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Topics in dyadic Dirichlet spaces

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Abstract. We investigate the function theory on function spaces on a dyadic tree which model Dirichlet spaces of holomorphic functions. Most of the specific questions addressed deal with Carleson measures on those spaces.

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1. Introduction

In a previous paper [ARS], together with Eric Sawyer, we studied Carleson measures and related topics for generalized Dirichlet spaces of holomorphic functions. One of our main tools there was a family of discrete models which, while considerably easier to work with, were faithful enough to the original situation so that results for the model problems could be applied in the continuous situation. Here we continue to study the function theory of such model spaces, the dyadic Dirichlet spaces of the title. We feel both that these spaces are intrinsically interesting and that understanding them better will help inform our study of spaces of holomorphic functions.

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We begin this introduction with a brief overview of some results in the holomorphic setting. After that we have a brief survey of the contents of the later sections.

1.1. Generalized Dirichlet spaces. Let $d\nu(z) := \left(1-|z|^2\right)^{-2}dxdy$ be the Möbius invariant measure on the unit disk $\mathbb{D}$ and let $\delta f(x) := (1-|z|^2)f'(z)$ be the gradient of a function f with respect to the hyperbolic geometry of $\mathbb{D}$. The classical Dirichlet space is the space of holomorphic functions f for which $\int_{\mathbb{D}}|\delta f|^2 d\nu < \infty$. More generally, for $\alpha \geq 0$, $1 < p < \infty$, set $\rho_\alpha(z) = (1-|z|^2)^\alpha$ and define the generalized Dirichlet space (a.k.a. generalized Besov spaces) $B_p(\alpha)$ to be the space of holomorphic functions f for which the seminorm

$$\|f\|^*_{\alpha,p} = \left(\int_{\mathbb{D}} |\delta f|^p \rho_\alpha d\nu\right)^{\frac{1}{p}}$$

is finite. Note that the seminorm on the spaces $B_p = B_p(0)$ is conformally invariant. We also have the norms $\|f\|_{\alpha,p} = \|f\|^*_{\alpha,p} + |f(0)|$.

We define the α-*hyperbolic* distance of z from the origin by

$$d_\alpha(z) = \int_{[0,z]} \rho_\alpha(w)^{1-p'} \frac{|dw|}{1-|w|^2}$$

where $[0,z]$ is the segment from 0 to z. $d_0(0,z) = d(0,z)$ is the (classical) hyperbolic distance from 0 to z. It is not difficult prove [ARS] that

$$|f(z) - f(0)| \leq C(\alpha,p) d_\alpha(z)^{1/p'} \|f\|^*_{\alpha,p} \tag{1.1}$$

where $C(\alpha,p)$ is a positive constant.

In the spaces $B_p(\alpha)$ we can pose natural questions such as the characterization of Carleson measures, of multipliers, of interpolating sequences, or of zero sets. Part of the motivation for these questions is that $B_2(1) = H^2$ is the classical analytic Hardy space. For H^2 the answers to these questions are known and are central to that theory as well as to much of commutative harmonic analysis. Also, the study of Carleson measures on the $B_p(\alpha)$ is the holomorphic counterpart of the study of trace inequalities for Sobolev spaces. That topic has been studied extensively and we refer the reader to [M], [KS1], [V], and [KV] for more information.

A positive measure μ on $\mathbb{D}$ is a *Carleson measure* for $B_p(\alpha)$ if for some constant $C(\mu)$

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C(\mu) \|f\|^p_{\alpha,p} \tag{1.2}$$

holds for all functions f holomorphic in $\mathbb{D}$.

For $a \in \mathbb{D}$, set

$$S(a) = \{z \in \mathbb{D} : 1-|z| \leq 2(1-|a|), |\arg(a\bar{z})| \leq 2\pi(1-|a|)\}$$

and, if I is an arc on $\partial\mathbb{D}$, let $S(I) = S((1-|I|/2\pi)e^{i\theta_I})$ where $e^{i\theta_I}$ is the midpoint of I. It was proved by Carleson [Car] for $H^2 = B_2(1)$ and then by Stegenga [Ste] for $B_2(\alpha)$, $\alpha \geq 1$, that μ is Carleson for $B_2(\alpha)$ if and only if for some $C(\mu)$

$$\mu(S(I)) \leq C(\mu)|I|^\alpha. \tag{1.3}$$

for all arcs I. However for $0 \le \alpha < 1$ this simple condition is necessary, but not sufficient. Stegenga showed that the correct condition for B_2 is the capacitary condition

$$\mu\left(\cup_{j=1}^n S(I_j)\right) \le C(\mu)(\log(\operatorname{cap}(\cup_{j=1}^n I_j))^{-1})^{-1} \tag{1.4}$$

whenever the I_j's are disjoint arcs of $\partial\mathbb{D}$. Here cap denotes logarithmic capacity. Stegenga also showed that similar conditions, with suitable capacities, also characterize the Carleson measures for $B_2(\alpha)$, $0 \le \alpha < 1$. J. Wang [Wang] had similar results for B_p and those results were obtained independently by Wu [Wu2]. We now describe those results.

For any open set $O \subset \partial\mathbb{D}$ we define the $B_p(\alpha)$-capacity of the set O by

$$\operatorname{cap}(O, B_p(\alpha)) = \inf\left\{\|f\|_{B_p(\alpha)} : \operatorname{Re} f \ge 1 \text{ on } O\right\}.$$

It is shown in [Wang] and [Wu2] that, in analogy with (1.4), μ is a Carleson measure for $B_2(\alpha)$, $0 \le \alpha < 1$, if and only if there is a $C_{p,\alpha}(\mu) > 0$ so that

$$\mu\left(\cup_{j=1}^n S(I_j)\right) \le C_{p,\alpha}(\mu) \operatorname{cap}\left(\cup_{j=1}^n I_j, B_p(\alpha)\right).$$

However for $\alpha = 1$ and $1 < p \le 2$ the condition is again

$$\mu(S(I)) \le C(\mu)|I|. \tag{1.5}$$

It isn't know if this condition is the correct one for $\alpha = 1$ and $p > 2$. (There is an unfortunate misprint in Theorem 1 (b) of [Wu2]. *In the notation used there*, the condition $\min(1, \alpha + 1) < p$ should be $\max(1, \alpha + 1) < p$.)

A simpler, "one box", condition was later found by Kerman and Sawyer [KS2]. We now recall their result in the special case $p = 2$. They showed that a necessary and sufficient condition which is valid in the range $0 \le \alpha \le 1$ is given by

$$\int_I \sup_{J:\theta \in J \subset I} \frac{\mu(S(J) \cap S(I))^2}{|J|\rho_\alpha(1 - |J|)} d\theta \le C(\mu)\mu(S(I)) \tag{1.6}$$

for all arcs I in $\partial\mathbb{D}$, the supremum being taken over arcs J in $\partial\mathbb{D}$. An interesting feature of (1.6) is that it "bridges" the gap between the simple (1.3) and the more complicated (1.4). Yet another characterization of the Carleson measures is available [ARS]. Namely, if $1 < p < \infty$ and p' is the conjugate index defined by $p^{-1} + p'^{-1} = 1$, then for $0 \le \alpha < 1$, a measure μ is Carleson if and only if

$$\int_{S(I)} \mu(S(z) \cap S(I))^{p'} \rho_\alpha(z)^{1-p'} d\nu(z) \le C(\mu)\mu(S(I)) \tag{1.7}$$

for all arcs I. In [ARS] it is shown that (1.7) is strictly stronger that (1.3) whenever $\alpha \ge 0$ and $1 < p < \infty$. A direct proof that the discrete versions of (1.6) and (1.7) are equivalent when $0 \le \alpha < 1$ will be given in Section 3.

A *multiplier* of $B_p(\alpha)$ is a function g, analytic on $\mathbb{D}$, such that the multiplication operator

$$M_g : f \mapsto gf$$

is bounded on $B_p(\alpha)$. We denote by $M(B_p(\alpha))$ the space of multipliers. It is not to difficult to show that $g \in M(B_p(\alpha))$ if and only if it belongs to H^∞ and the measure

$$d\mu_g(z) = |g'(z)|^p(1 - |z|^2)^{p-2}\rho_\alpha(z)dm(z) \tag{1.8}$$

is Carleson for $B_p(\alpha)$. Thus, each theorem characterizing the Carleson measures also provides a first step toward the characterization of the multiplier space.

The space X of holomorphic functions g for which $d\mu_g(z)$ is a Carleson measure for $B_2(0)$ appears to play a role in the theory of the Dirichlet space $B_2(0)$ similar to that of BMO in the theory of Hardy spaces $B_2(1)$. However analogs of classical facts about BMO such as the theorem of John and Nirenberg or Fefferman's result that the space originally defined by an oscillation condition can equivalently be described by a Carleson measure condition are not known for functions in X. In Section 6 we do develop one result of that sort for the dyadic model case.

