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Bounded geometry in relatively hyperbolic groups

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Abstract. If a group is relatively hyperbolic, the parabolic subgroups are virtually nilpotent if and only if there exists a hyperbolic space with bounded geometry on which it acts geometrically finitely.

This provides, via the embedding theorem of M. Bonk and O. Schramm, a very short proof of the finiteness of asymptotic dimension for such groups (which is known to imply Novikov conjectures).

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The class of relatively hyperbolic groups is an important class of groups encompassing hyperbolic groups, fundamental groups of geometrically finite orbifolds with pinched negative curvature, groups acting on CAT(0) spaces with isolated flats, and many other examples. It was introduced by M. Gromov in [G1] and developed by B. Bowditch, B. Farb, and other authors (e.g., [Bow, F]). There is now an interesting and rich literature on the subject.

A finitely generated group Γ is *hyperbolic relative* to a family of finitely generated subgroups $\mathcal{G}$ if it acts on a proper complete hyperbolic geodesic space X, preserving a family of disjoint open horoballs $\{B_p, p \in P\}$, finite up to the action of Γ, such that the stabilizers G_p of the B_p, $p \in P$ are exactly the elements of $\mathcal{G}$, and for all p, G_p acts cocompactly on the horospheres of B_p, and such that the action of Γ on $X \setminus (\bigcup_{p\in P} B_p)$ is cocompact (see [Bow]).

A space X satisfying the conditions of the definition is referred to as an *associated space* to Γ. Geometrically, one should think of the complement of the horoballs as the universal cover of the convex core of a geometrically finite hyperbolic manifold

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(or equivalently of the thick part of the manifold for Margulis decomposition), and should think of the horoballs as the covers of the cusps.

In many geometrical examples, the parabolic subgroups of Γ, that is, the elements of the family $\mathcal{G}$, are virtually nilpotent. The main examples are geometrically finite manifolds with pinched negative curvature (one can also mention limit groups [D1], groups with boundary homeomorphic to a Sierpinski curve or a 2-sphere [D2]). If the curvature on the manifold is allowed to collapse to $-\infty$, one can obtain other parabolic subgroups (especially nonamenable ones, see [GP, Prop. 0.3]). The difference between these two cases can be identified.

Let us say that a space X is *geometrically bounded* if there exists a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $R > 0$, every ball of radius R can be covered by $f(R)$ balls of radius 1, and every ball of radius 1 can be covered by $f(R)$ balls of radius $1/R$. In some sense, the function f measures the volume of balls. Such a function always exists when there is a cocompact action on X.

A difference between a complete simply connected manifold M of pinched negative curvature, and M' in which the curvature is not bounded below, is that M is geometrically bounded, whereas the volumes of a sequence of balls of same radius in M' may tend to infinity. This remark generalizes:

Theorem 0.1. *Let Γ be a finitely generated group, hyperbolic relative to a family $\mathcal{G}$ of finitely generated subgroups. Then, every element of $\mathcal{G}$ is virtually nilpotent if and only if there exists a space X associated to Γ that has bounded geometry.*

The purpose of this note is to prove this characterization, and explain how, in this case, one can deduce short proofs of significant results. Namely we prove that these groups have finite asymptotic dimension, a property with strong consequences.

The *asymptotic dimension* of a metric space is a quasiisometric invariant introduced by M. Gromov in [G3]. For an introduction, we refer to [R]. It is denoted $\mathrm{asdim}(X)$, for a space X, and is defined as follows: it is an integer, and it is less than $n \in \mathbb{N}$ if and only if for all $d > 0$ there exists a covering of X by subsets of uniformly bounded diameter, with d-multiplicity at most $n + 1$.

Here, the d-multiplicity of a covering is the minimal number m such that each ball of radius d intersects at most m elements of the covering.

For the classical examples, the notion gives what is expected, as for example, for Euclidean and hyperbolic spaces, $\mathrm{asdim}(\mathbb{R}^n) = \mathrm{asdim}(\mathbb{H}^n) = n$ (see [R] or [G3]).

