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Global well-posedness and scattering for the higher-dimensional energy-critical nonlinear Schrödinger equation for radial data

Terence Tao

Abstract. In any dimension $n \geq 3$, we show that spherically symmetric bounded energy solutions of the defocusing energy-critical nonlinear Schrödinger equation $iu_t + \Delta u = |u|^{\frac{4}{n-2}} u$ in $\mathbf{R} \times \mathbf{R}^n$ exist globally and scatter to free solutions; this generalizes the three and four-dimensional results of Bourgain, 1999a and 1999b, and Grillakis, 2000. Furthermore we have bounds on various spacetime norms of the solution which are of exponential type in the energy, improving on the tower-type bounds of Bourgain. In higher dimensions $n \geq 6$ some new technical difficulties arise because of the very low power of the nonlinearity.

Contents

1. Introduction

Let $n \geq 3$ be an integer. We consider solutions $u : I \times \mathbf{R}^n \to \mathbf{C}$ of the defocusing energy-critial nonlinear Schrödinger equation

$$iu_t + \Delta u = F(u) \tag{1}$$

on a (possibly infinite) time interval I, where $F(u) := |u|^{\frac{4}{n-2}} u$. We will be interested in the Cauchy problem for the equation (1), specifying initial data $u(t_0)$ for some

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$t_0 \in I$ and then studying the existence and long-time behavior of solutions to this Cauchy problem.

We restrict our attention to solutions for which the energy

$$E(u) = E(u(t)) := \int_{\mathbf{R}^n} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{n-2}{2n}|u(t,x)|^{\frac{2n}{n-2}}\, dx$$

is finite. It is then known (see, e.g., [4]) that for any given choice of finite energy initial data $u(t_0)$, the solution exists for times close to t_0, and the energy $E(u)$ is conserved in those times. Furthermore this solution is unique[1] in the class $C^0_t \dot H^1_x \cap L^{2(n+2)/(n-2)}_{t,x}$, and we shall always assume our solutions to lie in this class. The significance of the exponent in (1) is that it is the unique exponent which is *energy-critical*, in the sense that the natural scale invariance

$$u(t,x) \mapsto \lambda^{-(n-2)/2} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \tag{2}$$

of the equation (1) leaves the energy invariant; in other words, the energy $E(u)$ is a dimensionless quantity.

If the energy $E(u(t_0))$ is sufficiently small (smaller than some absolute constant $\varepsilon > 0$ depending only on n) then it is known (see [4]) that one has a unique global finite-energy solution $u : \mathbf{R} \times \mathbf{R}^n \to \mathbf{C}$ to (1). Furthermore we have the global-in-time *Strichartz bounds*

$$\|\nabla u\|_{L^q_t L^r_x(\mathbf{R}\times\mathbf{R}^n)} \leq C(q,r,n,E(u))$$

for all exponents (q,r) which are *admissible* in the sense that[2]

$$2 \leq q, r \leq \infty; \quad \frac{1}{q} + \frac{n}{2r} = \frac{n}{4}. \tag{3}$$

In particular, from Sobolev embedding we have the spacetime estimate

$$\|u\|_{L^{2(n+2)/(n-2)}_{t,x}(\mathbf{R}\times\mathbf{R}^n)} \leq M(n, E(u)) \tag{4}$$

for some explicit function $M(n,E) > 0$. Because of this and some further Strichartz analysis, one can also show scattering, in the sense that there exist Schwarz solutions u_+, u_- to the *free* Schrödinger equation $(i\partial_t + \Delta)u_\pm = 0$, such that

$$\|u(t) - u_\pm(t)\|_{\dot H^1(\mathbf{R}^n)} \to 0 \text{ as } t \to \pm\infty.$$

This can then be used to develop a small energy scattering theory (existence of wave operators, asymptotic completeness, etc.); see [3]. Also, one can show that the solution map $u(t_0) \to u(t)$ extends to a globally Lipschitz map in the energy space $\dot H^1(\mathbf{R}^n)$.

The question then arises as to what happens for large energy data. In [4] it was shown that the Cauchy problem is locally well posed for this class of data, so that we can construct solutions for short times at least; the issue is whether

[1] In fact, the condition that the solution lies in $L^{2(n+2)/(n-2)}_{t,x}$ can be omitted from the uniqueness result, thanks to the endpoint Strichartz estimate in [14] and the Sobolev embedding $\dot H^1_x \subseteq L^{2n/(n-2)}_x$; see [13], [8], [9] for further discussion. We thank Thierry Cazenave for this observation.

[2] Strictly speaking, the result in [4] did not obtain these estimates for the endpoint $q = 2$, but they can easily be recovered by inserting the Strichartz estimates from [14] into the argument in [4].

these solutions can be extended to all times, and whether one can obtain scattering results like before. It is well-known that such results will indeed hold if one could obtain the *a priori* bound (4) for all global Schwarz solutions u (see, e.g., [2]). It is here that the sign of the nonlinearity in (1) is decisive (in contrast to the small energy theory, in which it plays no role). Indeed, if we replaced the nonlinearity $F(u)$ by the focusing nonlinearity $-F(u)$ then an argument of Glassey [10] shows that large energy Schwarz initial data can blow up in finite time; for instance, this will occur whenever the potential energy exceeds the kinetic energy.

In the defocusing case, however, the existence of *Morawetz inequalities* allows one to obtain better control on the solution. A typical such inequality is

$$\int_I \int_{\mathbf{R}^n} \frac{|u(t,x)|^{2n/(n-2)}}{|x|}\, dxdt \leq C \left(\sup_{t\in I} \|u(t)\|_{\dot H^{1/2}(\mathbf{R}^n)} \right)^2$$

for all time intervals I and all Schwarz solutions $u : I \times \mathbf{R}^n \to \mathbf{C}$ to (1), where $C > 0$ is a constant depending only on n; this inequality can be proven by differentiating the quantity $\int_{\mathbf{R}^n} \mathrm{Im}\big(\frac{x}{|x|} \cdot \nabla u(t,x) \overline{u(t,x)} \big)\, dx$ in time and integrating by parts. This inequality is not directly useful for the energy-critical problem, as the right-hand side involves the Sobolev norm $\dot H^{1/2}(\mathbf{R}^n)$ instead of the energy norm $\dot H^1(\mathbf{R}^n)$. However, by applying an appropriate spatial cutoff, Bourgain [1, 2] and Grillakis [11] obtained the variant Morawetz estimate

$$\int_I \int_{|x| \leq A|I|^{1/2}} \frac{|u(t,x)|^{2n/(n-2)}}{|x|}\, dxdt \leq CA|I|^{1/2} E(u) \tag{5}$$

for all $A \geq 1$, where $|I|$ denotes the length of the time interval I; this estimate is more useful as it involves the energy on the right-hand side. For sake of self-containedness we present a proof of this inequality in Section 2.3.

The estimate (5) is useful for preventing concentration of $u(t,x)$ at the spatial origin $x = 0$. This is especially helpful in the *spherically symmetric case* $u(t,x) = u(t,|x|)$, since the spherical symmetry, combined with the bounded energy assumption can be used to show that u cannot concentrate at any other location than the spatial origin. Note that spatial concentration is the primary obstruction to establishing global existence for the critical NLS (1); see, e.g., [15] for some dicussion of this issue.

With the aid of (5) and several additional arguments, Bourgain [1, 2] and Grillakis [11] were able to show global existence of large energy spherically smooth solutions in the three-dimensional case $n = 3$. Furthermore, the argument in [1, 2] extends (with some technical difficulties) to the case $n = 4$ and also gives the spacetime bound (4) (which in turn yields the scattering and global well-posedness results mentioned earlier). However, the dependence of the constant $M(n, E(u))$ in (4) on the energy $E(u)$ given by this argument is rather poor; in fact it is an iterated tower of exponentials of height $O(E(u)^C)$. This is because the argument is based on an *induction on energy* strategy; for instance when $n = 3$ one selects a small number $\eta > 0$ which depends polynomially on the energy, removes a small component from the solution u to reduce the energy from $E(u)$ to $E(u) - \eta^4$, applies an induction hypothesis asserting a bound (4) for that reduced solution, and then glues the removed component back in using perturbation theory. The final argument gives a recursive estimate for $M(3,E)$ of the form

$$M(3,E) \leq C \exp\left(\eta^C M(3, E-\eta^4)^C\right)$$

for various absolute constants $C > 0$, and with $\eta = cE^{-C}$. It is this recursive inequality which yields the tower growth in $M(3,E)$. The argument of Grillakis [11] is not based on an induction on energy, but is based on obtaining $L^\infty_{t,x}$ control on u rather than Strichartz control (as in (4)), and it is not clear whether it can be adapted to give a bound on $M(3,E)$.

The main result of this paper is to generalize the result[3] of Bourgain to general dimensions, and to remove the tower dependence on $M(n,E)$, although we are still restricted to spherically symmetric data. As with the argument of Bourgain, a large portion of our argument generalizes to the non-spherically-symmetric case; the spherical symmetry is needed only to ensure that the solution concentrates at the spatial origin, and not at any other point in spacetime, in order to exploit the Morawetz estimate (5). In light of the recent result in [7] extending the three-dimensional results to general data, it seems in fact likely that at least some of the ideas here can be used in the non-spherically-symmetric setting; see Remark 3.9.

Theorem 1.1. *Let* $[t_-, t_+]$ *be a compact interval, and let*

$$u \in C^0_t \dot H^1([t_-, t_+] \times \mathbf{R}^n) \cap L^{2(n+2)/(n-2)}_{t,x}([t_-, t_+] \times \mathbf{R}^n)$$

be a spherically symmetric solution to (1) *with energy* $E(u) \leq E$ *for some* $E > 0$. *Then we have*

$$\|u\|_{L^{2(n+2)/(n-2)}_{t,x}([t_-,t_+]\times\mathbf{R}^n)} \leq C \exp(CE^C)$$

for some absolute constants C *depending only on* n (*and thus independent of* E, $t_\pm$, u).