By (1.1), if Z is subset of $\mathbb{D}$, the functional

$$T: \ f \mapsto \{(d_\alpha(z)+1)^{-1/p'} f(z) : z \in Z\}$$

maps *a priori* $B_p(\alpha)$ into l^∞. We say that Z is an *interpolating sequence* for $B_p(\alpha)$ if T maps $B_p(\alpha)$ into *and* onto l^p. Marshall and Sundberg [MS] and, independently, C. Bishop [Bi] found the following geometric characterization of the interpolating sequences. Consider, with $\alpha = 0$ and $p = 2$, the measure

$$d\mu_Z = \sum_{z \in Z} (1 + d_\alpha(z))^{1-p} \delta_z. \tag{1.9}$$

Then, Z is interpolating for B_2 if and only if μ_Z is Carleson for B_2 and the following separation condition holds: there are constants A and B so that for any distinct z and w in Z

$$d(z,0) \le A d(z,w) + B. \tag{1.10}$$

Note that the requirement that μ_Z is a Carleson measure is just a different way of saying that T is bounded. Marshall and Sundberg also proved that these two conditions are necessary and sufficient for Z to be interpolating for $M(B_2)$. By this, we mean that

$$U : f \mapsto \{f(z) : z \in Z\}$$

maps $M(B_p(\alpha))$ onto l^∞. J. Xiao [X] extended the characterization of the interpolating sequences to $0 < \alpha < 1$, for both $B_2(\alpha)$ and $M(B_2(\alpha))$. His conditions on Z are, as in [MS], that μ_Z is Carleson for $B_2(\alpha)$ and that the sequence Z is separated in the sense that

$$d(z,w) \ge C > 0$$

for all distinct z and w in Z. This latter separation condition is also the one which occurs in Carleson classical interpolating Theorem [Car] which corresponds to $\alpha = 1$. This and other results of Xiao show that, when $0 < \alpha < 1$, $B_2(\alpha)$ is similar to B_2 in some respects (Carleson measures), and to H^2 in others (interpolating sequences, $\overline{\partial}$-problems). An extension of the Marhall and Sundberg interpolating theorems to B_p, $1 < p < \infty$, was recently obtained by Böe [Bo]. He shows that a sequence Z is interpolating for B_p if and only if (1.9) and (1.10) hold, and that the interpolating sequences for B_p are exactly those which are interpolating for $M(B_p)$. A dyadic version of Böe's theorem will be given in Section 4.

There is another version of the problem of interpolating sequences which remains open. One can require the map T (resp., U) to be onto, although not necessarily defined on all $B_p(\alpha)$ (resp., $M(B_p(\alpha))$. Some results in this direction exist for B_2. It has been proved by Bishop [Bi] that a sequence Z in $\mathbb{D}$ is interpolating for B_2

in this weaker sense if, for each $z \in Z$, there is an analytic function h_z such that $\|h_z\|_{B_2} \le Cd(z)^{-1}$ and $h_z(w) = \delta_z(w)$ for all $w \in Z$. The condition is clearly also necessary to have interpolation. Bishop's theorem connects the problem of interpolation with that of characterizing the *zero sets* for B_2. A sequence Z in $\mathbb{D}$ is a zero set for B_2 if there is a nonzero function f in B_2 having zeros at all points of Z. Shapiro and Shields [SS] showed that a sufficient condition for Z to be a zero set is that

$$\sum_{z\in Z} d(z)^{-1} < \infty \tag{1.11}$$

It is not known if (1.11) is also necessary. Condition (1.11) is a special case of (1.3), with $p = 2$, $\alpha = 0$, $\mu = \mu_Z$ and $I = \partial\mathbb{D}$. Indeed, Bishop's condition asks for $Z - \{z\}$ to be a zero set for all $z \in Z$ and that there be natural uniform estimates for the functions with the required properties. Thus, although the two problems are related, it is in principle possible to find a geometric characterization of the sequences that are interpolating in the weak sense, without having to characterize the zero sets for B_2.

1.2. Contents. The remaining part of this note is devoted to the dyadic Dirichlet spaces, defined in Section 2. In Section 3, as part of an effort to understand the range of applicability of these various models, we introduce a discrete model for some of the $B_p(\alpha)$ which is different from the models used in [ARS]. More specifically, having considered a cancellation free model in [ARS] as well as a discrete harmonic model, here we consider a martingale model. In Section 4 we characterize the Carleson measures for our dyadic Dirichlet spaces. The condition turns out to be the same as for the cancellation free model. The sufficiency of the condition follows *a posteriori* from the results for the cancellation free model, the necessity of the condition, which was rather easy to verify for the cancellation free model, is more delicate here due to the paucity of test functions. The discrete analogs of condition (1.6) or of condition (1.7) characterize Carleson measures for our dyadic Dirichlet spaces, hence the conditions must be equivalent. However the equivalence is not transparent. It is straightforward that condition (1.7) implies condition (1.6); in Section 4 we give a direct proof of the opposite implication. In Section 5, we characterize the interpolating sequences for the dyadic Dirichlet spaces. In Section 6 we obtain a result for the dyadic Dirichlet space which models the relationship between BMO functions and bounded Hankel operators on the Hardy space.

2. Definitions and preliminary results

Let $\mathcal{D}$ be the index set

$$\mathcal{D} = \{(n, j) \in \mathbb{Z}\times\mathbb{Z} : n \ge 0,\ 1 \le j \le 2^n\}.$$

To each index $\alpha = (n, j) \in \mathcal{D}$, we associate an interval $I(\alpha) = [2^{-n}(j-1), 2^{-n}j]$. and we denote by $|I(\alpha)|$ the length of the interval $I(\alpha)$. We call $o = (0, 1)$ the *root* of $\mathcal{D}$. We endow $\mathcal{D}$ with a partial ordering and with a tree structure. We say that $\alpha < \beta$ if $I(\beta) \subset I(\alpha)$ and that there is an *edge* of $\mathcal{D}$ between α and β if either $\alpha < \beta$ or $\alpha > \beta$ and also $|I(\alpha)|\cdot|I(\beta)|^{-1} \in \{2, 1/2\}$. We define the distance $d(\alpha, \beta)$ between two points in $\mathcal{D}$ as the minimum number of edges in a path between α and β. We write $d(\alpha) = d(\alpha, o)$, the *level* of α. Given $\alpha \in \mathcal{D}\backslash\{o\}$, the *predecessor of*

α, α^{-1}, is the element β of level $d(\alpha)-1$ such that $\alpha > \beta$. The two *successors* of α, $\alpha \in \mathcal{D}$, α_- and α_+, are the elements β at level $d(\alpha)+1$ such that $\beta > \alpha$. As a convention, we let α_- to be the successor whose second coordinate is smaller. The map $\alpha \mapsto -\alpha$ is defined by $-o = o$, $-\alpha_\pm = \alpha_\mp$.

A function $h : \mathcal{D} \to \mathbb{R}$ is a (dyadic) *martingale* if

$$h(\alpha) = \frac{h(\alpha_-) + h(\alpha_+)}{2}$$

for all $\alpha \in \mathcal{D}$.

The *derivative* of a function $f : \mathcal{D} \to \mathbb{R}$ is defined by

$$Df(\alpha) = f(\alpha) - f(\alpha^{-1})$$

for $d(\alpha) \geq 1$, and by $Df(o) = f(o)$ at the root. For $\alpha \in \mathcal{D}$, let r_α be the function defined by

$$r_\alpha(\alpha_+) = 1, \; r_\alpha(\alpha_-) = -1,$$

r_α being zero otherwise. Then, h is a martingale if and only if for some choice of real numbers $a(\alpha)$, a_*,

$$Dh = a_*\delta_o + \sum_{\alpha \in \mathcal{D}} a(\alpha) r_\alpha$$

where δ_o is the unit mass at o. The *primitive* of $f : \mathcal{D} \to \mathbb{R}$, $\mathcal{I}f$, is

$$\mathcal{I}f(\alpha) = \sum_{\beta=o}^{\alpha} f(\beta)$$

It is a direct verification that $D\mathcal{I} = \mathcal{I}D = \mathrm{Id}$. In particular, a martingale can be reconstructed from its derivative.

The martingale spaces we are interested in are defined in terms of the size of the derivatives of the functions. Let $1 < p < \infty$ and $a \in \mathbb{R}$. The *Dirichlet space* $\mathcal{B}_p(a)$ is the space of those dyadic martingales h such that

$$\|h\|^p_{\mathcal{B}_p(a)} = \sum_{\alpha \in \mathcal{D}} |Dh(\alpha)|^p |I(\alpha)|^a < \infty.$$

(The choice of notation is to distinguish between these spaces and the spaces $B_p(a)$ in [ARS] which have a similar norm but for which the elements h are not required to be martingales.) In the remaining part of this section, we find reproducing kernels and duals of these spaces. We let $\mathcal{B}_p = \mathcal{B}_p(0)$.

