haar measure of $N_\Omega$. Directing the neighbourhoods of $e$ by reversed inclusion we obtain a net $(\omega_\Omega)$ in $M_1(G)$ which converges weakly to $\delta_e$.

Let $h$ be a bounded continuous $\mu$-harmonic function. We need to show that $h(xy) = h(x)$ for every $x \in G$ and $y \in G_\mu$. Now, when $\Omega$ is a neighbourhood of $e$, the function $\omega_\Omega * h$ is a bounded continuous $\mu$-harmonic function constant on the cosets of $N_\Omega$. Hence, $\omega_\Omega * h = h_\Omega \circ \pi_\Omega$ for a bounded continuous function $h_\Omega$ on $G/N_\Omega$. It is clear that $h_\Omega$ is $\pi_\Omega\mu$-harmonic where $\pi_\Omega\mu$ denotes the projection of $\mu$ onto $G/N_\Omega$. Moreover, $\pi_\Omega(G_\mu) = (G/N_\Omega)_{\pi_\Omega\mu}$. Therefore for $x \in G$ and $y \in G_\mu$, $(\omega_\Omega * h)(xy) = h_\Omega(\pi_\Omega(x)\pi_\Omega(y)) = h_\Omega(\pi_\Omega(x)) = (\omega_\Omega * h)(x)$. Since $(\omega_\Omega * h)(\cdot) = \int_G h(g^{-1}\cdot)\, \omega_\Omega(dg)$ and $\text{w-lim}_\Omega\, \omega_\Omega = \delta_e$, we conclude that $h(xy) = h(x)$. □

**Lemma 3.2.** *Let $(\mu_\alpha)$ be a net in $M_1(G)$. If for every neighbourhood $U$ of $e$ there exists $\varepsilon \in M_1(G)$ such that $\varepsilon(U) = 1$ and the net $(\mu_\alpha * \varepsilon)$ converges weakly, then the net $(\mu_\alpha)$ converges weakly.*

**Proof.** There exists a compactly supported $\nu \in M_1(G)$ such that the net $(\mu_\alpha * \nu)$ is weakly convergent, and, hence, tight. This implies that the net $(\mu_\alpha)$ itself is tight. Then by Prohorov's theorem, every subnet of $(\mu_\alpha)$ has a weak cluster point.

Therefore it suffices to show that the net $(\mu_\alpha)$ has a unique cluster point. But if $\mu'$ and $\mu''$ are cluster points of the net, then, due to our assumption, for every neighbourhood $U$ of $e$ there exists $\varepsilon \in M_1(G)$ such that $\varepsilon(U) = 1$ and $\mu' * \varepsilon = \mu'' * \varepsilon$. As in the proof of Lemma 3.1 we obtain a net $(\varepsilon_i)$ in $M_1(G)$ which converges weakly to $\delta_e$ and satisfies $\mu' * \varepsilon_i = \mu'' * \varepsilon_i$ for every $i$. Hence, $\mu' = \mu''$. □

**Lemma 3.3.** *Let $G$ be a totally disconnected locally compact group, $\tau \in \mathrm{Aut}(G)$, and $F$ a finite subset of $C(\tau)$. If $\nu \in M_1(G)$ and $\nu(F) = 1$ then the sequence $\nu * \tau\nu * \cdots * \tau^{n-1}\nu$ converges weakly to a probability measure $\rho$ such that $\nu * \tau\rho = \rho$.*

**Proof.** It is clear that if $\rho = \text{w-lim}_{n\to\infty} \nu * \tau\nu * \cdots * \tau^{n-1}\nu$ then $\nu * \tau\rho = \rho$. To see that the limit exists let $U$ be a compact open subgroup. Then there is $k \in \mathbb{N}$ such that for every $n \geq k$, $\tau^n(F) \subseteq U$. Let $\omega_U$ denote the normalized Haar measure of $U$. Then for $n \geq k$, $\tau^n\nu * \omega_U = \omega_U$. Hence, $\nu * \tau\nu * \cdots * \tau^{n-1}\nu * \omega_U$ converges to $\nu * \tau\nu * \cdots * \tau^{k-1}\nu * \omega_U$. By Lemma 3.2, $\nu * \tau\nu * \cdots * \tau^{n-1}\nu$ converges weakly. □

**Lemma 3.4.** *If $G$ is a locally compact group and $z \in G$ then $\overline{C(z)} \cap \langle z\rangle = \{e\}$.*

**Proof.** Suppose that $z^k \in \overline{C(z)}$ for some $k > 0$. Since $C(z) \subseteq C(z^k)$, we obtain $z^k \in \overline{C(z^k)}$. But when $U$ is a neighbourhood of $e$ in $\overline{C(z^k)}$, then $\overline{C(z^k)} = \overline{\bigcup_{n=1}^\infty z^{-kn}Uz^{kn}}$. This means that $\overline{C(z^k)}$ is either a *strange* group [22, Definition 1.1], or is compact. Since no locally compact group is strange [22, Theorem 1.8], $\overline{C(z^k)}$ is compact. As $z^k \in \overline{C(z^k)}$, it follows that $C(z) = \{e\}$. □