Because the bounds are independent of the length of the time interval $[t_-, t_+]$, it is a standard matter to use this theorem, combined with the local well-posedness theory in [4], to obtain global well-posedness and scattering conclusions for large energy spherically symmetric data; see [3, 2] for details.

Our argument mostly follows that of Bourgain [1, 2], but avoids the use of induction on energy using some ideas from other work [11, 7, 18]. We sketch the ideas informally as follows: following Bourgain, we choose a small parameter $\eta > 0$ depending polynomially on the energy, and then divide the time interval $[t_-, t_+]$ into a finite number of intervals $I_1, \ldots, I_J$, where on each interval the $L^{2(n+2)/(n-2)}_{t,x}$ norm is comparable to $c(\eta)$; the task is then to bound the number J of such intervals by $O(\exp(CE^C))$.

An argument of Bourgain based on Strichartz inequalities and harmonic analysis, which we reproduce here,[4] shows that for each such interval I_j, there is a "bubble" of concentration, by which we mean a region of spacetime of the form

$$\left\{(t,x) : |t-t_j| \leq c(\eta) N_j^{-2};\ |x-x_j| \leq c(\eta) N_j^{-1}\right\}$$

[3]We do not obtain regularity results, except in dimensions $n = 3, 4$, simply because the nonlinearity $|u|^{4/(n-2)}u$ is not smooth in dimensions $n \geq 5$. Because of this nonsmoothness, we will not rely on Fourier-based techniques such as Littlewood–Paley theory, $X^{s,b}$ spaces, or para-differential calculus, relying instead on the (ordinary) chain rule and some use of Hölder type estimates.

[4]For some results in the same spirit, showing that "bubbles" are the only obstruction to global existence, see [15].

inside the spacetime slab $I_j \times \mathbf{R}^n$ on which the solution u has energy[5] at least $c(\eta) > 0$. Here (t_j, x_j) is a point in $I_j \times \mathbf{R}^n$ and $N_j > 0$ is a frequency. The spherical symmetry assumption allows us to choose $x_j = 0$; there is also a lower bound $N_j \geq c(\eta)|I_j|^{1/2}$ simply because the bubble has to be contained inside the slab $I_j \times \mathbf{R}^n$. However, the harmonic analysis argument does not directly give an *upper bound* on the frequency N_j; thus the bubble may be much smaller than the slab.

In [1, 2] an upper bound on N_j is obtained by an *induction on energy* argument; one assumes for contradiction that N_j is very large, so the bubble is very small. Without loss of generality we may assume the bubble lies in the lower half of the slab $I_j \times \mathbf{R}^n$. Then when one evolves the bubble forward in time, it will have largely dispersed by the time it leaves $I_j \times \mathbf{R}^n$. Oversimplifying somewhat, the argument then proceeds by removing this bubble (thus decreasing the energy by a nontrivial amount), applying an induction hypothesis to obtain Strichartz bounds on the remainder of the solution, and then gluing the bubble back in by perturbation theory. Unfortunately it is this use of the induction hypothesis which eventually gives tower-exponential bounds rather than exponential bounds in the final result. Also there is some delicate playoff between various powers of η which needs additional care in four and higher dimensions.

Our main innovation is to obtain an upper bound on N_j by more direct methods, dispensing with the need for an induction on energy argument. The idea is to use Duhamel's formula, to compare u against the linear solutions $u_\pm(t) := e^{i(t-t_\pm)\Delta}u(t_\pm)$. We first eliminate a small number of intervals I_j in which the linear solutions $u_\pm$ have large $L^{2(n+2)/(n-2)}_{t,x}$ norm; the number of such intervals can be controlled by global Strichartz estimates for the free (linear) Schrödinger equation. Now let I_j be one of the remaining intervals. If the bubble occurs in the lower half of I_j then we[6] compare u with u_+, taking advantage of the dispersive properties of the propagator $e^{it\Delta}$ in our high-dimensional setting $n \geq 3$ to show that the error $u - u_+$ is in fact relatively smooth, which in turn implies the bubble cannot be too small. Similarly if the bubble occurs in the upper half of I_j we compare u instead with u_-. Interestingly, there are some subtleties in very high dimension ($n \geq 6$) when the nonlinearity $F(u)$ grows quadratically or slower, as it now becomes rather difficult (in the large energy setting) to pass from smallness of the nonlinear solution (in spacetime norms) to that of the linear solution or vice versa.

Once the bubble is shown to inhabit a sizeable portion of the slab, the rest of the argument essentially proceeds as in [1]. We wish to show that J is bounded, so suppose for contradiction that J is very large (so there are lots of bubbles). Then the Morawetz inequality (5) can be used to show that the intervals I_j must concentrate fairly rapidly at some point in time t_*; however one can then use localized mass conservation laws to show that the bubbles inside I_j must each shed a sizeable amount of mass (and energy) before concentrating at t_*. If J is large enough there is so much mass and energy being shed that one can contradict conservation of

[5] Actually, we will only seek to obtain lower bounds on potential energy here, but corresponding control on the kinetic energy can then be obtained by localized forms of the Sobolev inequality.

[6] Again, this is an oversimplification; we must also dispose of the nonlinear interactions of u with itself inside the interval I_j, but this can be done by some Strichartz analysis and use of the pigeonhole principle.

energy. To put it another way, the mass conservation law implies that the bubbles cannot contract or expand rapidly, and the Morawetz inequality implies that the bubbles cannot persist stably for long periods of time. Combining these two facts we can conclude that there are only a bounded number of bubbles.

It is worth mentioning that our argument is relatively elementary (compared against, e.g., [1, 2, 7]), especially in low dimensions $n = 3, 4, 5$; the only tools are (nonendpoint) Strichartz estimates and Sobolev embedding, the Duhamel formula, energy conservation, local mass conservation, and the Morawetz inequality, as well as some elementary combinatorial arguments. We do not need tools from Littlewood–Paley theory such as the para-differential calculus, although in the higher-dimensional cases $n \geq 6$ we will need fractional integration and the use of Hölder type estimates as a substitute for this para-differential calculus.

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2. Notation and basic estimates

We use $c, C > 0$ to denote various absolute constants depending only on the dimension n; as we wish to track the dependence on the energy, we will *not* allow these constants to depend on the energy E.

For any time interval I, we use $L^q_t L^r_x(I \times \mathbf{R}^n)$ to denote the mixed spacetime Lebesgue norm

$$\|u\|_{L^q_t L^r_x(I \times \mathbf{R}^n)} := \left(\int_I \|u(t)\|^q_{L^r(\mathbf{R}^n)} \, dt \right)^{1/q}$$

with the usual modifications when $q = \infty$.

We define the fractional differentiation operators $|\nabla|^\alpha := (-\Delta)^{\alpha/2}$ on $\mathbf{R}^n$. Recall that if $-n < \alpha < 0$ then these are fractional integration operators with an explicit form

$$|\nabla|^\alpha f(x) = c_{n,\alpha} \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n+\alpha}} \, dy \tag{6}$$

for some computable constant $c_{n,\alpha} > 0$ whose exact value is unimportant to us; see, e.g., [17]. We recall that the Riesz transforms $\nabla |\nabla|^{-1} = |\nabla|^{-1} \nabla$ are bounded on $L^p(\mathbf{R}^n)$ for every $1 < p < \infty$; again see [17].

2.1. Duhamel's formula and Strichartz estimates. Let $e^{it\Delta}$ be the propagator for the free Schrödinger equation $iu_t + \Delta u = 0$. As is well-known, this operator commutes with derivatives, and obeys the *energy identity*

$$\|e^{it\Delta} f\|_{L^2(\mathbf{R}^n)} = \|f\|_{L^2(\mathbf{R}^n)} \tag{7}$$

and the *dispersive inequality*

$$\|e^{it\Delta} f\|_{L^\infty(\mathbf{R}^n)} \leq C|t|^{-n/2} \|f\|_{L^1(\mathbf{R}^n)} \tag{8}$$

for $t \neq 0$. In particular we may interpolate to obtain the fixed-time estimates

$$\|e^{it\Delta} f\|_{L^p(\mathbf{R}^n)} \leq C|t|^{-n(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}(\mathbf{R}^n)} \tag{9}$$

for $2 \leq p \leq \infty$, where the dual exponent p' is defined by $1/p + 1/p'$.