Reproducing kernels. $\mathcal{B}_2$ is a Hilbert space with inner product

$$\langle f, g \rangle_{\mathcal{B}_2} = \sum_{\alpha \in \mathcal{D}} Df(\alpha)\overline{Dg(\alpha)}.$$

An orthonormal basis for $\mathcal{B}_2$ is provided by the functions $\mathcal{I}r_\beta$, $\beta \in \mathcal{D}$, where

$$r_\beta(\beta_-) = 2^{1/2}, \; r_\beta(\beta_+) = -2^{1/2}$$

$r_\beta(\gamma) = 0$, otherwise, and by the constant function $\delta_o = 1$. The reproducing kernel at α is a function $K_\alpha \in \mathcal{B}_2$ such that

$$\langle f, K_\alpha \rangle_{\mathcal{B}_2} = f(\alpha)$$

whenever $f \in \mathcal{B}_2$.

Proposition 1. *$\mathcal{B}_2$ has reproducing kernel K_α at all $\alpha \in \mathcal{D}$. The derivative of K_α is given by*

$$DK_\alpha(\beta) = \begin{cases} 1 & \text{if } \beta = o \\ 2^{-1} & \text{if } o < \beta \leq \alpha \\ -2^{-1} & \text{if } o < -\beta \leq \alpha \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

In particular, if $z \in \mathcal{D}$,

$$DK_z - DK_{z^{-1}} = \pm 1/2 r_{z^{-1}} \tag{2.2}$$

where the sign is $+$ or $-$ depending on $z = z_+^{-1}$ or $z = z_-^{-1}$.

Proof. The expression for DK_α is obtained by direct calculation starting with the fact that the reproducing kernel, K_α, can be built using (any) orthonormal basis.

$$K_\alpha(\cdot) = 1 + \sum_{\gamma \in \mathcal{D}} \mathcal{I}r_\gamma(\alpha)\mathcal{I}r_\gamma(\cdot).$$

□

Dual spaces. Let $a \in \mathbb{R}$ and $1 < p < \infty$. By Hölder's inequality, to each function $h \in \mathcal{B}_{p'}((1-p')a)$ we can associate a functional $\Lambda_h \in (\mathcal{B}_p(a))^*$, the dual of $\mathcal{B}_p(a)$, by

$$\Lambda_h(g) = \langle g, h \rangle_{\mathcal{B}_2} = \sum_{\alpha \in \mathcal{D}} Dg(\alpha)\overline{Dh(\alpha)}.$$

In fact, all elements of the dual of $\mathcal{B}_p(a)$ can be obtained this way.

Proposition 2. *The map $h \mapsto \Lambda_h$ is a bijection of $\mathcal{B}_{p'}((1-p')a)$ onto $(\mathcal{B}_p(a))^*$.*

Proof. Let $D\mathcal{B}_p(a) = \{Dh : h \in \mathcal{B}_p(a)\} \subseteq L^p(a)$, normed with the $L^p(a)$-norm. Consider the orthogonal projection Π of $l^2(\mathcal{D})$ onto $D\mathcal{B}_2$. Π can be computed by its action on δ_α, $\alpha \in \mathcal{D} - \{o\}$:

$$\Pi\delta_\alpha = \pm 1/2 r_{\alpha^{-1}}$$

the sign being $+$ (resp., $-$) if $\alpha = \alpha_+^{-1}$ (resp., $\alpha = \alpha_-^{-1}$), and $\Pi\delta_o = \delta_o$. One checks directly that $\Pi^2 = \Pi$ and Π is self-adjoint. One also easily verifies that Π is also a contraction of $L^{p'}((1-p')a)$ into $D\mathcal{B}_{p'}((1-p')a)$.

Now, let Λ be a continuous functional on $\mathcal{B}_p(a)$, which is isometric, through D, to a subspace of $L^p(a)$. By Hahn-Banach, it has a continuous extension to $L^p(a)$, hence,

$$\Lambda h = \langle Dh, f \rangle_{l^2}$$

for some $f \in L^{p'}((1-p')a)$. By the boundedness of Π, $\Pi f \in D\mathcal{B}_{p'}((1-p')a)$. Now, if $h \in \mathcal{B}^p(a)$ and $\alpha \in \mathcal{D}$,

$$\begin{aligned} \Lambda h &= \langle D\Lambda h, f \rangle_{l^2} = \langle \Pi D\Lambda h, f \rangle_{l^2} \\ &= \langle D\Lambda h, \Pi f \rangle_{l^2} = \Lambda_{\mathcal{I}\Pi f} h \end{aligned}$$

and $\mathcal{I}\Pi f \in \mathcal{B}^{p'}((1-p')a)$. □

In the continuous case, duality issues are more complicated and the analogue of the duality result above only holds when a belongs to a certain range of exponents. See [Bek], [L1], and [ARS].

3. Function spaces on dyadic trees

The function spaces on dyadic trees which we just introduced and as well as other similar spaces arise in a number of different contexts. We came to them as a model for certain function theoretic questions on the unit disk. It was our hope of course that the model would reflect basic properties of the function theory on the disk and yet be easier to work with. Here we wanted to note that if one is trying to understand harmonic or holomorphic functions on, say, the disk by using a tree model then there are several different approaches that are natural.

We regard a function $f_{\mathcal{D}}$ on the dyadic tree $\mathcal{D}$ as a model for the function $f_{\mathbb{D}}$ on the disk $\mathbb{D}$ by thinking of $f_{\mathcal{D}}(\alpha)$ where $\alpha = (n, j) \in \mathcal{D}$ as representing the values of $f_{\mathbb{D}}$ near the point $z_\alpha = (1 - 2^{-n}) \exp(2\pi i j/2^n)$. The operators D and $\mathcal{I}$ model differentiation and integration, etc. Such models give good representation of aspects of global behavior related to hyperbolic modulus of continuity estimates. For instance, (5.1) models (1.1). However, in such an approach it is not clear how to model the local cancellation properties of harmonic or holomorphic functions.

Of course one possibility is to ignore the issue. This corresponds to working not with the spaces we just described — martingales with derivatives in l^p — but rather working with the space of all functions on the tree with derivatives in l^p. This is the primary viewpoint taken in [ARS] and it served well there. Alternatively one can try to model the local mean value property of harmonic functions. One could work with functions on a tree which have local cancellation — for instance functions $f_{\mathcal{D}}$ with a local mean value property — the value of $f_{\mathcal{D}}$ at a point α is the average of the values of $f_{\mathcal{D}}$ at the nearest neighbors of α. These are the so-called harmonic functions on a tree. They were used in the final section of [ARS] as part of an attempt to understand why some of the results there failed in certain parameter ranges. Alternatively one can model the local mean value property by restricting attention to functions on the tree with the property that the value of $f_{\mathcal{D}}$ at α is the average of the values of $f_{\mathcal{D}}$ at the two successors of α; i.e., $f_{\mathcal{D}}$ is a dyadic martingale of the sort used in this paper. In fact there is a rich relationship between the harmonic analysis associated to martingales and the harmonic analysis associated to harmonic functions on trees. Much of that relationship is developed in [KPT] and [T].

One of the goals of this paper was to examine the extent to which the analysis of Carleson measures and interpolating sequences for space of all functions on the tree with derivatives in l^p and also for the space of harmonic functions on the tree space with derivatives in l^p, both carried out in [ARS], could be extended to the space of martingales with derivatives in l^p. The overall hope is to find more unity if the seemingly disparate answers to seemingly similar questions concerning characterization of Carleson measures. With that in mind it was satisfying to find that the spaces studied here have the same Carleson measures as those which were the main focus in [ARS].

We came to function theory on $\mathcal{D}$ as a way to model issues from analytic function theory. However function theory on trees is a subject with its own rich life. Issues of classical harmonic analysis in the context of function spaces on trees are developed in, among other places, [T], and [KPT]. However the methods of those papers don't apply well to the spaces $\mathcal{B}_p(a)$ with $a < 1$, which are our primary interest here. The approach to Carleson measures for function spaces on trees developed by Evans, Harris and Pick in [EHP] gives an alternative approach and an alternative resolution (in terms of capacities) to some of the results in [ARS]. We don't know if their approach can be adapted to the spaces $\mathcal{B}_p(a)$.

4. Carleson measures

4.1. Characterization of Carleson measures. Let $a \in \mathbb{R}$, $1 < p < \infty$. We say that a positive function μ on $\mathcal{D}$ is a *Carleson measure for* $\mathcal{B}_p(a)$ if $\mathcal{B}_p(a) \subset L^p(\mu)$, that is, if the following discrete Sobolev-Poincaré inequality holds for all $h \in \mathcal{B}_p(a)$,

$$\sum_{\alpha\in\mathcal{D}} |h(\alpha)|^p \mu(\alpha) \le C(\mu) \sum_{\alpha\in\mathcal{D}} |Dh(\alpha)|^p |I(\alpha)|^a. \tag{4.1}$$

In this section we give a geometric characterization of the Carleson measures for $\mathcal{B}_p(a)$, $0 \le a < 1$, and we discuss its relation with two different geometric conditions, corresponding to the characterization theorems in [KS2] and [Car]. With little effort, but at the expenses of brevity, these comparisons could be extended to cover the continuous case.

