

## The $E_2$ -term of the descent spectral sequence for continuous $G$ -spectra

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ABSTRACT. Given a profinite group  $G$  with finite virtual cohomological dimension, let  $\{X_i\}$  be a tower of discrete  $G$ -spectra, each of which is fibrant as a spectrum, so that  $X = \text{holim}_i X_i$  is a continuous  $G$ -spectrum, with homotopy fixed point spectrum  $X^{hG}$ . The  $E_2$ -term of the descent spectral sequence for  $\pi_*(X^{hG})$  cannot always be expressed as continuous cohomology. However, we show that the  $E_2$ -term is always built out of a certain complex of spectra, that, in the context of abelian groups, is used to compute the continuous cochain cohomology of  $G$  with coefficients in  $\text{lim}_i M_i$ , where  $\{M_i\}$  is a tower of discrete  $G$ -modules.

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### 1. Introduction

In this note,  $G$  always denotes a profinite group. Let  $H_c^*(G; M)$  denote the continuous cohomology of  $G$  with coefficients in the discrete  $G$ -module  $M$ . This cohomology is defined as the right derived functors of  $G$ -fixed points. Then we always assume that  $G$  has finite virtual cohomological dimension; that is, there exists an open subgroup  $H$  and a nonnegative integer  $m$ , such that  $H_c^s(H; M) = 0$ , for all discrete  $H$ -modules  $M$  and all  $s \geq m$ .

All of our spectra are Bousfield–Friedlander spectra of simplicial sets. In particular, a *discrete  $G$ -spectrum* is a  $G$ -spectrum such that each simplicial set  $X_k$  is a

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simplicial object in the category of discrete  $G$ -sets (thus, for any  $l \geq 0$ , the action map on the  $l$ -simplices,  $G \times (X_k)_l \rightarrow (X_k)_l$ , is continuous when  $(X_k)_l$  is regarded as a discrete space). The category of discrete  $G$ -spectra, with morphisms being  $G$ -equivariant maps of spectra, is denoted by  $\text{Spt}_G$ .

Discrete  $G$ -spectra are considered in more detail in [2], which shows (see [2, Theorem 3.6]) that  $\text{Spt}_G$  is a model category, where a morphism  $f$  in  $\text{Spt}_G$  is a weak equivalence (cofibration) if and only if  $f$  is a weak equivalence (cofibration) in  $\text{Spt}$ , the category of spectra. Given a discrete  $G$ -spectrum  $X$ , the *homotopy fixed point spectrum*  $X^{hG}$  is obtained as the total right derived functor of fixed points:  $X^{hG} = (X_{f,G})^G$ , where  $X \rightarrow X_{f,G}$  is a trivial cofibration and  $X_{f,G}$  is fibrant, all in  $\text{Spt}_G$ .

Let  $X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$  be a tower of discrete  $G$ -spectra, such that each  $X_i$  is a fibrant spectrum. As explained in [2, Lemma 4.4], there exists a tower  $\{X'_i\}$  of discrete  $G$ -spectra, such that there are weak equivalences

$$\text{holim}_i X_i \xrightarrow{\cong} \text{holim}_i X'_i \xleftarrow{\cong} \lim_i X'_i.$$

(In this paper,  $\text{holim}$  always denotes the version of the homotopy limit of spectra that is constructed levelwise in the category of simplicial sets, as defined in [1] and [10, 5.6].) Since the inverse limit of a tower of discrete  $G$ -sets is a topological  $G$ -space, and because  $\text{holim}_i X_i$  can be identified with  $\lim_i X'_i$ , in the eyes of homotopy,  $\text{holim}_i X_i$  is a *continuous  $G$ -spectrum*. Notice that, under this identification, the continuous  $G$ -action respects the topology of both  $G$  and all the  $X_i$  together. Continuous  $G$ -spectra and examples of such in chromatic stable homotopy theory are considered in [2].

