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Nonergodic actions, cocycles and superrigidity

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Abstract. This paper proves various results concerning nonergodic actions of locally compact groups and particularly Borel cocycles defined over such actions. The general philosophy is to reduce the study of the cocycle to the study of its restriction to each ergodic component of the action, while being careful to show that all objects arising in the analysis depend measurably on the ergodic component. This allows us to prove a version of the superrigidity theorems for cocycles defined over nonergodic actions.

Contents

1. Introduction

It is often the case that one has extensive information about each ergodic component of an action, and would like to piece this local information together (measurably) in order to obtain a global conclusion about the entire action. This note addresses a number of problems of this type, mostly dealing with cocycles. For example:

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1. If α and β are Borel cocycles, and the restriction of α to almost every ergodic component is cohomologous to the restriction of β, then α is cohomologous to β (see 3.6).
2. If α is a Borel cocycle, and the restriction of α to almost every ergodic component is cohomologous to a homomorphism cocycle, then α is cohomologous to a homomorphism cocycle (see 3.11).
3. If almost every ergodic component of a G-action has a certain standard Borel G-space X as a measurable quotient, then X is a measurable quotient of the entire action (see 5.4).

(We consider only Borel actions of second countable, locally compact groups on standard Borel spaces with a quasiinvariant probability measure.)

We also prove a superrigidity theorem for cocycles that applies to nonergodic actions (see 4.4). In fact our work was motivated by the discovery, during the writing of [FM], that no proof of any version of superrigidity for cocycles concerning cocycles over nonergodic actions, exists in the literature. For many applications to nonergodic actions, including those in [FM], other information concerning the action and cocycle allows one to, with some additional work make do with superrigidity theorems for cocycles which are defined over ergodic actions. However, the results in Section 4 allow some simplification of the arguments in Section 5 of [FM], and should have other applications as well.

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2. Some lemmas on measurability

This section records basic definitions and notation, and also proves that various natural constructions of sets, functions, and actions yield results that are measurable. Some of the conclusions are known (or even well-known), but others may be of independent interest.

2A. Properties of standard Borel spaces. We assume the basic theory of Polish spaces, standard Borel spaces and analytic Borel spaces, which can be found in a number of textbooks, such as [Ar, Chap. 3]. We recall the definition of a standard space and an analytic set.

Definition 2.1.

- A topological space is *Polish* if it is homeomorphic to a complete, separable metric space.
- A Borel space is *standard* if it is Borel isomorphic to a Polish topological space.
- The pair (S, μ) is a *standard Borel probability space* if S is a standard Borel space and μ is a probability measure on S.

- A subset A of a standard Borel space S is *analytic* if there exist:
  - a standard Borel space X,
  - a Borel map $\psi\colon X \to S$,

  such that $A = \psi(X)$ is the image of ψ.

Remark 2.2 ([Ar, Thm. 3.2.4, p. 67])**.** An analytic subset A of a standard Borel space S is *absolutely measurable*; that is, for any probability measure μ on S, there exist Borel subsets B_1 and B_2 of S with $B_1 \subset A \subset B_2$ and $\mu(B_2 \smallsetminus B_1) = 0$.

Theorem 2.3 (von Neumann Selection Theorem, cf. [Ar, Thm. 3.4.3, p. 77])**.** *Let:*

- *(Ω, ν) be a standard Borel probability space,*
- *L be a standard Borel space,*
- *$\mathcal{F}$ be an analytic subset of $\Omega \times L$, and*
- *$\Omega_{\mathcal{F}}$ be the projection of $\mathcal{F}$ to Ω.*

Then there are:

- *a conull, Borel subset Ω_0 of $\Omega_{\mathcal{F}}$, and*
- *a Borel function $\Phi\colon \Omega_0 \to L$,*

such that $\big(\omega, \Phi(\omega)\big) \in \mathcal{F}$, for all $\omega \in \Omega_0$.

Notation 2.4. Suppose (S, μ) is a standard Borel probability space, and (X, d) is a separable metric space.

1. We use $\mathbf{F}(S, X)$ to denote the space of measurable functions $f\colon S \to X$, where two functions are identified if they are equal almost everywhere. If X is complete, then $\mathbf{F}(S, X)$ is a Polish space, under the topology of convergence in measure (cf. [WZ, §4.4]). A metric can be given by
$$d_{\mathbf{F}}(f, g) = \min \left\{ \epsilon \geq 0 \;\middle|\; \mu\big\{ s \in S \mid d\big(f(s), g(s)\big) > \epsilon \big\} \leq \epsilon \right\}.$$
2. We use $\mathcal{B}(S)$ to denote the Boolean algebra of measurable subsets of S, where two subsets are identified if their symmetric difference has measure 0. It is well-known that this is a complete separable metric space, with metric
$$d_{\mathcal{B}}(A, B) = \mu(A \,\Delta\, B).$$

Remark 2.5. The σ-algebra of Borel subsets of $\mathbf{F}(S, X)$ is generated by the sets of the form
$$\Delta_{S_0, X_0, \epsilon} = \big\{ f \;\big|\; \mu\big(S_0 \cap f^{-1}(X_0)\big) < \epsilon \big\},$$
where S_0 and X_0 are Borel subsets of S and X, respectively, and $\epsilon > 0$. This implies that if (X', d') is Borel isomorphic to (X, d), then $\mathbf{F}(S, X')$ is Borel isomorphic to $\mathbf{F}(S, X)$.

The following is well-known:

Lemma 2.6 (cf. [Mr, Lem. 7.1.3, p. 215])**.** *Let:*

- *L and S be standard Borel spaces,*
- *μ be a probability measure on S,*
- *X be a separable metric space, and*
- *$f\colon L \times S \to X$ be Borel.*

Then:

1. *For each $\ell \in L$, the function $f_\ell\colon S \to X$, defined by $f_\ell(s) = f(\ell, s)$, is Borel.*
2. *The induced function $\check{f}\colon L \to \mathbf{F}(S, X)$, defined by $\check{f}(\ell) = f_\ell$, is Borel.*

In short, any Borel function from $L \times S$ to X yields a Borel function from L to $\mathbf{F}(S, X)$. The converse is true:

Lemma 2.7. *Let:*

- *L and S be standard Borel spaces,*
- *μ be a probability measure on S,*
- *(X, d) be a separable metric space, and*
- *$\phi\colon L \to \mathbf{F}(S, X)$ be a Borel function.*

Then there is a Borel function $\hat{\phi}\colon L \times S \to X$, such that, for each $\ell \in L$, we have

$$\hat{\phi}(\ell, s) = \phi(\ell)(s) \text{ for a.e. } s \in S. \tag{2.8}$$

Proof. For each $n \in \mathbb{N}$, let $\{D_n^i\}_{i=1}^{\infty}$ be a partition of $\mathbf{F}(S, X)$ into countably many (nonempty) Borel sets of diameter less than 2^{-n}, and choose $\phi_n^i \in D_n^i$. Define $\phi_n\colon L \times S \to X$ by

$$\phi_n(\ell, s) = \phi_n^i(s) \text{ if } \phi(\ell) \in D_n^i.$$

Then each ϕ_n is Borel.

By replacing each $\{D_n^i\}_{i=1}^{\infty}$ by the join of $\{D_1^i\}_{i=1}^{\infty}, \ldots, \{D_n^i\}_{i=1}^{\infty}$ we may assume that $\{D_{n+1}^i\}_{i=1}^{\infty}$ is a refinement of $\{D_n^i\}_{i=1}^{\infty}$. For each $m, n \in \mathbb{N}$ and each $\ell \in L$

$$\mu\big\{s \in S \mid d\big(\phi_m(\ell, s), \phi_n(\ell, s)\big) > 2^{-\min(m,n)}\big\} < 2^{-\min(m,n)}.$$

Thus, on each fiber $\{\ell\} \times S$, the sequence $\{\phi_n\}$ not only converges in measure, but converges quickly. There is no harm in assuming that X is complete; then $\{\phi_n\}$ converges pointwise a.e. Let $\hat{\phi}$ be the pointwise limit of $\{\phi_n\}$. (Define $\hat{\phi}$ to be constant on the set where $\{\phi_n\}$ does not converge.) Then $\hat{\phi}$ is Borel, and satisfies (2.8). □

Corollary 2.9. *Let:*

- *L and S be standard Borel probability spaces, and*
- *(X, d) be a separable metric space.*

Then $\mathbf{F}(L \times S, X)$ is naturally homeomorphic to $\mathbf{F}\big(L, \mathbf{F}(S, X)\big)$.

Proof. From the preceding two lemmas, we know that there is a natural bijection between the two spaces. Fubini's Theorem implies that a sequence converges in one of the spaces if and only if the corresponding sequence converges in the other space. □

Definition 2.10. Suppose (S, μ) is a standard Borel probability space.

