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Landstad's characterization for full crossed products

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Abstract. Full C^*-crossed products by actions of locally compact groups are characterized via the existence of suitable maximal coactions, in analogy with Landstad's characterization of reduced crossed products.

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1. Introduction

It (almost) goes without saying that two of the fundamental issues when dealing with C^*-dynamical systems are:

(1) When can a given C^*-algebra be realized as a crossed product by a given group?
(2) When is a given crossed product isomorphic to a given C^*-algebra?

For reduced crossed products by actions of locally compact groups, Landstad ([7]) answered question (1) in terms of the existence of suitable reduced coactions. An analogous solution for crossed products by reduced coactions was given by the second author in [8].

Answers to question (2) can be derived from a large class of results which, in various situations, show that an equivariant homomorphism between C^*-algebras is faithful if it is faithful on a certain subset of "fixed points" in its domain. Such results exist for actions of compact groups (folklore), normal coactions of locally

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compact groups ([10], [8]), graph C^*-algebras (gauge-invariant uniqueness theorems), and dual coactions on reduced crossed products (also probably folklore — see Theorem 4.1).

In this paper we answer questions (1) and (2) for full crossed products by actions of locally compact groups (Theorem 3.2 and Corollary 4.4, respectively). In each case, the answer involves the existence of a certain maximal coaction; this is natural, since maximal coactions are precisely those for which full-crossed-product duality holds. Also in each case, the answer is gotten by reducing to the situation of reduced crossed products and normal coactions. (See Section 2 for more background on coactions.) Although direct proofs almost certainly exist, this gives quicker results because the reduced theorems (3.1 and 4.1) are easily derived from existing work (mostly of Landstad) on reduced coactions.

In Section 5, we give an application which identifies $A^K \times_\beta G/K$, the crossed product of a fixed-point algebra by a quotient group, with a corner of $A \times_\alpha G$, for K compact. This result was motivated by our ongoing study of Hecke C^*-algebras in terms of projections, initiated in [4].

2. Preliminaries

Throughout this paper, A and B denote C^*-algebras, and G denotes a locally compact group. We adopt the conventions of [2] for actions and normal coactions, and of [1] for maximal coactions. To avoid ambiguity, we list here a few of our notational conventions.

Actions. If α is an action of a locally compact group G on a C^*-algebra A, we denote the reduced crossed product by $A \times_{\alpha,r} G$, the full crossed product by $A \times_\alpha G$, the canonical covariant homomorphism of (A, G) into the multiplier algebra $\mathcal{M}(A \times_{\alpha,r} G)$ by (i_A^r, i_G^r), the canonical covariant homomorphism of (A, G) into $\mathcal{M}(A \times_\alpha G)$ by (i_A, i_G), and the regular representation by $\Lambda\colon A \times_\alpha G \to A \times_{\alpha,r} G$. (Note that $i_A^r = \Lambda_\alpha \circ i_A$ and $i_G^r = \Lambda \circ i_G$.) When there is more than one action around, we will write i_G^α and i_G^β, for example, to distinguish the associated maps.

We identify G with its canonical image in $\mathcal{M}(C^*(G))$, and denote the left regular representation of G by λ and the compact operators on $L^2(G)$ by $\mathcal{K}$.

Coactions. We tacitly assume all our coactions are nondegenerate. If $\delta\colon A \to \mathcal{M}(A \otimes C^*(G))$ is a coaction of G on A, we denote the crossed product by $A \times_\delta G$ and the canonical covariant homomorphism of $(A, C_0(G))$ into $\mathcal{M}(A \times_\delta G)$ by (j_A, j_G). The integrated form of the covariant homomorphism

$$\big((\mathrm{id} \otimes \lambda) \circ \delta, 1 \otimes M\big)\colon (A, C_0(G)) \to \mathcal{M}(A \otimes \mathcal{K})$$

(where M is the multiplication representation of $C_0(G)$ on $L^2(G)$) is faithful.