Given the tower  $\{X_i\}$  and the continuous  $G$ -spectrum  $\text{holim}_i X_i$ , its *homotopy fixed point spectrum*,  $(\text{holim}_i X_i)^{hG}$ , is defined to be  $\text{holim}_i (X_i)^{hG}$ . This construction is called homotopy fixed points because it is equivalent to the usual definition when  $G$  is a finite group and it is the total right derived functor of fixed points in the appropriate sense (see [2, Remark 8.4]).

By [2, Theorem 8.8], thanks to the assumption of finite virtual cohomological dimension, there is a descent spectral sequence

$$(1.1) \quad E_2^{s,t} \Rightarrow \pi_{t-s}((\text{holim}_i X_i)^{hG}),$$

where

$$(1.2) \quad E_2^{s,t} = \pi^s \pi_t(\text{holim}_i (\Gamma_G^\bullet(X_i)_{f,G})^G)$$

(see the beginning of Section 2 for the meaning of  $\Gamma_G^\bullet$ ), and, if the tower of abelian groups  $\{\pi_t(X_i)\}$  satisfies the Mittag-Leffler condition for every integer  $t$ , then  $E_2^{s,t} \cong H_{\text{cont}}^s(G; \{\pi_t(X_i)\})$ , which is continuous cohomology in the sense of Jannsen. (This cohomology is obtained by taking the right derived functors of  $\lim_i (-)^G$ , a functor from towers of discrete  $G$ -modules to abelian groups; see [7].)

In expression (1.2), since  $\pi_t(\text{holim}_i (-))$  is not necessarily  $\lim_i \pi_t(-)$ , the  $E_2$ -term of descent spectral sequence (1.1), in general, can not be expressed as continuous cohomology, and, in general, it has no compact algebraic description. For example, as pointed out in [2, Remark 8.10], when  $G = \{e\}$ ,

$$H_{\text{cont}}^s(\{e\}; \{\pi_t(X_i)\}) = \lim_i \pi_t(X_i)$$

and  $E_2^{0,t} = \pi_t(\text{holim}_i X_i)$ , and these need not be isomorphic, due to the familiar  $\lim^1_i \pi_{t+1}(X_i)$  obstruction. However, in this note, we show that the  $E_2$ -term (1.2) can always be described in an interesting way.

In more detail, Theorem 2.3 gives a particular cochain complex  $\mathcal{C}^*$  for computing the continuous cochain cohomology of  $G$  for a topological  $G$ -module  $\lim_i M_i$ , where  $\{M_i\}$  is a tower of discrete  $G$ -modules. In Corollary 4.4, we show that the  $E_2$ -term of (1.2) can always be given by taking the cohomology of the homotopy groups of the complex  $\mathcal{C}^*$ , where  $\lim_i M_i$  is replaced by the continuous  $G$ -spectrum  $\text{holim}_i X_i$ , in an appropriate sense. This presentation of the  $E_2$ -term shows that  $E_2^{*,*}$  always takes into account the topology of the continuous  $G$ -spectrum, even when it cannot be expressed as continuous cohomology.

Section 3 of this note consists of a discussion of the construction of descent spectral sequence (1.1). Section 5 explains why two other potentially plausible interpretations of (1.2) fail to work.

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## 2. The pro-discrete cochain complex and continuous cohomology

We begin this section with some terminology. If  $\mathcal{C}$  is a category, then  $\mathbf{tow}(\mathcal{C})$  is the category of towers

$$C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \dots$$

in  $\mathcal{C}$ . The morphisms  $\{f_i\}$  are natural transformations such that each  $f_i$  is a morphism in  $\mathcal{C}$ . In this note, we will be working with  $\mathbf{tow}(\text{DMod}(G))$ , where  $\text{DMod}(G)$  is the category of discrete  $G$ -modules, and  $\mathbf{tow}(\text{Spt}_G)$ .