Suppose that the locally compact group $G$ acts on a locally compact space $\mathcal{X}$ so that the mapping $G \times \mathcal{X} \ni (g, x) \to gx \in \mathcal{X}$ is continuous. Given $\rho \in M_1(\mathcal{X})$ and $g \in G$ we write $g\rho$ for the measure $(g\rho)(\cdot) = \rho(g^{-1}\cdot)$. Given $\mu \in M_1(G)$ we denote by $\mu * \rho$ the measure $(\mu * \rho)(\cdot) = \int_G (g\rho)(\cdot)\,\mu(dg)$. Now, if $\rho = \mu * \rho$ then for every bounded continuous function $f\colon \mathcal{X} \to \mathbb{C}$, the function $h(g) = \int_{\mathcal{X}} f(gx)\,\rho(dx) = \int_{\mathcal{X}} f(x)\,(g\rho)(dx)$ is a bounded continuous $\mu$-harmonic function. Therefore in order to show that the Choquet–Deny theorem fails for $\mu$ it suffices to find $g \in G_\mu$ such that $g\rho \neq \rho$. This observation is being used in the proof of the next lemma.

**Lemma 3.5.** *Let $G$ be a totally disconnected locally compact group and $z$ an element of $G$ with $C(z) \neq \{e\}$. Let $g \in C(z) - \{e\}$, and $\nu = p\delta_g + (1-p)\delta_e$ where $p \in (0,1) - \{\frac{1}{2}\}$. Then the Choquet–Deny theorem is false for the measure $\mu = \nu * \delta_z$.*

**Proof.** Let $\tau$ denote the the inner automorphism $z \cdot z^{-1}$. By Lemma 3.3 the limit $\rho = \text{w-lim}_{n\to\infty} \nu * \tau\nu * \cdots * \tau^{n-1}\nu$ exists and satisfies $\nu * \tau\rho = \rho$. Moreover, $\rho(\overline{C(z)}) = 1$.

Note that $\langle z\rangle$ is necessarily infinite and discrete, so it is a closed subgroup of $G$ (isomorphic to $\mathbb{Z}$). Let $\pi\colon G \to G/\langle z\rangle$ denote the canonical mapping and let $\hat{\rho} = \pi\rho$. Then $\mu * \hat{\rho} = \pi(\mu * \rho) = \pi(\nu * \delta_z * \rho) = \pi(\nu * \tau\rho * \delta_z) = \pi\rho = \hat{\rho}$. Since $g \in G_\mu$, it suffices to show that $g\hat{\rho} \neq \hat{\rho}$.

Now, there exists a compact subgroup $U$ of $\overline{C(z)}$ such that $g \notin U$ but $\tau^j(g) \in U$ for every $j \geq 1$. Let $\omega_U$ be the normalized Haar measure of $U$. Then $\nu * \tau\nu * \cdots * \tau^{n-1}\nu * \omega_U = \nu * \omega_U = p(g\omega_U) + (1-p)\omega_U$. Thus $\rho * \omega_U = p(g\omega_U) + (1-p)\omega_U$ and $g(\rho * \omega_U) = p(g^2\omega_U) + (1-p)(g\omega_U)$. Since $\rho(\overline{C(z)}) = 1$ and by Lemma 3.4 $\overline{C(z)} \cap \langle z\rangle = \{e\}$, we obtain $\hat{\rho}(\pi(U)) = \rho(U\langle z\rangle) = \rho(U) = (\rho * \omega_U)(U) = 1 - p$ and $(g\hat{\rho})(\pi(U)) = (g\rho)(U\langle z\rangle) = (g\rho)(U) = (g(\rho * \omega_U))(U) = p(g^2\omega_U)(U) \neq 1 - p$. □

**Theorem 3.6.** *Let $G$ be a totally disconnected generalized $\overline{FC}$-group or a metrizable locally compact compactly generated totally disconnected metabelian group. Then the following conditions are equivalent*:

(a) *The Choquet–Deny theorem holds for $G$.*
(b) *The Choquet–Deny theorem holds for every $\mu \in M_1(G)$ with* $\operatorname{supp}\mu$ *of cardinality* 2.
(c) *$G$ is distal and has polynomial growth.*

**Proof.** (b)⇒(c): We first prove that $G$ is distal. When $G$ is metrizable, this is clear by Lemma 3.5, Theorems 2.9 and 2.15, and Proposition 2.1. Suppose that $G$ is a not necessarily metrizable generalized $\overline{FC}$-group. Note that it suffices to show that every neighbourhood of $e$ contains a compact normal subgroup $N$ such that $G/N$ is distal. But as $G$ is compactly generated, given a neighbourhood $U$ of $e$ there exists a compact normal subgroup $N \subseteq U$ such that $G/N$ is metrizable [11, Theorem 8.7]. Every probability measure on $G/N$ with support of cardinality 2 is the canonical image of a similar measure on $G$. Hence, by Lemma 3.1(a), Condition (b) must hold on $G/N$ and as $G/N$ is a generalized $\overline{FC}$-group, it is distal.

