

Weak mixing of maps with bounded cutting parameter

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ABSTRACT. In the class of Ornstein transformations the mixing property satisfies a 0-1 law. Here we consider Ornstein's construction with bounded cutting parameter. In fact, these latter transformations are not mixing, however it is proved that the weak mixing property occurs with probability one. Our situation is similar to the case of interval exchange transformations whose link with the cutting and stacking construction relies on a dynamical process called Rauzy induction.

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1. Introduction

In a nowadays classical work [11], Ornstein associates to every point ω in a probability space Ω a rank one transformation T_ω which he proves to be mixing for almost every ω . There are many extensions and generalizations of Ornstein's construction. Here we regard a class of natural examples of rank one maps of the interval for inspiration to consider a generalization of Ornstein's result.

It has been established by Veech [13] that interval exchanges are almost surely rank one, assuming the permutation is irreducible. The link between the cutting and stacking construction and interval exchanges is done through Rauzy induction which is a way to define first return induced transformation without increasing

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the number of exchanged intervals (see [12]). However Katok [10] proved that no interval exchange map is mixing. Katok's result remains at the present time, the only universal result about the spectrum of interval exchanges. In fact, he showed that every interval exchange map is α -rigid (a transformation T is said to be α -rigid, $0 < \alpha < 1$, if there exists an infinite sequence of integers $\{n_k\}_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} \mu(T^{n_k} A \cap A) \geq \alpha \mu(A)$, for every measurable set A). Naturally, one asks the following question:

Question 1.1. Does any interval exchange map have singular spectral type?

It is known that the stacking construction using a constant cutting parameter results in a map which is not mixing. In the class of interval exchanges, whether weak mixing is almost surely satisfied was still an open question (see [12]) until the recent work of Avilla and Forni [1]. We recall that Katok and Stepin [9] and Veech [13] have proved that for some permutations almost every interval exchange map is weak mixing, nevertheless in [4] it is shown that those permutations force the eigenvalue to be 1.

Here we will consider the class of Ornstein transformations with bounded cutting parameter. We prove that in this case the weak mixing property occurs with probability one.

It is clear that our result is in the same spirit of the result of [1]. One may hope that there is sufficient analogy between our construction and interval exchange maps for our methods to extend to this case and to obtain a unified proof.

Here we begin by considering a generalization of Ornstein's construction with less restrictions (namely in Ornstein's case, some nonexplicit cutting parameters satisfying some growth condition ensuring mixing are shown to exist, whereas here we fix them in advance). In this context, answering a question asked by J.-P. Thouvenot, we extend a result proved in [8] to the class of Ornstein transformations with bounded cutting parameter.

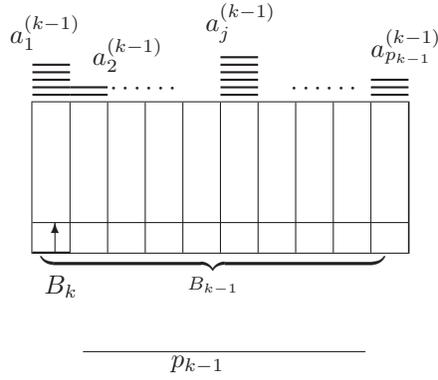
We will assume that the reader is familiar with the method of cutting and stacking for constructing rank one transformations.

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2. Construction of rank one transformations

Using the cutting and stacking method described in Friedman [7, 8], we can define inductively a family of measure preserving rank one transformations, as follows: let B_0 be the unit interval equipped with the Lebesgue measure. At the first stage we split B_0 into p_0 equal parts, add spacers and form a stack of height h_1 in the usual manner. At the k -th stage we divide the stack obtained at the $(k-1)$ -st stage into p_{k-1} equal columns, add spacers and obtain a new stack of height h_k . If during the k -th stage of our construction the number of spacers put above the j -th column of the $(k-1)$ -st stack is $a_j^{(k-1)} \geq 0$, $1 \leq j \leq p_k$, then we have

$$h_k = p_{k-1}h_{k-1} + \sum_{j=1}^{p_{k-1}} a_j^{(k-1)}.$$

Figure 1. k^{th} tower.