We observe *Duhamel's formula*: if $iu_t + \Delta u = F$ on some time interval I, then we have (in a distributional sense, at least)

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) - i\int_{t_0}^{t} e^{i(t-s)\Delta}F(s)\, ds \tag{10}$$

for all $t_0, t \in I$, where we of course adopt the convention that $\int_{t_0}^{t} = -\int_{t}^{t_0}$ when $t < t_0$. To estimate the terms on the right-hand side, we introduce the *Strichartz norms* $\dot S^k(I \times \mathbf{R}^n)$, defined for $k = 0$ as

$$\|u\|_{\dot S^0(I\times\mathbf{R}^n)} := \sup_{(q,r)\text{ admissible}} \|u\|_{L_t^q L_x^r(I\times\mathbf{R}^n)},$$

where admissibility was defined in (3), and then for general[7] k by

$$\|u\|_{\dot S^k(I\times\mathbf{R}^n)} := \big\||\nabla|^k u\big\|_{\dot S^0(I\times\mathbf{R}^n)}.$$

Observe that in the high-dimensional setting $n \geq 3$, we have $2 \leq r < \infty$ for all admissible (q,r), so have boundedness of Riesz transforms (and thus we could replace $|\nabla|^k$ by ∇^k for instance, when k is a positive integer. We note in particular that

$$\begin{aligned}\|\nabla^k u\|_{L_{t,x}^{2(n+2)/n}(I\times\mathbf{R}^n)} + \|\nabla^k u\|_{L_t^{2(n+2)/(n-2)} L_x^{2n(n+2)/(n^2+4)}(I\times\mathbf{R}^n)}& \\ + \|\nabla^k u\|_{L_t^\infty L_x^2(I\times\mathbf{R}^n)} &\leq C_k \|u\|_{\dot S^k(I\times\mathbf{R}^n)}\end{aligned} \tag{11}$$

for all positive integer $k \geq 1$. Specializing further to the $k = 1$ case we obtain

$$\|u\|_{L_{t,x}^{2(n+2)/(n-2)}(I\times\mathbf{R}^n)} + \|u\|_{L_t^\infty L_x^{2n/(n-2)}(I\times\mathbf{R}^n)} \leq C\|u\|_{\dot S^1(I\times\mathbf{R}^n)} \tag{12}$$

and in dimensions $n \geq 4$

$$\|u\|_{L_t^{2(n+2)/n} L_x^{2n(n+2)/(n^2-2n-4)}(I\times\mathbf{R}^n)} \leq C\|u\|_{\dot S^1(I\times\mathbf{R}^n)}. \tag{13}$$

We also define dual Strichartz spaces $\dot N^k(I \times \mathbf{R}^n)$, defined for $k = 0$ as the Banach space dual of $\dot S^0(I \times \mathbf{R}^n)$, and for general k as

$$\|F\|_{\dot N^k(I\times\mathbf{R}^n)} := \big\||\nabla|^k F\big\|_{\dot N^0(I\times\mathbf{R}^n)}$$

(or equivalently, $\dot N^k$ is the dual of $\dot S^{-k}$). From the first term in (11) and duality (and the boundedness of Riesz transforms) we observe in particular that

$$\|F\|_{\dot N^k(I\times\mathbf{R}^n)} \leq \|\nabla^k F\|_{L_{t,x}^{2(n+2)/(n+4)}(I\times\mathbf{R}^n)}. \tag{14}$$

We recall the *Strichartz inequalities*

$$\|e^{i(t-t_0)\Delta}u(t_0)\|_{\dot S^k(I\times\mathbf{R}^n)} \leq C\|u(t_0)\|_{\dot H^k(\mathbf{R}^n)} \tag{15}$$

and

$$\left\|\int_{t_0}^{t} e^{i(t-s)\Delta}F(s)\right\|_{\dot S^k(I\times\mathbf{R}^n)} \leq C\|F\|_{\dot N^k(I\times\mathbf{R}^n)}; \tag{16}$$

see, e.g., [14]; the dispersive inequality (9) of course plays a key role in the proof of these inequalities. While we include the endpoint Strichartz pair $(q,r) = (2, \frac{2n}{n-2})$

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[7]The homogeneous nature of these norms causes some difficulties in interpreting elements of these spaces as a distribution when $|k| \geq n/2$, but in practice we shall only work with $k = 0, 1$ and $n \geq 3$ and so these difficulties do not arise.

in these estimates, this pair is not actually needed in our argument. Observe that the constants C here are independent of the choice of interval I.

2.2. Local mass conservation. We now recall a local mass conservation law appearing for instance in [11]; a related result also appears in [1].

Let χ be a bump function supported on the ball $B(0,1)$ which equals one on the ball $B(0,1/2)$ and is nonincreasing in the radial direction. For any radius $R > 0$, we define the local mass $\mathrm{Mass}(u(t), B(x_0,R))$ of $u(t)$ on the ball $B(x_0,R)$ by

$$\mathrm{Mass}(u(t), B(x_0,R)) := \Big(\int \chi^2\Big(\frac{x-x_0}{R}\Big)|u(t,x)|^2\ dx\Big)^{1/2};$$

note that this is a nondecreasing function of R. Observe that if u is a finite energy solution (1), then

$$\partial_t |u(t,x)|^2 = -2\nabla_x \cdot \mathrm{Im}(\overline{u}\nabla_x u(t,x))$$

(at least in a distributional sense), and so by integration by parts

$$\partial_t \mathrm{Mass}(u(t), B(x_0,R))^2 = \frac{4}{R}\int \chi\Big(\frac{x-x_0}{R}\Big)(\nabla\chi)\Big(\frac{x-x_0}{R}\Big)\mathrm{Im}(\overline{u}\nabla_x u(t,x))\ dx$$

so by Cauchy–Schwarz

$$\begin{aligned}&|\partial_t \mathrm{Mass}(u(t), B(x_0,R))^2|\\ &\qquad \leq \frac{C}{R}\mathrm{Mass}(u(t), B(x_0,R))\left(\int_{R/2\leq |x-x_0|\leq R} |\nabla_x u(t,x)|^2\ dx\right)^{1/2}.\end{aligned}$$

If u has bounded energy $E(u) \leq E$, we thus have the approximate mass conservation law

$$|\partial_t \mathrm{Mass}(u(t), B(x_0,R))| \leq CE^{1/2}/R. \tag{17}$$

Observe that the same claim also holds if u solves the free Schrödinger equation $iu_t + \Delta u = 0$ instead of the nonlinear Schrödinger equation (1). Note that the right-hand side decays with R. This implies that if the local mass $\mathrm{Mass}(u(t), B(x_0,R))$ is large for some time t, then it can also be shown to be similarly large for nearby times t, by increasing the radius R if necessary to reduce the rate of change of the mass.

From Sobolev and Hölder (or by Hardy's inequality) we can control the mass in terms of the energy via the formula

$$|\mathrm{Mass}(u(t), B(x_0,R))| \leq CE^{1/2}R. \tag{18}$$

2.3. Morawetz inequality. We now give the proof of the Morawetz inequality (5); this inequality already appears in [1, 2, 11] in three dimensions, and the argument extends easily to higher dimensions, but for sake of completeness we give the argument here.

Using the scale invariance (2) we may rescale so that $A|I|^{1/2} = 1$. We begin with the local momentum conservation identity

$$\partial_t \mathrm{Im}(\partial_k u\overline{u}) = -2\partial_j \mathrm{Re}(\partial_k u\overline{\partial_j u}) + \frac{1}{2}\partial_k\Delta(|u|^2) - \frac{2}{n-2}\partial_k |u|^{2n/(n-2)}$$

where j,k range over spatial indices $1,\ldots,n$ with the usual summation conventions, and ∂_k is differentiation with respect to the x^k variable. This identity can be verified directly from (1); observe that when u is finite energy, both sides of this inequality make sense in the sense of distributions, so this identity can be justified in the finite energy case by the local well-posedness theory.[8] If we multiply the above identity by the weight $\partial_k a$ for some smooth, compactly supported weight $a(x)$, and then integrate in space, we obtain (after some integration by parts)

$$\begin{aligned}\partial_t \int_{\mathbf{R}^n} (\partial_k a)\mathrm{Im}(\partial_k u\overline{u}) =& 2\int_{\mathbf{R}^n} (\partial_j\partial_k a)\mathrm{Re}(\partial_k u\overline{\partial_j u}) \\ &+ \frac{1}{2}\int_{\mathbf{R}^n} (-\Delta\Delta a)|u|^2 \\ &+ \frac{2}{n-2}\int_{\mathbf{R}^n} \Delta a|u|^{2n/(n-2)}.\end{aligned}$$

We apply this in particular to the C_0^∞ weight $a(x) := (\varepsilon^2 + |x|^2)^{1/2}\chi(x)$, where χ is a bump function supported on $B(0,2)$ which equals 1 on $B(0,1)$, and $0 < \varepsilon < 1$ is a small parameter which will eventually be sent to zero. In the region $|x| \leq 1$, one can see from elementary geometry that a is a convex function (its graph is a hyperboloid); in particular, $(\partial_j\partial_k a)\mathrm{Re}(\partial_k u\overline{\partial_j u})$ is nonnegative. Further computation shows that

$$\Delta a = \frac{n-1}{(\varepsilon^2 + |x|^2)^{1/2}} + \frac{\varepsilon^2}{(\varepsilon^2 + |x|^2)^{3/2}}$$

and

$$-\Delta\Delta a = \frac{(n-1)(n-3)}{(\varepsilon^2 + |x|^2)^{3/2}} + \frac{6(n-3)\varepsilon^2}{(\varepsilon^2 + |x|^2)^{5/2}} + \frac{15\varepsilon^4}{(\varepsilon^2 + |x|^2)^{7/2}}$$

in this region; in particular $-\Delta\Delta a, \Delta a$ are positive in this region since $n \geq 3$. In the region $1 \leq |x| \leq 2$, a and all of its derivatives are bounded uniformly in ε, and so the integrals here are bounded by $O(E(u))$ (using (18) to control the lower-order term). Combining these estimates we obtain the inequality

$$\partial_t \int_{|x|\leq 2} (\partial_k a)\mathrm{Im}(\partial_k u\overline{u}) \geq c\int_{|x|\leq 1} \frac{|u(t,x)|^{2n/(n-2)}}{(\varepsilon^2 + |x|^2)^{1/2}}\, dx - CE(u).$$

Integrating this in time on I, and then using the fundamental theorem of calculus and the observation that a is Lipschitz, we obtain

$$\sup_{t\in I}\int_{|x|\leq 2} |\nabla u(t,x)|\,|u(t,x)|\, dx \geq c\int_I\int_{|x|\leq 1} \frac{|u(t,x)|^{2n/(n-2)}}{(\varepsilon^2 + |x|^2)^{1/2}}\, dx - CE(u)|I|.$$

By (18) and Cauchy–Schwarz the left-hand side is $O(E(u))$. Since $|I| = A^{-2} < 1$, we thus obtain

$$\int_I\int_{|x|\leq 1} \frac{|u(t,x)|^{2n/(n-2)}}{(\varepsilon^2 + |x|^2)^{1/2}}\, dx \leq CE(u).$$

Taking $\varepsilon \to 0$ and using monotone convergence, (5) follows.