For $z \in \mathcal{D}$, let $S(z) = \{w \in \mathcal{D} : w \ge z\}$ be the *Carleson box* with vertex z and let $M_z = \mu(S(z))$.

Theorem 1. *Let $0 \le a < 1$ and $1 < p < \infty$. Then, a measure μ on $\mathcal{D}$ is a Carleson measure for $\mathcal{B}_p(a)$ if and only if*

$$\sum_{w\ge z} M_w^{p'} |I(w)|^{a(1-p')} \le C(\mu) M_z. \tag{4.2}$$

In the proof of the theorem we need the following definition. Let $u \in \mathcal{D}$. A *geodesics* starting at u is a sequences $\xi = \{z_0, z_1, \dots, z_n, \dots\} \subset \mathcal{D}$ such that $z_0 = u$, $z_n > z_{n-1}$, $d(z_n, z_{n-1}) = 1$. It is well-known that the map $\xi \mapsto \cap_{n=0}^{\infty} I(z_n)$ is a map of the set of geodesics starting at u onto $I(u)$, which is one-to-one, but for the set of the dyadic rationals in $I(u)$.

Proof. Suppose that (4.2) holds. In [ARS] it is proved then, in greater generality, that $\mathcal{I}$ is bounded from $L^p(a)$ to $L^p(\mu)$. In particular, this shows that μ is Carleson.

Let μ be a Carleson measure. Testing (4.1) on $h \equiv 1$, we see that μ must be bounded. Also, μ is Carleson if and only if the identity Id is bounded from $\mathcal{B}_p(a)$ to $L^p(\mu)$. By duality, this is equivalent to the boundedness of

$$\Theta : L^{p'}(\mu) \to \mathcal{B}_{p'}((1-p')a)$$

where Θ is the (formal) adjoint of Id. Explicitly,

$$\Theta h(z) = \langle \Theta h, K_z \rangle_{\mathcal{B}_2} = \langle h, K_z \rangle_{L^2(\mu)} = \sum_{w\in\mathcal{D}} h(w) K_z(w) \mu(w).$$

Thus,

$$\sum_{z\in\mathcal{D}} \left| \sum_{w\in\mathcal{D}} h(w)(K_z(w) - K_{z^{-1}}(w))\mu(w) \right|^{p'} |I(z)|^{(1-p')a} \le C \sum_{w\in\mathcal{D}} |h(w)|^{p'} \mu(w).$$

As was noted in (2.2) the expression $K_z(w) - K_{z^{-1}}(w)$ has values $1/2$, if $w \geq z$, and $-1/2$, if $w \geq -z$. Testing this relation on functions of the form $h = \chi_{S(u)}$, $u \in \mathcal{D}$, we obtain the inequalities

$$M_u^{p'} |I(u)|^{(1-p')a} + \sum_{z>u} |M_z - M_{-z}|^{p'} |I(z)|^{(1-p')a} \leq CM_u. \tag{4.3}$$

To finish the proof, then, it suffices to show that, if $0 \leq a < 1$, then

(4.4)
$$\sum_{w \geq u} M_w^{p'} |I(w)|^{a(1-p')} \leq CM_u^{p'} |I(u)|^{(1-p')a} + C \sum_{z>u} |M_z - M_{-z}|^{p'} |I(z)|^{(1-p')a}.$$

Observe that, for all $a \in \mathbb{R}$ the right-hand side of (4.4) is controlled by the left-hand side.

Fix $\varepsilon > 0$ to be chosen later. On the geodesics $\xi = \{z_n : n \geq 0\}$ starting at u, define stopping times $t_0 = t_0(\xi) = 0$,

$$\begin{aligned} t_k &= t_k(\xi) \\ &= \inf \left\{ t > t_{k-1} : M_{z_t}^{p'} \left| I(z_t) \right|^{a(1-p')} > ((1+\varepsilon)/2)^{p'} M_{z_t^{-1}}^{p'} \left| I(z_t^{-1}) \right|^{a(1-p')} \right\} \\ &= \inf \left\{ t > t_{k-1} : M_{z_t}^{p'} > 2^{a(1-p')} ((1+\varepsilon)/2)^{p'} M_{z_t^{-1}}^{p'} \right\}. \end{aligned}$$

The inf might be infinite. Let b be a *k-stopping point* (that is, $b = z_{t_k}$ on some geodesic starting at u). Let $\mathcal{B}(b)$ be the set of the $(k+1)$-stopping points b' such that $b' > b$. Let $SP(u)$ be the set of the stopping points.

Claim.
$$\sum_{w \geq u} M_w^{p'} |I(w)|^{a(1-p')} \leq C \sum_{v \geq u, v \in SP(u)} M_v^{p'} |I(v)|^{a(1-p')}.$$

Let b be a k-stopping point. The claim is proved if we show that
$$\sum_{b \leq w < \mathcal{B}(b)} M_w^{p'} |I(w)|^{a(1-p')} \leq CM_b^{p'} |I(b)|^{a(1-p')}$$
where $w < \mathcal{B}(b)$ means that $w < v$ for all $k+1$-stopping points v.

Let n be a positive integer and let c be such that $b < c < \mathcal{B}(b)$, $d(b,c) = n$. If $b < -c < \mathcal{B}(b)$, then
$$\left(M_c^{p'} + M_{-c}^{p'} \right) |I(c)|^{a(1-p')} \leq 2 ((1+\varepsilon)/2)^{p'} M_{c^{-1}}^{p'} |I(c^{-1})|^{a(1-p')}$$
If $-c$ is not between b and $\mathcal{B}(b)$, then
$$M_c^{p'} |I(c)|^{a(1-p')} \leq ((1+\varepsilon)/2)^{p'} M_{c^{-1}}^{p'} |I(c^{-1})|^{a(1-p')}$$
Choose ε such that $2^{1-p'} (1+\varepsilon)^{p'} = 1 - \delta < 1$. Summing over all such c's and iterating,

$$\begin{aligned} \sum_{b<c<\mathcal{B}(b), d(b,c)=n} M_c^{p'} |I(c)|^{a(1-p')} &\leq (1-\delta) \sum_{b<c<\mathcal{B}(b), d(b,c)=n-1} M_c^{p'} |I(c)|^{a(1-p')} \\ &\leq \dots \\ &\leq (1-\delta)^n M_b^{p'} |I(b)|^{a(1-p')}. \end{aligned}$$

Summing over n, then over b, we obtain the claim.

Let now $b > u$ be a stopping point. By the definition of M_b and of the stopping times,

$$M_{-b} \le M_{b^{-1}} - M_b \le \left(\frac{2}{1+\varepsilon} 2^{a/p} - 1 \right) M_b$$

hence,

$$M_b - M_{-b} \ge 2 \left(1 - \frac{2^{a/p}}{1+\varepsilon} \right) M_b.$$

The right-hand side can be made greater than ηM_b, for some $\eta > 0$, if we can choose $\varepsilon > 0$ such that

$$2^{a/p} < 1 + \varepsilon < 2^{1/p}.$$

This is possible, since $a < 1$. Summing over all stopping points, we obtain

$$\sum_{b \ge u, b \in SP(u)} M_b^{p'} |I(b)|^{a(1-p')} \le C M_u^{p'} |I(u)|^{a(1-p')} + C \sum_{z > u} |M_z - M_{-z}|^{p'} |I(z)|^{a(1-p')}.$$

By the claim, we deduce (4.4). □

4.2. The equivalence of different conditions. When $a = 1$, the condition (4.2) in Theorem 1 is still sufficient, but no longer necessary. In fact, a measure μ is Carleson for $\mathcal{B}_2(1)$ if and only if the following Carleson type condition holds:

$$M_z \le C |I(z)|. \tag{4.5}$$

See [NT] for a short proof. In fact, as in the continuous case, a condition of Kerman-Sawyer type encompasses both the $a < 1$ and the $a = 1$ case. We give a direct proof of this in Theorems 2 and 3. In the theory of Carleson measures on the Hardy space, and the theory of the associated function space BMO, it is ubiquitous and often crucial that certain estimates appear to be self-improving. That is, the estimates imply further estimates that are, on casual inspection, strictly stronger. Such phenomena are less understood for the Carleson measures on the Dirichlet space. Theorem 2 is an example of such a phenomenon.