We now prove that $G$ is of polynomial growth. Suppose that $G$ is not of polynomial growth. By Proposition 2.8 and Theorem 2.15, $G$ has a compact open normal subgroup $N$ such that the quotient $G/N$ contains a finitely generated solvable subgroup $S$ of finite index (polycyclic when $G$ is an $\overline{FC}$-group and metabelian when $G$ is metabelian). By [10, Théorème I.4] $S$ is not of polynomial growth. Hence, by [21, Theorem 3.13 and its proof], $S$ supports a probability measure with a 2-element support for which the Choquet–Deny theorem fails. This implies that the Choquet–Deny theorem fails for a similar probability measure on $G$.

(c)⇒(a): When $N$ is a compact open normal subgroup, $G/N$ is a finitely generated group of polynomial growth, hence, the Choquet–Deny theorem holds for $G/N$. Since by Theorems 2.9 and 2.15, $G$ has arbitrarily small compact open normal subgroups, Lemma 3.1(b) yields the desired conclusion. □

## 4. On boundaries of random walks

It is well-known that the bounded $\mu$-harmonic functions can be represented, by means of a "Poisson formula", as bounded Borel functions on a certain "boundary space". Let us consider the bounded $\mu$-harmonic functions (on a general locally compact group $G$) as elements of $L^\infty(G)$, and let $\mathcal{H}_\mu$ denote the resulting subspace of $L^\infty(G)$. $\mathcal{H}_\mu$ is invariant under the usual left action of $G$ on $L^\infty(G)$ and for every absolutely continuous $\nu \in M_1(G)$ and every $h \in \mathcal{H}_\mu$, $\nu * h$ is a bounded (left uniformly) continuous $\mu$-harmonic function. When $G$ is locally compact second countable (lcsc), there exists a standard Borel $G$-space $\mathcal{X}$ with a $\sigma$-finite quasiinvariant measure $\alpha$ and an equivariant isometry $\Phi$ of $L^\infty(\mathcal{X},\alpha)$ onto $\mathcal{H}_\mu$ [18, §3]. $\Phi$ is given by the Poisson formula

$$(\Phi f)(g) = \int_{\mathcal{X}} f(gx)\,\rho(dx) \tag{4.1}$$

where $\rho$ is a probability measure on $\mathcal{X}$ satisfying $\mu*\rho = \rho$. The $G$-space $\mathcal{X}$, called the *$\mu$-boundary*, or *Poisson boundary*, is not unique. However, for any two $\mu$-boundaries $(\mathcal{X}',\alpha')$ and $(\mathcal{X}'',\alpha'')$, there exists an equivariant isomorphism between $L^\infty(\mathcal{X}',\alpha')$

and $L^\infty(\mathcal{X}'', \alpha'')$ (which implies that $(\mathcal{X}', \alpha')$ and $(\mathcal{X}'', \alpha'')$ are isomorphic up to sets of zero measure). The $\mu$-boundary can be always realized as a topological, compact metric $G$-space [18, §3].

When the Choquet–Deny theorem holds for $\mu$, the natural realization of the $\mu$-boundary is the homogeneous space $G/G_\mu$ where the "Poisson kernel" $\rho$ (cf. Eq. (4.1)) is the point measure $\delta_{G_\mu}$. When $G$ is a discrete (countable) group then the $\mu$-boundary is a homogeneous space if and only if the Choquet–Deny theorem is true for $\mu$ [21, Lemma 1.1 and the remark preceding Proposition 2.6]. The situation is different for continuous groups. When $G$ is an almost connected lcsc group then for every spread out probability measure on $G$ the $\mu$-boundary is a homogeneous space [17, Corollary 4.7]. It is well-known that the $\mu$-boundary of every spread out measure on a connected semisimple Lie group with finite centre is a compact homogeneous space [6, 2]. However, if the $\mu$-boundary of a spread out measure on an *amenable* lcsc group is a *compact* homogeneous space, then the Choquet–Deny theorem holds for $\mu$ (and the $\mu$-boundary is finite), see [21, Proposition 2.6] and [19, Lemma 2.3], or [2, Propositions IV.8 and IV.7].

Theorem 4.2 which we prove below applies, in particular, to every totally disconnected compactly generated lcsc group of polynomial growth. The result is that for such groups the $\mu$-boundary can be always realized as a homogeneous space, and, as a compact homogeneous space when $\mu$ is adapted; when $\mu$ is adapted and spread out the $\mu$-boundary is a singleton.

**Lemma 4.1.** *A probability measure $\mu$ on a locally compact group $G$ is a Choquet–Deny measure if and only if the restriction of $\mu$ to $G_\mu$ is a Choquet–Deny measure (on $G_\mu$).*

**Proof.** Let $\mu'$ denote the restriction of $\mu$ to $G_\mu$. The restriction of a $\mu$-harmonic function to $G_\mu$ is $\mu'$-harmonic; moreover, if $h$ is $\mu$-harmonic then for every $g \in G$ the left translate $(gh)(\cdot) = h(g^{-1}\cdot)$ is also $\mu$-harmonic. Hence, if $\mu'$ is Choquet–Deny then so is $\mu$. The converse is equally obvious when $G_\mu$ is open, because then every bounded continuous $\mu'$-harmonic function trivially extends to a bounded continuous $\mu$-harmonic function. However, in general, a technical argument is called for.