Remark 2.4. In [7], an interaction variant of this Morawetz inequality is used (superficially similar to the Glimm interaction potential as used in the theory of conservation laws), in which the weight $1/|x|$ is not present. In principle this allows

[8] For instance, one could smooth out the nonlinearity F (or add a parabolic dissipation term), obtain a similar law for smooth solutions to the smoothed out equation, and then use the local well-posedness theory, see, e.g., [4], to justify the process of taking limits.

for arguments such as the one here to extend to the nonradial setting. However the (frequency-localized) interaction Morawetz inequality in [7] is currently restricted to three dimensions, and has a less favorable numerology[9] than (5), so it seems that the arguments given here are insufficient to close the argument in the general case in higher dimensions. At the very least it seems that one would need to use more sophisticated control on the movement of mass across frequency ranges, as is done in [7].

3. Proof of Theorem 1.1

We now give the proof of Theorem 1.1. The spherical symmetry of u is used in only one step, namely in Corollary 3.5, to ensure that the solution concentrates at the spatial origin instead of at some other location.

We fix E, $[t_-, t_+]$, u. We may assume that the energy is large, $E > c > 0$, otherwise the claim follows from the small energy theory. From the bounded energy of u we observe the bounds

$$\|u(t)\|_{\dot{H}^1(\mathbf{R}^n)} + \|u(t)\|_{L^{2n/(n-2)}(\mathbf{R}^n)} \leq CE^C \tag{19}$$

for all $t \in [t_-, t_+]$.

We need some absolute constants $1 \ll C_0 \ll C_1 \ll C_2$, depending only on n, to be chosen later; we will assume C_0 to be sufficiently large depending on n, C_1 sufficiently large depending on C_0, n, and C_2 sufficiently large depending on C_0, C_1, n. We then define the quantity $\eta := C_2^{-1} E^{-C_2}$. Our task is to show that

$$\int_{t_-}^{t_+} \int_{\mathbf{R}^n} |u(t,x)|^{2(n+2)/(n-2)}\, dxdt \leq C(C_0, C_1, C_2) \exp\left(C(C_0, C_1, C_2) E^{C(C_0,C_1,C_2)} \right).$$

We may assume of course that

$$\int_{t_-}^{t_+} \int_{\mathbf{R}^n} |u(t,x)|^{2(n+2)/(n-2)}\, dxdt > 4\eta$$

since our task is trivial otherwise. We may then (by the greedy algorithm) subdivide $[t_-, t_+]$ into a finite number of disjoint intervals $I_1, \ldots, I_J$ for some $J \geq 2$ such that

$$\eta \leq \int_{I_j} \int_{\mathbf{R}^n} |u(t,x)|^{2(n+2)/(n-2)}\, dxdt \leq 2\eta \tag{20}$$

for all $1 \leq j \leq J$. It will then suffice to show that

$$J \leq C(C_0, C_1, C_2) \exp\left(C(C_0, C_1, C_2) E^{C(C_0,C_1,C_2)} \right).$$

We shall now prove various concentration properties of the solution on these intervals. We begin with a standard Strichartz estimate that bootstraps control on (20) to control on all the Strichartz norms (but we lose the gain in η):

[9] In the notation of Corollary 3.6, the interaction inequality in [7] would give a bound of the form $\sum_{I_j \subseteq I} |I_j|^{3/2} \leq C(\eta)(\max_{I_j \subseteq I} |I_j|)^{3/2}$, which is substantially weaker and in particular does not seem to easily give the conclusions in Corollary 3.7 or Proposition 3.8, because the exponent $3/2$ here is greater than 1, whereas the corresponding exponent $1/2$ arising from (5) is less than 1.

Lemma 3.1. *For each interval I_j we have*

$$\|u\|_{\dot{S}^1(I_j \times \mathbf{R}^n)} \leq CE^C.$$

Proof. From Duhamel (10), Strichartz (15), (16) and the equation (1) we have

$$\|u\|_{\dot{S}^1(I_j \times \mathbf{R}^n)} \leq C\|u(t_j)\|_{\dot{H}^1(\mathbf{R}^n)} + \|F(u)\|_{\dot{N}^1(I_j \times \mathbf{R}^n)}$$

for any $t_j \in I_j$. From (19), (14) we thus have

$$\|u\|_{\dot{S}^1(I_j \times \mathbf{R}^n)} \leq CE^C + \|\nabla F(u)\|_{L^{2(n+2)/(n+4)}(I_j \times \mathbf{R}^n)}.$$

But from the chain rule and Hölder we have (formally, at least)

$$\begin{aligned}\|\nabla F(u)\|_{L^{2(n+2)/(n+4)}(I_j \times \mathbf{R}^n)} &\leq C\| |u|^{4/(n-2)} |\nabla u| \|_{L^{2(n+2)/(n+4)}(I_j \times \mathbf{R}^n)} \\ &\leq C\|u\|^{4/(n-2)}_{L^{2(n+2)/(n-2)}_{t,x}(I_j \times \mathbf{R}^n)} \|\nabla u\|_{L^{2(n+2)/n}(I_j \times \mathbf{R}^n)} \\ &\leq C\eta^{2/(n+2)} \|u\|_{\dot{S}^1(I_j \times \mathbf{R}^n)}\end{aligned}$$

by (20), (11). Thus we have the formal inequality

$$\|u\|_{\dot{S}^1(I_j \times \mathbf{R}^n)} \leq CE^C + C\eta^{2/(n+2)} \|u\|_{\dot{S}^1(I_j \times \mathbf{R}^n)}.$$

If η is sufficiently small (by choosing C_2 large enough), then the claim follows, at least formally. To make the argument rigorous one can run a Picard iteration scheme that converges to the solution u (see, e.g., [4] for details) and obtain the above types of bounds uniformly at all stages of the iteration; we omit the standard details. □

Next, we obtain *lower* bounds on linear solution approximations to u on an interval where the $L^{2(n+2)/(n-2)}_{t,x}$ norm is small but bounded below.

Lemma 3.2. *Let $[t_1, t_2] \subseteq [t_-, t_+]$ be an interval such that*

$$\eta/2 \leq \int_{t_1}^{t_2} \int_{\mathbf{R}^n} |u(t,x)|^{2(n+2)/(n-2)} \, dxdt \leq 2\eta. \tag{21}$$

Then, if we define $u_l(t,x) := e^{i(t-t_l)\Delta} u(t_l)$ for $l = 1, 2$, we have

$$\int_{t_1}^{t_2} \int_{\mathbf{R}^n} |u_l(t,x)|^{2(n+2)/(n-2)} \, dxdt \geq c\eta^C$$

for $l = 1, 2$.

Proof. Without loss of generality it suffices to prove the claim when $l = 1$. In low dimensions $n = 3, 4, 5$ the lemma is easy; indeed an inspection of the proof of Lemma 3.1 reveals that we have the additional bound

$$\|u - u_1\|_{\dot{S}^1([t_1,t_2] \times \mathbf{R}^n)} \leq CE^C \eta^{2/(n+2)}$$

and hence by (12)

$$\|u - u_1\|_{L^{2(n+2)/(n-2)}_{t,x}([t_1,t_2] \times \mathbf{R}^n)} \leq CE^C \eta^{2/(n+2)}.$$

When $n = 3, 4, 5$ we have $2/(n+2) > (n-2)/2(n+2)$, and so the above estimates then show that $u - u_1$ is smaller than u in $L^{2(n+2)/(n-2)}_{t,x}([t_1, t_2] \times \mathbf{R}^n)$ norm if η is sufficienty small (i.e., C_2 is sufficiently large), at which point the claim follows from the triangle inequality (and we can even replace η^C by η).

In higher dimensions $n \geq 6$, the above simple argument breaks down. In fact the argument becomes considerably more complicated (in particular, we were only able to obtain a bound of η^C rather than the more natural η); the difficulty is that while the nonlinearity still decays faster than linearly as $u \to 0$, one of the factors is "reserved" for the derivative ∇u, for which we have no smallness estimates, and the remaining terms now decay linearly or worse, making it difficult to perform a perturbative analysis. The resolution of this difficulty is rather technical, so we defer the proof of the higher-dimensional case to an Appendix (Section 4) so as not to interrupt the flow of the argument. We remark however that the argument does not require any spherical symmetry assumption on the solution. □

Define the linear solutions u_-, u_+ on $[t_-, t_+] \times \mathbf{R}^n$ by $u_\pm(t) := e^{i(t-t_\pm)\Delta} u(t_\pm)$; these are the analogue of the scattering solutions for this compact interval $[t_-, t_+]$. From (19) and the Strichartz estimate (15), (12), we have

$$\int_{t_-}^{t_+} \int_{\mathbf{R}^n} |u_\pm(t,x)|^{2(n+2)/(n-2)} \, dxdt \leq CE^C.$$

Call an interval I_j *exceptional* if we have

$$\int_{I_j} \int_{\mathbf{R}^n} |u_\pm(t,x)|^{2(n+2)/(n-2)} \, dxdt > \eta^{C_1}$$

for at least one choice of sign $\pm$, and *unexceptional* otherwise. From the above global Strichartz estimate we see that there are at most $O(E^C/\eta^{C_1})$ exceptional intervals, which will be acceptable for us from definition of η. Thus we may assume that there is at least one unexceptional interval.

Unexceptional intervals will be easier to control than exceptional ones, because the homogeneous component of Duhamel's formula (10) is negligible, leaving only the inhomogeneous component to be considered. But as we shall see, this component enjoys some additional regularity properties. In particular, we now prove a concentration property of the solution on unexceptional intervals.

Proposition 3.3. *Let I_j be an unexceptional interval. Then there exists an $x_j \in \mathbf{R}^n$ such that*

$$\mathrm{Mass}\Big(u(t), B\big(x_j, C\eta^{-C}|I_j|^{1/2}\big)\Big) \geq c\eta^{CC_0}|I_j|^{1/2}$$

for all $t \in I_j$.

