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Equidistribution of small subvarieties of an abelian variety

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Abstract. We prove an equidistribution result for small subvarieties of an abelian variety which generalizes the Szpiro–Ullmo–Zhang theorem on equidistribution of small points.

Contents

1. Introduction

1.1. Notation. The following notation and conventions will be used throughout this paper:

K	a number field.
$\mathcal{O}_K$	the ring of integers of K.
A	an abelian variety defined over K.

We fix, for future use, a choice of an algebraic closure $\overline{K}$ of K and an embedding of $\overline{K}$ into $\mathbb{C}$.

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1.2. Heights of cycles. Let X be a smooth projective variety over K of dimension $N \geq 1$, and let $\mathcal{X}$ be a model for X over $\mathcal{O}_K$, i.e., an integral scheme projective and flat over Spec $\mathcal{O}_K$ whose generic fiber is X.

Let $\overline{\mathcal{L}}$ be a hermitian line bundle on $\mathcal{X}$. A hermitian metric is always assumed to be smooth and invariant under complex conjugation. We assume furthermore that $\mathcal{L}_K$ is ample, and that the curvature form $c_1(\overline{\mathcal{L}})$ satisfies $c_1(\overline{\mathcal{L}}) > 0$. (See [3] for a discussion of the curvature form associated to a hermitian line bundle).

By using arithmetic intersection theory, one defines the height of a cycle in Arakelov geometry as follows (see e.g., [5]):

Definition. The *height* of a nonzero effective cycle Y (of pure dimension) of X with respect to $\overline{\mathcal{L}}$ is

$$h_{\overline{\mathcal{L}}}(Y) \ := \ \frac{\widehat{c_1}(\overline{\mathcal{L}}|_{\overline{Y}})^{\dim Y+1}}{(\dim Y+1)c_1(\mathcal{L}|_Y)^{\dim Y}},$$

where $\overline{Y}$ is the (scheme-theoretic) Zariski closure of Y in $\mathcal{X}$.

For a detailed overview of all the properties of curvature forms, arithmetic Chern classes, and heights of arithmetic cycles which we will need, see [1] or [5]. Proofs of the relevant facts can be found in [3].

1.3. Canonical heights on abelian varieties. Let A/K be an abelian variety. Using the choice of an embedding of $\overline{K}$ into $\mathbb{C}$, we view $A(\overline{K})$ as a subset of $A(\mathbb{C})$. Let $\mathcal{A}$ be a model for A over Spec $\mathcal{O}_K$. Let $\overline{\mathcal{L}}$ be a hermitian line bundle on $\mathcal{A}$ such that $L := \mathcal{L}_K$ is symmetric and ample, and such that $\mathcal{L}$ is equipped with a *cubical metric* (see [4]). A cubical metric is one whose curvature form is translation-invariant; all such metrics are positive scalar multiples of one another. For simplicity of notation, we fix one such metric and call it "the" cubical metric on L.

Fix a nontrivial multiplication map on A (e.g., multiplication by 2). One can then construct from $(\mathcal{A}, \overline{\mathcal{L}})$ a sequence $(\mathcal{A}_n, \overline{\mathcal{L}}_n)_{n\geq 1}$ of models of (A, L), where each $\mathcal{L}_n$ is equipped with the cubical metric, in such a way that the sequence

$$h_{\overline{\mathcal{L}}_n}(Y) \ = \ \frac{\widehat{c_1}(\overline{\mathcal{L}}_n|_{\overline{Y}_n})^{\dim Y+1}}{(\dim Y+1)c_1(L|_Y)^{\dim Y}}$$

converges (uniformly in Y) to a nonnegative real number $\hat{h}_L(Y)$. (Here Y is a nonzero effective cycle of pure dimension on A, and $\overline{Y}_n$ is the Zariski closure of Y in $\mathcal{A}_n$. See [5] or [8] for details.) The canonical height $\hat{h}_L$ does not depend on the choice of cubical metrics, on $(\mathcal{A}, \mathcal{L})$, or on the sequence of models $(\mathcal{A}_n, \mathcal{L}_n)_{n\geq 1}$. For $x \in A(\overline{K})$, $\hat{h}_L(x)$ is the Néron-Tate canonical height of x with respect to L.

1.4. Statement of the main theorem. We need several definitions in order to state our main result. By a *variety* X over a field k, we mean an integral separated scheme of finite type over k. By a *subvariety* of X, we mean an integral closed subscheme of X.

Definitions.