- Let $\mathrm{Aut}_{[\mu]}(S)$ be the group of all equivalence classes of measure-class-preserving Borel automorphisms of (S, μ), where two automorphisms are equivalent if they are equal almost everywhere.
- Let $\mathbf{U}\big(L^2(S)\big)$ be the group of unitary operators on the Hilbert space $L^2(S)$, with the strong operator topology (that is, $T_n \to T$ if $\|T_n(f) - T(f)\| \to 0$, for every $f \in L^2(S)$, equivalently, the topology on $\mathbf{U}\big(L^2(S)\big)$ has a subbasis of open sets $\mathcal{U}(f, g, \epsilon) = \{T : \|Tf - g\| < \epsilon\}$). Note that $\mathbf{U}\big(L^2(S)\big)$ is a Polish space.
- There is a well-known embedding of $\mathrm{Aut}_{[\mu]}(S)$ in $\mathbf{U}\big(L^2(S)\big)$, given by

$$T_\phi(f)(s) = D_\phi(s)^{1/2}\, f\big(\phi^{-1}(s)\big),$$

for $\phi \in \mathrm{Aut}_{[\mu]}(S)$ and $f \in L^2(S)$, where D_ϕ is the Radon-Nikodym derivative of ϕ. This provides $\mathrm{Aut}_{[\mu]}(S)$ with the topology of a separable metric space, and thereby makes $\mathrm{Aut}_{[\mu]}(S)$ into a topological group.

Remark 2.11. Note that $\mathrm{Aut}_{[\mu]}(S)$ is *not* locally compact. On the other hand, $\mathrm{Aut}_{[\mu]}(S)$ is a closed subset of $\mathbf{U}\big(L^2(S)\big)$ (because it consists of the operators that map nonnegative functions to nonnegative functions [GGM, §3]), so it is a Polish space.

Proposition 2.12 (Ramsay, cf. [Ra, Cor. 3.4]). *If (S,μ) is a standard Borel probability space, then $\mathrm{Aut}_{[\mu]}(S)$ acts continuously on the Borel algebra $\mathcal{B}(S)$.*

Proof. We wish to show that if ϕ is close to ϕ_0 in $\mathrm{Aut}_{[\mu]}(S)$, and A is close to A_0 in $\mathcal{B}(S)$, then $\mu\big(\phi(A) \smallsetminus \phi_0(A_0)\big)$ and $\mu\big(\phi_0(A_0) \smallsetminus \phi(A)\big)$ are close to 0. Thus, letting ψ be χ_A and χ_{A_0}, it suffices to show that if $\psi \in L^2(S)$, with $\|\psi\| \le 1$, then $\big|\int_{\phi(A)} \psi^2\,d\mu - \int_{\phi_0(A_0)} \psi^2\,d\mu\big|$ is close to 0. To simplify the notation, we replace ϕ and ϕ_0 by their inverses in the following calculation:

We have

$$
\begin{aligned}
&\left|\int_{\phi^{-1}(A)} \psi^2\,d\mu - \int_{\phi_0^{-1}(A_0)} \psi^2\,d\mu\right| \\
&\quad = \left|\int_A T_\phi(\psi)^2\,d\mu - \int_{A_0} T_{\phi_0}(\psi)^2\,d\mu\right| \\
&\quad \le \left|\int_A T_\phi(\psi)^2\,d\mu - \int_A T_{\phi_0}(\psi)^2\,d\mu\right| + \left|\int_A T_{\phi_0}(\psi)^2\,d\mu - \int_{A_0} T_{\phi_0}(\psi)^2\,d\mu\right| \\
&\quad = \left|\int_A \big(T_\phi(\psi) + T_{\phi_0}(\psi)\big)\big(T_\phi(\psi) - T_{\phi_0}(\psi)\big)\,d\mu\right| \\
&\qquad\quad + \left|\int_A T_{\phi_0}(\psi)^2\,d\mu - \int_{A_0} T_{\phi_0}(\psi)^2\,d\mu\right| \\
&\quad \le 2\sqrt{\int_A \big(T_\phi(\psi) - T_{\phi_0}(\psi)\big)^2\,d\mu} \qquad \text{(Hölder's Inequality and } \|\psi\| \le 1) \\
&\qquad\quad + \int_{A\Delta A_0} T_{\phi_0}(\psi)^2\,d\mu \qquad \text{(integrand is 0 on } A \cap A_0).
\end{aligned}
$$

By definition of the topology on $\mathbf{U}\big(L^2(S)\big)$, the first term in the final expression is small whenever ϕ is close to ϕ_0 (since ψ is fixed). Because $T_{\phi_0}(\psi)^2$ is a fixed L^1 function, the second term is small whenever $A\Delta A_0$ has sufficiently small measure [Ru, Exer. 6.10(a)]; that is, whenever A is sufficiently close to A_0 in $\mathcal{B}(S)$. Thus, $\left|\int_{\phi^{-1}(A)} \psi^2\,d\mu - \int_{\phi_0^{-1}(A_0)} \psi^2\,d\mu\right|$ is close to 0, as desired. □

The fact that the action on $\mathcal{B}(S)$ is continuous at the empty set $\emptyset$ can be restated as follows:

Corollary 2.13. *If $\phi_n \to \phi$ in $\mathrm{Aut}_{[\mu]}(S)$, and A_n is a sequence of Borel subsets of S, such that $\mu(A_n) \to 0$, then $\mu\big(\phi_n(A_n)\big) \to 0$.*

In the following result, we assume that S is a separable metric space, so that we can speak of convergence in measure. This is a very mild assumption, because any standard Borel space is, by definition, Borel isomorphic to such a space.

Corollary 2.14. *Let $\phi_n \to \phi$ in $\mathrm{Aut}_{[\mu]}(S)$, with S a separable metric space. Then $\phi_n \to \phi$ in measure.*

Proof. By considering $\phi_n \circ \phi^{-1}$ (and using the fact that the action of ϕ on $\mathcal{B}(S)$ is continuous at $\emptyset$), we may assume that $\phi = \mathrm{Id}$. Let $\{A^i\}_{i=1}^{\infty}$ be a partition of S into countably many Borel sets, such that $\mathrm{diam}(A^i) < \epsilon$ for each i. Let

$$\Sigma_n^i = \{s \in A^i \mid d\big(\phi_n(s), s\big) > \epsilon\}.$$

Then, for each fixed i, we have

$$\mu\big(\phi_n(\Sigma_i^n)\big) \leq \mu\big(\phi_n(A^i) \smallsetminus A^i\big) \leq d\big(\phi_n(A^i), A^i\big) \to 0 \text{ as } n \to \infty.$$

Thus, $\phi_n(\Sigma_n^i) \to \emptyset$ in $\mathcal{B}(S)$. Because $\mathrm{Aut}_{[\mu]}(S)$ acts continuously on $\mathcal{B}(S)$, this implies that

$$\mu(\Sigma_n^i) \to 0 \text{ as } n \to \infty.$$

Therefore

$$\begin{aligned}\lim_{n\to\infty} \mu\{s \in S \mid d\big(\phi_n(s), s\big) > \epsilon\} &= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{\infty} \Sigma_n^i\right) \\ &\leq \inf_{k\in\mathbb{N}} \lim_{n\to\infty} \mu\left(\left(\bigcup_{i=1}^{k} \Sigma_n^i\right) \cup \big(S \smallsetminus (A^1 \cup \cdots \cup A^k)\big)\right) \\ &= \lim_{k\to\infty} \mu\big(S \smallsetminus (A^1 \cup \cdots \cup A^k)\big) \\ &= 0.\end{aligned}$$

So $\phi_n \to \mathrm{Id}$ in measure. □

Definition 2.15. There is a natural action of $\mathrm{Aut}_{[\mu]}(S)$ on $\mathbf{F}(S,X)$, defined by $\phi(f) = f \circ \phi^{-1}$, for $\phi \in \mathrm{Aut}_{[\mu]}(S)$ and $f \in \mathbf{F}(S,X)$.

Proposition 2.16. *If (S,μ) is a standard Borel probability space, and (X,d) is a separable metric space, then the natural action of $\mathrm{Aut}_{[\mu]}(S)$ on $\mathbf{F}(S,X)$ is continuous.*

Proof. Suppose $\phi_n \to \phi$ in $\mathrm{Aut}_{[\mu]}(S)$, and $f_n \to f$ in $\mathbf{F}(S,X)$. Note that

$$d(f_n\phi_n, f\phi) \leq d(f_n\phi_n, f\phi_n) + d(f\phi_n, f\phi).$$

We have $d(f_n\phi_n, f\phi_n) \to 0$, because

$$\begin{aligned}\mu\left\{s \;\middle|\; d\big(f_n\phi_n(s), f\phi_n(s)\big)\big) > \epsilon\right\} &= \mu\Big(\phi_n^{-1}\big\{s \mid d\big(f_n(s), f(s)\big) > \epsilon\big\}\Big) \\ &\to 0,\end{aligned}$$

using (2.13) and the fact that $f_n \to f$ in measure to get the final limit.

Because (S,μ) is standard, there is no harm in assuming that S is a complete, separable metric space. Then, by Lusin's Theorem, there is a (large) subset K of S, such that f is uniformly continuous on K. Let

$$A_n = \phi^{-1}(S \smallsetminus K) \cup \phi_n^{-1}(S \smallsetminus K).$$

From (2.13), we see that, by requiring $\mu(K)$ to be sufficiently large, we may ensure that

$$\mu(A_n) < \epsilon.$$

Choose $\delta > 0$, such that $d\big(f(s), f(t)\big) < \epsilon$, for all $s, t \in K$ with $d_S(s,t) < \delta$. We have

$$\begin{aligned}\limsup d(f\phi_n, f\phi) &= \limsup \mu\big\{ s \mid d\big(f\phi_n(s), f\phi(s)\big) > \epsilon \big\} \\ &\leq \limsup \mu(A_n) + \lim \mu\big\{ s \mid d\big(\phi_n(s), \phi(s)\big) > \delta \big\} \\ &\leq \epsilon + 0,\end{aligned}$$

using (2.14) to obtain the term "0" in the final expression.