For every action (A, α) of G there is a *dual coaction* $(A \times_\alpha G, \widehat{\alpha})$ of G, namely the integrated form of the covariant pair

$$\begin{aligned} a &\mapsto i_A(a) \otimes 1, & a &\in A, \\ s &\mapsto i_G(s) \otimes s, & s &\in G. \end{aligned}$$

There is also a coaction $(A \times_{\alpha,r} G, \widehat{\alpha}^n)$ of G which is the integrated form of the pair

$$\begin{aligned} a &\mapsto i_A^r(a) \otimes 1, & a &\in A, \\ s &\mapsto i_G^r(s) \otimes \lambda_s, & s &\in G. \end{aligned}$$

The regular representation Λ is equivariant with respect to $\widehat{\alpha}$ and $\widehat{\alpha}^n$.

For every coaction (A, δ) of G there is a *dual action* $(A \times_\delta G, \widehat{\delta})$ of G, such that for each $s \in G$ the automorphism $\widehat{\delta}_s$ is the integrated form of the covariant pair

$$\begin{aligned} a &\mapsto j_A(a), & a &\in A, \\ f &\mapsto j_G(\rho_s(f)), & f &\in C_0(G), \end{aligned}$$

where $\rho_s(f)(t) = f(ts)$.

The crossed product is functorial; in particular, for every equivariant homomorphism $\varphi\colon (A, \alpha) \to (B, \beta)$ between actions there is a unique equivariant homomorphism $\varphi \times G\colon (A \times_\alpha G, \widehat{\alpha}) \to (B \times_\beta G, \widehat{\beta})$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad\varphi\quad} & B \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\ \mathcal{M}(A \times_\alpha G) & \xrightarrow[\varphi \times G]{} & \mathcal{M}(B \times_\beta G) \end{array}$$

commute, and for every equivariant homomorphism $\varphi\colon (A, \delta) \to (B, \epsilon)$ between coactions there is a unique equivariant homomorphism

$$\varphi \times G\colon (A \times_\delta G, \widehat{\delta}) \to (B \times_\epsilon G, \widehat{\epsilon})$$

making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad\varphi\quad} & B \\ {\scriptstyle j_A}\downarrow & & \downarrow{\scriptstyle j_B} \\ \mathcal{M}(A \times_\delta G) & \xrightarrow[\varphi \times G]{} & \mathcal{M}(B \times_\epsilon G) \end{array}$$

commute.

For any coaction (A, δ), the *canonical surjection* is the map

$$\Phi_A := \big((\mathrm{id} \otimes \lambda) \circ \delta\big) \times (1 \otimes M) \times (1 \otimes \rho)$$

of $A \times_\delta G \times_{\widehat{\delta}} G$ onto $A \otimes \mathcal{K}$. δ is *maximal* if Φ_A is faithful, hence an isomorphism. Thus maximality means that crossed-product duality holds for full crossed products.

By [1, Theorem 3.3], every coaction (A, δ) has a *maximalization*; *i.e.*, there is a maximal coaction (A^m, δ^m) and an equivariant surjection $\vartheta\colon (A^m, \delta^m) \to (A, \delta)$ such that the induced map $\vartheta \times G\colon A^m \times_{\delta^m} G \to A \times_\delta G$ is an isomorphism. Moreover, the maximalization is essentially unique in the sense that if (C, ϵ) is another maximalization with equivariant surjection φ, then there is an equivariant isomorphism χ making the diagram

$$\begin{array}{ccc} (A^m, \delta^m) & \xrightarrow[\cong]{\quad\chi\quad} & (C, \epsilon) \\ & {\scriptstyle\vartheta}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & (A, \delta) & \end{array}$$

commute. We remark that the construction of the maximalization in [1] is not functorial — it involves a choice of a minimal projection in $\mathcal{K}$. Fischer ([3]) has given a functorial version, even for coactions of quantum groups, but we will not need this here.