1. A *torsion subvariety* of A is a translate of an abelian subvariety of A by a torsion point.

2. A sequence $(X_n)_{n\geq 1}$ of closed subvarieties of A is *small* if $\hat{h}_L(X_n) \to 0$ (as $n \to \infty$).
3. A sequence $(X_n)_{n\geq 1}$ of closed subvarieties of X is *generic in* X if it has no subsequence contained in a proper Zariski closed subset of X.
4. A sequence $(X_n)_{n\geq 1}$ of closed subvarieties of A is *strict* if it has no subsequence contained in a proper torsion subvariety of A.

Note that the subvarieties X_n are required to be defined over $\overline{K}$, but not necessarily over K.

The following result is a generalization of the Szpiro–Ullmo–Zhang/Ullmo/Zhang equidistribution theorem to sequences of small subvarieties of an abelian variety.

Theorem 1.1 (Strict Equidistribution)**.** *Let A/K be an abelian variety, let L be a symmetric ample line bundle on A, and let $\overline{L}$ denote L with the cubical metric. Let $(X_n)_{n\geq 1}$ be a small strict sequence of closed subvarieties of A. Then for every real-valued continuous function f on $A(\mathbb{C})$, we have*

$$\int_{A(\mathbb{C})} f\ \mu_n \longrightarrow \int_{A(\mathbb{C})} f\ \mu$$

as $n \to \infty$, where setting $d_n = \dim X_n$ and $g = \dim A$, we have

$$\mu_n = \frac{1}{c_1(L|_{X_n})^{d_n}} c_1(\bar{L})^{d_n} \delta_{X_n} \quad \textit{and} \quad \mu = \frac{c_1(\bar{L})^g}{c_1(L)^g}.$$

Remarks. 1. The first integral is the integral of f against the restriction of $c_1(\overline{L})^{d_n}/\deg_L(X_n)$ to $X_n(\mathbb{C})$. The second integral is the integral of f with respect to the Haar measure μ on $A(\mathbb{C})$, normalized to have total mass 1.

2. If $X_n = x_n$ is a point, i.e., if $d_n = 0$, note that

$$\int_{A(\mathbb{C})} f(x)\mu_n \ = \ \frac{1}{\#O(x_n)} \sum_{x\in O(x_n)} f(x),$$

where $O(x_n)$ is the orbit of x_n under the action of $\mathrm{Gal}(\overline{K}/K)$.

3. For notational convenience, we write

$$\mu_n \xrightarrow{w} \mu \ \text{ as } \ n \to \infty,$$

and say the sequence $(\mu_n)_{n\geq 1}$ of measures *weakly converges* to μ, if

$$\int_{A(\mathbb{C})} f\mu_n \to \int_{A(\mathbb{C})} f\mu$$

for every continuous function $f : A(\mathbb{C}) \to \mathbb{R}$. In this case, we say that the X_n's are *equidistributed* with respect to μ.

4. To get a feeling for what Theorem 1.1 says, consider the following simple example. Let E be an elliptic curve defined over $\mathbb{Q}$ and let $A = E \times E$. For each $n \geq 1$, let $E_n \subset A$ be the graph of the multiplication-by-n map on E. Then each E_n is a torsion subvariety of A defined over $\mathbb{Q}$ (in fact, E_n is $\mathbb{Q}$-isogenous to E).

   It is easy to see that $\deg(E_n) \to \infty$ as $n \to \infty$, and that $\bigcup_{n\geq 1} E_n$ is Zariski dense in A. Theorem 1.1 says something stronger than this, namely that as $n \to \infty$, the normalized Haar measure on E_n approximates the normalized Haar measure on A arbitrarily closely.

5. For related equidistribution results, see Theorem 1.1 of [2], Theorem 4.1 of [5], Theorem 2.3 of [6], and Theorem 1.1 of [9]. In addition, Pascal Autissier has recently obtained a proof of Theorem 2.2 of the present paper independently of the authors.

The basic idea behind our proof of Theorem 1.1 is to first approximate the height of each subvariety X_n by heights of points on X_n; the approximation is good when n is large because of Zhang's theorem of the successive minima and the assumption that $\hat{h}_L(X_n) \to 0$. We then apply the Szpiro–Ullmo–Zhang theorem (and its proof) to a suitable subsequence of these points. As in [5], we first prove the result under the stronger assumption that the sequence X_n is generic (as opposed to merely strict).