Proof. By time translation invariance and scale invariance (2) we may assume that $I_j = [0, 1]$. We subdivide I_j further into $[0, 1/2]$ and $[1/2, 1]$. By (20) and the pigeonhole principle and time reflection symmetry if necessary we may assume that

$$\int_{1/2}^{1} \int_{\mathbf{R}^n} |u(t,x)|^{2(n+2)/(n-2)} \, dxdt > \eta/2. \tag{22}$$

Since I_j is unexceptional, we have

$$\int_{0}^{1} \int_{\mathbf{R}^n} |u_-(t,x)|^{2(n+2)/(n-2)} \, dxdt \leq \eta^{C_1}. \tag{23}$$

By (23), (20) and the pigeonhole principle, we may find an interval $[t_* - \eta^{C_0}, t_*] \subset [0, 1/2]$ such that[10]

$$\int_{t_* - \eta^{C_0}}^{t_*} \int_{\mathbf{R}^n} |u(t,x)|^{2(n+2)/(n-2)}\ dxdt < C\eta^{C_0}. \tag{24}$$

and

$$\int_{\mathbf{R}^n} |u_-(t_* - \eta^{C_0}, x)|^{2(n+2)/(n-2)}\ dx \le C\eta^{C_1}. \tag{25}$$

Applying Lemma 3.2 to the time interval $[t_*, 1]$ we see that

$$\int_{t_*}^{1} \int_{\mathbf{R}^n} |(e^{i(t-t_*)\Delta} u(t_*))(x)|^{2(n+2)/(n-2)}\ dxdt \ge c\eta^{C}. \tag{26}$$

By Duhamel's formula (10) we have

$$\begin{aligned} e^{i(t-t_*)\Delta} u(t_*) = u_-(t) &- i \int_{t_* - \eta^{C_0}}^{t_*} e^{i(t-s)\Delta} F(u(s))\ ds \\ &- i \int_{t_-}^{t_* - \eta^{C_0}} e^{i(t-s)\Delta} F(u(s))\ ds. \end{aligned} \tag{27}$$

Since I_j is unexceptional, we have

$$\int_{t_*}^{1} \int_{\mathbf{R}^n} |u_-(t,x)|^{2(n+2)/(n-2)}\ dxdt \le \eta^{C_1}.$$

From (24) and Lemma 3.1, it is easy to see (using the chain rule and Hölder as in the proof of Lemma 3.1) that

$$\|F(u)\|_{\dot N^1([t_* - \eta^{C_0}, t_*] \times \mathbf{R}^n)} \le CE^C \eta^{cC_0}, \tag{28}$$

and hence by Strichartz (16)

$$\int_{t_*}^{1} \int_{\mathbf{R}^n} \left| \int_{t_* - \eta^{C_0}}^{t_*} e^{i(t-s)\Delta} F(u(s))\ ds \right|^{2(n+2)/(n-2)} (x)\ dxdt \le CE^C \eta^{cC_0}.$$

From these estimates and (26), we thus see from the triangle inequality (if C_0 is large enough, and η small enough (i.e., C_2 large enough depending on C_0)) that

$$\|v\|_{L^{2(n+2)/(n-2)}_{t,x}([t_*, 1] \times \mathbf{R}^n)} \ge c\eta^C \tag{29}$$

where v is the function

$$v := \int_{t_-}^{t_* - \eta^{C_0}} e^{i(t-s)\Delta} F(u(s))\ ds. \tag{30}$$

We now complement this lower bound on v with an upper bound. First observe from Lemma 3.1 that

$$\|u\|_{\dot S^1([t_*, 1] \times \mathbf{R}^n)} \le CE^C;$$

---

[10] In the low-dimensional case $n = 3, 4, 5$ we may skip this pigeonhole step. Indeed from (22), (23) and Duhamel we may conclude that $\int_{t_-}^{0} e^{i(t-s)\Delta} F(u(s))\ ds$ has large $L^{2(n+2)/(n-2)}_{t,x}$ norm on the slab $[1/2, 1] \times \mathbf{R}^n$; this is because the proof of Lemma 3.2 shows that the effect of the forcing terms arising from the time interval $[0, 1]$ are of size $O(\eta^{4/(n-2)})$, which is smaller than $\eta/2$ for $n = 3, 4, 5$; one then continues the proof from (29) onwards with only minor changes. However this simple argument does not seem to work in higher dimensions.

also from (19) and (15) we have

$$\|u_-\|_{\dot S^1([t_*,1]\times\mathbf{R}^n)} \leq CE^C.$$

Finally, from (28) and (16)

$$\left\|\int_{t_*-\eta^{C_0}}^{t_*} e^{i(t-s)\Delta}F(u(s))\,ds\right\|_{\dot S^1([t_*,1]\times\mathbf{R}^n)} \leq CE^C.$$

From the triangle inequality and (27) we thus have

$$\|v\|_{\dot S^1([t_*,1]\times\mathbf{R}^n)} \leq CE^C. \tag{31}$$

We shall need some additional regularity control on v. For any $h \in \mathbf{R}^n$, let $u^{(h)}$ denote the translate of u by h, i.e., $u^{(h)}(t,x) := u(t,x-h)$.

Lemma 3.4. *We have the bound*

$$\|v^{(h)} - v\|_{L_t^\infty L_x^{2(n+2)/(n-2)}([t_*,1]\times\mathbf{R}^n)} \leq CE^C\eta^{-CC_0}|h|^c$$

for all $h \in \mathbf{R}^n$.

Proof. First consider the high-dimensional case $n \geq 4$. We use (19), the chain rule and Hölder to observe that

$$\begin{aligned}\|\nabla F(u(s))\|_{L^{2n/(n+4)}(\mathbf{R}^n)} &\leq C\||u(s)|^{4/(n-2)}|\nabla u(s)|\|_{L^{2n/(n+4)}(\mathbf{R}^n)}\\ &\leq C\|u(s)\|_{L^{2n/(n-2)}(\mathbf{R}^n)}^{4/(n-2)}\|\nabla u(s)\|_{L^2(\mathbf{R}^n)}\\ &\leq CE^C,\end{aligned}$$

so by the dispersive inequality (9)

$$\|\nabla e^{i(t-s)\Delta}F(u(s))\|_{L^{2n/(n-4)}(\mathbf{R}^n)} \leq CE^C|t-s|^{-2}.$$

Integrating this for s in $[t_-, t_* - \eta^{C_0}]$ we obtain

$$\|\nabla v\|_{L_t^\infty L_x^{2n/(n-4)}([t_*,t_1]\times\mathbf{R}^n)} \leq CE^C\eta^{-CC_0};$$

interpolating this with (31), (11) we obtain

$$\|\nabla v\|_{L_t^\infty L_x^{2(n+2)/(n-2)}([t_*,t_1]\times\mathbf{R}^n)} \leq CE^C\eta^{-CC_0}.$$

The claim then follows (with $c = 1$) from the fundamental theorem of calculus and Minkowski's inequality.

Now consider the three-dimensional case $n = 3$. From (19), the fundamental theorem of calculus, and Minkowski's inequality we have

$$\|u^{(h)}(s) - u(s)\|_{L^2(\mathbf{R}^3)} \leq CE^C|h|,$$

while from the triangle inequality we have

$$\|u^{(h)}(s) - u(s)\|_{L^6(\mathbf{R}^3)} \leq CE^C,$$

and hence

$$\|u^{(h)}(s) - u(s)\|_{L^3(\mathbf{R}^3)} \leq CE^C|h|^{1/2}.$$

Since $F(u)$ is quintic in three dimensions, we thus have from Hölder and (19) that

$$\begin{aligned}\|F(u)^{(h)}(s) - F(u)(s)\|_{L^1(\mathbf{R}^3)} &\leq C\||u^{(h)}(s) - u(s)|(|u^{(h)}(s)| + |u(s)|)^4\|_{L^1(\mathbf{R}^3)} \\ &\leq C\|u^{(h)}(s) - u(s)\|_{L^3(\mathbf{R}^3)}\|u(s)\|^4_{L^6(\mathbf{R}^3)} \\ &\leq CE^C|h|^{1/2}.\end{aligned}$$

Integrating this for $s \in [t_-, t_* - \eta^{C_0}]$ using (8) we obtain

$$\|v^{(h)} - v\|_{L^\infty_{t,x}([t_*,1]\times\mathbf{R}^n)} \leq CE^C\eta^{-CC_0}|h|^{1/2}.$$

On the other hand, from (31), (12), and the triangle inequality we have

$$\|v^{(h)} - v\|_{L^\infty_t L^6_x([t_*,1]\times\mathbf{R}^n)} \leq CE^C\eta^{-CC_0}$$

and the claim follows by interpolation. □

We can average this lemma over all $|h| \leq r$, for some scale $0 < r < 1$ to be chosen shortly, to obtain

$$\|v_{av} - v\|_{L^\infty_t L^{2(n+2)/(n-2)}_x([t_*,1]\times\mathbf{R}^n)} \leq CE^C\eta^{-CC_0}r^c$$

where $v_{av}(x) := \int \chi(y)v(x+ry)\ dy$ for some bump function χ supported on $B(0,1)$ of total mass one. In particular by a Hölder in time we have

$$\|v_{av} - v\|_{L^{2(n+2)/(n-2)}_{t,x}([t_*,1]\times\mathbf{R}^n)} \leq CE^C\eta^{-CC_0}r^c.$$