A coaction (A, δ) *normal* if j_A is faithful; equivalently, if $(\mathrm{id} \otimes \lambda) \circ \delta$ is faithful. By [1, Proposition 2.2], normality is equivalent to crossed-product duality holding for reduced crossed products; *i.e.*, δ is normal if and only if the canonical surjection Φ_A factors through an isomorphism of the reduced double crossed product:

$$\begin{array}{ccc} A \times_\delta G \times_{\widehat{\delta}} G & \xrightarrow{\Phi_A} & A \otimes \mathcal{K}. \\ \Big\downarrow{\scriptstyle \Lambda} & \nearrow{\scriptstyle \cong} & \\ A \times_\delta G \times_{\widehat{\delta},r} G & & \end{array}$$

By [9, Proposition 2.6], every coaction (A, δ) has a *normalization*, *i.e.*, there is a normal coaction (A^n, δ^n) and an equivariant surjection $\psi_A \colon (A, \delta) \to (A^n, \delta^n)$ such that the induced map $\psi_A \times G \colon A \times_\delta G \to A \times_{\delta^n} G$ is an isomorphism. We have

$$\ker \psi_A = \ker j_A,$$

and in fact the construction in [9] uses $A^n = j_A(A)$.

Normalization is functorial; in particular, for every equivariant homomorphism $\varphi \colon (A, \delta) \to (B, \epsilon)$ between coactions there is a unique equivariant homomorphism φ^n making the diagram

$$\begin{array}{ccc} (A, \delta) & \xrightarrow{\varphi} & (B, \epsilon) \\ \Big\downarrow{\scriptstyle \psi_A} & & \Big\downarrow{\scriptstyle \psi_B} \\ (A^n, \delta^n) & \xrightarrow[\varphi^n]{} & (B^n, \epsilon^n) \end{array}$$

commute, and if φ is an isomorphism then so is φ^n ([1, Lemma 2.1]).

If (B, α) is an action of G, then the coaction $(B \times_\alpha G, \widehat{\alpha})$ is maximal, the coaction $(B \times_{\alpha,r} G, \widehat{\alpha}^n)$ is normal, and the regular representation

$$\Lambda \colon (B \times_\alpha G, \widehat{\alpha}) \to (B \times_{\alpha,r} G, \widehat{\alpha}^n)$$

is both a maximalization of $(B \times_{\alpha,r} G, \widehat{\alpha}^n)$ and a normalization of $(B \times_\alpha G, \widehat{\alpha})$ ([2, Proposition A.61], [1, Proposition 3.4]).

3. Characterization

We begin with a version, in modern language, of Landstad's characterization, including a version of his uniqueness clause.

Theorem 3.1 (Landstad)**.** *Let A be a C^*-algebra and let G be a locally compact group. There exist an action (B, G, α) and an isomorphism $\theta \colon A \to B \times_{\alpha,r} G$ if and only if there exist a normal coaction δ of G on A and a strictly continuous unitary homomorphism $u \colon G \to \mathcal{M}(A)$ such that*

$$\delta(u_s) = u_s \otimes s \quad \text{for all } s \in G.$$

Given δ and u as above, B, α, and θ can be chosen such that θ is δ-$\widehat{\alpha}^n$ equivariant and $\theta \circ u = i_G^{\alpha,r}$; moreover, if (C, G, β) is another action and $\sigma \colon A \to C \times_{\beta,r} G$ is a

δ-$\widehat{\beta}^n$ equivariant isomorphism such that $\sigma \circ u = i_G^{\beta,r}$, then there is an isomorphism $\varphi\colon (B,\alpha) \to (C,\beta)$ such that $(\varphi \times_r G) \circ \theta = \sigma$.

Proof. We must show that what Landstad actually proved in [7] implies the above theorem. First of all, Landstad worked with *reduced* coactions, which involve the reduced group C^*-algebra $C_r^*(G)$ rather than $C^*(G)$. However, as shown in [9], (nondegenerate) reduced coactions are in 1-1 correspondence with normal (full) coactions, with the same crossed products, so we can safely state Landstad's results in terms of normal coactions. Careful examination of Landstad's proof of [7, Theorem 3] shows that, given δ and u as in our hypotheses, there exist an action (B,α) and an equivariant isomorphism

$$\theta\colon (A,\delta) \xrightarrow{\cong} (B \times_{\alpha,r} G, \widehat{\alpha}^n)$$

such that $\theta \circ u = i_G^{\alpha,r}$. Conversely, given B, α, and θ as in our hypotheses, one can easily construct δ and u from $\widehat{\alpha}^n$ and $i_G^{\alpha,r}$ using θ.