Corollary 0.2. *Let Γ a finitely generated group that is hyperbolic relative to a family of virtually nilpotent groups. Then, the asymptotic dimension of Γ is finite.*

Shortly after our preprint was first posted on arXiv, D. Osin announced a result in [O] that generalizes ours, for groups hyperbolic relative to a family of groups of finite asymptotic dimension. He uses completely different methods. The interest of our method, we believe, is its simplicity and rapidity, and that we show a general property in the spirit of the Margulis Lemma(s).

Once Theorem 0.1 is established, Corollary 0.2 follows from the embedding theorem of M. Bonk and O. Schramm [BS], stating, in particular, that any geometrically bounded Gromov-hyperbolic geodesic space is quasiisometric to some convex subspace of some hyperbolic space $\mathbb{H}^n$. Such a space has asymptotic dimension at most n. Applying this to the space associated to Γ given by Theorem 0.1, we get that

Γ acts properly discontinuously by isometries on a space that has finite asymptotic dimension. Therefore, it has finite asymptotic dimension itself.

It is worth noting that G. Yu proved in [Y] the coarse Baum-Connes conjecture for proper metric spaces with finite asymptotic dimension. G. Carlsson and B. Goldfarb [CG] proved the integral Novikov conjecture under such a hypothesis.

In order to prove Theorem 0.1, we make use of a certain space X associated to Γ, that is constructed by B. Bowditch in [Bow] when he proves that certain definitions are equivalent. In this model, we first prove, by a growth argument, that, if the horospheres have polynomial growth for their length metric, then the horoballs are geometrically bounded. Then we prove it for the whole space X, using the cocompactness of the action on the complement.

The converse finds its roots in a claim of M. Gromov [G3, p. 150] titled "Generalised and Weakened Margulis Lemma". We prove it by giving an upper bound to the growth of the volume of a space X associated to Γ, and deduce that a group acting properly discontinuously on an horosphere must have polynomial growth.

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1. Preliminaries

Let Γ be a finitely generated group. We note by gr the growth function of Γ, i.e., for all $R > 0$, $\mathrm{gr}(R)$ is the cardinality of a ball of radius R. It is a well-known theorem of M. Gromov that a finitely generated group is virtually nilpotent if and only if it has polynomial growth (*i.e.*, $\mathrm{gr}(R) \leq CR^p$ for some constants C and p). We formulate this latter condition in a slightly different way in the following lemma:

Lemma 1.1. *Given a finitely generated group Γ with a word metric, the following are equivalent:*

A1) *For all $\epsilon < 1$ there is a constant $N = N(\epsilon)$ such that all balls $B(R)$ of radius R can be covered by at most N balls of radius ϵR.*

A2) *The growth of Γ is polynomial.*

Before giving the proof let us recall a result that characterises polynomial growth.

Theorem 1.2 (H. Bass [Bas])**.** *A group G has polynomial growth if and only if there exist constants K_1, K_2, p such that for all R, one has $K_1 R^p \leq \mathrm{gr}(R) \leq K_2 R^p$.*

Proof of Lemma 1.1. If one assumes A1, then, for $\epsilon = \frac{1}{2}$, there exists N such that for all k and R, one has $\mathrm{gr}(R) \leq N \times \mathrm{gr}(\epsilon R) \leq N^k \mathrm{gr}(\epsilon^k R)$. Choose k such that $1 \leq \epsilon^k R \leq 2$. One has

$$\left(e^{k \log(\epsilon)}\right)^{\frac{\log(N)}{\log(\epsilon)}} \leq R^{-\frac{\log(N)}{\log(\epsilon)}},$$

hence $N^k \leq R^{-\log(N)/\log(\epsilon)}$. Therefore, $\mathrm{gr}(R) \leq \mathrm{gr}(2) \times R^{-\log(N)/\log(\epsilon)}$, what we wanted.

Conversely if one assumes that A1 is not satisfied, then we claim that there is $\epsilon \in (0,1)$, such that for all $N > 0$, there exists R and a ball B of radius R containing N disjoint ball of radius $\epsilon R/4$.