Theorem 2. *If $1 < p < \infty$ and $0 \le a < 1$, then the following are equivalent:*

$$\sum_{w \ge z} M_w^{p'} |I(w)|^{a(1-p')} \le C M_z; \tag{ARS}$$

$$\int_{I(z)} \sup_{x \in I(w) \subseteq I(z)} \left(M_w^{p'} |I(w)|^{a(1-p')-1} \right) dx \le C M_z. \tag{KS}$$

Theorem 3. *If $1 < p < \infty$, the following are equivalent:*

$$\int_{I(z)} \sup_{x \in I(w) \subseteq I(z)} \left(M_w^{p'} |I(w)|^{-p'} \right) dx \le C M_z; \tag{KS}$$

$$M_z \le C |I(z)| \tag{Car}$$

Proof of Theorem 2. The implication (ARS) $\Longrightarrow$ (KS) is elementary and it holds for all a's. For $z \in \mathcal{D}$, let $S_n(z) = \{w \in s(z) : d(z,w) = n\}$.

$$\begin{aligned}\sum_{w\geq z} M_w^{p'}|I(w)|^{a(1-p')} &= \sum_{n=0}^{\infty}\sum_{S_n(z)} M_w^{p'}|I(w)|^{a(1-p')}\\ &= \sum_{n=0}^{\infty}\sum_{w\in S_n(z)}\int_{I(w)} M_w^{p'}|I(w)|^{a(1-p')-1}dx\\ &= \int_{I(z)}\sum_{\substack{z\leq w\\ x\in I(w)}} M_w^{p'}|I(w)|^{a(1-p')-1}dx\\ &\geq \int_{I(z)}\sup_{x\in I(w)\subseteq I(z)}\left(M_w^{p'}|I(w)|^{a(1-p')-1}\right)dx.\end{aligned}$$

The converse, (KS) $\Longrightarrow$ (ARS), says that the inclusion $l^1 \subset l^\infty$ in the last chain of inequalities can somewhat be reversed. Let $z \in \mathcal{D}$ and define stopping times on the geodesics $\xi = \{z_n : n \geq 0\}$ starting at z: $t_0 = t_0(\xi) = 0$,

$$t_k = t_k(\xi) = \inf\left\{t > t_{k-1} : M_{z_t}^{p'}|I(z_t)|^{a(1-p')-1} > M_{z_{t_{k-1}}}^{p'}|I(z_{t_{k-1}})|^{a(1-p')-1}\right\}.$$

Let $SP(z)$ be the set of the stopping points on the geodesics starting at z.

Claim 1. *Let $\gamma = a(p'-1)+1$. Then*

$$\sum_{w\geq z} M_w^{p'}|I(w)|^{1-\gamma} \leq C \sum_{\substack{w\geq z\\ w\in SP(z)}} M_w^{p'}|I(w)|^{1-\gamma}.$$

Proof of Claim 1. Let $k \geq 0$, c be a k-stopping point and $\Gamma(n) = \{w \in S(c) : d(c,w) = n,\ w < SP(k+1)\}$. The last requirement means that $w < \xi$, whenever ξ is a $k+1$-stopping point. Then, if $b \in \Gamma(n)$,

$$\begin{aligned}M_w^{p'} &\leq \left(\frac{|I(w)|}{|I(c)|}\right)^{\gamma} M_c^{p'}\\ &= 2^{-n\gamma}M_c^{p'}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{w\in\Gamma(n)} M_w^{p'}|I(w)|^{1-\gamma} &= |I(c)|^{1-\gamma}2^{n(\gamma-1)}\sum_{w\in\Gamma(n)} M_w^{p'}\\ &\leq |I(c)|^{1-\gamma}2^{n(\gamma-1)}M_c^{p'-1}2^{-n\gamma\frac{p'-1}{p'}}\sum_{w\in\Gamma(n)} M_w\\ &\leq |I(c)|^{1-\gamma}M_c^{p'}2^{n\left(\gamma-1-\gamma\frac{p'-1}{p'}\right)}\\ &\leq |I(c)|^{1-\gamma}M_c^{p'}2^{n\left(\frac{\gamma}{p'}-1\right)}\end{aligned}$$

Now, since $a < 1$, $\frac{\gamma}{p'} - 1 < 0$. Thus, summing over $n \geq 1$, we have that

$$\sum_{c\leq w<SP(k+1)} M_w^{p'}|I(w)|^{1-\gamma} \leq C|I(c)|^{1-\gamma}M_c^{p'}.$$

Summing over all stopping points c, we obtain the desired inequality. $\square$

By Claim 1, the theorem is proved if we show the inequality

$$\sum_{w\geq z} M_w^{p'}|I(w)|^{-a(p'-1)} \leq C\int_{I(z)} \sup_{x\in I(w)\subseteq I(z)} \left(M_w^{p'}|I(w)|^{-a(p'-1)-1}\right)dx. \tag{4.6}$$

Recall that the map $\xi = \{z_0 = z, z_1, \dots, z_n, \dots\} \mapsto x(\xi) = \cap_{n=0}^{\infty} I(z_n)$ maps the set of geodesics starting at z onto $I(z)$ and it is one-to-one, but for the dyadic rationals $x \in I(z)$. Since the latter has measure zero, from the viewpoint of measure theory we can identify geodesics and points in $I(z)$. In particular, the stopping times t_k can be thought of as functions on $I(z)$. We write $x > w$ if $x = x(\xi)$ and w is a point on the geodesic ξ. The proof of (4.6) is based on the following:

Claim 2. *Let c be a k-stopping point and let*

$$E(c) = \{x \in I(c) : t_{k+1}(x) = +\infty\}.$$

Then, there exists $\varepsilon \in (0, 1)$ such that

$$|E(c)| \geq \varepsilon |I(c)|.$$

Proof that Claim 2 implies (4.6). Any two sets in $\{E(c) : c \in SP(z)\}$ have intersection with zero measure, and $\sup_{z\leq w<x}\left(M_w^{p'}|I(w)|^{-\gamma}\right)$ is achieved for $w = c$ on $E(c)$. Hence

$$\begin{aligned}\int_{I(z)} \sup_{z\leq w<x}\left(M_w^{p'}/|I(w)|^{\gamma}\right)dx &\geq \sum_{c\in SP(z)} \int_{E(c)} M_c^{p'}|I(c)|^{-\gamma} \\ &= \sum_{c\in SP(z)} |E(c)| M_c^{p'}|I(c)|^{-\gamma} \\ &\geq \varepsilon \sum_{c\in SP(z)} M_c^{p'}|I(c)|^{1-\gamma}\end{aligned}$$

as desired. □

Proof of Claim 2. Let $SP_k(z)$ be the set of the k-stopping points and let $c \in SP_k(z)$. Then,

$$I(c) - E(c) = \cup_{b\in SP_{k+1}(z)\cap S(c)} I(b)$$

the union being disjoint. For any such b, by the definition of the stopping times,

$$M_b^{p'}|I(b)|^{-\gamma} > M_c^{p'}|I(c)|^{-\gamma}.$$

By a scaling argument, we can assume $|I(c)| = M_c = 1$. Let $q = p'/\gamma > 1$, since $a < 1$, and let $\beta \in SP_{k+1}(z) \cap S(c)$ be such that $M_\beta = \max\{M_b : SP_{k+1}(z) \cap S(c)\}$.

$$\begin{aligned}1 &\geq M_\beta^q + \sum_{\substack{b\neq\beta \\ b\in SP_{k+1}(z)\cap S(c)}} M_b^q \\ &\geq M_\beta^q + \sum_{\substack{n\neq\beta \\ b\in SP_{k+1}(z)}} |I(b)|.\end{aligned}$$

We consider three cases:

(1) If $\sum_{n\neq\beta,\ b\in SP_{k+1}(z)} |I(b)| \leq 1/4$, since, *a priori*, $|I(\beta)| \leq 1/2$,

$$|I(c) - E(c)| \leq 3/4|I(c)|.$$

(2) Fix $A > 0$ such that $3/4 < A^q < 1$ (and hence $A < 1$). If

$$\sum_{b\neq\beta,\ b\in SP_{k+1}(z)\cap S(c)} |I(b)| \geq 1/4, \text{ and}$$

$$M_\beta > A \sum_{b\in SP_{k+1}(z)\cap S(c)} M_b$$

then

$$\begin{aligned}\sum_{b\in SP_{k+1}(z)\cap S(c)} |I(b)| &\leq \sum_{b\in SP_{k+1}(z)\cap S(c)} M_b^q \\ &\leq \left(\sum_{b\in SP_{k+1}(z)\cap S(c)} M_b\right)^q \\ &\leq A^{-q} M_\beta^q \\ &\leq A^{-q}\left(1 - \sum_{\substack{b\neq\beta \\ b\in SP_{k+1}(z)\cap S(c)}} |I(b)|\right) \\ &\leq \frac{3}{4}A^{-q}.\end{aligned}$$

Hence,

$$|I(c) - E(c)| \leq \frac{3}{4}A^{-q}|I(c)|.$$

(3) If $M_\beta \leq A\sum_{b\in SP_{k+1}(z)\cap S(c)} M_b$, then

$$\begin{aligned}\sum_{b\in SP_{k+1}(z)\cap S(c)} |I(b)| &\leq \sum_{b\in SP_{k+1}(z)\cap S(c)} M_b^q \\ &\leq (1-\varepsilon)\left(\sum_{b\in SP_{k+1}(z)\cap S(c)} M_b\right)^q \\ &\leq 1-\varepsilon\end{aligned}$$

where $\varepsilon > 0$. The gain in Hölder's inequality is due to the assumption and to the following lemma, whose easy proof is left to the reader. □

Lemma 1. *Let $q > 1$ and $0 < K < 1$. There exists ε, $0 < \varepsilon < 1$, such that for all choices of $0 \leq x_n \leq K$, if $\sum_{n\geq 0} x_n = 1$, then*

$$\sum_{n\geq 0} x_n^q \leq 1-\varepsilon.$$

□

Proof of Theorem 3. To show that (KS) implies (Car), it suffices to minorize the supremum in (KS) with $M_z^{p'}|I(z)|^{-p'}$.