However, Landstad's uniqueness clause is stated differently from ours. Translating into our notation, his uniqueness is in the form of a characterization of the image of B in the multipliers of the reduced crossed product. Specifically, it states that

$$\begin{aligned} i_B^r(B) = \{c \in \mathcal{M}(B \times_{\alpha,r} G) \mid\ & \widehat{\alpha}^n(c) = c \otimes 1, \\ & s \mapsto \operatorname{Ad} i_G^{\alpha,r}(s)(c) \text{ is norm continuous,} \\ & \text{and } c i_G^{\alpha,r}(f), i_G^{\alpha,r}(f)c \in B \times_{\alpha,r} G \text{ for all } f \in C_c(G)\}. \end{aligned}$$

It follows that

$$\begin{aligned} \theta^{-1} \circ i_B^r(B) = \{a \in \mathcal{M}(A) \mid\ & \delta(a) = a \otimes 1, \\ & s \mapsto \operatorname{Ad} u_s(a) \text{ is norm continuous,} \\ & \text{and } au(f), u(f)a \in A \text{ for all } f \in C_c(G)\}. \end{aligned}$$

We must verify that this implies our uniqueness clause. Suppose we also have an action (C,β) and an isomorphism $\sigma\colon (A,\delta) \xrightarrow{\cong} (C \times_{\beta,r} G, \widehat{\beta}^n)$ such that $\sigma \circ u = i_G^{\beta,r}$. Then we have

$$\begin{aligned} \sigma^{-1} \circ i_C^r(C) = \{a \in \mathcal{M}(A) \mid\ & \delta(a) = a \otimes 1, \\ & s \mapsto \operatorname{Ad} u_s(a) \text{ is norm continuous,} \\ & \text{and } au(f), u(f)a \in A \text{ for all } f \in C_c(G).\}. \end{aligned}$$

Thus $\theta^{-1}(i_B^r(B))$ and $\sigma^{-1}(i_C^r(C))$ are the same subalgebra of $\mathcal{M}(A)$, so

$$\varphi := {i_C^r}^{-1} \circ \sigma \circ \theta^{-1} \circ i_B^r$$

is an isomorphism of B onto C; it follows from the definitions that φ is equivariant for the actions α and β, and that $\varphi \times_r G = \sigma \circ \theta^{-1}$. □

Here is our version of Landstad's characterization for full crossed products:

Theorem 3.2. *Let A be a C^*-algebra and let G be a locally compact group. There exist an action (B,G,α) and an isomorphism $\theta\colon A \to B \times_\alpha G$ if and only if there exist a maximal coaction δ of G on A and a strictly continuous unitary homomorphism $u\colon G \to \mathcal{M}(A)$ such that*

$$\delta(u_s) = u_s \otimes s \quad \text{for all } s \in G.$$

Given δ and u as above, B, α, and θ can be chosen such that θ is δ-$\widehat{\alpha}$ equivariant and $\Lambda_\alpha \circ \theta \circ u = i_G^{\alpha,r}$, where $\Lambda_\alpha \colon B \times_\alpha G \to B \times_{\alpha,r} G$ is the regular representation; moreover, if (C, G, β) is another action and $\sigma \colon A \to C \times_\beta G$ is a δ-$\widehat{\beta}$ equivariant isomorphism such that $\Lambda_\beta \circ \sigma \circ u = i_G^{\beta,r}$, then there is an isomorphism $\varphi \colon (B, \alpha) \to (C, \beta)$ such that $\Lambda_\beta \circ (\varphi \times G) \circ \theta = \Lambda_\beta \circ \sigma$.

Proof. For the forward implication, δ and u are easily constructed from $\widehat{\alpha}$ and i_G^α using θ. For the reverse implication, let $\psi \colon (A, \delta) \to (A^n, \delta^n)$ be the normalization. Then $u^n := \psi \circ u \colon G \to \mathcal{M}(A^n)$ is a strictly continuous unitary homomorphism such that

$$\delta^n(u_s^n) = u_s^n \otimes s,$$

so by Theorem 3.1 there exist an action (B, α) and an isomorphism we will call $\theta^n \colon (A^n, \delta^n) \to (B \times_{\alpha,r} G, \widehat{\alpha}^n)$ such that $\theta^n \circ u^n = i_G^{\alpha,r}$. Note that

$$\theta^n \circ \psi \colon (A, \delta) \to (B \times_{\alpha,r} G, \widehat{\alpha}^n)$$

is maximalization of $(B \times_{\alpha,r} G, \widehat{\alpha}^n)$.