If  $A$  is an abelian group with the discrete topology, let  $\text{Map}_c(G, A)$  be the abelian group of continuous maps from  $G$  to  $A$ . If  $X$  is a spectrum, one can also define  $\text{Map}_c(G, X)$ , where the  $l$ -simplices of the  $k$ th simplicial set  $(\text{Map}_c(G, X)_k)_l$  are given by  $\text{Map}_c(G, (X_k)_l)$ , where  $(X_k)_l$  is given the discrete topology.

Consider the functor

$$\Gamma_G: \text{Spt}_G \rightarrow \text{Spt}_G, \quad X \mapsto \Gamma_G(X) = \text{Map}_c(G, X),$$

where the action of  $G$  on  $\text{Map}_c(G, X)$  is induced on the level of sets by  $(g \cdot f)(g') = f(g'g)$ , for  $g, g' \in G$  and  $f \in \text{Map}_c(G, (X_k)_l)$ , for each  $k, l \geq 0$ . As explained in [2, Definition 7.1], the functor  $\Gamma_G$  forms a triple and there is a cosimplicial discrete  $G$ -spectrum  $\Gamma_G^\bullet X$ . Also, it is clear that  $\Gamma_G: \text{DMod}(G) \rightarrow \text{DMod}(G)$  can be defined as above, so that, given a discrete  $G$ -module  $M$ ,  $\Gamma_G^\bullet M$  is a cosimplicial discrete  $G$ -module.

We do not claim any originality in the following definition.

**Definition 2.1.** Let  $\{X_i\}$  be an object in  $\mathbf{tow}(\text{DMod}(G))$  or in  $\mathbf{tow}(\text{Spt}_G)$ . Then the *pro-discrete cochain complex* is defined to be the complex

$$\mathcal{C}^*(G; \{X_i\}) = \lim_i (\Gamma_G^* X_i)^G,$$

where  $\Gamma_G^* X_i$  is the canonical complex associated to  $\Gamma_G^\bullet X_i$ , where, if  $\{X_i\}$  is in  $\mathbf{tow}(\mathbf{Spt}_G)$ , then the complex lives in the stable homotopy category. The pro-discrete cochain complex is a complex of abelian groups or spectra, respectively, and the limit and colimit are both formed in abelian groups or in spectra (not the stable homotopy category), respectively.

Let  $M$  be any topological  $G$ -module (that is, an abelian Hausdorff topological group that is a  $G$ -module, with a continuous  $G$ -action). Then the *continuous cochain cohomology of  $G$  with coefficients in  $M$* ,  $H_{\text{cts}}^*(G; M)$ , is the cohomology of a cochain complex that has the form

$$(2.2) \quad M \rightarrow \text{Map}_c(G, M) \rightarrow \text{Map}_c(G^2, M) \rightarrow \dots$$

(see [9, p. 106] for details). We note that, by [7, Theorem (2.2)], if  $\{M_i\}$  is a tower of discrete  $G$ -modules that satisfies the Mittag-Leffler condition, then

$$H_{\text{cts}}^s(G; \varinjlim M_i) \cong H_{\text{cont}}^s(G; \{M_i\}),$$

for all  $s \geq 0$ , but, in general, these two versions of continuous cohomology need not be isomorphic. Also, if  $M$  is a discrete  $G$ -module, then  $H_{\text{cts}}^s(G; M) = H_c^s(G; M)$ .

Now we show that the pro-discrete cochain complex can be used to compute continuous cochain cohomology.