Thus if we choose $r := \eta^{CC_0}$ for some large enough C, and η is sufficiently small, we see from (29) that

$$\|v_{av}\|_{L^{2(n+2)/(n-2)}_{t,x}([t_*,1]\times\mathbf{R}^n)} \geq c\eta^C.$$

On the other hand, by Hölder and Young's inequality

$$\begin{aligned}\|v_{av}\|_{L^{2n/(n-2)}_{t,x}([t_*,1]\times\mathbf{R}^n)} &\leq C\|v_{av}\|_{L^\infty_t L^{2n/(n-2)}_x([t_*,1]\times\mathbf{R}^n)} \\ &\leq C\|v\|_{L^\infty_t L^{2n/(n-2)}_x([t_*,1]\times\mathbf{R}^n)} \\ &\leq CE^C\end{aligned}$$

by (31), (11). Thus by Hölder we have

$$\|v_{av}\|_{L^\infty_{t,x}([t_*,1]\times\mathbf{R}^n)} \geq c\eta^C E^{-C}.$$

Thus we may find a point $(t_j, x_j) \in [t_*, 1] \times \mathbf{R}^n$ such that

$$\left|\int \chi(y)v(t_j, x_j + ry)\ dy\right| \geq c\eta^C E^{-C},$$

and in particular by Cauchy–Schwarz

$$\mathrm{Mass}(v(t_j), B(x_j, R)) \geq c\eta^C E^{-C}r^C.$$

for all $R \geq r$. Observe from (30) that v solves the free Schrödinger equation on $[t_* - \eta^{C_0}, 1]$, and has energy $O(E^C)$ by (31), (11). Thus by (17) we have

$$\mathrm{Mass}(v(t_* - \eta^{C_0}), B(x_j, R)) \geq c\eta^C E^{-C}r^C$$

for all $t \in [t_*, 1]$, if we set $R := C\eta^{-C}E^Cr^{-C}$ for some appropriate constants C. From Duhamel's formula (10) (or (27)) we have

$$u(t_* - \eta^{C_0}) = u_-(t_* - \eta^{C_0}) - iv(t_* - \eta^{C_0}).$$

From (25) and Hölder we have

$$\text{Mass}\left(u_-(t_*-\eta^{C_0}), B(x_j,R)\right) \leq CR^C\eta^{CC_1}.$$

Thus if we choose C_1 sufficiently large depending on C_0 (recalling that $r = \eta^{CC_0}$ and $R = C\eta^{-C}E^Cr^{-C}$), and assume η sufficiently small depending polynomially on E, we have

$$\text{Mass}\left(u(t_*-\eta^{C_0}), B(x_j,R)\right) \geq c\eta^C E^{-C} r^C.$$

By another application of (17) we thus have

$$\text{Mass}\left(u(t), B(x_j,\eta^{-CC_0})\right) \geq c\eta^{-CC_0}$$

for all $t \in [0,1]$, and Proposition 3.3 follows. □

We now exploit the radial symmetry of u to place the concentration point x_j at the origin. This is the only place where the spherical symmetry assumption is used.

Corollary 3.5. *Let I_j be an unexceptional interval, and assume that the solution u is spherically symmetric. Then we have*

$$\text{Mass}\left(u(t), B\left(0, C\eta^{-CC_0}|I_j|^{1/2}\right)\right) \geq c\eta^{CC_0}|I_j|^{1/2}$$

for all $t \in I_j$.

Proof. We again rescale $I_j = [0,1]$. Let x_j be as in Proposition 3.3. Fix $t \in [0,1]$. If $|x_j| = O(\eta^{-C'C_0})$ for some C' depending only on n then we are done. Now suppose that $|x_j| \geq \eta^{-C'C_0}$. Then if C' is big enough, we can find $\eta^{-cC'}$ rotations of the ball $B(x_j, C\eta^{-CC_0})$ which are disjoint. On each one of these balls, the mass of $u(t)$ is at least $c\eta^{CC_0}$ by the spherical symmetry assumption; by Hölder this shows that the $L^{2n/(n-2)}$ norm of $u(t)$ on these balls is also $c\eta^{CC_0}$. Adding this up for each of the $\eta^{-cC'C_0}$ balls, we obtain a contradiction to (19) if $C'C_0$ is large enough. Thus we have $|x_j| = O(\eta^{-C'C_0})$ and the claim follows. □

From this corollary and Hölder we see that

$$\int_{|x|\leq R} \frac{|u(t,x)|^{2n/(n-2)}}{|x|}\, dxdt \geq c\eta^{CC_0}|I_j|^{-1/2}$$

whenever $t \in I_j$ for some unexceptional interval I_j, and $R \geq C\eta^{-CC_0}|I_j|^{1/2}$. In particular we have

$$\int_{I_j}\int_{|x|\leq R} \frac{|u(t,x)|^{2n/(n-2)}}{|x|}\, dxdt \geq c\eta^{CC_0}|I_j|^{1/2}.$$

Combining this with (5) and the bounded energy we obtain the following combinatorial bound on the distribution of the intervals I_j.

Corollary 3.6. *Assume that the solution u is spherically symmetric. For any interval $I \subseteq [t_-,t_+]$, we have*

$$\sum_{1\leq j\leq J: I_j\subseteq I} |I_j|^{1/2} \leq C\eta^{-C(C_0,C_1)}|I|^{1/2}.$$

(note we can use η^{-C} to absorb any powers of the energy which appear; also, note that the $O(C\eta^{-C_1})$ exceptional intervals cause no difficulty).

This bound gives quite strong control on the possible distribution of the intervals I_j, for instance we have

Corollary 3.7. *Assume that the solution u is spherically symmetric. Let $I = \bigcup_{j_1 \leq j \leq j_2} I_j$ be a union of consecutive intervals. Then there exists $j_1 \leq j \leq j_2$ such that $|I_j| \geq c\eta^{C(C_0,C_1)}|I|$.*

Proof. From the preceding corollary we have

$$C\eta^{-C(C_0,C_1)}|I|^{1/2} \geq \sum_{j_1 \leq j \leq j_2} |I_j|^{1/2} \geq \sum_{j_1 \leq j \leq j_2} |I_j| (\sup_{j_1 \leq j \leq j_2} |I_j|)^{-1/2}.$$

Since $\sum_{j_1 \leq j \leq j_2} |I_j| = |I|$, the claim follows. □

We now repeat a combinatorial argument[11] of Bourgain [1] to show that the intervals I_j must now concentrate at some time t_*:

Proposition 3.8. *Assume that the solution u is spherically symmetric. Then there exists a time $t_* \in [t_-, t_+]$ and distinct unexceptional intervals $I_{j_1}, \ldots, I_{j_K}$ for some $K > c\eta^{C(C_0,C_1)} \log J$ such that*

$$|I_{j_1}| \geq 2|I_{j_2}| \geq 4|I_{j_3}| \geq \cdots \geq 2^{K-1}|I_{j_K}| \tag{32}$$

and such that $\mathrm{dist}(t_*, I_{j_k}) \leq C\eta^{-C(C_0,C_1)}|I_{j_k}|$ for all $1 \leq k \leq K$.

Proof. We run the algorithm from Bourgain [1]. We first recursively define a nested sequence of intervals $I^{(k)}$, each of which is a union of consecutive unexceptional I_j, as follows. We first remove the $O(\eta^{-C_1})$ exceptional intervals from $[t_-, t_+]$, leaving $O(\eta^{-C_1})$ connected components. One of these, call it $I^{(1)}$, must be the union of $J_1 \geq c\eta^{C_1} J$ consecutive unexceptional intervals. By Corollary 3.7, there exists an $I_{j_1} \subseteq I^{(1)}$ such that $|I_{j_1}| \geq c\eta^{CC_0}|I^{(1)}|$, so in particular $\mathrm{dist}(t, I_{j_1}) \leq C\eta^{-CC_0}|I_{j_1}|$ for all $t \in |I^{(1)}|$. Now we remove I_{j_1} from $I^{(1)}$, and more generally remove all intervals I_j from $I^{(1)}$ for which $|I_j| > |I_{j_1}|/2$. There can be at most $C\eta^{-CC_0}$ such intervals to remove, since I_{j_1} was so large. If $J_1 \leq C\eta^{-CC_0}$ then we set $K = 1$ and terminate the algorithm. Otherwise, we observe that the remaining connected components of $I^{(1)}$ still contain at least $c\eta^{CC_0} J$ intervals, and there are $O(\eta^{-CC_0})$ such components. Thus by the pigeonhole principle we can find one of these components, $I^{(2)}$, which is the union of $J_2 \geq c\eta^{CC_0} J_1$ intervals, each of which must have length less than or equal to $|I_{j_1}|/2$ by construction. Now we iterate the algorithm, using Corollary 3.7 to locate an interval I_{j_2} in $I^{(2)}$ such that $|I_{j_2}| \geq c\eta^{CC_0}|I^{(2)}|$, and then removing all intervals of length $> |I_{j_2}|/2$ from $I^{(2)}|$. If

[11] It seems of interest to remove the logarithm in this proposition, since this would make our final estimate polynomial in the energy instead of exponential. It seems however one cannot achieve this purely on the strength of the Morawetz estimate (5) and the mass conservation law (17), as the control on the intervals I_j provided by these two estimates does not preclude the possibility for the energy to concentrate on a Cantor set of times of dimension less than 1/2, which can use up an exponential number of intervals before the local mass conservation begins to conflict with energy conservation. One possibility is to combine the Morawetz inequality (5) with the interaction Morawetz inequalities in [7], although those inequalities are in some sense even weaker and thus less able to control the total number of intervals. We remark that for the cubic NLS in three dimensions, the known bounds are polynomial in the energy and mass [5, 6], but this is because the equation is H^1-subcritical and L^2-supercritical, which force the lengths $|I_j|$ of the intervals to be bounded both above and below. See [16] for a related discussion.

the number of intervals in $|I^{(2)}|$ is $O(\eta^{-CC_0})$, we terminate the algorithm, otherwise we can pass as before to a smaller interval $I^{(3)}$ which is a union of $J_3 \geq c\eta^{CC_0} J_2$ intervals. We can continue in this manner for K steps for some $K > c\eta^{C(C_0,C_1)} \log J$ until we run out of intervals. The claim then follows by choosing t_* to be an arbitrary time in $I^{(K)}$. □