Let us first consider the case that $G$ is second countable. Let $h'$ be a bounded continuous $\mu'$-harmonic function. As $G$ is second countable, the canonical projection $\pi\colon G \to G/G_\mu$ admits a Borel cross-section $\kappa$. Since for every $g \in G$, $\kappa(\pi(g))^{-1}g \in G_\mu$, we can define a function $h\colon G \to \mathbb{C}$ by $h(g) = h'\big(\kappa(\pi(g))^{-1}g\big)$. $h$ is a bounded (in general, discontinuous) $\mu$-harmonic function.

Let $(\varepsilon_n)$ be a sequence of absolutely continuous probability measures on $G$ converging weakly to $\delta_e$. Then the sequence $(\varepsilon_n * h)$ converges in the weak* topology of $L^\infty(G)$ to $h$. Since $\varepsilon_n * h$ is a bounded continuous $\mu$-harmonic function and $\mu$ is a Choquet–Deny measure, it follows that there exists a bounded Borel function $\hat{h}\colon G/G_\mu \to \mathbb{C}$ such that $h = \hat{h} \circ \pi$ $\lambda$-a.e., where $\lambda$ is the Haar measure of $G$.

Now, the mapping $\varphi\colon (G/G_\mu) \times G_\mu \to G$ given by $\varphi(x, g) = \kappa(x)g$ is a Borel isomorphism. Moreover, if $\nu$ is a $\sigma$-finite quasiinvariant measure on $G/G_\mu$ and $\lambda'$ the Haar measure of $G_\mu$, then the measure $\varphi(\nu \times \lambda') = (\nu \times \lambda') \circ \varphi^{-1}$ is equivalent to the Haar measure $\lambda$ of $G$. Consequently,

$$0 = \int_{(G/G_\mu)\times G_\mu} |h' \circ \varphi - \hat{h} \circ \pi \circ \varphi| \, d(\nu \times \lambda')$$
$$= \int_{G/G_\mu} \left[ \int_{G_\mu} |(h \circ \varphi)(x,g) - (\hat{h} \circ \pi \circ \varphi)(x,g)| \; \lambda'(dg) \right] \nu(dx)$$
$$= \int_{G/G_\mu} \left[ \int_{G_\mu} |h'(g) - \hat{h}(x)| \; \lambda'(dg) \right] \nu(dx).$$

Thus for $\nu$-a.e. $x \in G/G_\mu$, $\int_{G_\mu} |h'(g) - \hat{h}(x)| \, \lambda'(dg) = 0$. Hence, as $h'$ is continuous, it is constant.

Consider now the general case that $G$ is not necessarily second countable. Observe that due to the regularity of $\mu$ and local compactness of $G$, $G_\mu$ is $\sigma$-compact and, hence, there is also an open $\sigma$-compact subgroup $G_1$ with $\mu(G_1) = 1$. Since $G_1$ is open it is clear that the restriction of $\mu$ to $G_1$ is a Choquet–Deny measure. Hence, we may assume that $G$ itself is $\sigma$-compact. By Lemma 3.1(b) it suffices to show that every neighbourhood $U$ of $e$ in $G_\mu$ contains a compact normal subgroup $N$ such that the projection of $\mu'$ onto $G_\mu/N$ is Choquet–Deny. But by [11, Theorem 8.7] there exists a compact normal subgroup $K$ of $G$ such that $K \cap G_\mu \subseteq U$ and $G/K$ is second countable. Let $\pi_K \colon G \to G/K$ denote the canonical homomorphism. Since $(G/K)_{\pi_K\mu} = \pi_K(G_\mu)$, combining Lemma 3.1(a) with what we just proved for second countable groups, we conclude that the restriction of $\pi_K\mu$ to $\pi_K(G_\mu)$ is a Choquet–Deny measure. As $\pi_K(G_\mu)$ is canonically isomorphic to $G_\mu/(K \cap G_\mu)$, it follows that the projection of $\mu'$ onto $G_\mu/(K \cap G_\mu)$ is a Choquet–Deny measure. □

**Theorem 4.2.** *Let $\mu \in M_1(G)$ where $G$ is a lcsc group. If $G$ contains a compact normal subgroup $K$ such that the projection of $\mu$ onto $G/K$ is a Choquet–Deny measure, then the $\mu$-boundary can be realized as a homogeneous space; when $\mu$ is adapted, the $\mu$-boundary can be realized as a compact homogeneous space on which $K$ acts transitively.*