To see this, consider a ball B' of radius R' that is a counterexample of A1 (for given $\epsilon < 1$ and $N - 1$), and let choose $R = \frac{R'}{1-\epsilon/2}$, and B the ball of same center

as B' and radius R. Consider $B_1, \dots, B_N$, N balls of radius $\epsilon R'/2$ in B. Assume they are not disjoint. By assumption, the balls of same centers but of radius $\epsilon R'$ do not fill B', we choose x be a point in the complement of this union, in B'. Hence, if two of the balls $B_1 \dots B_N$ intersect, one can exchange one of them with a ball of same radius centered in x, and this one intersects no other, being at distance at least $\epsilon R' + 1$ from any other center. After at most $N - 1$ of these moves, one has a family of disjoint balls of radius $\epsilon R'/2 = \epsilon R \times (2 - \epsilon)/4 \geq \epsilon R/4$ in B', and by the triangle inequality, they all lie in B. This proves the claim.

Assuming that Γ has polynomial growth, it satisfies Bass' Theorem 1.2. It follows from the claim that $\mathrm{gr}(R) \geq (K_1 R^p \epsilon^p / 4^p) \times (N + 1)$. On the other hand, $\mathrm{gr}(R) \leq K_2 R^p$. Therefore, $K_2 \geq K_1(\epsilon/4)^p \times (N + 1)$. As this is true for all N this yields a contradiction. □

2. Polynomial growth for groups and bounded geometry for horoballs

We recall constructions that can be found in the work of Bowditch [Bow] that associate an hyperbolic horoball to every group G, on which G acts, cocompactly on horospheres. We remark that if G has polynomial growth then this space has bounded geometry.

In the upper half plane model of the hyperbolic plane $\mathbb{H}^2$ let us note $T_t = [0, 1] \times [t, \infty)$, for all $t \geq 1$.

Let K be a connected graph. Let $\mathcal{C}(K)$ be the space $K \times [1, \infty)$, with a length metric ρ that induces an isometry between T_1 and $e \times [1, \infty)$, for every edge e. It is shown in [Bow] that $\mathcal{C}(K)$ is a proper hyperbolic metric space in which every two rays are asymptotic. Its Gromov boundary consists in a single point a.

For all $t \geq 1$, let K_t be the horosphere $K \times \{t\}$, at distance $t - 1$ from K. Note that K_t with its induced metric is isometric to (K, d_t) where $d_t = e^{1-t} d_1$, and d_1 is the graph metric of K.

Proposition 2.1. *There exists constants $A, B > 0$ and $\alpha, \beta > 0$ depending only on the constant of hyperbolicity of $(\mathcal{C}(K), \rho)$ such that for all $t \geq 0$ and for all $x, y \in K_t$, $B \exp(\beta \rho(x, y)) \leq d_t(x, y) \leq A \exp(\alpha \rho(x, y))$.*

Proof. The lower bound is classical and true in any horosphere of any hyperbolic space. The upper bound follows from the fact that, if $[x, y]$ is a geodesic segment of K_t, then its convex hull in $(\mathcal{C}(K), \rho)$ is, by construction, isometric to the region of the upper half plane $[0, d_t(x, y)] \times [t, +\infty)$, where the result is classical. □

We denote by π_t the orthogonal projection map on K_t. For all $t \geq t' \geq 0$ and for all $x, y \in K_{t'}$ we have $d_{t'}(x, y) = \exp(t - t') d_t(\pi_t(x), \pi_t(y))$. Note that the projection map π_t sends the balls of $K_{t'}$ to the balls of K_t.

Let G be a finitely generated group given with a preferred set of generators and let K_G be its associated Cayley graph. We consider the space $\mathcal{C}(K_G) = \mathcal{C}(G)$.

Proposition 2.2. *If G has polynomial growth then $\mathcal{C}(G)$ has bounded geometry.*

Proof. We identify G with the set of vertices of a Cayley graph $K_G = K_1$. We first note that K_1 also satisfies the property A1 of Lemma 1.1. In fact it suffices to show that A1 is satisfied for $\epsilon = 1/2$, since by iteration each ball $B(R)$ of radius R in the Cayley graph can be covered by at most $N(1/2)^n$ balls of radius $(1/2)^n R$, where n

is the first integer with $(1/2)^n < \epsilon$. Thus $N(\epsilon) = N(1/2)^n$ gives the result. Now when $R \geq 1$ the statement is justified by Lemma 1.1, since $B(R)$ can be entirely covered by N' balls centered at vertices of the Cayley graph and of radius $R/2$. When $R < 1$, $B(R)$ is covered by $\mathrm{gr}(1) + 1$ balls of radius $R/2$. Thus by setting $N(1/2) = \max\{N', \mathrm{gr}(1) + 1\}$ we prove the claim. Note also that this property is invariant by homothety, thus for all t, (K_t, d_t) satisfies A1.