Suppose that (Car) holds and let $\varphi(w) = M_w|I(w)|^{-1}$. Observe that φ is a *supermartingale*:

$$\varphi(w) \geq \frac{\varphi(w_-) + \varphi(w_+)}{2}.$$

Let

$$\mathcal{M}_z\varphi(x) = \sup_{z \leq w < x} \varphi(w).$$

The theorem is proved if we show

$$\frac{1}{|I(z)|} \int_{I(z)} (\mathcal{M}_z\varphi(x))^{p'} \, dx \leq C(p)\varphi(z)\|\varphi\|_{l^\infty(\mathcal{D})}^{p'-1}. \tag{4.7}$$

To prove (4.7), we use a simple interpolation argument. Observe that $\mathcal{M}_z\varphi(x) \leq \|\varphi\|_{l^\infty(\mathcal{D})}$ whenever $x \in I(z)$.

For fixed $\lambda > 0$, $\{\mathcal{M}_z\varphi(x) > \lambda\}$ is the disjoint union of intervals $I(w)$, such that $w \geq z$ and $M_w|I(w)|^{-1} > \lambda$. Let $J(\lambda)$ be the set of such w's. Then,

$$\begin{aligned}
\lambda|I(z)|^{-1}|\{x : \mathcal{M}_z\varphi(x) > \lambda\}| &= \lambda|I(z)|^{-1} \sum_{w \in J(\lambda)} |I(w)| \\
&\leq \lambda|I(z)|^{-1} \sum_{w \in J(\lambda)} \frac{M_w}{\lambda} \\
&\leq |I(z)|^{-1} M_z = \varphi(z).
\end{aligned}$$

Then,

$$\begin{aligned}
|I(z)|^{-1} \int_{I(z)} \mathcal{M}_z\varphi(x)^{p'} dx &= p' \int_0^\infty |I(z)|^{-1}|\{x : \mathcal{M}_z\varphi(x) > \lambda\}|\lambda^{p'-1} d\lambda \\
&= p' \int_0^{\|\mathcal{M}_z\varphi\|_{l^\infty(\mathcal{D})}} |I(z)|^{-1}|\{x : \mathcal{M}_z\varphi(x) > \lambda\}|\lambda^{p'-1} d\lambda \\
&\leq \frac{p'}{p'-1}\varphi(z)\|\mathcal{M}_z\varphi\|_{l^\infty(\mathcal{D})}^{p'-1} \\
&= \frac{p'}{p'-1}\varphi(z)\|\varphi\|_{l^\infty(\mathcal{D})}^{p'-1}.
\end{aligned}$$

□

5. An interpolation theorem

In this section, we characterize the interpolating sequences for the martingales in $\mathcal{B}_p$, $1 < p < \infty$. The results can be easily extended to $\mathcal{B}_p(a)$, $0 \leq a < 1$.

The definition of interpolating sequences below is justified by the following discrete analogue of (1.1).

Lemma 2. *For all functions $f : \mathcal{D} \to \mathbb{R}$,*

$$|f(z) - f(w)| \leq d(z, w)^{1/p'} \|Df\|_{l^p}. \tag{5.1}$$

Proof. It is a simple consequence of Hölder's inequality. □

We say that a sequence $Z \subset \mathcal{D}$ is *interpolating* for $\mathcal{B}_p$ if the map $T : \mathcal{B}_p \to l^p(Z)$,

$$T : h \mapsto \left\{ d(z)^{-\frac{1}{p'}} h(z) : z \in Z \right\} \tag{5.2}$$

is into and onto.

Theorem 4. *A sequence Z is interpolating for $\mathcal{B}_p$ if and only if:*

(i) *The measure*

$$\mu_Z = \sum_{z \in Z} d(z)^{1-p} \delta_z$$

is Carleson for $\mathcal{B}_p$.

(ii) *There is a constant $A > 0$ such that, for all $z, w \in Z$, $z \neq w$,*

$$Ad(z,w) \geq d(z). \tag{5.3}$$

Proof. The proof is similar to that of Theorem 26 in [ARS], and we give only a detailed outline of it. In [ARS] it is proved that (i) and (ii) are necessary and sufficient to solve the interpolation problem with functions having derivatives in l^p, such functions not necessarily being dyadic martingales.

Hence, it suffices to show that if Z satisfies (i) and (ii), then Z is interpolating. As in [ARS], we show that there exists a family of functions $\{h_z : z \in Z\}$ in $\mathcal{B}_p$ such that:

(a) $h_z(w) = \delta_z(w)$, for $z, w \in Z$.

(b) $\|h_z\|_{\mathcal{B}_p} \leq Cd(z)^{-1/p'}$.

(c) If z and w are distinct points in Z, then the supports of h_z and h_w are disjoint.

Given (a)–(c), it can be easily verified that an explicit solution to the equation $Th = g \in l^p(z)$ is given by the linear extension operator

$$h(x) = \mathcal{E}g(x) = \sum_{z \in Z} g(z) d(z)^{1/p'} h_z(x).$$

The rest of the proof consists in the construction of the functions h_z.

First, we endow Z with a suitable tree-like structure. Assume, for convenience, that $o \in Z$. For any $X \subset \mathcal{D}$ let $\widetilde{X}$ be the smallest subtree of $\mathcal{D}$ containing X; i.e., $\widetilde{X} = \{w \in \mathcal{D} \mid w \leq x, \text{for some } x \in X\}$. We now proceed inductively. Let $z_0 = o$, $\widetilde{Z}_0 = \{o\}$. Now suppose that Z_{n-1} and $\widetilde{Z}_{n-1}$ have already been formed. Pick $z_n \in Z_{n-1}^c$, the complement of Z_{n-1}, such that

$$d(o, z_n) = \min\{d(o,z) | z \in Z_{n-1}^c\}. \tag{5.4}$$

One easily verifies that $Z_n \subseteq Z_{n+1}$, $\bigcup_{n \geq 0} Z_n = Z$, and $\bigcup_{n \geq 0} \widetilde{Z}_n = \widetilde{Z}$.

Let $\xi_n = \max\{x \in \widetilde{Z}_{n-1} |\ x \leq z_n\}$. The map $z_n \mapsto \xi_n = \gamma(z_n)$ defines a function $\gamma \colon Z \to \mathcal{D}$. We call $\gamma(z)$ the *landing point* of z on $\widetilde{Z}$.

The following lemma is proved in [ARS], but we repeat its proof for the convenience of the reader.

Lemma 3. *Let A be the constant in* (5.3). *Then*

$$d(z, \gamma(z)) \geq (2A)^{-1} d(z). \tag{5.5}$$

Proof. (5.5) is obvious for $z = o$. Otherwise, $z = z_n$ for some n and $\gamma(z) \in [0, z_j]$ for some $j < n$. By the construction of Z_n, $d(z_n) \geq d(z_j)$. Hence,

$$\begin{aligned} 2d(z_n, \gamma(z_n)) &\geq d(z_n, \gamma(z_n)) + d(\gamma(z_n), z_j) \\ &= d(z_n, z_j) \geq A^{-1} d(z_n). \end{aligned}$$

□

In the remaining part of the proof, we assume that, if $z \in Z$, then $d(z, \gamma(z)) \geq 4$. We can make this assumption upon removing a finite number of points from Z, and this can be shown not to affect the interpolating properties of Z. Let $I = [x, y]$ be an interval in $\mathcal{D}$, $x < y$. That is, $I = \{x = x_1, x_2, \dots, x_n = y\} \subset \mathcal{D}$, $x_1 < \dots < x_n$ and $d(x_j, x_{j+1}) = 1$. Define $\psi_I \in \mathcal{B}_p$,

$$D\psi_I = \sum_{\xi = x^{-1}}^{y^{-1}} \varepsilon(\xi) r_\xi$$

where the coefficients $\varepsilon(\xi) \in \{1, -1\}$ are chosen in such a way that $D\psi_I|_I \equiv 1$. The function h_z will be defined recursively.

$$Dh_z^0 = r(z, \alpha(z))^{-1} D\psi_{[z, \alpha(z)]}.$$

The support of $D\psi_I$ is $I \cup (-I)$, where $-I = \{-x : x \in I\}$.

We will need to divide the intervals $I = [\gamma(z), z]$. For $z \in Z$, consider points $\alpha(z)$ and $\beta(z)$ in I, where $\gamma(z) < \beta(z) < \alpha(z) < w$, $\beta(z) = \alpha(z)^{-1}$ and $\alpha(z)$ is minimum in I with the property that

$$\frac{d(\alpha(z), z) + 1}{d(\gamma(z), z)} \leq 1/2. \tag{5.6}$$

That is, $[\gamma(z), \beta(z)]$ and $[\alpha(z), z]$ have comparable length.