Since $\epsilon > 0$ is arbitrary, we conclude that $d(f_n\phi_n, f\phi) \to 0$. □

We will use the following easy observation:

Lemma 2.17. *If A is any Borel subset of any standard Borel probability space (S, μ), then the integration functional $I \colon \mathbf{F}(S, \mathbb{R}^{\geq 0}) \to \mathbb{R} \cup \{\infty\}$, defined by $I(f) = \int_A f \, d\mu$, is Borel.*

Proof. Let χ_A be the characteristic function of A. It is easy to see that the map $f \mapsto \chi_A f$ is a continuous function on $F(S, \mathbb{R}^{\geq 0})$, so we may (and will) assume $A = S$.

For each $n \in \mathbb{N}$, choose a continuous function $\xi_n \colon \mathbb{R}^{\geq 0} \to [0, n]$, such that

$$\xi_n(x) = \begin{cases} x & \text{if } x \leq n, \\ 0 & \text{if } x \geq n+1. \end{cases}$$

Note, for each $f \in \mathbf{F}(S, \mathbb{R}^{\geq 0})$, that the composition $\xi_n \circ f$ is bounded by n, and $\xi_n \circ f \uparrow f$ pointwise.

Because ξ_n is uniformly continuous (being a continuous function with compact support), it is easy to see that the map $\mathbf{F}(S, \mathbb{R}^{\geq 0}) \to \mathbf{F}(S, [0, n])$, defined by $f \mapsto \xi_n \circ f$, is continuous. Furthermore, the restriction of I to $\mathbf{F}(S, [0, n])$ is continuous. Thus, the function

$$I_n \colon \mathbf{F}(S, \mathbb{R}^{\geq 0}) \to \mathbb{R}^{\geq 0}, \text{ defined by } I_n(f) = I(\xi_n \circ f),$$

is continuous. The Monotone Convergence Theorem implies that

$$I(f) = \lim_{n \to \infty} I_n(f),$$

so I is a pointwise limit of continuous functions. Therefore, I is Borel. □

2B. Ergodic decomposition and near actions. A proof of the following folklore theorem has been provided by G. Greschonig and K. Scmidt [GS]. The statement here is slightly stronger than [GS, Thm. 1.1] (because Theorem 2.19(1) is more precise than [GS, Thm. 1.1(3)]), but it follows immediately from [GS, Thm. 5.2], by letting Ω be a $p_*\mu$-conull Borel subset of $\operatorname{Prob}(X)$, $X' = p^{-1}(\Omega)$, $\psi(x) = p(x)$, and $\xi(\omega) = \omega$.

Notation 2.18. For any standard Borel space X, we use $\operatorname{Prob}(X)$ to denote the space of all probability measures on X. It is well-known that $\operatorname{Prob}(X)$ is a standard Borel space, under an appropriate weak* topology.

Theorem 2.19 (Ergodic Decomposition [GS])**.** *Let:*

- *G be a locally compact second countable group,*
- *(X, μ) be a standard Borel probability space, and*
- *$\rho \colon G \times X \to X$ be a Borel action, such that μ is quasiinvariant.*

Then there exist:

- *a standard Borel probability space* (Ω, ν)*,*
- *a conull G-invariant Borel subset* X' *of* X*,*
- *a G-invariant Borel map* $\psi\colon X' \to \Omega$*, and*
- *a Borel map* $\xi\colon \Omega \to \operatorname{Prob}(X')$*,*

such that:

1. $\xi(\omega)\big(\psi^{-1}(\omega)\big) = 1$ *for each* $\omega \in \Omega$.
2. $\mu = \int_\Omega \xi(\omega)\, d\nu(\omega)$.
3. *For each* $\omega \in \Omega$*,* $\xi(\omega)$ *is quasiinvariant and ergodic.*

Remark 2.20. To simplify the notation in the conclusion of Theorem 2.19, we will often assume that the space X can be written as a Cartesian product $X = \Omega \times S$, such that:

- μ is the product measure on $X = \Omega \times S$.
- $\psi(\omega, s) = \omega$, for a.e. $(\omega, s) \in X$.

(For example, we make this assumption in the statement of Proposition 2.22.) The following decomposition theorem of V. A. Rohlin [Ro] asserts that (up to isomorphism) the general case is a countable union of examples of this type.

Proposition 2.21 (Rohlin)**.** *Assume the notation of Theorem* 2.19*. There is a partition of* Ω *into countably many Borel sets* $\Omega_1, \Omega_2, \ldots$ (*some of these sets may be empty*)*, such that, for each* k*, there exist:*

1. *a conull subset* X'_k *of* $\psi^{-1}(\Omega_k)$*,*
2. *a standard Borel probability space* (S_k, μ_k)*,*
3. *a* $(\psi_*\mu \times \mu_k)$*-conull subset* $(\Omega_k \times S_k)'$ *of* $\Omega_k \times S_k$*, and*
4. *a measure-class-preserving Borel isomorphism* $\theta\colon X'_k \to (\Omega_k \times S_k)'$*,*

such that $\psi(x) = \pi_1\big(\theta(x)\big)$ *for each* $x \in X'_k$*, where* $\pi_1(\omega, s) = \omega$.

Proof. Say that two standard Borel probability spaces (S_1, μ_1) and (S_2, μ_2) are *of the same type* if there exists a measure-class-preserving Borel isomorphism from a conull subset of S_1 onto a conull subset of S_2 [Ro, pp. 10–11]. This is obviously an equivalence relation. It has only countably many equivalence classes [Ro, p. 18]. Thus, there is a decomposition $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots$, such that if ω and ω' belong to the same Ω_k, then the fibers $\big(\psi^{-1}(\omega), \xi(\omega)\big)$ and $\big(\psi^{-1}(\omega'), \xi(\omega')\big)$ are of the same type [Ro, (I), p. 41]. This implies that, modulo sets of measure 0, $\psi^{-1}(\Omega_k)$ is Borel isomorphic to the Cartesian product $\Omega_k \times \psi^{-1}(\omega)$, for any $\omega \in \Omega_k$ [Ro, p. 42]. □

Proposition 2.22. *Let:*

- G *be a second countable, locally compact group,*
- (Ω, ν) *and* (S, μ) *be standard Borel probability spaces,*
- Ω' *be a conull Borel subset of* Ω*, and*
- $\rho\colon G \times (\Omega \times S) \to \Omega \times S$ *be a Borel action of* G *on* $\Omega \times S$*, such that, for each* $\omega \in \Omega'$*, the probability measure* μ_ω *on* $\{\omega\} \times S$ (*induced by the natural isomorphism with* S) *is quasiinvariant.*

Then:

1. *There are Borel maps* $\rho_S\colon G \times \Omega' \times S \to S$ *and* $\rho_{\mathrm{Aut}}\colon G \times \Omega' \to \operatorname{Aut}_{[\mu]}(S)$*, defined by*

$$\rho(g, \omega, s) = \big(\omega, \rho_S(g, \omega, s)\big) = \big(\omega, \rho_{\mathrm{Aut}}(g, \omega)(s)\big) \text{ for a.e. } s \in S \tag{2.23}$$

for each $(g,\omega) \in G \times \Omega'$.

2. *There is a Borel function* $D\colon G \times \Omega \times S \to \mathbb{R}^{\geq 0}$, *such that*
$$\int_S D(g,\omega,s) f(s)\, d\mu(s) = \int_S f\big(\rho_S(g,\omega,s)\big)\, d\mu(s)$$
for every $g \in G$, $\omega \in \Omega$, *and* $f \in \mathbf{F}(S,\mathbb{R}^{\geq 0})$.

Remark 2.24. The function D in conclusion (2) above is a *fiberwise* Radon-Nikodym derivative. In particular, in the case when Ω is a one point set, conclusion (2) implies that the Radon-Nikodym derivative is a Borel function on $G \times S$.

Proof of Proposition 2.22. Equation (2.23) determines a well-defined function $\rho_{\mathrm{Aut}}\colon G \times \Omega' \to \mathbf{F}(S,S)$. Because G is a group, we know that $\rho_{\mathrm{Aut}}(g,\omega)$ is (essentially) a Borel automorphism. Because μ_ω is quasiinvariant, we see that $\rho_{\mathrm{Aut}}(g,\omega)$ is measure-class preserving. Thus, $\rho_{\mathrm{Aut}}(g,\omega) \in \mathrm{Aut}_{[\mu]}(S)$, so ρ_{Aut} is actually a map into $\mathrm{Aut}_{[\mu]}(S)$. Thus, in order to complete the proof of (1), it only remains to show that ρ_{Aut} is Borel. For this, we will use the conclusion of (2), so let us establish the latter.