Proceeding in this way we get a rank one transformation T on a certain measure space (X, \mathcal{B}, ν) which may be finite or σ -finite depending on the number of spacers added.

The construction of a rank one transformation thus needs two parameters: a cutting and stacking parameter $(p_k)_{k=0}^\infty$ and a spacers parameter $\left(\left(a_j^{(k)}\right)_{j=1}^{p_k}\right)_{k=0}^\infty$. We define

$$T \stackrel{\text{def}}{=} T\left(p_k, \left(a_j^{(k)}\right)_{j=1}^{p_k}\right)_{k=0}^\infty.$$

3. Ornstein's class of transformations

In Ornstein's construction, the p_k 's are rapidly increasing and the number of spacers, $a_i^{(k)}$, $1 \leq i \leq p_k - 1$, is chosen stochastically in a certain way (subject to certain bounds). This may be organized in various ways as noted by Bourgain [3], in fact, let (t_k) be a sequence of positive integers such that $2t_k \leq h_k$. We choose now independently, using the uniform distribution on the set $X_k = \{-t_k, \dots, t_k\}$, the numbers $(x_{k,i})_{i=1}^{p_k-1}$, and x_{k,p_k} is chosen deterministically in \mathbb{N} . We set, for $1 \leq i \leq p_k$,

$$a_i^{(k)} = 2t_k + x_{k,i} - x_{k,i-1}, \text{ with } x_{k,0} = 0.$$

Then one sees that

$$h_{k+1} = p_k(h_k + 2t_k) + x_{k,p_k}.$$

So the deterministic sequences of positive integers $(p_k)_{k=0}^\infty$ and $(x_{k,p_k})_{k=0}^\infty$ completely determine the sequence of heights $(h_k)_{k=1}^\infty$. The total measure of the resulting measure space is finite if

$$\sum_{k=1}^\infty \frac{t_k}{h_k} + \sum_{k=1}^\infty \frac{x_{k,p_k}}{p_k h_k} < \infty.$$

We will assume that this requirement is satisfied.

We thus have a probability space of Ornstein transformations $\prod_{k=1}^\infty X_k^{p_k-1}$ equipped with the natural probability measure $\mathbb{P} \stackrel{\text{def}}{=} \otimes_{k=1}^\infty P_k$, where $P_k \stackrel{\text{def}}{=} \otimes_{i=1}^{p_k-1} \mathcal{U}_i$,

with \mathcal{U}_l the uniform probability on X_l . We denote this space by $(\Omega, \mathcal{A}, \mathbb{P})$. The projection of Ω onto the i -th coordinate space of $\Omega_k \stackrel{\text{def}}{=} X_k^{p_k-1}$, $1 \leq i \leq p_k - 1$ is $x_{k,i}$. Naturally each point $\omega = (\omega_k = (x_{k,i}(\omega))_{i=1}^{p_k-1})_{k=1}^\infty$ in Ω defines the spacers and therefore a rank one transformation which we denote by $T_{\omega,x}$, where $x = (x_{k,p_k})_{k=1}^\infty$ is *admissible*, i.e.,

$$\sum_{k=1}^{\infty} \frac{x_{k,p_k}}{p_k h_k} < \infty.$$

The above construction gives a more general definition of the random construction due to Ornstein.

We recall that an automorphism is said to be *totally ergodic* if all its nonzero powers are ergodic. It is shown in [5] that the classical Ornstein transformations are almost surely totally ergodic using the fact that a measure preserving automorphism is totally ergodic if and only if no root of unity other than 1 is an eigenvalue. In fact it is proved in [5] that, for a fixed $z \in \mathbb{T} \setminus \{1\} \equiv [0, 1) \setminus \{0\}$, $\{\omega : z \text{ is an eigenvalue of } T_\omega\}$ is a null measure set. Later, in [6], by Van der Corput's inequality and Bernstein's inequality on the derivative of a trigonometric polynomial combined with the ingredients of [5] and [2], we obtain a null measure set N such that for all $\omega \notin N$, T_ω has no eigenvalue other than 1 provided that for infinitely many n 's we have $t_n = p_n$ (p_n goes to ∞ , as n goes to ∞), and this implies the almost sure weak mixing property.