Let t_* and $I_{j_1}, \ldots, I_{j_k}$ be as in the above proposition. From Proposition 3.3 we recall that

$$\text{Mass}\left(u(t), B\left(x_{j_k}, C\eta^{-C(C_0)}|I_{j_k}|^{1/2}\right)\right) \geq c\eta^{C(C_0)}|I_{j_k}|^{1/2}$$

for all $t \in I_{j_k}$. Applying (17) and adjusting the constants c, C as necessary we thus see that

$$\text{Mass}(u(t_*), B_k) \geq c\eta^{C(C_0,C_1)}|I_{j_k}|^{1/2},$$

where each B_k is a ball $B_k := B\left(x_{j_k}, C\eta^{-C(C_0,C_1)}|I_{j_k}|^{1/2}\right)$. On the other hand, from (18) we observe that

$$\text{Mass}(u(t_*), B_k) \leq C\eta^{-C(C_0,C_1)}|I_{j_k}|^{1/2}.$$

Let $N := C_2 \log(1/\eta)$. If we choose this constant C_2 large enough, we thus see from the above mass bounds and (32) that

$$\sum_{k+N \leq k' \leq K} \int_{B_{k'}} |u(t_*, x)|^2 \, dx \leq \frac{1}{2} \int_{B_k} |u(t_*, x)|^2 \, dx,$$

and hence

$$\int_{B_k \setminus (\bigcup_{k+N \leq k' \leq K} B_{k'})} |u(t_*, x)|^2 \geq c\eta^{C(C_0,C_1)}|I_{j_k}|.$$

Applying Hölder's inequality,[12] we thus obtain

$$\int_{B_k \setminus (\bigcup_{k+N \leq k' \leq K} B_{k'})} |u(t_*, x)|^{2n/(n-2)} \geq c\eta^{C(C_0,C_1)}.$$

Summing this in k and telescoping, we obtain

$$\int_{\mathbf{R}^n} |u(t_*, x)|^{2n/(n-2)} \geq c\eta^{C(C_0,C_1)} K/N.$$

Using (19) we thus obtain

$$K \leq C\eta^{-C(C_0,C_1)} N E^C \leq C(C_0, C_1, C_2)\eta^{-C(C_0,C_1)}.$$

Since $K > c\eta^{C(C_0,C_1)} \log J$, we obtain

$$J \leq \exp\left(C\eta^{-C(C_0,C_1)}\right) \leq \exp\left(C(C_0, C_1, C_2)E^{C(C_0,C_1,C_2)}\right)$$

as desired. This proves Theorem 1.1.

[12] An alternate approach here is to use the spherical symmetry to move the balls to be centered at the origin, and apply Hardy's inequality, see [1, 11]. However this approach shows that one does not need the spherical symmetry assumption to conclude the argument provided that one has a concentration result similar to Proposition 3.8.

Remark 3.9. One can use Proposition 3.3 to improve the bounds obtained in [7] in the nonradial case, as one no longer needs to use the induction hypothesis to obtain concentration bounds on the solution. It may also be possible to use a variant of the techniques here to also obtain the reverse Sobolev inequality. However the remaining portion of the arguments seem to require a heavier use of the induction hypothesis (in order to obtain certain frequency localization properties of the energy), and so we were unable to fully remove the tower-type bounds from the result in [7].

4. Appendix: Proof of Lemma 3.2 in high dimensions

We now give the rather technical proof of Lemma 3.2 in the high-dimensional case $n \geq 6$; the idea is to find an iteration scheme which converges acceptably after the first few terms, leaving us to estimate a finite number of iterates (which we can estimate by more inefficient means). We differentiate (1) and use the chain rule to obtain the equation

$$(i\partial_t + \Delta)\nabla u = V_1 \nabla u + V_2 \overline{\nabla u}$$

where $V_1 := \frac{n}{n-2}|u|^{\frac{4}{n-2}}$ and $V_2 := \frac{2}{n-2}|u|^{\frac{4}{n-2}}\frac{u^2}{|u|^2}$. From (21) we have

$$\|V_1\|_{L^{(n+2)/2}_{t,x}([t_1,t_2]\times\mathbf{R}^n)} + \|V_2\|_{L^{(n+2)/2}_{t,x}([t_1,t_2]\times\mathbf{R}^n)} \leq C\eta^c, \tag{33}$$

which by (14), (11), and Hölder implies in particular that

$$\|V_1 w + V_2 \overline{w}\|_{\dot{N}^0([t_1,t_2]\times\mathbf{R}^n)} \leq C\eta^c \|w\|_{\dot{S}^0([t_1,t_2]\times\mathbf{R}^n)}. \tag{34}$$

From Duhamel's formula (10) we have $\nabla u = \nabla u_1 + A\nabla u$, where A is the (real) linear operator

$$Aw(t) := -i\int_{t_1}^t e^{i(t-s)\Delta}(V_1(s)w(s) + V_2(s)\overline{w}(s))\, ds.$$

From Strichartz (16) and (34) we see that

$$\|Aw\|_{\dot{S}^0([t_1,t_2]\times\mathbf{R}^n)} \leq C\eta^c \|w\|_{\dot{S}^0([t_1,t_2]\times\mathbf{R}^n)}; \tag{35}$$

thus for η sufficiently small, A is a contraction on $\dot{S}^0([t_1,t_2]\times\mathbf{R}^n)$. Also, from Strichartz (15) and (19) we see that

$$\|\nabla u_1\|_{\dot{S}^0([t_1,t_2]\times\mathbf{R}^n)} \leq CE^C. \tag{36}$$

Thus for some absolute constant M (depending only on n), we see that we have the Neumann series approximation

$$\left\|\nabla u - \sum_{m=0}^M A^m \nabla u_1\right\|_{\dot{S}^0([t_1,t_2]\times\mathbf{R}^n)} \leq \eta$$

(for instance), assuming that η is sufficiently small depending (polynomially) on the energy. Now introduce the spacetime norm

$$\|w\|_X := \||\nabla|^{-1}w\|_{L^{2(n+2)/(n-2)}_{t,x}([t_1,t_2]\times\mathbf{R}^n)}$$

where $|\nabla|^{-1} := (-\Delta)^{-1/2}$. From (12) (and the boundedness of Riesz transforms) we observe that

$$\|w\|_X \leq C\|w\|_{\dot{S}^0([t_1,t_2]\times\mathbf{R}^n)} \tag{37}$$

and hence
$$\left\| \nabla u - \sum_{m=0}^{M} A^m \nabla u_1 \right\|_X \leq C\eta.$$
On the other hand, from (21) and Calderón-Zygmund theory we have
$$\|\nabla u\|_X \geq c\eta^{(n-2)/2(n+2)}.$$
Thus by the triangle inequality, we have
$$\|A^m \nabla u_1\|_X \geq c\eta^{(n-2)/2(n+2)}$$
for some $0 \leq m \leq M$, again assuming that η is sufficiently small.

Ideally we would now like the operator A to be bounded on X. We do not know if this is true; however we have the following weaker (and technical) version of this fact which suffices for our application.

Lemma 4.1. *For any $w \in \dot S^0([t_1, t_2] \times \mathbf{R}^n)$, we have the estimate*
$$\|Aw\|_X \leq CE^C \|w\|^{1-\theta}_{\dot S^0([t_1,t_2]\times\mathbf{R}^n)} \|w\|^\theta_X$$
for some absolute constant $0 < \theta < 1$ (depending only on n).

Assuming this lemma for the moment, we apply it together with (35), (36), (37) we obtain a bound of the form
$$\|A^m \nabla u_1\|_X \leq C_m \eta^{-C_m} E^{C_m} \|\nabla u_1\|_X^{\theta_m}$$
for some constants $C_m, \theta_m > 0$. Together with our lower bound on $\|A^m \nabla u_1\|_X$ this gives $\|\nabla u_1\|_X \geq c\eta^C$ (assuming η sufficiently small depending on E, and allowing constants to depend on the fixed constant M), and Lemma 3.2 follows (again using the boundedness of Riesz transforms).

It remains to prove Lemma 4.1. The point is to take advantage of one of the (many) refinements of the Sobolev embedding used[13] to prove (37); we shall use an argument based on Hedberg's inequality. We will not attempt to gain powers of η here (since the Neumann series step has in some sense fully exploited those gains already) and so shall simply discard all such gains that we encounter.

We make the *a priori* assumption that w is smooth and rapidly decreasing; this can be removed by the usual limiting argument. We normalize $\|w\|_{\dot S^0([t_1,t_2]\times\mathbf{R}^n)} := 1$, and write $\alpha := \|w\|_X$, thus $\alpha \leq C$ by (37). Our task is to show that $\|Aw\|_X \leq C\alpha^c$.

Observe from (35) and (13) that
$$\||\nabla|^{-1} Aw\|_{L_t^{2(n+2)/n} L_x^{2n(n+2)/(n^2-2n-4)}([t_1,t_2]\times\mathbf{R}^n)} \leq C$$
and hence by Hölder it will suffice to show that
$$\||\nabla|^{-1} Aw(t)\|_{L^{2n/(n-2)}(\mathbf{R}^n)} \leq C\alpha^c$$
for all $t \in [t_1, t_2]$.

[13]More precisely, we need a statement to the effect that the Sobolev theorem is only sharp if one of the "wavelet coefficients" of the function is extremely large (close to its maximal size). The argument below could be reformulated as an interpolation inequality (of Gagliardo–Nirenberg type) for Triebel–Lizorkin spaces, but we have elected to give a direct argument that does not rely on too much external machinery.