On the other hand, $\Lambda_\alpha \colon (B \times_\alpha G, \widehat{\alpha}) \to (B \times_{\alpha,r} G, \widehat{\alpha}^n)$ is also a maximalization of $(B \times_{\alpha,r} G, \widehat{\alpha}^n)$. Thus, by uniqueness of maximalizations, there exists an isomorphism θ making the diagram

$$\begin{array}{ccc} (A, \delta) & \xrightarrow{\theta} & (B \times_\alpha G, \widehat{\alpha}) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\Lambda_\alpha} \\ (A^n, \delta^n) & \xrightarrow[\theta^n]{} & (B \times_{\alpha,r} G, \widehat{\alpha}^n) \end{array}$$

commute, and we have $\Lambda_\alpha \circ \theta \circ u = \theta^n \circ \psi \circ u = \theta^n \circ u^n = i_G^{\alpha,r}$.

For the uniqueness, suppose we have another action (C, G, β) and an isomorphism $\sigma \colon (A, \delta) \xrightarrow{\cong} (C \times_\beta G, \widehat{\beta})$ such that $\Lambda_\beta \circ \sigma \circ u = i_G^{\beta,r}$. Since the regular representation $\Lambda_\beta \colon (C \times_\beta G, \widehat{\beta}) \to (C \times_{\beta,r} G, \widehat{\beta}^n)$ is a normalization, by functoriality of normalization we have an isomorphism σ^n making the diagram

$$\begin{array}{ccc} (A, \delta) & \xrightarrow[\cong]{\sigma} & (C \times_\beta G, \widehat{\beta}) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\Lambda_\beta} \\ (A^n, \delta^n) & \xrightarrow[\sigma^n]{\cong} & (C \times_{\beta,r} G, \widehat{\beta}^n) \end{array}$$

commute. Thus (with $u^n = \psi \circ u$ as above) we have

$$\sigma^n \circ u^n = \sigma^n \circ \psi \circ u = \Lambda_\beta \circ \sigma \circ u = i_G^{\beta,r},$$

so by Theorem 3.1 there is an isomorphism $\varphi \colon (B, \alpha) \xrightarrow{\cong} (C, \beta)$ such that

$$\Lambda_\beta \circ (\varphi \times G) \circ \theta = (\varphi \times_r G) \circ \theta^n = \sigma^n = \Lambda_\beta \circ \sigma. \qquad \square$$

4. Fidelity

We begin with a fidelity criterion for homomorphisms of reduced crossed products. This result is probably folklore; we could not find it in the literature, so we include a proof for completeness.

Theorem 4.1. *Let (B, α) be an action of a locally compact group G, and let $\varphi\colon B \times_{\alpha,r} G \to A$ be a surjective homomorphism. Then φ is an isomorphism if and only if $\varphi \circ i_B^r$ is faithful and there exists a normal coaction δ of G on A such that φ is $\widehat{\alpha}^n - \delta$ equivariant.*

Proof. For the forward implication, $\varphi \circ i_B^r$ is clearly faithful, and δ is easily constructed from $\widehat{\alpha}^n$ using φ. We adapt to our purposes a trick of Landstad [6] for the reverse implication. Let (π, u) be a covariant homomorphism of (B, G, α) into $\mathcal{M}(A)$ such that

$$\pi \times u = \varphi \circ \Lambda_\alpha\colon B \times_a G \to A.$$

Since the coaction δ is normal, the homomorphism

$$\tau := (\mathrm{id} \otimes \lambda) \circ \delta\colon A \to \mathcal{M}(A \otimes \mathcal{K})$$

is faithful. Elementary computations using equivariance of $\pi \times u$ with respect to $\widehat{\alpha}$ and δ show that