We note that it is an easy exercise to show that the spectral properties satisfy the *Zero-One law*.

4. Ornstein transformations with bounded cutting parameter

Here we assume that the cutting parameter $(p_k)_{k \geq 0}$ is bounded. Next we state our main result.

Theorem 4.1. *Let $x = (x_{k,p})_{k \in \mathbb{N}}$ be admissible sequences of positive integers: i.e.,*

$$\sum_{k=1}^{\infty} \frac{x_{k,p_k}}{p_k h_k} < \infty.$$

Then $\mathbb{P}(\mathcal{W}) = 1$, where \mathcal{W} is the set of ω for which T_ω is weak mixing.

First we remark that it is an easy exercise to see that the weak mixing property occurs with probability 1 if the series $\sum_{k=1}^{\infty} \frac{1}{t_k}$ diverges. In fact, one can show that Chacon's pattern occurs for infinitely many values of k with probability 1. Hence we assume that the series $\sum_{k=1}^{\infty} \frac{1}{t_k}$ converges.

A standard argument yields that for any rank one transformation

$$T \left(p_k, \left(a_j^{(k)} \right)_{j=1}^{p_k} \right)_{k=0}^\infty$$

with bounded cutting parameter, if $\lambda = e^{2i\pi\alpha}$ is an eigenvalue, then

$$\lambda^{h_k + a_1^{(k)}} \xrightarrow[k \rightarrow \infty]{} 1 \quad \text{or} \quad \|(h_k + a_1^{(k)})\alpha\| \xrightarrow[k \rightarrow \infty]{} 0,$$

where $\|x\| = d(x, \mathbb{Z})$.

Let \mathcal{N} be the subsequence of positive integers (n_k) and put

$$G(\mathcal{N}) \stackrel{\text{def}}{=} \{\lambda = e^{2i\pi\alpha} \in \mathbb{T} : \|n_k\alpha\| \xrightarrow[k \rightarrow \infty]{} 0\}.$$

4.1. Some general facts about $G(\mathcal{N})$ when $\left(\frac{n_{k+1}}{n_k}\right)_{k \in \mathbb{N}}$ is bounded. Let ε be a positive number and L a positive integer. Put

$$A_{(n_k), L}^{(\varepsilon)} = \{\lambda \in [0, 1) : \|n_k\lambda\| < \varepsilon, \forall n_k > L\}.$$

Observe that we have

$$(1) \quad G(\mathcal{N}) = \bigcap_{\varepsilon > 0} \bigcup_{L \in \mathbb{N}} A_{(n_k), L}^{(\varepsilon)}.$$

Now, from this observation, we shall study the properties of $G(\mathcal{N})$ when the sequence $\left(\frac{n_{k+1}}{n_k}\right)$ is bounded. We note, first, that we have the following lemma:

Lemma 4.2. *Assume that there exists a positive number M such that*

$$\frac{n_{k+1}}{n_k} < M, \text{ for any } k \in \mathbb{N}.$$

Then, for any ε less than $\frac{1}{4M}$ we have $|A_{(n_k), L}^{(\varepsilon)}| \leq n_{k_0}$, where k_0 is the smallest positive integer such that $n_{k_0} > L$ and $|A_{(n_k), L}^{(\varepsilon)}|$ is the cardinal of $A_{(n_k), L}^{(\varepsilon)}$.

Proof. Observe that

$$(2) \quad A_{(n_k), L}^{\varepsilon} = \bigcap_{n_k \geq L} B_{n_k}^{\varepsilon},$$

where $B_{n_k}^{\varepsilon} = \{\lambda \in [0, 1) : \|n_k\lambda\| < \varepsilon\}$. We deduce from the definition of k_0 and (2) that

