

An ergodic sum related to the approximation by continued fractions

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ABSTRACT. To each irrational number x is associated an infinite sequence of rational fractions $\frac{p_n}{q_n}$, known as the convergents of x . Consider the functions $q_n|q_n x - p_n| = \theta_n(x)$. We shall primarily be concerned with the computation, for almost all real x , of the ergodic sum

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \log \theta_k(x) = -1 - \frac{1}{2} \log 2 \approx -1.34657.$$

Each irrational number x has a unique infinite, regular continued fraction expansion of the form

$$x = [a_1; a_2, a_3 \dots] = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}$$

where the a_i are integers and $a_i > 0$ for $i > 1$. To x is associated an infinite sequence of rational fractions $\frac{p_n}{q_n} = [a_1; a_2, \dots, a_n]$, in lowest terms, known as the convergents of x . Define the functions $\theta_n(x)$ by the identity

$$\left| x - \frac{p_n}{q_n} \right| = \frac{\theta_n(x)}{q_n^2}.$$

Important metrical results on the $\theta_n(x)$ are proved in the papers [3],[5] and [7]. Since the convergents satisfy the following well-known inequality, usually attributed to Dirichlet,

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2},$$

we have $0 < \theta_n(x) < 1$.

It does not seem to have been observed that for almost all x

$$(1) \quad \lim_{n \rightarrow \infty} \frac{\log \theta_n(x)}{n} = 0.$$

To begin we supply a proof of this fact which was suggested by A. Rockett.

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From the sequence of inequalities (see [9]),

$$\frac{1}{q_n(q_n + q_{n+1})} \leq \left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}}$$

we get

$$\frac{1}{n} \log \left(\frac{q_n}{q_n + q_{n+1}} \right) \leq \frac{1}{n} \log \theta_n(x) \leq \frac{1}{n} \log \left(\frac{q_n}{q_{n+1}} \right).$$

The result then follows easily from the Khintchine–Lévy Theorem, which asserts that for almost all x

$$(2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log q_n = \frac{\pi^2}{12 \log 2}, \quad (\text{see [2] or [9]}).$$

Now we look at the limiting average of the functions $\log \theta_n(x)$. While this average resembles those, such as $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \theta_k(x)$, computed in [3] and [5], its evaluation is complicated by the fact that $\log x$ is not continuous on the interval $[0,1]$. As a result, knowledge of the distribution function for $\theta_n(x)$ is not sufficient to prove the theorem. As in [3], we work with a form of the natural automorphic extension of the Gauss transform, derived from the extension originally given by Nakada [8].

Theorem 1. *For almost all x*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \log \theta_k(x) = -1 - \frac{1}{2} \log 2 \approx -1.34657.$$

Let $\Lambda = ((0, 1) \setminus \mathbb{Q}) \times [0, 1]$ and define the map $\tilde{S} : \Lambda \rightarrow \Lambda$ by

$$\tilde{S}(s, t) = \left(\frac{1}{s} - \left[\frac{1}{s} \right], \frac{1}{t + \left[\frac{1}{s} \right]} \right)$$

where $[x]$ is the greatest integer function. Let ν be the probability measure with density $m(s, t) = \frac{1}{\log 2} (1 + st)^{-2}$. It was first observed by Nakada [8] that the dynamical system $(\Lambda, B, \nu, \tilde{S})$ is ergodic. See also [1].

Consider the related self-mapping \tilde{T} of $\Omega = ((0, 1) \setminus \mathbb{Q}) \times [-\infty, -1]$ defined by

$$\tilde{T}(x, y) = \left(\frac{1}{x} - \left[\frac{1}{x} \right], \frac{1}{y} - \left[\frac{1}{x} \right] \right).$$

Let $\varphi : \Lambda \rightarrow \Omega$ be the invertible function given by $\varphi(s, t) = (s, -\frac{1}{t})$. It is clear that φ maps Λ onto Ω and that $\tilde{T} = \varphi \circ \tilde{S} \circ \varphi^{-1}$.

The measure $\mu = \varphi^* \nu$ is defined by

$$\mu(D) = \frac{1}{\log 2} \int_{\varphi^{-1}(D)} \frac{1}{(1+st)^2} dsdt,$$

where D is a borel subset of Ω . It follows by an application of the chain rule that μ has the density $p(x, y) = \frac{1}{\log 2}(x - y)^{-2}$. As constructed, μ is invariant under the action of \tilde{T} and φ defines an isomorphism between the dynamical systems $(\Lambda, B, \nu, \tilde{S})$ and $(\Omega, B, \mu, \tilde{T})$. It follows that $(\Omega, B, \mu, \tilde{T})$ is an ergodic dynamical system. The ergodicity of \tilde{T} is central to the proof of Theorem 1 that follows.

Let $T(x) = \frac{1}{x} - [\frac{1}{x}]$ be the classical Gauss map and let $\pi(x_1, x_2) = x_1$ be projection on the first factor. Then independent of y , $\pi \circ \tilde{T}(x, y) = T(x)$ and $\pi^*(\mu)$ is the classical Gauss measure, which is an invariant measure for T , with density $g(x) = \frac{1}{\log 2} \frac{1}{1+x}$.