**Theorem 2.3.** *If  $\{M_i\}$  is a tower of discrete  $G$ -modules, then*

$$H_{\text{cts}}^s(G; \varinjlim M_i) \cong H^s[\mathcal{C}^*(G; \{M_i\})].$$

**Proof.** As explained in [9, p. 106], for a topological  $G$ -module  $M$ , the chain complex in (2.2) is defined by taking the  $G$ -fixed points of the complex

$$(2.4) \quad X^*(G; M) = [\text{Map}_c(G, M) \rightarrow \text{Map}_c(G^2, M) \rightarrow \dots],$$

where  $X^n(G; M) = \text{Map}_c(G^{n+1}, M)$  has a  $G$ -action that is defined by

$$(g \cdot f)(g_1, \dots, g_{n+1}) = g \cdot f(g^{-1}g_1, \dots, g^{-1}g_{n+1}).$$

Now let  $M$  be a discrete  $G$ -module. Then it is a standard fact that the cochain complex  $(X^*(G, M))^G$  is naturally isomorphic as a complex to the cochain complex  $(\Gamma_G^* M)^G$ . This isomorphism uses the fact that the abelian group of  $n$ -cochains of  $(\Gamma_G^* M)^G$  is isomorphic to  $\text{Map}_c(G^{n+1}, M)^G$ , where  $\text{Map}_c(G^{n+1}, M)$  has a  $G$ -action that is given by

$$(g \cdot f)(g_1, g_2, g_3, \dots, g_{n+1}) = f(g_1g, g_2, g_3, \dots, g_{n+1}).$$

Since

$$(X^n(G; \varinjlim M_i))^G \cong \varinjlim ((X^n(G; M_i))^G) \cong \varinjlim (\Gamma_G^{n+1} M_i)^G,$$

we have:

$$H_{\text{cts}}^s(G; \varinjlim M_i) = H^s[(X^*(G; \varinjlim M_i))^G] = H^s[\varinjlim (\Gamma_G^* M_i)^G],$$

where we used the aforementioned fact that  $(X^*(G, M_i))^G$  and  $(\Gamma_G^* M_i)^G$  are naturally isomorphic cochain complexes. □

### 3. Constructing the descent spectral sequence

In this section, we review how descent spectral sequence (1.1) is constructed and we compare it with a spectral sequence whose  $E_2$ -term is always Jannsen’s continuous cohomology.

Given a tower  $\{X_i\}$  of discrete  $G$ -spectra, such that each  $X_i$  is a fibrant spectrum, by [2, Remark 7.8, Definition 8.1],

$$(\operatorname{holim}_i X_i)^{hG} = \operatorname{holim}_i \operatorname{holim}_{\Delta}(\Gamma_G^\bullet(X_i)_{f,G})^G.$$

Thus,

$$(\operatorname{holim}_i X_i)^{hG} \cong \operatorname{holim}_{\Delta}(\operatorname{holim}_i(\Gamma_G^\bullet(X_i)_{f,G})^G),$$

and descent spectral sequence (1.1) is the conditionally convergent homotopy spectral sequence

$$\lim_{\Delta}^s \pi_t(\operatorname{holim}_i(\Gamma_G^\bullet(X_i)_{f,G})^G) \Rightarrow \pi_{t-s}(\operatorname{holim}_{\Delta}(\operatorname{holim}_i(\Gamma_G^\bullet(X_i)_{f,G})^G))$$

(see [2, Theorem 8.8]).

In the above context, there is another spectral sequence that is natural to consider. Since

$$(\operatorname{holim}_i X_i)^{hG} \cong \operatorname{holim}_{\Delta \times \{i\}}(\Gamma_G^\bullet(X_i)_{f,G})^G,$$

there is a conditionally convergent homotopy spectral sequence

$$\lim_{\Delta \times \{i\}}^s \pi_t((\Gamma_G^\bullet(X_i)_{f,G})^G) \Rightarrow \pi_{t-s}(\operatorname{holim}_{\Delta \times \{i\}}(\Gamma_G^\bullet(X_i)_{f,G})^G) \cong \pi_{t-s}((\operatorname{holim}_i X_i)^{hG}),$$

such that

$$\lim_{\Delta \times \{i\}}^s \pi_t((\Gamma_G^\bullet(X_i)_{f,G})^G) \cong H_{\operatorname{cont}}^s(G; \{\pi_t(X_i)\})$$

(see [5, Proposition 3.1.2]). This spectral sequence is closely related to the  $\ell$ -adic descent spectral sequence of algebraic  $K$ -theory (see [10], [8]).