(2) Let $\{A_n\}$ be a countable, dense subset of the Borel algebra $\mathcal{B}(S)$, and define
$$D^{\Delta} = \bigcap_{n \in \mathbb{N}} \left\{ (g,\omega,f) \;\middle|\; \mu\big(\rho_{\mathrm{Aut}}(g,\omega)(A_n)\big) = \int_{A_n} f\, d\mu \right\} \subset G \times \Omega' \times \mathbf{F}(S,\mathbb{R}^{\geq 0}).$$
For each $g \in G$ and $\omega \in \Omega'$, the fiber $\{s \in S \mid (g,\omega,s) \in D^{\Delta}\}$ of D^{Δ} consists precisely of the Radon-Nikodym derivative of the transformation $\rho_{\mathrm{Aut}}(g,\omega)$. Therefore, D^{Δ} is the graph of a function $\check{D}\colon G \times \Omega' \to \mathbf{F}(S,\mathbb{R}^{\geq 0})$, and $\check{D}(g,\omega)$ is the Radon-Nikodym derivative of $\rho_{\mathrm{Aut}}(g,\omega)$.

Note:

- Because the Borel map $(g,\omega,s) \mapsto \big(g,\omega,\rho_S(g,\omega,s)\big)$ is injective, we know that it maps $G \times \Omega' \times A_n$ (or any other Borel set) to a Borel subset of $G \times \Omega' \times S$. So Fubini's Theorem implies that $\mu\Big(\rho_S\big((g,\omega) \times A_n\big)\Big)$ is a Borel function of (g,ω).
- Lemma 2.17 implies that $\int_{A_n} f\, d\mu$ is a Borel function of f.

Therefore D^{Δ} is a Borel set, so the corresponding function $\check{D}$ is a Borel function. Then the desired Borel function $D\colon (G \times \Omega') \times S \to \mathbb{R}^{\geq 0}$ is obtained by applying Lemma 2.7.

(1) Given L^2 functions $f, g\colon S \to \mathbb{R}$, and any $\epsilon > 0$, define
$$\Theta\colon G \times \Omega' \times S \to \mathbb{R}^{\geq 0}$$
by
$$\begin{aligned}\Theta(g,\omega,s) &= \big|T_{\rho_{\mathrm{Aut}}(g,\omega)} f(s) - g(s)\big|^2 \\ &= \big|D(g,\omega,s)^{1/2} f\big(\rho_{\mathrm{Aut}}(g,\omega)^{-1}(s)\big) - g(s)\big|^2,\end{aligned}$$
where $D(g,\omega,s)$ is given by 2.22(2). Then Θ is a Borel function on $G \times \Omega' \times S$, so Fubini's Theorem implies that
$$\left\{ (g,\omega) \in G \times \Omega' \;\middle|\; \int_S \Theta(g,\omega,s) < \epsilon^2 \right\}$$

is Borel. In other words, if $\mathcal{U}(f,g,\epsilon)$ is any basic open set in $\mathbf{U}\big(L^2(S)\big)$, then $\rho_{\mathrm{Aut}}^{-1}\big(\mathcal{U}(f,g,\epsilon)\big)$ is Borel. So ρ_{Aut} is Borel. □

Corollary 2.25. *Let:*

- *G, Ω, S, Ω', and ρ be as in Proposition 2.22, and*
- *X be a complete, separable metric space.*

Then the action ρ induces a Borel map $\rho_{\mathbf{F}}\colon G\times\Omega'\times\mathbf{F}(S,X)\to\mathbf{F}(S,X)$, defined by

$$\rho_{\mathbf{F}}(g,\omega,f)(s)=f\big(\rho_S(g,\omega,s)\big)\quad \textit{for a.e. } s. \tag{2.26}$$

Proof. The existence of an abstract function $\rho_{\mathbf{F}}$ satisfying (2.26) is not an issue. Because ρ_{Aut} is Borel (see 2.22(1)), and the natural action of $\mathrm{Aut}_{[\mu]}(S)$ on $\mathbf{F}(S,X)$ is continuous (see 2.16), we know that $\rho_{\mathbf{F}}$ is Borel. □

2C. Borel cocycles. We assume the basic theory of Borel cocycles, as in [Mr, §7.2] or [Z2, §4.2].

Definition 2.27 ([Z2, Defns. 4.2.1 and 4.2.2])**.** Suppose H is a topological group, and $\rho\colon G\times X\to X$ is a Borel action of a locally compact group G on a standard Borel probability space X with quasiinvariant measure μ.

1. A Borel function $\alpha\colon G\times X\to H$ is a Borel *cocycle* (for the action ρ) if, for all $g_1,g_2\in G$, we have

$$\alpha(g_1g_2,x)=\alpha\big(g_1,\rho(g_2,x)\big)\,\alpha(g_2,x)\quad \text{for a.e. } x\in X. \tag{2.28}$$

2. The cocycle α is *strict* if the equality in (2.28) holds for *all* $x\in X$, not merely almost all.
3. Two Borel cocycles $\alpha,\beta\colon G\times X\to H$ are *cohomologous* if there is a Borel function $\phi\colon X\to H$, such that, for each $g\in G$, we have

$$\beta(g,x)=\phi\big(\rho(g,x)\big)\,\alpha(g,x)\,\phi(x)^{-1}\quad \text{for a.e. } x\in X.$$

   This is an equivalence relation.

Remark 2.29. There is usually no harm in assuming that a Borel cocycle is strict, because any Borel cocycle is equal a.e. to a strict Borel cocycle [Z2, Thm. B.9, p. 200]. More precisely, if α is a Borel cocycle, then there is a strict Borel cocycle α', such that, for each $g\in G$, $\alpha(g,x)=\alpha'(g,x)$ for a.e. $x\in X$.

Lemma 2.30. *Let:*

- *(S,μ) be a standard Borel probability space,*
- *X, Y, and Z be complete, separable, locally compact metric spaces,*
- *$\tau\colon X\times Y\to Z$ be a continuous function.*

Then the induced map $\tau^{\mathbf{F}}\colon\mathbf{F}(S,X)\times\mathbf{F}(S,Y)\to\mathbf{F}(S,Z)$, defined by

$$\tau^{\mathbf{F}}(\phi,\psi)(s)=\tau\big(\phi(s),\psi(s)\big),$$

is continuous.

Proof. Given sequences $\phi_n\to\phi\in\mathbf{F}(S,X)$, $\psi_n\to\psi\in\mathbf{F}(S,Y)$, and $\epsilon>0$, Lusin's Theorem gives us a compact subset K of S, such that ϕ and ψ are continuous on K, and $\mu(K)>1-\epsilon$. Because X and Y are locally compact, we may let K_X and K_Y

be compact neighborhoods of $\phi(K)$ and $\psi(K)$ in X and Y, respectively. Because τ is uniformly continuous on $K_X \times K_Y$, there is some $\delta > 0$, such that

$$\begin{aligned}
\limsup_{n\to\infty} \mu \left\{ s \in S \;\middle|\; d\Big(\tau\big(\phi_n(s), \psi_n(s)\big), \tau\big(\phi(s), \psi(s)\big)\Big) > \epsilon \right\} \\
\leq \limsup_{n\to\infty} \mu\{ s \in S \mid \phi_n(s) \notin K_X \text{ or } \psi_n(s) \notin K_Y \} \\
+ \limsup_{n\to\infty} \mu\{ s \in S \mid d\big(\phi_n(s), \phi(s)\big) > \delta \} \\
+ \limsup_{n\to\infty} \mu\{ s \in S \mid d\big(\psi_n(s), \psi(s)\big) > \delta \} \\
\leq 2\epsilon + 0 + 0.
\end{aligned}$$

Because $\epsilon > 0$ is arbitrary, we conclude that $\tau^{\mathbf{F}}(\phi_n, \psi_n) \to \tau^{\mathbf{F}}(\phi, \psi)$. □

Corollary 2.31 (cf. [Mr, Rmk. after Lem. 7.2.1, p. 217])**.** *Let:*

- *G, Ω, S, Ω', and ρ be as in Proposition* 2.22*,*
- *H be a locally compact, second countable group,*
- *X be a complete, separable metric space,*
- *$\tau \colon H \times X \to X$ be a continuous action of H on X, and*
- *$\alpha \colon G \times (\Omega \times S) \to H$ be a Borel cocycle.*

Then the function $\rho^{\alpha,\tau} \colon G \times \Omega' \times \mathbf{F}(S, X) \to \mathbf{F}(S, X)$, defined by

$$\rho^{\alpha,\tau}(g, \omega, \phi)(s) = \tau\Big(\alpha(g, \omega, s)^{-1}, \phi\big(\rho_S(g, \omega, s)\big)\Big),$$

is Borel.