For irrational $x = [0; a_1, a_2 \dots]$, the Gauss map T acts as a shift on continued fraction expansions with $T(x) = [0; a_2, a_3 \dots]$. Even when x is rational, T acts as a shift on the finite continued fraction expansion and the iterates are defined until $T^n(x) = 0$. We assume henceforth that $x = [0; a_1, a_2 \dots]$ is irrational. The iterates $\tilde{T}^n(x)$ are then defined for all positive integers n . If $y = -[a_{-1}; a_{-2}, \dots] \in (-\infty, -1]$, with a possibly finite continued fraction expansion and x is as above then $\tilde{T}^n(x, y) = (\hat{x}, \hat{y})$ where $\hat{x} = [0; a_{n+1}, a_{n+2} \dots]$ and $\hat{y} = -[a_n; a_{n-1}, a_{n-2}, \dots, a_1, a_{-1} \dots]$.

Define the function

$$F(x, y) = \log \left(\frac{1}{x} - \frac{1}{y} \right) = \log \left(\frac{y - x}{xy} \right).$$

We show that F is integrable on $\Omega = (0, 1) \times (-\infty, -1]$ with respect to the density p . It shall soon be clear that F is very useful for computing the quantity $\log \theta_n(x)$.

$$\begin{aligned} & \int_{\Omega} \log \left(\frac{y - x}{xy} \right) p(x, y) \, dx dy \\ &= \frac{1}{\log 2} \int_{-\infty}^{-1} \int_0^1 \frac{\log \left(\frac{y-x}{xy} \right)}{(x - y)^2} \, dx dy \\ &= \frac{1}{\log 2} \int_{-\infty}^{-1} \frac{1 + \log(1 - y) - \log(-y)}{y(y - 1)} \, dy \\ &= \frac{1}{\log 2} \lim_{h \rightarrow \infty} \left[\log(-y) + \log(1 - y) + \frac{1}{2}(\log(-y))^2 \right. \\ & \quad \left. + \frac{1}{2}(\log(1 - y))^2 - \log(-y) \log(1 - y) \right] \Big|_{-h}^{-1} \\ &= \frac{1}{\log 2} \left[\left(\log 2 + \frac{1}{2}(\log 2)^2 \right) \right. \\ & \quad \left. - \lim_{h \rightarrow \infty} \left(\frac{\frac{1}{2} \log(-y) - \frac{1}{2} \log(1 - y) + 1}{\frac{1}{\log(-y)}} + \frac{\frac{1}{2} \log(1 - y) - \frac{1}{2} \log(-y) + 1}{\frac{1}{\log(1 - y)}} \right) \right] \end{aligned}$$

where the last limit is zero by L'Hospital's rule.

Since F is μ -integrable and $\tilde{T}^n(x, y)$ is defined on a set of full measure for all $n \geq 0$, it is a direct consequence of the Birkhoff Ergodic Theorem (see [4] or [2]) that for almost all $(x, y) \in \Omega$

$$(3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n F(\tilde{T}^i(x, y)) = \int_{\Omega} F(x, y) p(x, y) \, dx dy.$$

This value was just computed to be $1 + \frac{1}{2} \log 2$.

It is known, see, e.g., [2], that if two numbers α, β have continued fractions expansions which agree in their first n digits, then $|\alpha - \beta| < 2^{-n+1}$. Thus for $y, y' \in [-\infty, -1]$, $|\pi_2 \circ \tilde{T}^n(x, y) - \pi_2 \circ \tilde{T}^n(x, y')| < 2^{-n+1}$.

Our next task is to prove that if the equality (3) holds for a given (x, y) then it holds for (x, y') for all $y' \in [-\infty, -1]$. In essence, the equality is true for almost all x independent of y . Fix x and suppose that the equality (3) holds for $(x, y) \in \Omega$. Let $y' \in [-\infty, -1]$. To simplify the computation write $\tilde{T}^i(x, y) = (x_i, y_i)$ and $\tilde{T}^i(x, y') = (x_i, y'_i)$. Keep in mind that y_i and y'_i are negative numbers. Then

$$(4) \quad \begin{aligned} \frac{1}{n} \sum_{i=1}^n \left| F(\tilde{T}^i(x, y)) - F(\tilde{T}^i(x, y')) \right| &= \frac{1}{n} \sum_{i=1}^n \left| \log \left(\frac{y_i - x_i}{x_i y_i} \right) - \log \left(\frac{y'_i - x_i}{x_i y'_i} \right) \right| \\ &= \frac{1}{n} \sum_{i=1}^n \left| \left(\log(x_i - y_i) - \log(x_i) - \log(-y_i) \right) \right. \\ &\quad \left. - \left(\log(x_i - y'_i) - \log(x_i) - \log(-y'_i) \right) \right| \\ &\leq \frac{1}{n} \sum_{i=1}^n \left| \log \frac{x_i - y_i}{x_i - y'_i} \right| + \frac{1}{n} \sum_{i=1}^n \left| \log \frac{y'_i}{y_i} \right|. \end{aligned}$$