We see that we have two spectral sequences with abutment  $\pi_*((\operatorname{holim}_i X_i)^{hG})$ . As pointed out in the Introduction, if the tower  $\{\pi_t(X_i)\}$  satisfies the Mittag-Leffler condition for every integer  $t$ , then the spectral sequences have isomorphic  $E_2$ -terms. However, as stated in the Introduction, the case  $G = \{e\}$  shows that these two spectral sequences can have different  $E_2$ -terms, so that the spectral sequences can be different from each other.

Since the second spectral sequence described above has an  $E_2$ -term that always has a nice algebraic description, it is natural to ask what is the value of descent spectral sequence (1.1). We will see that, because the descent spectral sequence is the homotopy spectral sequence of a cosimplicial spectrum, in certain cases it can be compared with an Adams-type spectral sequence that is strongly convergent.

Let  $n \geq 1$  and let  $p$  be a prime. Let  $K(n)$  be the  $n$ th Morava  $K$ -theory spectrum with  $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ , where the degree of  $v_n$  is  $2(p^n - 1)$ . Also, let  $E_n$  denote the Lubin–Tate spectrum, where  $E_{n*} = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$ , where the degree of  $u$  is  $-2$ , and the complete power series ring over the Witt vectors is in degree zero.

Let  $Z$  be a  $K(n)$ -local spectrum and suppose that there is an augmentation  $Z \rightarrow \operatorname{holim}_i (\Gamma_G^\bullet(X_i)_{f,G})^G$ . If the associated complex of spectra

$$* \rightarrow Z \rightarrow \operatorname{holim}_i \operatorname{Map}_c(G, (X_i)_{f,G})^G \rightarrow \operatorname{holim}_i \operatorname{Map}_c(G, \operatorname{Map}_c(G, (X_i)_{f,G}))^G \rightarrow \dots$$

is a  $K(n)$ -local  $E_n$ -resolution of  $Z$  (for the definition of this, see [4, Appendix A]), then descent spectral sequence (1.1) is isomorphic to the strongly convergent  $K(n)$ -local  $E_n$ -Adams spectral sequence with abutment  $\pi_*(Z)$  (see [4, Proposition A.5, Corollary A.8]). Thus, the descent spectral sequence is strongly convergent and the map  $Z \rightarrow \operatorname{holim}_\Delta \operatorname{holim}_i (\Gamma_G^\bullet(X_i)_{f,G})^G \cong (\operatorname{holim}_i X_i)^{hG}$  is a weak equivalence ([4, Corollary A.8]).

In this way, in [3, Chapter 10], the author showed that, given a finite spectrum  $X$ , the descent spectral sequence for  $\pi_*((E_n \wedge X)^{hG})$  is strongly convergent and isomorphic to the  $K(n)$ -local  $E_n$ -Adams spectral sequence with abutment  $\pi_*(E_n^{dhG} \wedge X)$ , where  $G$  is a closed subgroup of the extended Morava stabilizer group  $G_n$  and  $E_n^{dhG}$  is the spectrum constructed by Devinatz and Hopkins in [4] ( $E_n^{dhG}$  is denoted by  $E_n^{hG}$  in [4]).

#### 4. The $E_2$ -term and the pro-discrete cochain complex

In this section, we show that the  $E_2$ -term of (1.2) can be built out of the same complex that computes continuous cochain cohomology. More precisely, given a continuous  $G$ -spectrum  $\operatorname{holim}_i X_i$ , there exists a tower  $\{X'_i\}$  of discrete  $G$ -spectra, such that

$$(4.1) \quad E_2^{s,t} \cong H^s[\pi_t(\mathcal{C}^*(G; \{X'_i\}))].$$

We find the expression on the right-hand side in (4.1) interesting for the following reason. The homotopy fixed point spectrum is defined with respect to a continuous action of  $G$  on the spectrum. Thus, homotopy fixed points take into account the topology of the spectrum. Similarly, since the  $E_2$ -term is built out of the pro-discrete cochain complex of spectra, the  $E_2$ -term is always taking into account the topology of the spectrum.