$$A_{(n_k), L}^{\varepsilon} = \bigcap_{k \geq k_0} B_{n_k}^{\varepsilon}.$$

But $B_{n_k}^{\varepsilon}$ is the union of the intervals centered on $\frac{j}{n_k}$, $0 \leq j \leq n_k - 1$ and of length $\frac{2\varepsilon}{n_k}$. It follows that if some interval I from $B_{n_k}^{\varepsilon}$ centered on some $\frac{j}{n_k}$ has a nonempty intersection with two different intervals from $B_{n_{k+1}}^{\varepsilon}$, then we must have

$$(3) \quad \frac{1 - 2\varepsilon}{n_{k+1}} \leq \frac{2\varepsilon}{n_k}.$$

It follows from (3) that

$$\frac{2\varepsilon}{1 - 2\varepsilon} \geq \frac{1}{M} \text{ hence } 4\varepsilon \geq \frac{1}{M},$$

which yields a contradiction. Now, Let x, x' be in $A_{(n_k), L}^{\varepsilon}$. Assume that x, x' are in the same interval from $B_{n_{k_0}}^{\varepsilon}$. Then from the above we deduce by induction that

x, x' are in the same interval from $B_{n_k}^\varepsilon$, for any $k \geq k_0$. It follows that $x = x'$ and this yields that $|A_{(n_k), L}^{(\varepsilon)}| \leq n_{k_0}$. The proof of the lemma is complete. \square

Corollary 4.3. *If $\left(\frac{n_{k+1}}{n_k}\right)$ is bounded then $G(n_k)$ is countable.*

We have also the following lemma:

Lemma 4.4. *Let $\mathcal{N} = \{n_k, k \in \mathbb{N}\}$ and $\mathcal{N}' = \{n'_k, k \in \mathbb{N}\}$ be two sequences of positive integers such that there exist infinitely many k for which $n'_k = n_k + 1$. Then $G(\mathcal{N}) \cap G(\mathcal{N}') = \{1\}$.*

Proof. Straightforward. (Chacon's argument!) \square

5. Application and proof of theorem

First, set

$$\begin{aligned} n_k(\omega) &\stackrel{\text{def}}{=} h_k + x_{k,1}(\omega), \\ \mathcal{N}(\omega) &\stackrel{\text{def}}{=} \{n_k(\omega), k \in \mathbb{N}\}, \end{aligned}$$

$\omega \in \Omega$, and observe that we have, for any $\omega \in \Omega$,

$$\frac{n_{k+1}}{n_k} \leq p + 1.$$

Let \mathcal{F} be the σ -algebra generated by the random variables $\{x_{2k}, k \in \mathbb{N}\}$ and the event $x_{2k+1} \in A$, where A is any atom from the partition \mathcal{P}_{2k+1} given by

$$\mathcal{P}_{2k+1} = \left\{ \left\{ -\frac{t_{2k+1}}{2}, -\frac{t_{2k+1}}{2} + 1 \right\}, \dots \right\}.$$

The proof of the theorem will follow easily from the following lemma:

Lemma 5.1. *The conditionnal probability knowing \mathcal{F} satisfies*

$$\mathbb{P}_{\mathcal{F}}(G(\mathcal{N}(\omega))) = \{1\} = 1.$$

Proof. Observe that

$$G(\mathcal{N}(\omega)) \subseteq G(\mathcal{N}_2(\omega)),$$

where $\mathcal{N}_2 = \{n_{2k}, k \in \mathbb{N}\}$. Fix a fiber ϕ in \mathcal{F} . It follows that the sequence $\mathcal{N}_2 = \{n_{2k}\}$ is fixed. But, then if for some $\omega \in \phi$, λ is an eigenvalue of T_ω then λ is in $G(\mathcal{N}_2)$ and this last set is countable by the corollary. We deduce that there are countably many possible eigenvalues, for any ω in the fibre ϕ .

Let $\lambda \neq 1$ and assume that there exists ω^0 in the fiber such that $\lambda \in G(\mathcal{N}(\omega^0))$. It follows from Lemma 4.4 that for any ω such that $\lambda \in (\mathcal{N}(\omega))$ we have $\omega_k^0 \neq \omega_k$ for only finitely many k . We deduce that

$$\mathbb{P}_{|\phi}(\lambda \in G(\mathcal{N}(\omega))) = 0.$$

The proof of the lemma is complete. \square

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