By time translation invariance we may set $t = 0$. Write $v := Aw(0)$. We now use a variant of Hedberg's inequality. From (35) and (11) we have

$$\|v\|_{L^2_x} \leq C; \tag{38}$$

if we let M denote the Hardy–Littlewood maximal function

$$Mv(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |v(y)|\, dy,$$

then by the Hardy–Littlewood maximal inequality (see [17]) we have

$$\|(Mv)^{n-2/n}\|_{L^{2n/(n-2)}_x} = \|Mv\|^{(n-2)/n}_{L^2_x} \leq C.$$

It thus suffices to prove the pointwise Hedberg-type inequality

$$||\nabla|^{-1} v|(x) \leq C\alpha^c (Mv(x))^{(n-2)/n}.$$

We may translate so that $x = 0$. Set $R := (Mv(0))^{-2/n}$ (the case $Mv(0) = 0$ being trivial), thus

$$\int_{B(0,r)} |v(y)|\, dy \leq CR^{-n/2} r^n$$

for all $r > 0$. From (38) and Cauchy–Schwarz we also have

$$\int_{B(0,r)} |v(y)|\, dy \leq Cr^{n/2}$$

and thus

$$\int_{B(0,r)} |v(y)|\, dy \leq Cr^{n/2} \left(1 + \frac{r}{R}\right)^{-n/2}. \tag{39}$$

By (6), it suffices to show that

$$\left| \int_{\mathbf{R}^n} \frac{v(y)}{|y|^{n-1}}\, dy \right| \leq C\alpha^c R^{1-\frac{n}{2}}.$$

By the above estimates, we see that we can prove this estimate in the regions $|y| \leq \alpha^{c_0} R$ and $|y| \geq \alpha^{-c_0} R$, even if we place the absolute values inside the integral, where $0 < c_0 \ll 1$ is an absolute constant to be chosen shortly. Thus it will suffice to estimate the remaining region $\alpha^{c_0} R \leq |y| \leq \alpha^{-c_0} R$. Partitioning the integral via smooth cutoffs, we see that it suffices (if c_0 was chosen sufficiently small) to show that

$$\left| \int_{\mathbf{R}^n} v(y)\varphi(y/r)\, dy \right| \leq C\alpha^c r^{\frac{n}{2}},$$

for all $r > 0$, where φ is a real-valued bump function. One may verify from dimensional analysis that this estimate (as well as the hypotheses) are invariant under the scaling

$$w(t,x) \mapsto \lambda^{-n/2} w(t/\lambda^2, x/\lambda); \quad v(t,x) \mapsto \lambda^{-n/2} w(t/\lambda^2, x/\lambda)$$

and so we may take $r = 1$. Since $v = Aw(0)$, we thus reduce to proving that

$$|\langle Aw(0), \varphi\rangle| \leq C\alpha^c.$$

Expanding out the definition of A and using duality, we can write this as

$$\left| \int_{t_1}^0 \int_{\mathbf{R}^n} (V_1(t)w(t) + V_2(t)\overline{w}(t)) e^{it\Delta}\varphi\, dxdt \right| \leq C\alpha^c. \tag{40}$$

From (33), (11) we have

$$\|V_1 w + V_2\overline{w}\|_{L^{2(n+2)/(n+4)}_{t,x}([t_1,0]\times\mathbf{R}^n)} \leq C,$$

while a direct computation[14] of $e^{-it\Delta}\varphi$ shows that

$$\|e^{-it\Delta}\varphi\|_{L^{2(n+2)/n}_{t,x}([t_1,-\tau]\times\mathbf{R}^n)} \leq C\tau^{-c}$$

for all $\tau > 1$. Thus if we set $\tau = C\alpha^{-c_0}$ for some small c_0 to be chosen later, we see that the portion of (40) arising from $[t_1, -\tau]$ is acceptable, and it suffices to then prove the bound on $[-\tau, 0]$. In fact we will prove the fixed time estimates

$$\left|\int_{\mathbf{R}^n} (V_1(t)w(t) + V_2(t)\overline{w(t)})e^{it\Delta}\varphi\, dx\right| \leq CE^C(1+t)^C \||\nabla|^{-1}w(t)\|^c_{L^{2(n+2)/(n-2)}(\mathbf{R}^n)}$$

for all $t \in [-\tau, 0]$, which proves the claim if c_0 is sufficiently small, thanks to Hölder's inequality and the hypothesis $\|w\|_X = \alpha$.

Fix t. We shall just prove this inequality for $V_2\overline{w}$, as the corresponding estimate for $V_1 w$ is similar. Because of the negative derivative on w on the right-hand side, we shall need some regularity control on V_2. Note that V_2 behaves like $|u|^{4/(n-2)}$; since $4/(n-2) \leq 1$, the standard fractional chain rule is not easy to apply. Instead, we will work in Hölder-type spaces,[15] which are more elementary. As with Lemma 3.4, we let $u^{(h)}$ denote the translate $u^{(h)}(x) := u(x-h)$ of u by h for any $h \in \mathbf{R}^n$; similarly define $V_2^{(h)}$, etc. From (19), the fundamental theorem of calculus, and Minkowski's inequality we have

$$\|u^{(h)}(t) - u(t)\|_{L^2(\mathbf{R}^n)} \leq CE^C|h|.$$

Since the function $z \mapsto |z|^{\frac{4}{n-2}}\frac{z^2}{|z|^2}$ is Hölder continuous of order $4/n-2$, we thus have the pointwise inequality

$$|V_2^{(h)}(t) - V_2(t)| \leq C|u^{(h)} - u|^{4/(n-2)}$$

which gives the Hölder type bounds

$$\|V_2^{(h)}(t) - V_2(t)\|_{L^{(n-2)/2}(\mathbf{R}^n)} \leq CE^C|h|^{4/(n-2)}.$$

From (11) and the normalization $\|w\|_{\dot{S}^0([t_1,t_2]\times\mathbf{R}^n)} = 1$ we have $\|w(t)\|_{L^2_x} \leq C$, and hence by Hölder (and the decay of $e^{-it\Delta}\varphi$ in space)

$$\left|\int_{\mathbf{R}^n} (V_2(t) - V_2^{(h)})\overline{w(t)}e^{it\Delta}\varphi\, dx\right| \leq CE^C(1+t)^C|h|^{4/(n-2)}.$$

Similarly, from (19) we have

$$\|V_2^{(h)}(t)\|_{L^{n/2}(\mathbf{R}^n)} \leq C\|u(t)\|^{4/(n-2)}_{L^{2n/(n-2)}(\mathbf{R}^n)} \leq CE^C; \tag{41}$$

a direct computation also shows that

$$\|e^{it\Delta}\varphi - (e^{it\Delta}\varphi)^{(h)}\|_{L^{2n/(n-4)}(\mathbf{R}^n)} \leq C(1+t)^C|h| \leq C(1+t)^C|h|^{4/(n-2)}$$

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[14] Indeed, one just needs to note that $e^{-it\Delta}\varphi$ is bounded in L^2_x and decays in L^∞_x like $O(t^{-n/2})$ to verify this claim.

[15] Using Hölder spaces rather than Sobolev spaces costs an epsilon of regularity (see, e.g., [17] for a discussion) but for our purposes any nonzero amount of regularity will suffice. The reader may recognize the arguments below as that of splitting a product into paraproducts; however we are avoiding the use of standard paraproduct theory as it does not interact well with nonlinear maps such as $u \mapsto V_2$ which may only be Hölder continuous of order $4/(n-2) < 1$.

for $|h| \leq 1$ (say), and so by Hölder again

$$\left| \int_{\mathbf{R}^n} V_2^{(h)} \overline{w(t)} \left(e^{it\Delta}\varphi - \left(e^{it\Delta}\varphi\right)^{(h)} \right) \, dx \right| \leq CE^C (1+t)^C |h|^{4/(n-2)}.$$

Combining this with the previous estimate we obtain

$$\left| \int_{\mathbf{R}^n} \overline{w(t)} \left(V_2 e^{it\Delta}\varphi - \left(V_2 e^{it\Delta}\varphi\right)^{(h)} \right) \, dx \right| \leq CE^C (1+t)^C |h|^{4/(n-2)},$$

or equivalently that

$$\left| \int_{\mathbf{R}^n} \left(\overline{w(t)} - \overline{w(t)}^{(-h)} \right) V_2 e^{it\Delta}\varphi \, dx \right| \leq CE^C (1+t)^C |h|^{4/(n-2)}.$$

We can average this over all $|h| \leq r$, where the radius $0 < r < 1$ will be chosen later, to obtain

$$\left| \int_{\mathbf{R}^n} \left(\overline{w(t)} - \overline{w_{av}(t)} \right) V_2 e^{it\Delta}\varphi \, dx \right| \leq CE^C (1+t)^C r^{4/(n-2)} \tag{42}$$

where $w_{av}(t,x) := \int \chi(y) w(t, x+ry)$ for some bump function χ of total mass 1. On the other hand, from the Hörmander multiplier theorem (see [17]) and some Fourier analysis we see that

$$\|w_{av}(t)\|_{L^{2(n+2)/(n-2)}(\mathbf{R}^n)} \leq Cr^{-C} \left\| |\nabla|^{-1} w(t) \right\|_{L^{2(n+2)/(n-2)}(\mathbf{R}^n)},$$

and by combining this with (41) and decay estimates on $e^{it\Delta}\varphi$ we obtain

$$\left| \int_{\mathbf{R}^n} \overline{w_{av}(t)} V_2 e^{it\Delta}\varphi \, dx \right| \leq CE^C (1+t)^C r^{-C} \left\| |\nabla|^{-1} w(t) \right\|_{L^{2(n+2)/(n-2)}(\mathbf{R}^n)}.$$

Combining this with (42) and optimizing in r we obtain Lemma 4.1, and thus Lemma 3.2.

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DEPARTMENT OF MATHEMATICS, UCLA, LOS ANGELES CA 90095-1555
tao@math.ucla.edu http://www.math.ucla.edu/~tao/

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