By [6, VI, Proposition 1.3],  $\mathbf{tow}(\operatorname{Spt}_G)$  is a model category, where  $\{f_i\}$  is a weak equivalence (cofibration) if and only if each  $f_i$  is a weak equivalence (cofibration) in  $\operatorname{Spt}_G$ .

**Theorem 4.2.** *The  $E_2$ -term (1.2) of descent spectral sequence (1.1) has the form*

$$(4.3) \quad E_2^{s,t} \cong \pi^s \pi_t(\lim_i (\Gamma_G^\bullet X'_i)^G),$$

where  $\{X_i\} \rightarrow \{X'_i\}$  is a trivial cofibration with  $\{X'_i\}$  fibrant, all in  $\mathbf{tow}(\operatorname{Spt}_G)$ .

**Proof.** Let  $\{X'_i\}$  be as stated in the theorem. By [6, VI, Remark 1.5], each  $X'_i$  is fibrant and each map  $X'_i \rightarrow X'_{i-1}$  is a fibration, all in  $\operatorname{Spt}_G$ .

For any  $k \geq 0$ , we consider the expression

$$\operatorname{holim}_i ((\Gamma_G^\bullet X'_i)^G)^k = \operatorname{holim}_i (\operatorname{Map}_c(G, \operatorname{Map}_c(G, \dots, \operatorname{Map}_c(G, X'_i) \dots)))^G,$$

where  $\operatorname{Map}_c(G, -)$  appears  $k + 1$  times. By [2, Section 3], the forgetful functor  $U: \operatorname{Spt}_G \rightarrow \operatorname{Spt}$ ,  $\operatorname{Map}_c(G, -): \operatorname{Spt} \rightarrow \operatorname{Spt}_G$ , where  $\operatorname{Map}_c(G, X) = \Gamma_G(X)$ , and the functor  $(-)^G: \operatorname{Spt}_G \rightarrow \operatorname{Spt}$  all preserve fibrations. Thus,  $\{X'_i\}$  is a tower of

fibrations of fibrant spectra, all in  $\text{Spt}$ . This implies that  $\{\text{Map}_c(G, X'_i)\}$  is a tower of fibrations of fibrant spectra, in  $\text{Spt}_G$ , and hence, in  $\text{Spt}$ . By iteration,

$$\{\text{Map}_c(G, \text{Map}_c(G, \dots, \text{Map}_c(G, X'_i) \dots))\}$$

is a tower of fibrations of fibrant spectra, in  $\text{Spt}_G$ , so that

$$\{(\text{Map}_c(G, \text{Map}_c(G, \dots, \text{Map}_c(G, X'_i) \dots))\}^G\}$$

is a tower of fibrations of fibrant spectra in  $\text{Spt}$ . Therefore, the canonical map

$$\lim_i ((\Gamma_G^\bullet X'_i)^G)^k \rightarrow \text{holim}_i ((\Gamma_G^\bullet X'_i)^G)^k$$

is a weak equivalence.