**Proof.** Denote by $\pi \colon G \to G/K$ the canonical homomorphism.

Suppose that $\mu$ is adapted and let $(\mathcal{X}, \alpha)$ be the $\mu$-boundary realized as a standard Borel $G$-space. Let $f \in L^\infty(\mathcal{X}, \alpha)$ be invariant under the action of $K$. Then the corresponding $\mu$-harmonic function $h = \Phi f \in \mathcal{H}_\mu$ (cf. Eq. (4.1)) is also invariant under the (left) action of $K$. Hence, $h = \hat{h} \circ \pi$ where $\hat{h} \in \mathcal{H}_{\pi\mu}$. Since $\pi\mu$ is adapted and the Choquet–Deny theorem holds on $G/K$, it follows that $h$ is constant. Thus so is $f$. As $\mathcal{X}$ is a standard Borel $G$-space this implies that $K$ acts ergodically on $\mathcal{X}$, and, hence, $\alpha$ is carried on an orbit of $K$ [33, Corollary 2.1.21 and Proposition 2.1.10]. Consequently, the $\mu$-boundary can be realized as a compact homogeneous space of $G$ on which $K$ acts transitively.

When $\mu$ is not necessarily adapted, let $\mu'$ denote the restriction of $\mu$ to $G_\mu$ and let $(\mathcal{X}', \alpha')$ be a realization of the $\mu'$-boundary as a standard Borel $G_\mu$-space. By Lemma 4.1 the restriction of $\pi\mu$ to $\pi(G_\mu) = (G/K)_{\pi\mu}$ is a Choquet–Deny measure. Since $G_\mu/(G_\mu \cap K) \cong \pi(G_\mu)$, it follows that the projection of $\mu'$ onto $G_\mu/(G_\mu \cap K)$ is a Choquet–Deny measure. As $\mu'$ is adapted, we may assume that

$\mathcal{X}'$ is a homogeneous space of $G_\mu$ (on which $G_\mu \cap K$ acts transitively). Now, by [20, Proposition 3.5 and Remark 3.9] the $\mu$-boundary can be realized as the skew product $\mathcal{X} = G/G_\mu \times_\gamma \mathcal{X}'$ (the $G$-space induced from the $G_\mu$-space $\mathcal{X}'$ [33, p. 75]), where $\gamma\colon G \times G/G_\mu \to G_\mu$ is the cocycle associated with a Borel cross section of the canonical projection of $G$ on $G/G_\mu$. It follows that $G$ acts transitively on $\mathcal{X}$. This means that the $\mu$-boundary can be realized as a homogeneous space of $G$. □

**Corollary 4.3.** *Let $G$ be a totally disconnected compactly generated lcsc group of polynomial growth. Then for every $\mu \in M_1(G)$ the $\mu$-boundary can be realized as a homogeneous space of $G$; when $\mu$ is adapted, the $\mu$-boundary can be realized as a compact homogeneous space.*

The next corollary can be regarded as a generalization of the implication (c)⇒(a) of Theorem 3.6. Contrary to the proof of Theorem 3.6, the proof of Corollary 4.4 does not rely on equicontinuity of $\mathrm{Inn}(G)$.

**Corollary 4.4.** *Let $G$ be a locally compact group containing a compact normal subgroup $K$ such that the Choquet–Deny theorem holds for $G/K$ and* $\mathrm{Inn}(G)$ *acts distally on $K$. Then the Choquet–Deny theorem holds for $G$.*

**Proof.** It is not difficult to see that if a locally compact group $G$ contains a compact normal subgroup $K$ such that the Choquet–Deny theorem holds for $G/K$ and $\mathrm{Inn}(G)$ acts distally on $K$, then the same is true for every closed subgroup and every quotient of $G$. Let $\mu \in M_1(G)$. To show that $\mu$ is a Choquet–Deny measure it suffices to show that the restriction, $\mu'$, of $\mu$ to $G_\mu$ is Choquet–Deny. By Lemma 3.1(b), to show the latter it is enough to show that every neighbourhood of $e$ in $G_\mu$ contains a compact normal subgroup $N$ such that the projection of $\mu'$ onto $G_\mu/N$ is Choquet–Deny. But as $G_\mu$ is $\sigma$-compact, every neighbourhood of $e$ in $G_\mu$ contains a compact normal subgroup with second countable quotient. Hence, it is enough to prove that if a lcsc group $G$ contains a compact normal subgroup $K$ such that the Choquet–Deny theorem holds for $G/K$ and $\mathrm{Inn}(G)$ acts distally on $K$, then every adapted probability measure on $G$ is a Choquet–Deny measure.

For such $G$ and $\mu$, by Theorem 4.2, the $\mu$-boundary has the form $G/H$ where $K$ acts transitively on $G/H$, i.e., $G = KH$. Let $\rho$ denote the Poisson kernel. Note that due to the identity $\rho = \mu * \rho$ and adaptedness of $\mu$, it suffices to show that $\rho$ is a point measure (this will imply that $G/H$ is a singleton).

Now, by [14, Proposition 2.8] there exists a sequence $(h_n)$ in $G$ such that the sequence $(h_n\rho)$ converges weakly to a point measure $\delta_{x_0}$. Since $G = KH$ and $K$ is compact, we may assume that $h_n \in H$ for all $n$. Next, by [22, Lemma 2.8] we may assume that there is a Borel set $B \subseteq G/H$ such that $\rho(B) = 1$ and $\lim_{n\to\infty} h_n x = x_0$ for every $x \in B$. It is enough to show that $B$ is a singleton.