There is no loss of generality in supposing that $x_i - y_i \geq x_i - y'_i$, since the absolute value of the log of the quotient in the first sum of (4) is the same either way the inequality goes. Then by an earlier observation

$$0 < (x_i - y_i) - (x_i - y'_i) = y'_i - y_i < 2^{-i+1}.$$

Since $x_i - y'_i > 1$,

$$\frac{x_i - y_i}{x_i - y'_i} < 1 + 2^{-i+1} (x_i - y'_i)^{-1} < 1 + 2^{-i+1}.$$

Now take logs and apply the standard estimate that comes from the alternating series for $\log x$ to get

$$\log \frac{x_i - y_i}{x_i - y'_i} < \log(1 + 2^{-i+1}) < 2^{-i+1}.$$

It follows that the first sum in (4) converges to zero as n goes to ∞ . By a similar argument the same conclusion can be reached for the second sum in (4). This shows that if the equality (3) holds for some $(x, y) \in \Omega$ then it holds for any $(x, y') \in \Omega$.

We are now close to completing the the proof of Theorem 1. Two identities from the classical theory will link the above to our main theorem. If $x = [0; a_1, a_2 \dots]$ then

$$(5) \quad [a_n; a_{n-1}, \dots, a_1] = \frac{q_n}{q_{n-1}} \quad (\text{see [9]})$$

and

$$(6) \quad \theta_n(x) = \left(\frac{1}{T^n(x)} + \frac{q_{n-1}}{q_n} \right)^{-1} \quad (\text{see [6, p. 29, (11)]}).$$

Given $x \in (0, 1)$, let $(x_0, y_0) = \tilde{T}(x, \infty) = (T(x), -[1/x]) \in \Omega$. As above define $\tilde{T}^i(x_0, y_0) = (x_i, y_i)$. If $x = [0; a_1, a_2 \dots]$ then for $i > 0$

$$(x_{i-1}, y_{i-1}) = ([0; a_{i+1}, a_{i+2} \dots], -[a_i; a_{i-1}, a_{i-2}, \dots, a_1]) = \left(T^i(x), -\frac{q_i}{q_{i-1}} \right)$$

where we have used (5) above. From (6) and the definition of F ,

$$\begin{aligned} -F(\tilde{T}^i(x_0, y_0)) &= -\log \left(\frac{1}{x_i} - \frac{1}{y_i} \right) \\ &= \log \left(\frac{1}{T^{i+1}(x)} + \frac{q_i}{q_{i+1}} \right)^{-1} \\ &= \log \theta_{i+1}(x). \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log \theta_{i+1}(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n -F(\tilde{T}^i(x_0, y_0))$$

converges to $-1 - \frac{1}{2} \log 2$ for almost all $x_0 \in (0, 1)$, independent of y_0 , and consequently for almost all $x \in (0, 1)$. The proof of Theorem 1 is complete.

References

- [1] R. L. Adler and L. Flatto, *Cross-section maps for geodesic flows*. I, Ergodic theory and dynamical systems, II (College Park, Md., 1979/1980), Progress in Math., **21**, Birkhäuser, Boston, 1980, 103–161, [MR 0670077](#) (84h:58113), [Zbl 0496.58009](#).
- [2] P. Billingsley, *Ergodic theory and information*, J. Wiley & Sons, New York-London, 1965, [MR 0192027](#) (33 #254), [Zbl 0141.16702](#).
- [3] W. Bosma, H. Jager and F. Wiedijk, *Some metrical observations on the approximation by continued fractions*, Indag. Math. **45** (1983), 281–299, [MR 0718069](#) (85f:11059), [Zbl 0519.10043](#).
- [4] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic Theory*, Grundlehren der Mathematischen Wissenschaften, **245**, Springer-Verlag, Berlin-Heidelberg-New York, 1982, [MR 0832433](#) (87f:28019), [Zbl 0493.28007](#).
- [5] H. Jager, *The distribution of certain sequences connected with the continued fraction*, Indag. Math. **89** (1986), 61–69, [MR 0834320](#) (87g:11092), [Zbl 0588.10061](#).
- [6] J. F. Koksma, *Diophantische approximationen*, Julius Springer, Berlin 1936, [MR 0344200](#) (49 #8940), [Zbl 0012.39602](#).
- [7] D. Knuth, *The distribution of continued fraction approximations*, J. Number Theory **19** (1984), 443–448, [MR 0769794](#) (86d:11058), [Zbl 0547.10030](#).
- [8] H. Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. **4** (1981), 399–426, [MR 0646050](#) (83k:10095), [Zbl 0479.10029](#).
- [9] A. Rockett and P. Szűsz, *Continued fractions*, World Sci. Pub., Singapore, 1992, [MR 1188878](#) (93m:11060), [Zbl 0925.11038](#).

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