Since  $\{((\Gamma_G^\bullet X'_i)^G)^k\}$  and  $\{((\Gamma_G^\bullet (X_i)_{f,G})^G)^k\}$  are towers of fibrant spectra, there is a zigzag of weak equivalences

$$\lim_i ((\Gamma_G^\bullet X'_i)^G)^k \rightarrow \text{holim}_i ((\Gamma_G^\bullet X'_i)^G)^k \leftarrow \text{holim}_i ((\Gamma_G^\bullet X_i)^G)^k \rightarrow \text{holim}_i ((\Gamma_G^\bullet (X_i)_{f,G})^G)^k,$$

where  $\Gamma = \Gamma_G$ . This zigzag of weak equivalences implies that

$$\pi^s \pi_t (\lim_i ((\Gamma_G^\bullet X'_i)^G)) \cong \pi^s \pi_t (\text{holim}_i ((\Gamma_G^\bullet (X_i)_{f,G})^G)). \quad \square$$

**Corollary 4.4.** *Let  $\{X'_i\}$  be as in Theorem 4.2. Then there is an isomorphism*

$$E_2^{s,t} \cong H^s[\pi_t(\mathcal{C}^*(G; \{X'_i\}))],$$

where  $E_2^{s,t}$  is the  $E_2$ -term of (1.2).

**Remark 4.5.** By Theorem 2.3,  $H^s[\mathcal{C}^*(G; \{\pi_t(X'_i)\})] \cong H_{\text{cts}}^s(G; \lim_i \pi_t(X_i))$ .

## 5. The failure of other possible descriptions of the $E_2$ -term

After studying the expression in (4.3) further, one recalls that  $\lim_i (-)^G$  is the functor used to define  $H_{\text{cont}}^s(G; -)$ , and, if  $M$  is any discrete  $G$ -module, then

$$0 \rightarrow M \rightarrow \Gamma_G^* M$$

is a  $(-)^G$ -acyclic resolution of  $M$ , so that  $H^s[(\Gamma_G^* M)^G] = H_c^s(G; M)$ .

Let  $\{M_i\}$  be a tower of discrete  $G$ -modules. If

$$\{0\} \rightarrow \{M_i\} \rightarrow \{\Gamma_G^* M_i\}$$

is a  $\lim_i (-)^G$ -acyclic resolution of  $\{M_i\}$  in  $\mathbf{tow}(\text{DMod}(G))$ , then

$$H^s[(\lim_i (-)^G)(\{\Gamma_G^* M_i\})] = H_{\text{cont}}^s(G; \{M_i\}).$$

This would imply that  $E_2^{s,t} \cong H^s[\pi_t(\lim_i (\Gamma_G^* X'_i)^G)]$  is computed by taking the cohomology of the homotopy groups of a complex of spectra in the stable homotopy category, that, in the context of abelian groups, computes continuous cohomology. This would be an interesting presentation of the  $E_2$ -term.

However, it is not hard to show that

$$\{0\} \rightarrow \{M_i\} \rightarrow \{\Gamma_G^* M_i\}$$

need not be a  $\lim_i (-)^G$ -acyclic resolution of  $\{M_i\}$  in  $\mathbf{tow}(\text{DMod}(G))$ , so that the above interpretation of the  $E_2$ -term does not work out. For example, by [7, (2.1)], there is a short exact sequence

$$0 \rightarrow \lim_i^1 H_c^{s-1}(G; \Gamma_G M_i) \rightarrow H_{\text{cont}}^s(G; \{\Gamma_G M_i\}) \rightarrow \lim_i H_c^s(G; \Gamma_G M_i) \rightarrow 0,$$

for each  $s \geq 0$ , where  $H_c^{-1}(G; -) = 0$ . Therefore, when  $s \geq 1$ ,  $H_c^s(G; \Gamma_G M_i) = 0$ , so that, for all  $s \geq 2$ ,  $H_{\text{cont}}^s(G; \{\Gamma_G M_i\}) = 0$ . But, the short exact sequence also implies that

$$H_{\text{cont}}^1(G; \{\Gamma_G M_i\}) \cong \lim_i^1 M_i,$$

which need not vanish. Thus,  $\{\Gamma_G M_i\}$ , the first object in the complex  $\{\Gamma_G^* M_i\}$ , need not be  $\lim_i(-)^G$ -acyclic in  $\mathbf{tow}(\mathbf{DMod}(G))$ .