Proof. Because the map $\rho_{\mathbf{F}}$ of Corollary 2.25 is Borel, the action ρ does not affect the measurability of $\rho^{\alpha,\tau}$, so it may be ignored. Furthermore, by replacing Ω with $G \times \Omega'$, we may assume that G is trivial and $\Omega' = \Omega$; in particular, G may be ignored. Thus:

- α is a Borel map from $\Omega \times S$ to H, and
- $\rho^{\alpha,\tau} \colon \Omega \times \mathbf{F}(S, X) \to \mathbf{F}(S, X)$ is defined by

$$\rho^{\alpha,\tau}(\omega, \phi)(s) = \tau\big(\alpha(\omega, s)^{-1}, \phi(s)\big).$$

Then, because $\breve{\alpha} \colon \Omega \to \mathbf{F}(S, X)$ and $\tau^{\mathbf{F}}$ are Borel (see 2.6(2) and 2.30), we conclude that $\rho^{\alpha,\tau}$ is a composition of Borel functions. Therefore, it is Borel. □

Although we do not need the following result in this paper, we include the proof to provide a convenient reference.

Corollary 2.32 ([Mr, Rmk. after Lem. 7.2.1, p. 217])**.** *Let:*

- *G and H be second countable, locally compact groups,*
- *(S, μ) be a standard Borel probability space,*
- *$\rho \colon G \times S \to S$ be a Borel action of G on S, such that μ is quasiinvariant,*
- *$\alpha \colon G \times S \to H$ be a strict Borel cocycle, and*
- *$\tau \colon H \times X \to X$ be a continuous action of H on X.*

Then:

1. *The function $\breve{\alpha} \colon G \to \mathbf{F}(S, H)$, induced by α (see* 2.6*), is continuous.*
2. *The α-twisted action $\zeta_{\rho,\tau,\alpha}$ of G on $\mathbf{F}(S, H)$ is continuous (see Defn.* 5.1(1)*).*

Proof. Let $\mathbf{1}\colon G \times S \to H$ be the trivial cocycle, defined by $\mathbf{1}(g,s) = e$ (the identity element of H). For convenience, let $\mathbf{F} = \mathbf{F}(S,H)$.

Step 1. *The action* $\rho^{\mathbf{1},\alpha}$ *is continuous.* From Proposition 2.22(1) (with Ω consisting of a single point), we know that ρ induces a Borel function $\rho_{\mathrm{Aut}}\colon G \to \mathrm{Aut}_{[\mu]}(S)$. This function is a homomorphism, and any measurable homomorphism into a second countable topological group is continuous [Z2, Thm. B.3, p. 198], so we conclude that ρ_{Aut} is continuous. Because the action of $\mathrm{Aut}_{[\mu]}(S)$ on $\mathbf{F}$ is continuous (see 2.16), we conclude that $\rho^{\mathbf{1},\alpha}$ is continuous.

Step 2. $\breve{\alpha}$ *is continuous.* Note that $\mathbf{F}$ is a (second countable) topological group under pointwise multiplication (cf. 2.30). Furthermore, $\rho^{\mathbf{1},\alpha}$ is a continuous action of G on $\mathbf{F}$ by automorphisms, so we may form the semidirect product $G \ltimes \mathbf{F}$. The cocycle identity implies that the Borel function $\alpha^{G\ltimes\mathbf{F}}\colon G \to G \ltimes \mathbf{F}$, defined by $\alpha^{G\ltimes\mathbf{F}}(g,f) = \big(g, \breve{\alpha}(g)\big)$ is a homomorphism. Because (as noted above) measurable homomorphisms are continuous, we conclude that $\breve{\alpha}$ is continuous.

Step 3. $\rho^{\tau,\alpha}$ *is continuous.* Because $\rho^{\tau,\alpha}(g,f) = \tau^{\mathbf{F}}\big(\breve{\alpha}(g), \rho^{\mathbf{1},\alpha}(g,f)\big)$, this conclusion can be obtained by combining Steps 1 and 2 with Lemma 2.30. □

3. Restricting cocycles to ergodic components

This section uses the von Neumann Selection Theorem (2.3) to obtain information about a cocycle from its restrictions to ergodic components. The main result is (3.4); the others are corollaries.

Notation 3.1. Throughout this section we fix the following notations and conventions:

- G, Ω, S, Ω' and ρ are as in Proposition 2.22 (or, equivalently, as in Theorem 3.4 below).
- H is a locally compact, second countable group.
- $\alpha\colon G \times (\Omega \times S) \to H$ is a strict Borel cocycle.

For $\omega \in \Omega'$, define $\rho_\omega\colon G \times S \to S$ and $\alpha_\omega\colon G \times S \to H$ by

$$\rho_\omega(g,s) = \rho_S(g,\omega,s) \quad \text{and} \quad \alpha_\omega(g,s) = \alpha(g,\omega,s),$$

where $\pi_S(*,s) = s$.

Definition 3.2 ([Ra, Lem. 3.1], [Z1, Defn. 3.1])**.** Suppose G is a locally compact second countable group, and (S,μ) is a standard Borel probability space. A Borel map $\rho\colon G \times S \to S$ is a *near action* of G on S if:

- for all $g_1, g_2 \in G$, we have $\rho(g_1 g_2, s) = \rho\big(g_1, \rho(g_2,s)\big)$ for a.e. $s \in S$,
- $\rho(e,s) = s$ for a.e. $s \in S$, and
- each $g \in G$ preserves the measure class of μ.

Note that the definition of a cocycle (2.27) can be applied to near actions, not only actions.

Let us record the following elementary observation. Part (1) follows easily from the assumption that μ_ω is quasiinvariant. The other two parts are consequences of the first.

Lemma 3.3. *Assume the setting of Notation* 3.1*, and let* $\omega \in \Omega'$*. Then:*

1. $\rho(\omega,s) \in \{\omega\} \times S$*, for a.e.* $s \in S$*.*

2. *ρ_ω is a near action of G on S.*
3. *α_ω is a Borel cocycle for ρ_ω.*

Theorem 3.4. *Let:*

- *G and H be second countable, locally compact groups,*
- *(Ω, ν) and (S, μ) be standard Borel probability spaces,*
- *Ω' be a conull subset of Ω,*
- *$\rho\colon G \times (\Omega \times S) \to \Omega \times S$ be a Borel action of G on $\Omega \times S$, such that, for each $\omega' \in \Omega$, the probability measure μ_ω on $\{\omega\} \times S$ (induced by the natural isomorphism with S) is quasiinvariant,*
- *$\alpha\colon G \times (\Omega \times S) \to H$ be a strict Borel cocycle,*
- *$\mathcal{F}$ be an analytic subset of $\Omega \times \mathbf{F}(G \times S, H)$, and*
- *$\Omega_{\mathcal{F}} = \{\omega \in \Omega' \mid \alpha_\omega$ is cohomologous to a cocycle in $\mathcal{F}_\omega\}$, where*

$$\mathcal{F}_\omega = \{f \in \mathbf{F}(G \times S, H) \mid (\omega, f) \in \mathcal{F}\}.$$

Then:

1. *$\Omega_{\mathcal{F}}$ is analytic.*
2. *α is cohomologous to a Borel cocyle $\beta\colon G \times (\Omega \times S) \to H$, such that $\beta_\omega \in \mathcal{F}_\omega$, for a.e. $\omega \in \Omega_{\mathcal{F}}$.*

Proof. The cocycle α defines a Borel map $\check{\alpha}\colon \Omega \to \mathbf{F}(G \times S, H)$ (see 2.6(2)). Define

$$\delta\colon \Omega' \times \mathbf{F}(G \times S, H) \times \mathbf{F}(S, H) \to \mathbf{F}(G \times S, H)$$

by

$$\delta(\omega, \phi, f)(g, s) = f\big(\rho_\omega(g, s)\big)\, \phi(g, s)\, f(s)^{-1}. \tag{3.5}$$

We claim that δ is Borel.

- We know that the map $\rho_{\mathbf{F}}$ (defined in Corollary 2.25) is Borel. This induces a Borel map $\check{\rho}_{\mathbf{F}}\colon \Omega' \times \mathbf{F}(S, H) \to \mathbf{F}\big(G, \mathbf{F}(S, H)\big)$ (see 2.6). From Corollary 2.9, we see that we may think of this as a map into $\mathbf{F}(G \times S, H)$. Thus, the first factor on the right-hand side of (3.5) represents a Borel function from $\Omega' \times \mathbf{F}(S, H)$ into $\mathbf{F}(G \times S, H)$.
- The second factor on the right-hand side of (3.5) represents the identity function on $\mathbf{F}(G \times S, H)$, and the term $f(s)$ represents the inclusion of $\mathbf{F}(S, H)$ into $\mathbf{F}(G \times S, H)$. These are obviously Borel maps into $\mathbf{F}(G \times S, H)$.

Because pointwise multiplication and pointwise inversion are continuous operations on $\mathbf{F}(G \times S, H)$ (see 2.30), we conclude that δ is Borel, as claimed.

Therefore, the function

$$\sigma\colon \Omega' \times \mathbf{F}(S, H) \to \Omega' \times \mathbf{F}(G \times S, H),$$

defined by

$$\sigma(\omega, f) = \Big(\omega, \delta\big(\omega, \check{\alpha}(\omega), f\big)\Big),$$

is Borel, so $\sigma^{-1}(\mathcal{F})$ is analytic. Then, because $\Omega_{\mathcal{F}}$ is the projection of $\sigma^{-1}(\mathcal{F})$ to Ω', we conclude that $\Omega_{\mathcal{F}}$ is analytic. This establishes (1).