2. Generic equidistribution

Let X be a closed subvariety of dimension $N \geq 1$ of A. The following result is a special case of Zhang's "theorem of the successive minima"(see [9] for details):

Theorem 2.1. *Define*

$$\lambda_1(X) := \sup_Z \inf_{x \in X - Z} \hat{h}_L(x),$$

where Z runs over the set of all proper closed subsets of X, and x runs over all $\overline{K}$-valued points of $X - Z$. Then

$$\lambda_1(X) \ \geq \ \hat{h}_L(X) \ \geq \ \frac{1}{N+1}\lambda_1(X).$$

Definition. Let $\overline{\mathcal{L}}$ be a hermitian line bundle on $\mathcal{A}$. If f is a real-valued C^∞-function on $A(\mathbb{C})$, define

$$\overline{\mathcal{L}}(f) \ := \ \overline{\mathcal{L}} \otimes (\mathcal{O}_{\mathcal{A}}, e^{-f})$$

to be the tensor product of $\overline{\mathcal{L}}$ with the trivial bundle, endowed with the metric given by $\|1\|(P) = e^{-f(P)}$.

Theorem 2.2 (Generic Equidistribution)**.** *Let A/K be an abelian variety, and let L be a symmetric ample line bundle on A. Let $(X_n)_{n\geq 1}$ be a small generic sequence of closed subvarieties of A. Then, for every real-valued continuous function f on $A(\mathbb{C})$, we have*

$$\int_{A(\mathbb{C})} f(x)\mu_n \longrightarrow \int_{A(\mathbb{C})} f(x)\mu$$

as $n \to \infty$, where $d_n = \dim X_n$, $\mu_n = \frac{1}{c_1(L|_{X_n})^{d_n}} c_1(\overline{L})^{d_n}\delta_{X_n}$, $g = \dim A$, $\mu = \frac{c_1(\overline{L})^g}{c_1(L)^g}$, and $\overline{L}$ is L with the cubical metric.

Proof. Enumerate the countably many subvarieties $(Z_n)_{n\geq 1}$ of A defined over $\overline{K}$. Since $(X_n)_{n\geq 1}$ is generic, we may assume, without loss of generality, $X_n \not\subset Z_1 \cup \cdots \cup Z_n$. By the definition of $\lambda_1(X_n)$, we can find (for each $n \geq 1$) an infinite sequence $(x_{n,m})_{m\geq 1}$ in X_n such that:

(i) For each $m \geq 1$, $x_{n,m} \notin \bigcup_{1\leq i\leq n} Z_i$.

(ii) $|\hat{h}_L(x_{n,m}) - \lambda_1(X_n)| < \frac{1}{n}$ for all $m \geq 1$.

(iii) For each $n \geq 1$, $\lim_{m\to\infty} \hat{h}_L(x_{n,m}) = \lambda_1(X_n)$.

By choosing a bijection between $\mathbf{N}^2$ and $\mathbf{N}$, we may consider the doubly-indexed sequence $(x_{n,m})$ as a sequence indexed by the natural numbers. Property (i) guarantees that the resulting sequence $(x_{n,m})$ is generic in A. Furthermore, since $\hat{h}_L(X_n) \to 0$ by assumption, it follows from the theorem of the successive minima that $\lambda_1(X_n) \to 0$ as $n \to \infty$. Using this observation, properties (ii) and (iii) easily imply that $\lim_{n,m\to\infty} \hat{h}_L(x_{n,m}) = 0$, i.e., that the sequence $(x_{n,m})$ is small.

Define

$$\alpha_{n,m} := \frac{1}{\#O(x_{n,m})} \sum_{x \in O(x_{n,m})} f(x),$$

where $O(x_{n,m})$ is the orbit of $x_{n,m}$ under the action of $\mathrm{Gal}(\overline{K}/K)$. By choosing a subsequence of $(x_{n,m})_{m\geq 1}$ if necessary, we may assume that $\lim_{m\to\infty} \alpha_{n,m}$ exists for all $n \geq 1$. Note that every subsequence of a small (resp. generic) sequence is small (resp. generic).

Approximating f by C^∞-functions if necessary, we may assume that f is a C^∞-function. Let $\lambda > 0$ be a real number. Note that $c_1(\overline{\mathcal{L}}_l(\lambda f)) > 0$ if $\lambda > 0$ is small enough. We then note, for $l \geq 1$, that

$$\begin{aligned} h_{\overline{\mathcal{L}}_l(\lambda f)}(x_{n,m}) &= h_{\overline{\mathcal{L}}_l}(x_{n,m}) + \lambda\alpha_{n,m}; \quad \text{and} \\ \liminf_{m\to\infty} h_{\overline{\mathcal{L}}_l(\lambda f)}(x_{n,m}) &\geq h_{\overline{\mathcal{L}}_l(\lambda f)}(X_n) \qquad \text{([5], Proposition 2.1)} \\ &= h_{\overline{\mathcal{L}}_l}(X_n) + \lambda \int_{A(\mathbb{C})} f(x)\mu_n + O(\lambda^2), \end{aligned}$$

where the last equality follows from [1, Proof of Proposition 2.9]. Here the O-constant is independent of l and n, and $\lambda > 0$ is sufficiently small.