$$\tau \circ (\pi \times u) = ((\pi \times u) \otimes \lambda) \circ \widehat{\alpha} = (\pi \otimes 1) \times (u \otimes \lambda),$$

which is unitarily equivalent to the homomorphism $\mathrm{Ind}\,\pi$ induced from π (see, *e.g.*, [2, Lemma A.18]). Since

$$\pi = (\pi \times u) \circ i_B = \varphi \circ \Lambda_\alpha \circ i_B = \varphi \circ i_B^r$$

is faithful by hypothesis, $\mathrm{Ind}\,\pi$ is a regular representation. Thus

$$\tau \circ (\pi \times u) = \tau \circ \varphi \circ \Lambda_\alpha$$

has the same kernel as Λ_α, so $\tau \circ \varphi$ is faithful, hence so is φ. □

In order to deduce an analogous fidelity result for full crossed products, we will need the following lemmas:

Lemma 4.2. *Let (A, G, δ) be a coaction, with normalization*

$$(A, \delta) \xrightarrow{\psi} (A^n, \delta^n).$$

Then ψ is 1-1 on

$$\mathcal{M}(A)^\delta := \{a \in \mathcal{M}(A) \mid \delta(a) = a \otimes 1\}.$$

Proof. By [2, Proposition A.61], we can take $\psi = (\mathrm{id} \otimes \lambda) \circ \delta$, so that for $a \in \mathcal{M}(A)^\delta$ we have

$$\psi(a) = (\mathrm{id} \otimes \lambda)(\delta(a)) = (\mathrm{id} \otimes \lambda)(a \otimes 1) = a \otimes 1,$$

which is 0 if and only if a is. □

Lemma 4.3. *Let (C, G, ϵ) and (A, G, δ) be maximal coactions, and let*

$$(C, \epsilon) \xrightarrow{\varphi} (A, \delta)$$

be a surjective equivariant homomorphism. If the normalization φ^n is an isomorphism, then so is φ.

Proof. By [1, Lemma 2.1], we have a commutative diagram of normalizations

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & A \\ {\scriptstyle\psi_C}\downarrow & & \downarrow{\scriptstyle\psi_A} \\ C^n & \xrightarrow[\varphi^n]{} & A^n. \end{array}$$

Applying functoriality of crossed products, we get a commutative diagram

$$\begin{array}{ccc} C\otimes\mathcal{K} & \xrightarrow{\varphi\otimes\mathrm{id}} & A\otimes\mathcal{K} \\ {\scriptstyle\Phi_C}\uparrow\cong & & \cong\uparrow{\scriptstyle\Phi_A} \\ C\times_\epsilon G\times_{\widehat{\epsilon}} G & \xrightarrow{\varphi\times G\times G} & A\times_\delta G\times_{\widehat{\delta}} G \\ {\scriptstyle\psi_C\times G\times G}\downarrow\cong & & \cong\downarrow{\scriptstyle\psi_A\times G\times G} \\ C^n\times_{\epsilon^n} G\times_{\widehat{\epsilon^n}} G & \xrightarrow{\varphi^n\times G\times G} & A^n\times_{\delta^n} G\times_{\widehat{\delta^n}} G, \end{array}$$

where the lower two vertical arrows are isomorphisms by definition of normalization, and the upper two vertical arrows (the canonical surjections) are isomorphisms because ϵ and δ are maximal. Assuming that φ^n (and hence $\varphi^n\times G\times G$) is an isomorphism, it follows that $\varphi\otimes\mathrm{id}$ is an isomorphism, and therefore so is φ. □

Corollary 4.4. *Let (B,α) be an action of a locally compact group G, and let $\varphi\colon B\times_\alpha G\to A$ be a surjective homomorphism. Then φ is an isomorphism if and only if $\varphi\circ i_B$ is faithful and there exists a maximal coaction δ of G on A such that φ is $\widehat{\alpha}$-δ equivariant.*