The von Neumann Selection Theorem (2.3) implies that there is a Borel function $\hat{\Phi}\colon \Omega' \to \mathbf{F}(S, H)$, such that $\sigma\big(\omega, \hat{\Phi}(\omega)\big) \in \mathcal{F}$, for a.e. $\omega \in \Omega_{\mathcal{F}}$. Corresponding to $\hat{\Phi}$, there is a Borel function $\Phi\colon \Omega' \times S \to H$ (see 2.7). Letting

$$\beta(g, \omega, s) = \delta\big(\omega, \check{\alpha}(\omega), \hat{\Phi}(\omega)\big)(g, s) = \Phi\big(\rho(g, \omega, s)\big)\, \alpha(g, \omega, s)\, \Phi(\omega, s)^{-1},$$

we obtain (2). □

Corollary 3.6. *Let:*

- *G and H be locally compact, second countable groups,*
- *(X, μ) be a standard Borel probability space,*
- *$\rho\colon G \times X \to X$ be a Borel action, such that μ is quasiinvariant,*
- *$\psi\colon X' \to \Omega$ be the corresponding ergodic decomposition (see 2.19), and*
- *$\alpha, \beta\colon G \times X \to H$ be Borel cocycles.*

For each $\omega \in \Omega$, let α_ω and β_ω be the restrictions of α and β to $G \times \psi^{-1}(\omega)$. Then:

1. *There is a conull Borel subset Ω' of Ω, such that, for each $\omega \in \Omega'$, the maps α_ω and β_ω are Borel cocycles.*
2. *$\{\omega \in \Omega' \mid \alpha_\omega$ is cohomologous to $\beta_\omega\}$ is an analytic subset of Ω.*
3. *If α_ω is cohomologous to β_ω, for a.e. $\omega \in \Omega'$, then α is cohomologous to β.*

Proof. From Proposition 2.21 (and Rem. 2.20), we may assume the notation of Theorem 3.4. By changing α and β on a null set, we may assume these cocycles are strict (see 2.29).

Conclusion (1) is immediate from Lemma 3.3(3).

Recall that β induces a Borel function $\check{\beta}\colon \Omega' \to \mathbf{F}(G \times S, H)$, defined by $\check{\beta}(\omega) = \beta_\omega$ (see 2.6(2)). Let

$$\mathcal{F} = \{(\omega, \check{\beta}(\omega) \mid \omega \in \Omega'\} \subset \Omega' \times \mathbf{F}(G \times S, H).$$

Because $\mathcal{F}$ is an analytic subset (in fact, it is closed), (2) is immediate from 3.4(1).

Assume, now, that α_ω is cohomologous to β_ω, for a.e. $\omega \in \Omega'$. From 3.4(2) and Fubini's Theorem, we conclude that α is cohomologous to a cocycle $\widetilde{\alpha}$, such that for a.e. $g \in G$,

(3.7) for a.e. $x \in \Omega' \times S$, we have $\widetilde{\alpha}(g, x) = \beta(g, x)$.

From the cocycle identity, one easily concludes that (3.7) must hold for every $g \in G$, not merely almost every g. Therefore $\widetilde{\alpha}$ is (obviously) cohomologous to β. By transitivity, then α is also cohomologous to β; this establishes (3). □

Definition 3.8. Suppose $\rho\colon G \times X \to X$ is a Borel action with quasiinvariant measure, and H is a locally compact second countable group.

1. The *trivial cocycle* $\mathbf{1}_{G\times X}\colon G \times X \to H$ is defined by $\mathbf{1}_{G\times X}(g, x) = e$.
2. A Borel cocycle $\alpha\colon G \times X \to H$ is a *coboundary* if it is cohomologous to the trivial cocycle.

Corollary 3.9. *Let G, H, X, ρ, ψ, Ω, α, α_ω, and Ω' be as in Corollary 3.6. Then:*

1. *$\{\omega \in \Omega' \mid \alpha_\omega$ is a coboundary$\}$ is an analytic subset of Ω.*
2. *If α_ω is a coboundary, for a.e. $\omega \in \Omega'$, then α is a coboundary.*

Proof. Let β be the trivial cocycle, and apply Corollary 3.6. □

Definition 3.10. Recall that a Borel cocycle $\alpha\colon G \times S \to H$ is a *constant* (or *homomorphism*) cocycle if $\alpha(g, s)$ is essentially independent of s, for each $g \in G$.

Corollary 3.11. *Let G, H, X, ρ, ψ, Ω, α, α_ω, and Ω' be as in Corollary 3.6. If α_ω is cohomologous to a constant cocycle, for a.e. $\omega \in \Omega'$, then α is cohomologous to a constant cocycle.*

Proof. Similar to the proof of Corollary 3.6(3), but with $\mathcal{F} = \Omega' \times \mathrm{Const}$, where

$$\mathrm{Const} = \{f \colon G \times S \to H \mid f(g,s) \text{ is essentially independent of } s\}. \qquad \square$$

Notation 3.12. We use $\mathrm{Cpct}(H)$ to denote the set of compact subgroups of a locally compact, second countable group H. It is well-known that this is a complete, separable metric space, under the Hausdorff metric

$$d(K_1, K_2) = \max_{k_1 \in K_1} \mathrm{dist}(k_1, K_2) + \max_{k_2 \in K_2} \mathrm{dist}(K_1, k_2),$$

where dist is any metric on H.

Corollary 3.13. *Let G, H, X, ρ, ψ, Ω, α, α_ω, and Ω' be as in Corollary* 3.6. *If, for a.e. $\omega \in \Omega'$, the cocycle α_ω is cohomologous to a cocycle whose essential range is contained in a compact subgroup of H, then there are:*

- *a Borel function $\kappa \colon \Omega' \to \mathrm{Cpct}(H)$, and*
- *a Borel cocycle β that is cohomologous to α,*

such that the essential range of β_ω is contained in $\kappa(\omega)$, for a.e. $\omega \in \Omega'$.

Proof. As in the proof of Corollary 3.6, we assume the notation of Theorem 3.4, and we assume the cocycle α is strict. Let

$$\mathcal{F}^+ = \{(f, K) \in \mathbf{F}(G \times S, H) \times \mathrm{Cpct}(H) \mid \mathrm{EssRg}(f) \subset K\},$$

and let $\mathcal{F}$ be the projection of $\mathcal{F}^+$ to $\mathbf{F}(G \times S, H)$. Then $\mathcal{F}^+$ is closed, so $\mathcal{F}$ is analytic. Applying Theorem 3.4(2) yields a Borel cocycle β, cohomologous to α, such that $\beta_\omega \in F$, for a.e. $\omega \in \Omega'$. Now the Borel function κ is obtained from the von Neumann Selection Theorem (2.3). $\square$

Corollary 3.14. *Let G, H, X, ρ, ψ, Ω, α, α_ω, and Ω' be as in Corollary* 3.6. *If, for a.e. $\omega \in \Omega'$, there are a compact subgroup K_ω of H and a Borel cocycle β_ω, cohomologous to α_ω, such that:*

a. *the essential range of β_ω is contained in the normalizer $N_H(K_\omega)$, and*
b. *the induced cocycle $\overline{\beta_\omega} \colon G \times S \to N_H(K_\omega)/K_\omega$ is a homomorphism cocycle,*

then there are:

1. *a Borel cocycle β, cohomologous to α, and*
2. *a Borel function $\kappa \colon \Omega \to \mathrm{Cpct}(H)$,*

such that (a) *and* (b) *hold with $\beta_\omega = \check{\beta}(\omega)$ and $K_\omega = \kappa(\omega)$, for a.e. $\omega \in \Omega'$.*

Proof. Let

$$\mathcal{F}^+ = \left\{ (f, K) \in \mathbf{F}\big(G, \mathbf{F}(S, H)\big) \times \mathrm{Cpct}(H) \;\middle|\; \begin{array}{l} \text{for a.e. } g \in G, \ \exists h \in N_H(K), \\ \text{such that } \mathrm{EssRg}\big(f(g)\big) \subset h\,K \end{array} \right\}.$$

Then $\mathcal{F}^+$ is closed, so the proof is similar to that of Corollary 3.13. (Recall that $\mathbf{F}\big(G, \mathbf{F}(S, H)\big)$ is naturally homeomorphic to $\mathbf{F}(G \times S, H)$ (see 2.9).) $\square$

4. Superrigidity for nonergodic actions

One main application of the results in this paper is to prove general versions of superrigidity for cocycles for nonergodic actions of certain groups. We now define this class of groups. Let I be a finite index set and for each $i \in I$, we let k_i be a local field of characteristic zero and $\mathbb{G}_i$ be a connected simply connected semisimple algebraic k_i-group. We first define groups G_i, and then let $G = \prod_{i \in I} G_i$. If k_i is non-Archimedean, $G_i = \mathbb{G}_i(k_i)$ the k_i-points of $\mathbb{G}_i$. If k_i is Archimedean, then G_i is either $\mathbb{G}_i(k_i)$ or its topological universal cover. (This makes sense, since when $\mathbb{G}_i$ is simply connected and k_i is Archimedean, $\mathbb{G}_i(k_i)$ is topologically connected.) We assume that the k_i-rank of any simple factor of any $\mathbb{G}_i$ is at least two.

We will need one assumption on the cocycles we consider.