Upon further consideration of the expression in (4.3), one notices that, for any  $k, l, m \geq 0$ ,

$$((\lim_i(\Gamma_G^{m+1} X'_i)^G)_k)_l = \lim_i(\Gamma_G^{m+1}((X'_i)_k)_l)^G \cong \text{Map}_c(G^m, \lim_i((X'_i)_k)_l)$$

is an isomorphism of sets. If one could promote this isomorphism to

$$(5.1) \quad \lim_i(\Gamma_G^{m+1} X'_i)^G \cong \text{Map}_c(G^m, \lim_i X'_i),$$

then one could use this to interpret the expression in (4.3) as being the cohomology of homotopy groups applied to the complex of continuous cochains with target (“coefficients”) the continuous  $G$ -spectrum  $\lim_i X'_i$ .

But notice that, in this interpretation, the expression  $\text{Map}_c(G^m, \lim_i X'_i)$  does not have the desired meaning. For isomorphism (5.1) to hold,  $\lim_i X'_i$  must be a spectrum whose simplicial sets have simplices with the pro-discrete topology. But, as a Bousfield–Friedlander spectrum, in the construction  $\text{Map}_c(G^m, \lim_i X'_i)$ ,  $\lim_i X'_i$  consists of simplicial sets whose simplices all have the discrete topology, by default. This conflict means that this interpretation also fails to work.

## References

- [1] Bousfield, A. K.; Kan, D. M. Homotopy limits, completions and localizations. Lecture Notes in Mathematics, 304. Springer-Verlag, Berlin, 1972. [MR0365573](#) (51 #1825), [Zbl 0259.55004](#).
- [2] Davis, Daniel G. Homotopy fixed points for  $L_{K(n)}(E_n \wedge X)$  using the continuous action, *J. Pure Appl. Algebra* **206** (2006), no. 3, 322–354.
- [3] Davis, Daniel G. The Lubin–Tate spectrum and its homotopy fixed point spectra. Thesis. Northwestern University, May 9, 2003.
- [4] Devinatz, Ethan S.; Hopkins, Michael J. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology* **43** (2004), no. 1, 1–47. [MR2030586](#) (2004i:55012), [Zbl 1047.55004](#).
- [5] Geisser, Thomas; Hesselholt, Lars. Topological cyclic homology of schemes. Algebraic K-Theory (Seattle, WA, 1997) (W. Raskind and C. Weibel, eds.), 41–87. Proc. Sympos. Pure Math., 67. Amer. Math. Soc., Providence, RI, 1999. [MR1743237](#) (2001g:19003), [Zbl 0953.19001](#).
- [6] Goerss, Paul G.; Jardine, John F. Simplicial homotopy theory. Birkhäuser Verlag, Basel, 1999. [MR1711612](#) (2001d:55012), [Zbl 0949.55001](#).
- [7] Jannsen, Uwe. Continuous étale cohomology. *Math. Ann.* **280** (1988), no. 2, 207–245. [MR929536](#) (89a:14022), [Zbl 0649.14011](#).
- [8] Mitchell, Stephen A. Hypercohomology spectra and Thomason’s descent theorem. Algebraic K-Theory (Toronto, ON, 1996) (V. Snaith, ed.), 221–277. Fields Inst. Commun., 16. Amer. Math. Soc., Providence, RI, 1997. [MR1466977](#) (99f:19002), [Zbl 0888.19003](#).
- [9] Neukirch, Jürgen; Schmidt, Alexander; Wingberg, Kay. Cohomology of Number Fields. Grundlehren der Mathematischen Wissenschaften 323. Springer-Verlag, Berlin, 2000. [MR1737196](#) (2000j:11168), [Zbl 0948.11001](#).

- [10] Thomason, R. W. Algebraic  $K$ -theory and étale cohomology, *Ann. Sci. École Norm. Sup.* (4) **18** (1985), no. 3, 437–552. [MR826102](#) (87k:14016), [Zbl 0596.14012](#).

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