Fix $n \geq 1$ and $\varepsilon > 0$. Then for m sufficiently large, we have:

$$h_{\overline{\mathcal{L}}_l}(x_{n,m}) + \lambda\alpha_{n,m} \;\geq\; h_{\overline{\mathcal{L}}_l}(X_n) + \lambda \int_{A(\mathbb{C})} f(x)\mu_n + O(\lambda^2) - \varepsilon.$$

Letting $l \to \infty$, we have:

$$\hat{h}_L(x_{n,m}) - \hat{h}_L(X_n) + \lambda\alpha_{n,m} \;\geq\; \lambda \int_{A(\mathbb{C})} f(x)\mu_n + O(\lambda^2) - \varepsilon.$$

Now let $m \to \infty$, and we obtain (since $\epsilon > 0$ is arbitrary):

$$\lambda_1(X_n) - \hat{h}_L(X_n) + \lambda \lim_{m\to\infty} \alpha_{n,m} \;\geq\; \lambda \int_{A(\mathbb{C})} f(x)\mu_n + O(\lambda^2). \tag{1}$$

On the other hand, the Szpiro–Ullmo–Zhang/Ullmo/Zhang equidistribution theorem ([5], [6], and [9]), applied to the small generic sequence $(x_{n,m})$, implies that $\lim_{n,m\to\infty} \alpha_{n,m}$ exists, and that

$$\lim_{n,m\to\infty} \alpha_{n,m} \;=\; \int_{A(\mathbb{C})} f(x)\mu. \tag{2}$$

Taking $\limsup_{n\to\infty}$ in (1), we have:

$$\lambda \lim_{n,m\to\infty} \alpha_{n,m} \;\geq\; \lambda \limsup_{n\to\infty} \int_{A(\mathbb{C})} f(x)\mu_n + O(\lambda^2).$$

Now divide both sides by $\lambda > 0$, and let $\lambda \to 0^+$. We obtain:

$$\lim_{n,m\to\infty} \alpha_{n,m} \geq \limsup_{n\to\infty} \int_{A(\mathbb{C})} f(x)\mu_n. \tag{3}$$

Replacing f by $-f$, we see that:

$$\lim_{n,m\to\infty} \alpha_{n,m} \leq \liminf_{n\to\infty} \int_{A(\mathbb{C})} f(x)\mu_n. \tag{4}$$

It then follows from (3) and (4) that $\lim_{n\to\infty} \int_{A(\mathbb{C})} f(x)\mu_n$ exists and that

$$\lim_{n\to\infty} \int_{A(\mathbb{C})} f(x)\mu_n = \lim_{n,m\to\infty} \alpha_{n,m}.$$

We conclude from (2) that

$$\lim_{n\to\infty} \int_{A(\mathbb{C})} f(x)\mu_n = \int_{A(\mathbb{C})} f(x)\mu,$$

as desired. □

3. Strict equidistribution

The following result is a consequence of two results of Zhang: the generalized Bogomolov conjecture (see [9]) and the theorem of the successive minima. The proof is similar to the proof of Theorem 2.2.

Theorem 3.1. *Let X be a nontorsion subvariety of A. Then there is an $\epsilon > 0$ such that the set*

$$\bigcup \{Y : Y \text{ is a closed subvariety of } X \text{ such that } \hat{h}_L(Y) \leq \epsilon\}$$

is not Zariski dense in X.

Proof. Suppose, for the sake of contradiction, that $(Y_n)_{n\geq 1}$ is a sequence of distinct closed subvarieties of X which is small (i.e., $\hat{h}_L(Y_n) \to 0$) and generic in X (i.e., no subsequence is contained in a proper Zariski closed subset of X). Then, proceeding as in the proof of Theorem 2.2, we can construct an infinite sequence $(y_k)_{k\geq 1}$ of points in X such that $\{y_k \in X : k \geq 1\}$ is Zariski dense in X and $\hat{h}_L(y_k) \to 0$. But Corollary 3 of [9] then implies that X is a torsion subvariety of A, a contradiction. □

Now we are ready to prove Theorem 1.1 (Strict Equidistribution Theorem).

Proof of Theorem 1.1. By Theorem 2.2, it suffices to show that the small and strict sequence $(X_n)_{n\geq 1}$ is generic. Let X' be the Zariski closure of $\bigcup_k X_{n_k}$ for any subsequence $(X_{n_k})_{k\geq 1}$ of $(X_n)_{n\geq 1}$. By Theorem 3.1, X' must be a torsion subvariety of A. Since $(X_n)_{n\geq 1}$ is strict, it follows that $X' = A$, so that $(X_n)_{n\geq 1}$ is generic as desired. □

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