Definition 4.1. Let D be a locally compact group, (S,μ) a standard probability measure space on which D acts preserving μ and H be a normed topological group. We call a cocycle $\alpha\colon D\times S\to H$ over the D action *D-integrable* if for any compact subset $M \subset D$, the function $Q_{M,\alpha}(x) = \sup_{m\in M} \ln^+ \|\alpha(m,x)\|$ is in $L^1(S)$ (recall that $\ln^+ x = \max(\ln x, 0)$).

Any continuous cocycle over a continuous action on a compact topological space is automatically D-integrable. We remark that a cocycle over a cyclic group action is D-integrable if and only if $\ln^+ \|(\alpha(\pm 1, x)\|$ is in $L^1(S)$.

We first recall the superrigidity theorems from [FM] for ergodic actions.

Theorem 4.2. *Let G be as above, let (S,μ) be a standard probability measure space and let H be the k points of a k-algebraic group where k is a local field of characteristic* 0*. Assume G acts ergodically on S preserving μ. Let $\alpha\colon G\times S\to H$ be a G-integrable Borel cocycle. Then α is cohomologous to a cocycle β where $\beta(g,x) = \pi(g)c(g,x)$. Here $\pi\colon G\to H$ is a continuous homomorphism and $c\colon G\times S\to C$ is a cocycle taking values in a compact group centralizing $\pi(G)$.*

Theorem 4.3. *Let G, S, H and μ be as Theorem* 4.2 *and let $\Gamma < G$ be a lattice. Assume Γ acts ergodically on S preserving μ. Assume $\alpha\colon \Gamma\times S\to H$ is a Γ-integrable, Borel cocycle. Then α is cohomologous to a cocycle β where $\beta(\gamma,x) = \pi(\gamma)c(\gamma,x)$. Here $\pi\colon G\to H$ is a continuous homomorphism of G and $c\colon \Gamma\times X\to C$ is a cocycle taking values in a compact group centralizing $\pi(G)$.*

To state nonergodic versions of the above theorems, we will need a Borel structure on the space of homomorphisms for G to H. Given G as above and H as in Theorem 4.2, it is well-known that there are only finitely many conjugacy classes of homomorphisms $\pi\colon G\to H$. We choose a set $\Pi = \{\pi_i\}$ of representatives and endow it with the discrete topology, so as to be able to consider measurable maps to Π.

Given a group D acting on a standard probability measure space (X,μ), we denote by Ω the space of ergodic components of the action and let $p\colon S\to\Omega$ be the natural projection. We now state the general versions of the superrigidity theorems above.

Theorem 4.4. *Let G be as above, let (X,μ) be a standard probability measure space and let H be the k points of a k-algebraic group where k is a local field of characteristic* 0*. Assume G acts on S preserving μ. Let $\alpha\colon G\times S\to H$ be a G-integrable Borel cocycle. Then there exist measurable maps $\pi\colon \Omega\to\Pi, \kappa\colon \Omega\to \mathrm{Cpct}(H)$ and $\phi\colon S\to H$ with $\kappa(p(x))\subset Z_H(\pi(p(x))$ almost everywhere such that*

$$\alpha(g,x) = \phi(gx)^{-1}\beta(g,x)\phi(x)$$

where $\beta(g,x) = \pi(p(x))(g)c(g,x)$. Here $c\colon G\times S\to H$ is a measurable cocycle with $c(g,x)\in\kappa(p(x))$ almost everywhere.

Theorem 4.5. *Let G, S, H and μ be as Theorem* 4.4 *and let $\Gamma < G$ be a lattice. Assume Γ acts on X preserving μ. Assume $\alpha\colon \Gamma\times S\to H$ is a Γ-integrable, Borel* *cocycle. Then there exist measurable maps* $\pi\colon \Omega\to\Pi$, $\kappa\colon \Omega\to \mathrm{Cpct}(H)$ *and* $\phi\colon S\to H$ *with* $\kappa(p(x))\subset Z_H(\pi(p(x)))$ *such that* $\alpha(\gamma,x) = \phi(\gamma x)^{-1}\beta(\gamma,x)\phi(x)$ *where* $\beta(\gamma,x) = \pi(p(x))(\gamma)c(\gamma,x)$. *Here* $c\colon \Gamma\times S\to H$ *is a measurable cocycle with* $c(\gamma,x)\in\kappa(p(x))$ *almost everywhere.*

Proof of Theorems 4.4 and 4.5. These are an immediate consequence of Theorems 4.2 and 4.3, Corollary 3.14, and Proposition 2.21. Moreover, one can also prove these results by using the proof of Theorems 4.2 and 4.3 from [FM], Proposition 2.21 and Corollary 5.6 below. □

There are also versions of Theorems 4.4 and 4.5 which do not require that we assume the cocycle is G-integrable and versions, in that context, where the class of G considered can be somewhat broader, i.e., G of rank at least 2, with some/all simple factors of rank 1. To remove the G-integrability assumption requires assumptions on the algebraic hull of the cocycle, while weakening the rank assumption requires both assumptions on the algebraic hull and the assumption that the G action on each ergodic component of (X,μ) is weakly irreducible. These assumptions are less natural in the nonergodic setting, so we leave it to the interested reader to formulate and prove such results, using Theorems 3.6 and 3.7 of [FM] in place of Theorems 4.2 and 4.3 above.

5. Equivariant maps on ergodic components

This section uses von Neumann Selection Theorem (2.3) to prove that if almost every ergodic component of a G-action has a fixed standard Borel G-space X as a measurable quotient, then X is a measurable quotient of the entire action. Actually, the conclusion is proved in a more general setting that includes twisting by cocycles (see 5.4). This yields a corollary (5.6) that obtains information about a cocycle from the algebraic hulls of its restrictions to ergodic components.

Definition 5.1 ([Mr, §3.2.0, pp. 216–217])**.** Suppose:

- G and H are locally compact, second countable groups,
- S and X are Borel spaces,
- $\rho\colon G\times S\to S$ and $\tau\colon H\times X\to X$ are Borel actions,
- μ is a probability measure on S, and
- $\alpha\colon G\times S\to H$ is a Borel cocycle.

Then:

1. We define an action $\zeta_{\rho,\tau,\alpha}\colon G\times \mathbf{F}(S,X)\to\mathbf{F}(S,X)$ by
$$\zeta_{\rho,\tau,\alpha}(g,\phi)(s) = \tau\Big(\alpha(g,s),\phi\big(\rho(g^{-1},s)\big)\Big).$$
We may refer to $\zeta_{\rho,\tau,\alpha}$ as the *α-twisted action* of G on $\mathbf{F}(S,X)$. It is Borel (see 2.31).
2. A function $\phi\colon S\to X$ is *essentially* (ρ,τ,α)*-equivariant* if, for each $g\in G$, we have
$$\phi\big(\rho(g,s)\big) = \tau\big(\alpha(g,s),\phi(s)\big) \text{ for a.e. } s\in S.$$
In other words, a Borel function $\phi\colon S\to X$ is essentially (ρ,τ,α)-equivariant if and only if it represents a fixed-point of the α-twisted action of G on $\mathbf{F}(S,X)$.

Proposition 5.2. *Let:*

- *G, H, X, ρ, ψ, α, and α_ω be as in Corollary* 3.6*,*
- *ρ_ω be the restriction of ρ to $G \times \psi^{-1}(\omega)$, for each $\omega \in \Omega$,*
- *(Y, d) be a complete, separable metric space,*
- *$\tau \colon H \times Y \to Y$ be a continuous action of H on Y,*
- *Ω' be a conull Borel subset of Ω, such that α_ω is a Borel cocycle, for each $\omega \in \Omega'$, and*
- $\Omega^X = \left\{ \omega \in \Omega' \;\middle|\; \begin{array}{c} \text{there is a Borel map } \phi_\omega \colon \psi^{-1}(\omega) \to Y \\ \text{that is essentially } (\rho_\omega, \tau, \alpha_\omega)\text{-equivariant} \end{array} \right\}.$

Then:

1. *Ω^X is analytic.*
2. *There is an essentially (ρ, τ, α)-equivariant Borel map $\phi \colon X \to Y$ if and only if Ω^X is conull in Ω.*

Proof. From Proposition 2.21 (and Rem. 2.20), we may assume the notation of Theorem 3.4. Recall that ρ, τ, and α induce Borel maps:

- $\rho_{\mathbf{F}} \colon G \times \Omega \times \mathbf{F}(S, X) \to \mathbf{F}(S, X)$ (see 2.25),
- $\tau^{\mathbf{F}} \colon \mathbf{F}(S, H) \times \mathbf{F}(S, X) \to \mathbf{F}(S, X)$ (see 2.30), and
- $\breve{\alpha} \colon G \times \Omega \to \mathbf{F}(S, H)$ (see 2.6).

Let G_0 be a countable, dense subset of G, and define

$$\mathcal{F} = \bigl\{ (\omega, f) \in \Omega \times \mathbf{F}(S, X) \mid \rho_{\mathbf{F}}(g, \omega, f) = \tau^{\mathbf{F}}\bigl(\breve{\alpha}(g, \omega), f\bigr) \text{ for all } g \in G_0 \bigr\}.$$

Then $\mathcal{F}$ is an analytic set (in fact, it is Borel, because $\rho_{\mathcal{F}}$, $\tau^{\mathcal{F}}$, and $\breve{\alpha}$ are Borel and G_0 is countable).