Consider the compact homogeneous space $K/(K \cap H)$. The formulas $h^\bullet k = hkh^{-1}$ and $h_\bullet k(K \cap H) = hkh^{-1}(K \cap H)$, $k \in K$, define actions of $H$ on $K$ and $K/(K \cap H)$, respectively. Clearly, the $_\bullet$-action is a factor of the $^\bullet$-action. As $^\bullet$ is distal, so is $_\bullet$ [5, Corollary 6.10, p. 52].

Let $x_1, x_2 \in B$. Write $x_1 = k_1 H$ and $x_2 = k_2 H$ with $k_1, k_2 \in K$. Then $\lim_{n\to\infty} h_n x_j = \lim_{n\to\infty} h_n k_j h_n^{-1} H = x_0$ for $j = 1, 2$. Since the compact homogeneous spaces $G/H$ and $K/(K \cap H)$ are isomorphic as $K$-spaces, we then obtain $\lim_{n\to\infty} h_{n\bullet} k_1(K \cap H) = \lim_{n\to\infty} h_{n\bullet} k_2(K \cap H)$. Since $_\bullet$ is distal, $k_1(K \cap H) = k_2(K \cap H)$ and, hence, $x_1 = x_2$. Therefore $B$ is a singleton. □

**Example 4.5.** Recall that $G$ is called an IN group if it contains a compact neighbourhood of $e$, invariant under $\mathrm{Inn}(G)$. By [8, Theorem 2.5] every locally compact IN group $G$ contains a compact normal subgroup $N$ such that $G/N$ is a SIN group. When $G$ is almost connected, it follows from Iwasawa's theorem on automorphisms of compact groups [13, Theorem 1] that the natural image of $\mathrm{Inn}(G)$ in $\mathrm{Aut}(N)$ is compact (in the usual topology). Hence, the Ascoli theorem for automorphism groups [9, Theorem 4.1] yields that $\mathrm{Inn}(G)$ acts equicontinuously on $N$. But by [14, Corollary 6.5] and [8, Theorem 2.9] the Choquet–Deny theorem holds for every almost connected locally compact SIN group (see also [14, Corollary 6.6]). Thus Corollary 4.4 yields that the Choquet–Deny theorem is true for every locally compact almost connected IN group.

**Example 4.6.** Let $\tau$ be the automorphism of the torus $\mathbb{T}^3$, defined by $\tau(x, y, z) = (x, xy, xyz)$. Then $\tau$ is distal but not equicontinuous. By Corollary 4.4 the Choquet–Deny theorem is true for the 3-step nilpotent group $\mathbb{T}^3 \times_\tau \mathbb{Z}$.

**Example 4.7.** Let $\tau$ be the shift $\tau((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$ on the compact abelian group $\mathbb{Z}_2^{\mathbb{Z}}$, and let $G = \mathbb{Z}_2^{\mathbb{Z}} \times_\tau \mathbb{Z}$. Since $C(\tau) = \{x \in \mathbb{Z}_2^{\mathbb{Z}}\,;$ there exists $k \in \mathbb{Z}$ with $x_i = 0$ for every $i \geq k\}$, $\mathrm{Inn}(G)$ does not act distally on $\mathbb{Z}_2^{\mathbb{Z}} \times \{0\}$. By Theorem 3.6 the Choquet–Deny theorem is not true for $G$.

Let $\mu \in M_1(G)$ be adapted. According to Theorem 4.2 the $\mu$-boundary has the form $G/H$ where $G = (\mathbb{Z}_2^{\mathbb{Z}} \times \{0\})H$. It is not difficult to see that a closed subgroup $H \subseteq G$ satisfies $G = (\mathbb{Z}_2^{\mathbb{Z}} \times \{0\})H$ if and only if there is a closed $\tau$-invariant subgroup $T \subseteq \mathbb{Z}_2^{\mathbb{Z}}$ and $g \in G$ such that $gHg^{-1} = T \times \mathbb{Z}$. We may therefore assume that $H = T \times \mathbb{Z}$ where $T$ is a closed $\tau$-invariant subgroup. Now, the formula $(x, y)(zT) = x\tau^y(z)T$, $x, z \in \mathbb{Z}_2^{\mathbb{Z}}$, $y \in \mathbb{Z}$, defines an action of $G$ on $\mathbb{Z}_2^{\mathbb{Z}}/T$ under which $\mathbb{Z}_2^{\mathbb{Z}}/T$ becomes a homogeneous space of $G$, isomorphic to $G/H$. Thus for an adapted $\mu \in M_1(G)$, the $\mu$-boundary can be realized as one of the $G$-spaces $\mathbb{Z}_2^{\mathbb{Z}}/T$, where $T$ is a closed $\tau$-invariant subgroup of $\mathbb{Z}_2^{\mathbb{Z}}$.