Now, fix $z \in Z$. The function h_z will be defined inductively. Let $S_0 = [z, \alpha(z)] \cup (-[z, \alpha(z)])$. Define

$$Dh_z = (d(\alpha(z), z) + 1)^{-1} D\psi_{[\alpha(z), z]}.$$

on S_0 and $h_z = 0$ on $[o, \beta(z)]$.

Now, let w_1 be a point in Z such that $\gamma(w_1) \in S_0$. Let $\delta(w_0)$ be the minimum element in $[\gamma(w_1), w_1] - S_0$. Then, $d(\gamma(w_1), \delta(w_1)) \leq 2$. On the set $[\delta(w_1), \beta(w_1)] \cup (-[\delta(w_1), \beta(w_1)])$, define

$$Dh_z = A_1 D\psi_{[\delta(w_1), \beta(w_1)]}$$

where A_1 is chosen in such a way that

$$0 = h_z((\delta(w_1))^{-1}) + A_1(d(\delta(w_1), \beta(w_1)) + 1).$$

Let $S_1 = S_0 \cup [\delta(w_1), \beta(w_1)] \cup (-[\delta(w_1), \beta(w_1)])$. We can proceed inductively. Suppose that Dh_z has been defined on S_{n-1}. Let $w_n \in Z$ be such that $\gamma(w_n) \in S_{n-1}$. Let $\delta(w_n)$ be minimum in $[\gamma(w_n), w_n] - S_{n-1}$ and let

$$S_n = S_{n-1} \cup [\delta(w_n), \beta(w_n)] \cup (-[\delta(w_n), \beta(w_n)]).$$

It can be proved, inductively, that $d(\gamma(w_n), \delta(w_n)) \leq 2$. On the set $[\delta(w_n), \beta(w_n)] \cup (-[\delta(w_n), \beta(w_)])$, define Dh_z by

$$Dh_z = A_n D\psi_{[\delta(w_n), \beta(w_n)]}$$

where A_n is chosen in such a way that

$$0 = h_z((\delta(w_n))^{-1}) + A_n(d(\delta(w_n), \beta(w_n)) + 1).$$

The procedure could run for infinitely many w_n. Let $S_\infty = \bigcup_1^\infty S_n$ and set $Dh_z = 0$ on $\mathcal{D} - S_\infty$.

Observe that, for all n, $w_n > \beta(z)$. Also, Dh_z has been defined in such a way that $|h_z| \le 1$. Since, by Lemma 3, $d(\delta(w_n), \beta(w_n))$ is comparable with $d(w_n)$, we have $|A_n| \le Cd(w_n)$ for some universal constant C.

Now, by the way it was constructed, h_z has the properties (a) and (c). By the estimate on A_n, we have that

$$\|h_z\|_{\mathcal{B}_p}^p \le C \left(d(z)^{1-p} + \sum_{w \in Z,\ w \ge \beta(z)} d(w)^{1-p} \right).$$

To prove (b), then, it suffices to show that

$$\sum_{w \in Z,\ w \ge \beta(z)} d(w)^{1-p} \le Cd(\beta(z))^{1-p}. \tag{5.7}$$

since $d(\beta(z))$ is comparable with $d(z)$, still by Lemma 3. Now, in [ARS], Lemma 29, it is proved that (5.7) is a consequence of the fact that μ_Z is a Carleson measure. □

6. Boundedness of Hankel forms

For a function b on the dyadic tree $\mathcal{D}$ we define $\mathcal{I}^* f$ by

$$\mathcal{I}^* b(w) = \sum_{z \ge w} b(z).$$

$\mathcal{I}^*$ is the formal adjoint of the operator $\mathcal{I}$ acting on $l^2(\mathcal{D})$. That is, one checks directly that if h and k are functions on $\mathcal{D}$ with finite support then

$$\langle \mathcal{I} h, k \rangle_{l^2(\mathcal{D})} = \langle h, \mathcal{I}^* k \rangle_{l^2(\mathcal{D})}.$$

We are interested in knowing for which functions b_2 we have the following estimate for F, G with DF, $DG \in l^2(\mathcal{D})$:

$$\left| \sum F(z) G(z) b_2(z) \right| \le c_b \|DF\|_{l^2(\mathcal{D})} \|DG\|_{l^2(\mathcal{D})}. \tag{6.1}$$

(The reason for the subscript will be given shortly.) The answer will be in terms of the measure with density $|\mathcal{I}^* b_2(w)|^2$; that is the measure $\mu_{|\mathcal{I}^* b_2|^2}$ on $\mathcal{D}$ defined by $\mu_{|\mathcal{I}^* b_2|^2}(S) = \sum_{w \in S} |\mathcal{I}^* b_2(w)|^2$. We will prove the following:

Theorem 5. *Given b_2, there is a constant c_{b_2} so that* (6.1) *holds for all F, G if and only if $\mu_{|\mathcal{I}^* b_2|^2}$ is a Carleson measure for B_2; i.e., if and only if $\mu_{|\mathcal{I}^* b_2|^2}$, satisfies condition* (4.1) *with $p = 2$, $a = 0$.*

Before going to the proof we give some background.

The question of when (6.1) holds can be viewed as a discrete model problem for the question of boundedness of Hankel forms on the Dirichlet space. Recall that a Hankel form on the Hardy space, $B_2(1)$, is a bilinear form defined on holomorphic functions on the disk. The form is determined by a *symbol function,* b, which we may take to be holomorphic. In that case the form is given by

$$H_b(F,G) = \int_{\mathbb{D}} F(z)G(z)\overline{b'(z)}dxdy$$

and to say that the form is bounded is to say that there is a c_b so that

$$\left| \int_{\mathbb{D}} F(z)G(z)\overline{b'(z)}dxdy \right| \le c_b \left\| F \right\|_{B_2(1)} \left\| G \right\|_{B_2(1)}.$$

(This is the classical Hankel form with symbol b. The derivative of b appears because we have chosen to use the bilinear pairing of the Bergman space. A similar comment applies in (6.3) below.) To emphasize the analogy with (6.1) we write this out in full under the assumption $F(0) = G(0) = 0$. We must have

$$\left| \int_{\mathbb{D}} F(z)G(z)\overline{b'(z)}dxdy \right| \\ \le c_b \left(\int_{\mathbb{D}} \left| F'(z) \right|^2 (1-|z|^2)dxdy \right)^{1/2} \left(\int_{\mathbb{D}} \left| G'(z) \right|^2 (1-|z|^2)dxdy \right).$$

In this case it is classical that the necessary and sufficient condition for such an estimate is that the measure

$$d\mu = \left| b'(z) \right|^2 (1-|z|^2)\, dxdy \tag{6.2}$$

be a Carleson measure for the Hardy space, or, equivalently b be in BMO. For this as well as analogous results for the Bergman space see [Z]. Based on analogies with the Hardy space as well as results for related problems in [RW] one conjectures that a similar result holds for the Dirichlet space, $B_2(0)$; that is, that a necessary and sufficient condition for an estimate of the form

$$\left| \int_{\mathbb{D}} F(z)G(z)\overline{b''(z)}dxdy \right| \le c_b \left\| F \right\|_{B_2(0)} \left\| G \right\|_{B_2(0)},$$

or, equivalently, with $F(0) = G(0) = 0$,

$$\left| \int_{\mathbb{D}} F(z)G(z)\overline{b''(z)}dxdy \right| \le c_b \left(\int_{\mathbb{D}} \left| F'(z) \right|^2 dxdy \right)^{1/2} \left(\int_{\mathbb{D}} \left| G'(z) \right|^2 dxdy \right) \tag{6.3}$$

is that

$$d\mu = \left| b'(z) \right|^2 dxdy$$

be a Carleson measure for the Dirichlet space. However it is not known if this is true. The theorem just stated is a model result for the conjecture using the model for the Dirichlet space studied in [ARS]. The subscript on b_2 is to emphasize that the b_2 in the theorem is a model for $\overline{b''}$ in the conjecture. Thus $\mathcal{I}^* b_2$, in some sense a primitive of b_2, is a model of b'. We should note that our theorem gives a result for the discrete space B_2 of [ARS] not for the discrete *martingale* space $\mathcal{B}_2$ considered earlier in this paper.

Hankel operators are considered in a range of contexts in function theoretic operator theory. Similar questions have also been considered in contexts where there is no holomorphy. In [MV] Maz'ya and Verbitsky study the conditions on functions b_2 defined on $\mathbb{R}^n$ which insure that there are estimates of the form

$$\left|\int_{\mathbb{R}^n} F(x)G(x)b_2(x)dx\right| \leq c_b \left(\int_{\mathbb{R}^n} |F'(x)|^2\, dx\right)^{1/2} \left(\int_{\mathbb{R}^n} |G'(x)|^2\, dx\right)^{1/2} \tag{6.4}$$

which are formally similar to (6.3). To oversimplify, their result is that for an appropriate "integral operator", Int, a necessary and sufficient condition for (6.4) is that $|\mathrm{Int}(b_2)(x)|^2\, dx$ is, in our language, a Carleson measure for the Sobolev space.