(1) Ω^X is the projection of the analytic set $\mathcal{F}$ to Ω.

($2 \Rightarrow$) For a.e. $\omega \in \Omega$, the map ϕ_ω, defined by $\phi_\omega(s) = \phi(\omega, s)$, is essentially $(\rho_\omega, \tau, \alpha_\omega)$-equivariant.

($2 \Leftarrow$) Because $\mathcal{F}$ is analytic, we may apply the von Neumann Selection Theorem (2.3). By assumption, $\Omega_{\mathcal{F}}$ is conull in Ω, so we conclude that there is a Borel function $\phi \colon \Omega \to \mathbf{F}(S, X)$, such that $\bigl(\omega, \phi(\omega)\bigr) \in \mathcal{F}$, for a.e. $\omega \in \Omega$. Lemma 2.7 provides us with a corresponding Borel function $\hat{\phi} \colon \Omega \times S \to X$. By applying Fubini's Theorem, we see, for each $g \in G_0$, that

$$\hat{\phi}\bigl(\rho(g, \omega, s)\bigr) = \tau\bigl(\alpha(g, \omega, s), \hat{\phi}(\omega, s)\bigr) \text{ for a.e. } (\omega, s) \in \Omega \times S. \tag{5.3}$$

Let

$$H = \{ g \in G \mid (5.3) \text{ holds} \}.$$

Then H is the stabilizer of $\hat{\phi}$ under the α-twisted action of G on $\mathbf{F}(\Omega \times S, X)$ (see Defn. 5.1(1)). Because the action is Borel, we know that H is a closed subgroup of G. On the other hand, H is dense, because it contains G_0. Therefore, $H = G$, which means that $\hat{\phi}$ is essentially (ρ, τ, α)-equivariant. □

Any continuous homomorphism $\pi \colon G \to H$ determines a "constant" cocycle $\pi^\times \colon G \times X \to H$, defined by $\pi^\times(g, x) = \pi(g)$ (cf. 3.10). In the statement of the following corollary, we ignore the distinction between π and $\pi^\times$.

Corollary 5.4. *Let:*

- *G, H, X, ρ, ψ, ρ_ω, Y, and τ be as in Proposition* 5.2*, and*
- *$\pi \colon G \to H$ be a continuous homomorphism.*

There exists an essentially (ρ, τ, π)*-equivariant Borel map* $\phi\colon X \to Y$ *if and only if there exists an essentially* (ρ_ω, τ, π)*-equivariant Borel map* $\phi_\omega\colon \psi^{-1}(\omega) \to Y$ *for a.e.* $\omega \in \Omega$.

Definition 5.5 ([Z2, Defn. 9.2.2])**.** If:

- G acts ergodically on X,
- $\alpha\colon G \times X \to H$ is a Borel cocycle,
- $\mathbb{F}$ is a local field, and
- H is the $\mathbb{F}$-points of an algebraic group over $\mathbb{F}$,

then there exists a Zariski-closed subgroup L of H, such that:

1. α is cohomologous to a cocycle taking values in L.
2. α is *not* cohomologous to a cocycle taking values in any proper Zariski-closed subgroup of H.

The subgroup L is unique up to conjugacy. It is called the *algebraic hull* of α.

Corollary 5.6. *Let:*

- (X, μ) *be a standard Borel probability space,*
- $\rho\colon G \times X \to X$ *be a Borel action, such that* μ *is quasiinvariant,*
- $\psi\colon X' \to \Omega$ *be the corresponding ergodic decomposition* (*see* 2.19)*,*
- $\mathbb{F}$ *be a local field,*
- H *be the* $\mathbb{F}$*-points of an algebraic group over* $\mathbb{F}$*, and*
- $\alpha\colon G \times X \to H$ *be a Borel cocycle.*

Then α *is cohomologous to a Borel cocyle* $\beta\colon G \times X \to H$*, such that, for a.e.* $\omega \in \Omega$*, the Zariski closure of the range of* β_ω *is equal to the algebraic hull of* β_ω (*where* β_ω *is the restriction of* β *to* $G \times \psi^{-1}(\omega)$)*.*

Proof. Choose Ω' as in 3.6(1) (with $\beta = \alpha$). Chevalley's Theorem [Bo, Thm. 5.1, p. 89] implies there is a countable collection $\{(\tau_i, V_i)\}_{i=0}^{\infty}$ of rational representations of H, such that every Zariski-closed subgroup of H is the stabilizer of some point in some projective space $\mathbb{P}(V_i)$. For $c, d, i \in \mathbb{N}$, define:

- $\Omega_{c,d} = \left\{ \omega \in \Omega' \;\middle|\; \begin{array}{c}\text{the algebraic hull of } \alpha_\omega \text{ is } d\text{-dimensional}\\ \text{and has exactly } c \text{ connected components}\end{array} \right\}$,
- $Y^i_{c,d} = \left\{ v \in \mathbb{P}(V_i) \;\middle|\; \begin{array}{c}\dim \operatorname{Stab}_H(v) = d, \text{ and}\\ \operatorname{Stab}_H(v) \text{ has exactly } c \text{ connected components}\end{array} \right\}$,
- $Y_{c,d} = \coprod_{i=0}^{\infty} Y^i_{c,d}$ (disjoint union), and
- $\tau\colon H \times Y_{c,d} \to Y_{c,d}$ by $\tau(h, x) = \tau_i(h)x$ if $x \in \mathbb{P}(V_i)$.

Each $Y^i_{c,d}$ is Borel (see 5.7 below), so $Y_{c,d}$ is a standard Borel space. Therefore, combining the Cocycle Reduction Lemma [Z2, Lem. 5.2.11, p. 108] with Proposition 5.2(1) and Rem. 2.2 implies that $\Omega_{c,d}$ is absolutely measurable. Thus, we may let $\Omega'_{c,d}$ be a conull, Borel subset of $\Omega_{c,d}$, for each d.

There is no harm in assuming $\Omega = \Omega'_{c,d}$, for some c and d. Then there is an essentially $(\rho_\omega, \tau, \alpha_\omega)$-equivariant Borel map from $\psi^{-1}(\omega)$ to $Y_{c,d}$, for a.e. $\omega \in \Omega$, so Corollary 5.2(2) implies that there is an essentially (ρ, τ, α)-equivariant Borel map from X to $Y_{c,d}$. This means that α is cohomologous to a cocycle β, such that the essential range of β_ω is contained in an algebraic group whose dimension and number of connected components are no more than those of the algebraic hull of β_ω. By changing β on a set of measure 0, we may assume that the entire range of β_ω is contained in this subgroup. So the desired conclusion follows from the minimality (and uniqueness) of the algebraic hull of β_ω. □

The following observation, used in the proof of Corollary 5.6 above, must be well-known, but the authors do not know of a reference.

Lemma 5.7. *Suppose:*

- *G is a real Lie group,*
- *M is a Polish space, and*
- *$\rho\colon G \times M \to M$ is a continuous action, such that the stabilizer $\operatorname{Stab}_G(m)$ has only finitely many connected components, for each $m \in M$.*

Then:

1. $\dim \operatorname{Stab}_G(m)$ *is a Borel function of $m \in M$.*
2. *For each compact subgroup K of G,*
$$\{m \in M \mid K \text{ is conjugate to a maximal compact subgroup of } \operatorname{Stab}_G(m)\}$$
is Borel.
3. *The dimension of the maximal compact subgroup of $\operatorname{Stab}_G(m)$ is a Borel function of m.*
4. *The number of connected components of $\operatorname{Stab}_G(m)$ is a Borel function of m.*
5. *For each d and c,*
$$\left\{ m \in M \;\middle|\; \begin{array}{c} \dim \operatorname{Stab}_G(m) = d, \text{ and} \\ \operatorname{Stab}_G(m) \text{ has exactly } c \text{ connected components} \end{array} \right\}$$
is Borel.

Proof. (1) It is well-known (and easy to see) that $\dim \operatorname{Stab}_G(m)$ is an upper semicontinuous function of $m \in M$.

(2) The fixed-point set M^K is closed, and G is σ-compact, so $\rho(G, M^K)$ is a countable union of closed sets. Therefore $\rho(G, M^K)$ is Borel. Now

$$\rho(G, M^K) = \{m \in M \mid \operatorname{Stab}_G(m) \text{ contains a conjugate of } K\},$$

so

$$\left\{ m \in M \;\middle|\; \begin{array}{c} K \text{ is conjugate to a maximal} \\ \text{compact subgroup of } \operatorname{Stab}_G(m) \end{array} \right\} = \rho(G, M^K) \smallsetminus \bigcup_{K' \supset K} \rho(G, M^{K'}).$$

Any real Lie group has only countably many conjugacy classes of compact subgroups [Ad, Prop. 10.12], so the union is countable. Therefore, this is a Borel set.

(3, 4) Immediate from (2). (Recall that the number of components of $\operatorname{Stab}_G(m)$ is the same as the number of components of any of its maximal compact subgroups.)

(5) Combine (1) and (4). □

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