Let for $k = 1, 2, \ldots,$ $S_k = \{x \in \mathbb{Z}_2^{\mathbb{Z}}\ \ \tau^k(x) = x\}$. Then $S_k$ is a closed $\tau$-invariant subgroup of $\mathbb{Z}_2^{\mathbb{Z}}$ and it can be shown that $T$ is a closed $\tau$-invariant subgroup of $\mathbb{Z}_2^{\mathbb{Z}}$ if and only if $T = \mathbb{Z}_2^{\mathbb{Z}}$ or $T$ is a $\tau$-invariant subgroup of $S_k$ for some $k$. In particular, proper $\tau$-invariant subgroups are finite. Let $\mathcal{T}$ denote the class of closed $\tau$-invariant subgroups of $\mathbb{Z}_2^{\mathbb{Z}}$. The $G$-spaces $\mathbb{Z}_2^{\mathbb{Z}}/T$, $T \in \mathcal{T}$ are mutually nonisomorphic and each of them is an equivariant image of $\mathbb{Z}_2^{\mathbb{Z}} = \mathbb{Z}_2^{\mathbb{Z}}/\{e\}$.

One can construct a family $\mu_T$, $T \in \mathcal{T}$, of discrete probability measures on $G$ such that $\mathbb{Z}_2^{\mathbb{Z}}/T$ is the $\mu_T$-boundary for every $T \in \mathcal{T}$. We refrain from going into the details here as this would require a longer digression into the theory of the $\mu$-boundaries. A more difficult question concerns determining, when $\mu \in M_1(G)$ is given, which of the spaces $\mathbb{Z}_2^{\mathbb{Z}}/T$ is the $\mu$-boundary. In particular, one would like to know for which $\mu \in M_1(G)$ the $\mu$-boundary is a singleton. In addition to the case of spread out measures, this is so for every adapted probability measure which induces a recurrent random walk on $\mathbb{Z} \cong G/(\mathbb{Z}_2^{\mathbb{Z}} \times \{0\})$. We do not know of any relevant conditions that are both sufficient and necessary.

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**Note added in proof.** Since this paper was completed the first named author proved that the results of Baumgartner and Willis [4] remain true for nonmetrizable groups. As a consequence, Proposition 2.1, Lemmas 2.2 and 2.3, Corollary 2.4, and Theorems 2.7, 2.15, and 3.6 of the present work remain true without the assumption that $G$ be metrizable.

## References


[1] ABELS, HERBERT. Distal automorphism groups of Lie groups. J. Reine Angew. Math. **329** (1981), 82–87. MR0636446 (83i:22013), Zbl 0463.22006.

[2] AZENCOTT, ROBERT. Espaces de Poisson des groupes localement compacts. Lecture Notes in Mathematics, Vol. 148. *Springer, Berlin*, 1970. MR0501376 (58 #18748), Zbl 0208.15302.

[3] BAGLEY, R. W.; WU, T. S. Maximal compact normal subgroups and pro-Lie groups. *Proc. Amer. Math. Soc.* **93** (1985), 373–376. MR0770558 (86c:22005), Zbl 0576.22008.

[4] BAUMGARTNER, UDO; WILLIS, GEORGE A. Contraction groups and scales of automorphisms of totally disconnected locally compact groups. *Israel J. Math.* **142** (2004), 221–248. MR2085717 (2005f:22009), Zbl 1056.22001.

[5] BERGLUND, JOHN F.; JUNGHENN, HUGO D.; MILNES, PAUL. Analysis on semigroups. Function spaces, compactifications, representations. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. *John Wiley & Sons, New York*, 1989. MR0999922 (91b:43001), Zbl 0727.22001.

[6] FURSTENBERG, HARRY. A Poisson formula for semisimple Lie groups. *Ann. of Math.* (2) **77** (1963), 335–386. MR0146298 (26 #3820), Zbl 0192.12704.

[7] FURSTENBERG, HARRY. Boundary theory and stochastic processes on homogeneous spaces. *Harmonic analysis on homogeneous spaces* (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), 193–229. *Amer. Math. Soc., Providence, R.I.*, 1973. MR0352328 (50 #4815), Zbl 0289.22011.

[8] GROSSER, SIEGFRIED; MOSKOWITZ, MARTIN. Compactness conditions in topological groups. *J. Reine Angew. Math.* **246** (1971), 1–40. MR0284541 (44 #1766), Zbl 0219.22011.

[9] GROSSER, SIEGFRIED; MOSKOWITZ, MARTIN. On central topological groups. *Trans. Amer. Math. Soc.* **27** (1967), 317–340. MR0209394 (35 #292), Zbl 0145.03305.

[10] GUIVARC'H, YVES. Croissance polynômiale et périodes des fonctions harmoniques. Bull. Soc. Math. France **101** (1973), 333–379. MR0369608 (51 #5841), Zbl 0294.43003.

[11] HEWITT, EDWIN; ROSS, KENNETH A. Abstract harmonic analysis. Vol. I. Structure of topological groups, integration theory, group representations. Second edition. Grundlehren der Mathematischen Wissenschaften, 115. *Springer-Verlag, Berlin-New York*, 1979. MR0551496 (81k:43001), Zbl 0837.43002.