Let R be a number, and let R' be equal to R or 1. We want to cover any ball of radius R' in $\mathcal{C}(G)$ by a controlled number of balls of radius R'/R.

Consider a ball B of radius R' in $\mathcal{C}(G)$. Let K_t be an horosphere intersecting it. Then $K_t \cap B$ has diameter at most $2R'$ in $\mathcal{C}(G)$ and hence is contained in a ball of K_t of radius $A\exp(2\alpha R')$ for the metric d_t, which is homothetic (with factor $\exp(t-1)$) to (K_1, d_1). By the remark above this intersection can be covered by $N(R', R, A, B, \alpha, \beta)$ balls (note that this number does not depend on t) of radius $B\exp(\beta R'/2R)$ of (K_t, d_t), which are contained in balls of $\mathcal{C}(G)$ of same center, and radius at most $R'/(2R)$ for the ambiant metric.

To cover the entire ball B by balls of radius 1, it is enough to perform this on $4R'$ regularly spaced horospheres (at distance $1/(2R')$ from each other) intersecting B. One gets a number depending only on R, of balls of radius R'/R that cover B. □

3. Proof of Theorem 0.1

We recall now some of the results and constructions given by Bowditch in [Bow]. Given a group Γ hyperbolic relative to the family $\mathcal{G}$ and a space X associated to Γ, he shows that there is a family of disjoint Γ-invariant, quasiconvex horoballs H_p based at parabolic points $p \in \partial X$ with following properties:

- ⋆ There are only finitely many orbits of horoballs in the family, and the elements of $\mathcal{G}$ are exactly the stabilizers of the different horoballs.
- ⋆ The quotient of an horosphere based at p (i.e., the frontier in X of an horoball based at p) by the stabiliser of p in Γ is compact.
- ⋆ The quotient of $X \setminus \bigcup_p \mathrm{int}(H_p)$ by Γ is compact.

The proof of these statements can be found in [Bow, Chapter 6] under Lemma 6.3 and Proposition 6.13. Moreover he proves that there exists another associated space to Γ where the horoballs can be chosen to be isometric to the space $\mathcal{C}(G)$ where G is a maximal parabolic subgroup in $\mathcal{G}$ ([Bow] Chapter 3, Lemma 3.7 and Theorem 3.8). We will refer for the rest of the paper to this particular space as Bowditch's space. In general a space X associated to a relatively hyperbolic group Γ can be different from it.

Proof of Theorem 0.1. We first prove that if the parabolic subgroups are virtually nilpotent then Bowditch's space is geometrically bounded. Indeed, as Γ acts on $X \setminus \cup_p \mathrm{int}\, H_p$ cocompactly X has bounded geometry if and only if each horoball in X has uniformly bounded geometry. On the other hand since there are only finitely many orbits of horoballs it suffices to show that they all have bounded geometry. But in this particular space, each horoball H_p is isometric to some $\mathcal{C}(G)$ where G is the stabiliser of p in Γ, and therefore, by Proposition 2.2, this follows.

We turn to the converse. Given a relatively hyperbolic group Γ we assume that there exists a space X of bounded geometry associated to Γ. Denote its metric

by ρ. We consider a family of disjoint Γ-invariant horoballs H_p based at parabolic points with the above properties, and we note Σ_p their horospheres. By assumption each horoball H_p has bounded geometry.

For all parabolic points p, let G_p be the parabolic group associated. It acts cocompactly on the horosphere Σ_p. Let us consider $\mathcal{O}_p$ an orbit of G_p in Σ_p, with a metric d_p induced by a word metric on G_p. Then $(\mathcal{O}_p, d_p)$ is quasiisometric to G_p, and note that it is s-separated, for a certain constant s that can be chosen to be 2, up to rescaling X once and for all (note that being geometrically bounded is preserved by rescaling the metric). Thus to show that G_p has polynomial growth it is sufficient to show that $(\mathcal{O}_p, d_p)$ has polynomial growth. It is classical that the distortion of $\mathcal{O}_p$ in X is at least exponential, since it lies on a horosphere.