Rather than work with (6.1) we will work with the associated linear operator. Also, having noted various analogies, we drop the subscript on b.Recall that $D\mathcal{I} = \mathcal{I}D = \mathrm{Id}$. Hence, setting $f = DF$ and $g = DG$, (6.1) is equivalent to

$$|\langle (\mathcal{I}f)\,(\mathcal{I}g), b\rangle_{l^2}| \leq c_b \, \|f\|_{l^2}\, \|g\|_{l^2}\, .$$

Although we cannot freely assume that b is positive, it is elementary to reduce to the case of f, g, and b real valued and we now consider that case. In particular $|\mathcal{I}^*b|^2 = (\mathcal{I}^*b)^2$. Also, it suffices to consider the case in which f, g, and b have finite support; we now assume that. We have $\langle (\mathcal{I}f)\,(\mathcal{I}g), b\rangle = \langle (\mathcal{I}g), b\,(\mathcal{I}f)\rangle = \langle g, \mathcal{I}^*(b\,(\mathcal{I}f))\rangle$. Being able to estimate this for all g is equivalent to knowing the boundedness of the operator $f \to H_b f := \mathcal{I}^*(b\,(\mathcal{I}f))$ on $l^2(\mathcal{D})$. We now prove:

Theorem 6. *H_b is bounded on l^2 if and only if $\mu_{|\mathcal{I}^*b|^2}$ is a Carleson measure for $\mathcal{B}_2$; that is it satisfies* (4.2) *with $a = 0$, $p = 2$.*

We begin with a formula for summation by parts in this context:

Lemma 4. *Suppose h and k are functions on $\mathcal{D}$, then*

$$\mathcal{I}^*(h(\mathcal{I}k)) = (\mathcal{I}k)(\mathcal{I}^*h) + \mathcal{I}^*(k(\mathcal{I}^*h)) - k\mathcal{I}^*h.$$

Proof of the lemma.

$$\begin{aligned}
\mathcal{I}^*(h(\mathcal{I}k))(z) &= \sum_{x\geq z} h(x)\left(\sum_{o\leq y\leq x} k(y)\right)\\
&= \sum_{o\leq y\leq x}\sum_{x\geq z} h(x)k(y)\\
&= \sum_{o\leq y\leq z}\sum_{x\geq z} h(x)k(y) + \sum_{y>z}\sum_{x\geq y} h(x)k(y)\\
&= \sum_{o\leq y\leq z}\sum_{x\geq z} h(x)k(y) + \sum_{y\geq z}\sum_{x\geq y} h(x)k(y) - k(z)\sum_{x\geq z} h(x)\\
&= \sum_{o\leq y\leq z} (\mathcal{I}^*h)(z)k(y) + \sum_{y\geq z} (\mathcal{I}^*h)(y)k(y) - k(z)(\mathcal{I}^*h)(z)\\
&= (\mathcal{I}^*h)(z)\,(\mathcal{I}k)\,(z) + \mathcal{I}^*\left((\mathcal{I}^*h)k\right)(z) - k(z)(\mathcal{I}^*h)(z).
\end{aligned}$$

□

Proof of Theorem 6. For convenience we set $B = \mathcal{I}^*b$. By the previous lemma we have

$$H_b f = \mathcal{I}^*(b\,(\mathcal{I}f)) = B\,(\mathcal{I}f) + \mathcal{I}^*(fB) - fB. \tag{6.5}$$

We consider the summands separately. If μ_{B^2} is a Carleson measure then, by the very definition,

$$\|B(\mathcal{I}f)\|_{l^2}^2 = \sum_{\mathcal{D}} |(\mathcal{I}f)|^2 B^2 = \|(\mathcal{I}f)\|_{l^2(\mathcal{D},\mu_{B^2})}^2 \le c\|f\|^2.$$

Hence in this case the operator $f \to B(\mathcal{I}f)$ is bounded. The operator $f \to \mathcal{I}^*(fB)$ is the adjoint of that operator and hence is also bounded. By applying the definition of a Carleson measure with the function f equal to the point mass at o we find that $B \in l^2$, hence by Cauchy-Schwarz the third term is bounded. Adding we find that H_b is bounded.

Suppose now that H_b is bounded. By computing the norm of H_b applied to the point mass at o we find that $B \in l^2$ and hence the third term is also bounded, and hence so is the sum of the first two terms. We will work with that sum. That is we will consider $\widetilde{H}_b$ defined by $\widetilde{H}_b f = B(\mathcal{I}f) + \mathcal{I}^*(fB)$.We need to show that μ_{B^2} is a Carleson measure. By Theorem 3 of [ARS] that is equivalent to showing that there is a constant c_b so that for all w in $\mathcal{D}$

$$\sum_{x\ge w}\left(\sum_{z\ge x} B^2(z)\right)^2 \le c\sum_{t\ge w} B^2(t). \tag{6.6}$$

Pick and fix $w \in \mathcal{D}$, $w \ne o$ and set $S := \{x \in \mathcal{D} : x \ge w\}$. Let χ be the characteristic function of S and set $f = \chi B$. We are assuming $\widetilde{H}_b$ is bounded. Thus

$$c\sum_{t\in S} B^2(t) = c\|f\|^2 \ge \left\|\widetilde{H}_b f\right\|^2.$$

Thus we will be finished if we can show that for some $c > 0$

$$\sum_{x\in S}\left(\sum_{z\ge x} B^2(z)\right)^2 \le c\left\|\widetilde{H}_b f\right\|^2.$$

Set $k = \#\{z \in \mathcal{D} : o \ge z > w\}$. Direct computation gives

$$\begin{aligned}
\left\|\widetilde{H}_b f\right\|^2 &= \|\mathcal{I}^*(b)(\mathcal{I}f) + \mathcal{I}^*(f(\mathcal{I}^*b))\|^2 \\
&= \|B(\mathcal{I}f) + \mathcal{I}^*(fB)\|^2 \\
&= \|\mathcal{I}^*(fB)\|^2 + \|B(\mathcal{I}f)\|^2 + 2\langle B(\mathcal{I}f), \mathcal{I}^*(fB)\rangle \\
&= \left\|\mathcal{I}^*(\chi B^2)\right\|^2 + \|B\mathcal{I}(\chi B)\|^2 + 2\left\langle B\mathcal{I}(\chi B), \mathcal{I}^*(\chi B^2)\right\rangle \\
&= \sum_{x\in S}\left(\sum_{z\ge x} B^2\right)^2 + k\mathcal{I}^*(B^2)(w) \\
&\qquad + \|B\mathcal{I}(\chi B)\|^2 + 2\left\langle B\mathcal{I}(\chi B), \mathcal{I}^*(\chi B^2)\right\rangle.
\end{aligned}$$

The first term on the last line is the one we wish to dominate by the left-hand side. The third term on the last line is nonnegative. Hence we will be finished if we show that

$$0 \le k\mathcal{I}^*(B^2)(w) + 2\left\langle B\mathcal{I}(\chi B), \mathcal{I}^*(\chi B^2)\right\rangle. \tag{6.7}$$

We have

$$\begin{aligned}
2\left\langle B\mathcal{I}\left(\chi B\right), \mathcal{I}^*(\chi B^2)\right\rangle &= 2\left\langle \mathcal{I}\{B\mathcal{I}\left(\chi B\right)\}, \chi B^2\right\rangle \\
&= 2\sum_{x\in S}\mathcal{I}\{B\mathcal{I}\left(\chi B\right)\}(x)B^2(x) \\
&= 2\sum_{x\in S}\left\{\sum_{x\geq y\geq w}B(y)\mathcal{I}\left(\chi B\right)(y)\right\}B^2(x) \\
&= 2\sum_{x\in S}\left\{\sum_{x\geq y\geq w}B(y)\sum_{y\geq z\geq w}B(z)\right\}B^2(x) \\
&= 2\sum_{x\in S}\left\{\sum_{x\geq y\geq z\geq w}B(y)B(z)\right\}B^2(x) \\
&= \sum_{x\in S}\left\{\left(\sum_{x\geq y\geq w}B(y)\right)^2 - \left(\sum_{x\geq y\geq w}B^2(y)\right)\right\}B^2(x) \\
&\geq -\sum_{x\in S}\left(\sum_{x\geq y\geq w}B^2(y)\right)B^2(x) \\
&= -\left[\left(\mathcal{I}^*B^2\right)(w)\right]^2.
\end{aligned}$$

Because $k \geq 1$ in (6.7) the estimate in (6.7) holds and we are finished with the cases $w \neq o$.

The case $w = o$ follows by elementary estimates from the fact that we have (6.6) for $w = o_+$ and $w = o_-$ and the fact that the total measure if finite, i.e., that we have a bound on $\|B\|_{l^2}$. This last fact follows from applying the operator to a point mass at o. □

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