[12] HOFMANN, KARL H.; LIUKKONEN, JOHN R.; MISLOVE, MICHAEL W. Compact extensions of compactly generated nilpotent groups are pro-Lie. *Proc. Amer. Math. Soc.* **84** (1982), 443–448. MR0640250 (84e:22007), Zbl 0489.22006.

[13] IWASAWA, KENKICHI. On some types of topological groups. *Ann. of Math.* (2) **50** (1949), 507–558. MR0029911 (10,679a), Zbl 0034.01803.

[14] JAWORSKI, WOJCIECH. Ergodic and mixing probability measures on [SIN] groups. *J. Theoret. Probab.* **17** (2004), 741–759. MR2091559 (2005g:60016), Zbl 1070.60006.

[15] JAWORSKI, WOJCIECH. Countable amenable identity excluding groups. *Canadian Math. Bull.* **47** (2004), 215–228. MR2059416 (2005c:47013), Zbl 1062.22010.

[16] JAWORSKI, WOJCIECH. On shifted convolution powers and concentration functions in locally compact groups. *Probability on algebraic structures* (Gainesville, FL, 1999), 23–41, Contemp. Math., 261, *Amer. Math. Soc., Providence, RI*, 2000. MR1787148 (2002h:60010), Zbl 0968.60015.

[17] JAWORSKI, WOJCIECH. Random walks on almost connected locally compact groups: boundary and convergence. *J. d'Analyse Math.* **74** (1998), 235–273. MR1631666 (99i:60014), Zbl 0908.60071.

[18] JAWORSKI, WOJCIECH. The asymptotic $\sigma$-algebra of a recurrent random walk on a locally compact group. *Israel J. Math.* **94** (1995), 201–219. MR1394575 (98g:60011), Zbl 0848.43001.

[19] JAWORSKI, WOJCIECH. Strong approximate transitivity, polynomial growth, and spread out random walks on locally compact groups. *Pacific J. Math.* **170** (1995), 517–533. MR1363877 (96k:43002), Zbl 0849.60007.

[20] Jaworski, Wojciech. On the asymptotic and invariant $\sigma$-algebras of random walks on locally compact groups. *Probability Theory and Related Fields* **101** (1995), 147–171. MR1318190 (95m:60013), Zbl 0815.60063.
[21] Jaworski, Wojciech. Strongly approximately transitive group actions, the Choquet–Deny theorem, and polynomial growth. *Pacific J. Math.* **165** (1994), 115–129. MR1285567 (95e:22015), Zbl 0881.22004.
[22] Jaworski, Wojciech; Rosenblatt, Joseph; Willis, George. Concentration functions in locally compact groups. *Math. Ann.* **305** (1996), 673–691. MR1399711 (97k:43001), Zbl 0854.43001.
[23] Kaĭmanovich, V. A.; Vershik, A. M. Random walks on discrete groups: boundary and entropy. *Ann. Probab.* **11** (1983), 457–490. MR0704539 (85d:60024), Zbl 0641.60009.
[24] Losert, V. On the structure of groups with polynomial growth. II. *J. London Math. Soc.* (2) **63** (2001), 640–654. MR1825980 (2002f:22007), Zbl 1010.22008.
[25] Macbeath, A. M.; Świerczkowski, S. On the set of generators of a subgroup. *Indag. Math.* **21** (1959), 280–281. MR0106960 (21 #5690), Zbl 0118.26503.
[26] Raja, C. Robinson Edward. On classes of $p$-adic Lie groups. *New York J. Math.* **5** (1999), 101–105. MR1703206 (2000f:22014), Zbl 0923.22006.
[27] Raja, C. Robinson Edward. Growth, recurrence, and the Choquet–Deny theorem for $p$-adic Lie groups. *Math. Z.* **251** (2005), 827–847. MR2190145 (2006k:60012), Zbl 1077.22013.
[28] Raugi, Albert. A general Choquet–Deny theorem for nilpotent groups. *Ann. Inst. H. Poincaré Probab. Statist.* **40** (2004), 677–683. MR2096214 (2005h:60016), Zbl 1063.60006.
[29] Rosenblatt, Joseph. A distal property of groups and the growth of connected locally compact groups. *Mathematika* **26** (1979), 94–98. MR0557132 (81c:22014), Zbl 0402.22002.
[30] Rosenblatt, Joseph. Ergodic and mixing random walks on locally compact groups. *Math. Ann.* **257** (1981), 31–42. MR0630645 (83f:43002), Zbl 0451.60011.
[31] Willis, George. Totally disconnected, nilpotent, locally compact groups *Bull. Austral. Math. Soc.* **55** (1997), 143–146. MR1428537 (98d:22001), Zbl 0893.22001.
[32] Willis, George. The structure of totally disconnected locally compact groups. *Math. Ann.* **300** (1994), 341–363. MR1299067 (95j:22010), Zbl 0811.22004.
[33] Zimmer, Robert J. Ergodic theory and semisimple Lie groups. Birkhäuser, Boston 1984.

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