Proof. The forward implication is straightforward. For the reverse implication, by [1, Lemma 2.1], the equivariant homomorphism φ fits into a commutative diagram

$$\begin{array}{ccc} (B\times_\alpha G,\widehat{\alpha}) & \xrightarrow{\varphi} & (A,\delta) \\ {\scriptstyle\Lambda_\alpha}\downarrow & & \downarrow{\scriptstyle\psi} \\ (B\times_{\alpha,r} G,\widehat{\alpha}^n) & \xrightarrow[\varphi^n]{} & (A^n,\delta^n), \end{array}$$

where the vertical arrows are normalizations. It is easily verified that $\varphi\circ i_B(B)\subseteq\mathcal{M}(A)^\delta$, so ψ is 1-1 on the range of $\varphi\circ i_B$ by Lemma 4.2. Since $\varphi\circ i_B$ is faithful, so is

$$\varphi^n\circ i_B^r=\varphi^n\circ\Lambda_\alpha\circ i_B=\psi\circ\varphi\circ i_B.$$

Thus Theorem 4.1 tells us that φ^n is an isomorphism, and it follows from Lemma 4.3 that φ is also an isomorphism. □

5. Application

We apply Corollary 4.4 to the following situation: let α be an action of G on A, and let K be a compact normal subgroup of G. Normalize the Haar measure on G so that K has measure 1, making the characteristic function χ_K a projection in

$C^*(G)$. Let A^K denote the fixed-point subalgebra of A under K. Then the action of G leaves A^K invariant, so there is a unique action β of G/K on A^K such that $\beta_{sK}(a) = \alpha_s(a)$ for all $a \in A^K$. Let $p = i_G(\chi_K)$, which is a projection in $\mathcal{M}(A \times_\alpha G)$ which commutes with $i_A(A^K)$. (Recall that (i_A, i_G) denotes the canonical covariant homomorphism of (A, G, α) into $\mathcal{M}(A \times_\alpha G)$.) Routine calculations show that the maps

$$\sigma\colon A^K \to A \times_\alpha G \qquad \tau\colon G/K \to \mathcal{M}(A \times_\alpha G)$$
$$\sigma(a) = i_A(a)p \qquad \tau(sK) = i_G(s)p$$

give a covariant homomorphism (σ, τ) of $(A^K, G/K, \beta)$ to $\mathcal{M}(A \times_\alpha G)$. Moreover, the integrated form $\theta := \sigma \times \tau$ is a surjection of $A^K \times_\beta G/K$ onto $p(A \times_\alpha G)p$.

Corollary 5.1. *With the above hypotheses and notation, the map*

$$\theta\colon A^K \times_\beta G/K \to p(A \times_\alpha G)p$$

is an isomorphism.

Proof. It remains to show that θ is injective, and to do this we aim to apply Corollary 4.4. The dual coaction $\widehat{\alpha}$ of G on $A \times_\alpha G$ is maximal by [1, Proposition 3.4]. Moreover, $\widehat{\alpha}$ restricts to a coaction $\widehat{\alpha}| := (\mathrm{id} \otimes q) \circ \widehat{\alpha}$ of G/K on $A \times_\alpha G$ (where $q\colon C^*(G) \to C^*(G/K)$ denotes the quotient homomorphism), and by [5, Corollary 7.2] this coaction is also maximal.

The projection p is invariant under the coaction $\widehat{\alpha}$, hence under $\widehat{\alpha}|$, so by the elementary lemma below the restriction δ of $\widehat{\alpha}|$ to the corner $p(A \times_\alpha G)p$ is a maximal coaction. Routine calculations verify that $\delta \circ \theta = (\theta \otimes \mathrm{id}) \circ \widehat{\beta}$, so θ is $\widehat{\beta}$-δ equivariant. $\theta \circ i_B = \sigma$ is evidently injective, so θ is injective by Corollary 4.4. □

Lemma 5.2. *Let δ be a maximal coaction of G on A, and let p be an invariant projection in $\mathcal{M}(A)$. Then $\epsilon := \delta|_{pAp}$ is a maximal coaction of G on the corner pAp.*

Proof. Let $B := pAp$. Routine calculations show that ϵ is a coaction of G on B. For maximality, as in the proof of [1, Proposition 3.5], the canonical surjection $\Phi_B\colon B \times_\epsilon G \times_{\widehat{\epsilon}} G \to B \otimes \mathcal{K}$ can be identified with a restriction of Φ_A, and therefore Φ_B is injective because Φ_A is. □

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