Let $x_0 \in \mathcal{O}_p$, and let us consider $R > 0$ and $B_R(x_0)$ the ball of radius R of X centered at x_0. Let $f(R)$ be the cardinality of $\mathcal{O}_p \cap B_R(x_0)$. By minoration by an exponential of distances on the horosphere, one deduces that $\mathcal{O}_p \cap B_R(x_0)$ contains a ball of $(\mathcal{O}_p, d_p)$ of center x_0 and radius $A\exp(\alpha R)$, where A and α depend only on X and H_p. Let us denote $gr_{\mathcal{O}}$ the growth function of $\mathcal{O}_p$. Thus one has $gr_{\mathcal{O}}(A\exp(\alpha R)) \leq f(R)$.

We now remark that the definition of bounded geometry allows one to map a N-regular tree T on an 1-dense image in X by a 2-Lipschitz map $\pi : T \to X$, where N is the constant required to cover a ball of radius 2 by balls of radius 1. Indeed it suffices to map the neighbors of a vertex v of the tree to the centers of balls of radius 1 covering a ball of radius 2 centered at $\pi(v)$.

The growth function of a N-regular tree is exponential. Therefore, the number of disjoint balls of radius 1 that belong to a ball of radius R in X is bounded exponentially depending only on R and X, and hence the function f is at most an exponential, since it counts 2-separated elements in the ball $B_R(x_0)$. Let us say that $f(R) \leq B\exp(\beta(R))$. From this and the fact $gr_{\mathcal{O}}(A\exp(\alpha R)) \leq f(R)$, one computes that $gr_{\mathcal{O}}(t) \leq B(t/A)^{\beta/\alpha}$, which is polynomial. □

References

[Bas] H. Bass, *The degree of polynomial growth of finitely generated nilpotent groups*, Proc. London Math. Soc. (3) **25** (1972), 603–614, MR 0379672 (52 #577), Zbl 0259.20045.

[BS] M. Bonk and O. Schramm, *Embeddings of Gromov hyperbolic spaces*, Geom. Funct. Anal. **10** (2000), 266–306, MR 1771428 (2001g:53077), Zbl 0972.53021.

[Bow] B.H. Bowditch, *Relatively hyperbolic groups*, preprint, Southampton, 1997.

[CG] G. Carlsson and B. Goldfarb, *The integral K-theoretic Novikov conjecture for groups with finite asymptotic dimension*, Invent. Math. **157** (2004), 405–418, MR 2076928.

[D1] F. Dahmani, *Combination of convergence groups*, Geom. & Topol. **7** (2003), 933–963, MR 2026551, Zbl 1037.20042.

[D2] F. Dahmani *Parabolic groups acting on one-dimensional compact spaces*, to appear in Internat. J. Alg. Comp. special issue dedicated to R. Grigorchuk's 50th birthday.

[F] B. Farb, *Relatively hyperbolic groups*, Geom. Funct. Anal. **8** (1998), 810–840, MR 1650094 (99j:20043), Zbl 0985.20027.

[GP] D. Gaboriau and F. Paulin, *Sur les immeubles hyperboliques*, Geom. Dedicata **88** (2001), 153–197, MR 1877215 (2002k:51014), Zbl 1020.51007.

[G1] M. Gromov, *Hyperbolic groups*, Essays in geometric group theory, Math. Sci. Res. Inst. Publ. Springer, New York, **8** (1987), 75–263, MR 0919829 (89e:20070), Zbl 0634.20015.

[G2] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–73, MR 0623534 (83b:53041), Zbl 0474.20018.

[G3] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser. **182**, Cambridge Univ. Press, Cambridge, 1993, MR 1253544 (95m:20041), Zbl 0841.20039.

[O] D. Osin, *Asymptotic dimension of relatively hyperbolic groups*, preprint, 2004.

[R] J. Roe, *Lectures on coarse geometry*, University Lecture Series, **31**, American Mathematical Society, Providence, 2003, MR 2007488 (2004g:53050).

[Y] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. **139** (2000), 201–240, MR 1728880 (2000j:19005), Zbl 0956.19004.

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