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Examples and Counterexamples to Almost-Sure Convergence of Bilateral Martingales

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Abstract. Given a stationary process $(X_p)_{p\in\mathbb{Z}}$ and an event $B \in \sigma(X_p,\, p \in \mathbb{Z})$, we study the almost sure convergence as n and m go to infinity of the "bilateral" martingale

$$\mathbb{E}\left[\mathbf{1}_B \mid X_{-n}, X_{-n+1}, \ldots, X_{m-1}, X_m\right].$$

We show that almost sure convergence holds in some classical examples such as i.i.d. or Markov processes, as well as for the natural generator of Chacon's transformation. However, we also prove that in every aperiodic dynamical system with finite entropy, there exists a generating process and a measurable set B for which the almost sure convergence of the bilateral martingale does not hold.

Contents

1. Introduction

1.1. Bilateral martingales and reliable processes. In this work, we consider a stationary process $(X_p)_{p\in\mathbb{Z}}$ taking values in a finite alphabet, and an event B which is measurable with respect to the σ-algebra $\sigma(X_p,\, p \in \mathbb{Z})$. We observe a finite

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number of coordinates of the process: $X_{-n}, X_{-n+1}, \dots, X_0, \dots, X_{m-1}, X_m$ (where m and n are always positive integers), and we are interested in the conditional probability of B given this observation. More precisely, we wonder whether the two-parameter martingale (henceforth called *bilateral martingale*)

$$\mathbb{E}\left[\mathbf{1}_B \mid X_{-n}, X_{-n+1}, \dots, X_{m-1}, X_m\right] \tag{1}$$

converges almost surely to $\mathbf{1}_B$ as n and m go to infinity.

Let us remark that if two increasing sequences of integers (n_k) and (m_k) are given, both going to infinity, the classical martingale convergence theorem ensures the almost sure convergence of $\mathbb{E}\left[\mathbf{1}_B \mid X_{-n_k}, X_{-n_k+1}, \dots, X_{m_k-1}, X_{m_k}\right]$ to $\mathbf{1}_B$ when $k \longrightarrow \infty$, but the negligible set outside which this convergence holds may depend on the sequences (n_k) and (m_k). The problem stated here amounts to ask whether it is possible to find a negligible set outside which convergence holds *for each choice of the sequences* (n_k) *and* (m_k).

We will say that the process $(X_p)_{p\in\mathbb{Z}}$ is *reliable* if for each B in $\sigma(X_p,\, p \in \mathbb{Z})$ the bilateral martingale (1) converges almost surely to $\mathbf{1}_B$ as n and m go to infinity. On the contrary, the process will be said *unreliable* if there exists an event B such that with positive probability, it is possible to find two misleading sequences (n_k) and (m_k) increasing to infinity, for which the conditional probability of B given the observation of $X_{-n_k}, X_{-n_k+1}, \dots, X_{m_k-1}, X_{m_k}$ is always far apart from $\mathbf{1}_B$.

Unreliable processes: too much information kills information? Although we have not been able to find a simple explicit example in this class, Theorem 3.1 shows the existence of many different unreliable processes. For such a process, if B is a measurable set for which the bilateral martingale (1) does not converge almost surely, we can ask whether it is possible to replace the conditonal probability $\mathbb{E}\left[\mathbf{1}_B \mid X_{-n}, X_{-n+1}, \dots, X_{m-1}, X_m\right]$ with another $\sigma(X_{-n}, \dots, X_m)$-measurable function $f_{n,m}$, such that

$$f_{n,m} \xrightarrow[n,m\to\infty]{} \mathbf{1}_B \qquad \text{(a.s.)}.$$

But such a function is easy to find: just set $k_{n,m} \stackrel{\text{def}}{=} n \wedge m$, and

$$f_{n,m} \stackrel{\text{def}}{=} \mathbb{E}\left[\mathbf{1}_B \;\middle|\; X_{-k_{n,m}}, \dots, X_{k_{n,m}}\right].$$

Then we are in the following paradoxical situation: a reliable way to estimate whether the event B is realized or not consists in systematically disregarding a part of the information in our possession!

1.2. Some known results about almost-sure convergence of two-parameter martingales. If $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is an increasing sequence of σ-algebras in a probability space, it is well known that every $(\mathcal{F}_n)$-martingale which is bounded in L^1 converges almost surely. But if we consider a family $(\mathcal{F}_{n,m})_{(n,m)\in\mathbb{N}^2}$ of σ-algebras with $\mathcal{F}_{n_1,m_1} \subset \mathcal{F}_{n_2,m_2}$ whenever $n_1 \le n_2$ and $m_1 \le m_2$, such a general result about almost sure convergence of $(\mathcal{F}_{n,m})$-martingales does not exist. Cairoli ([1]) has studied the particular case where $\mathcal{F}_{n,m}$ can be written $\mathcal{G}_n \vee \mathcal{H}_m$, with $(\mathcal{F}_n)$ and $(\mathcal{G}_n)$ increasing filtrations and $\bigvee_n \mathcal{G}_n$ is independent of $\bigvee_m \mathcal{H}_m$. Under this hypothesis, if $(M_{n,m})$ is an $(\mathcal{F}_{n,m})$-martingale which satisfies

$$\sup_{n,m} \mathbb{E}\left[|M_{n,m}| \log^+ |M_n, m|\right] < +\infty, \tag{2}$$

then $M_{n,m}$ converges almost surely. Note: this result still holds for a martingale which is indexed by $\mathbb{N}^d$, providing that we replace condition (2) by

$$\sup_{n_1,\dots,n_d} \mathbb{E}\left[|M_{n_1,\dots,n_d}| \left(\log^+ |M_{n_1,\dots,n_d}|\right)^{d-1}\right] < +\infty.$$

However, even in this case of a product filtration, there exist $\mathcal{F}_{n,m}$-martingales which are bounded in L^1 and which do not converge almost surely.

The bilateral martingale (1) being always between 0 and 1, there is no problem of integrability in our study. But if no structure condition on the filtration $(\mathcal{F}_{n,m})$ is required, Dubins and Pitman ([4]) have proved that even an $(\mathcal{F}_{n,m})$-martingale which is bounded in L^∞ can fail to converge almost surely. Indeed, we will give in this work some examples of bilateral martingales which diverge with positive probability.

For other results about convergence of multiparameter martingales, the reader is invited to consult [5] and [6].

1.3. Stationary processes defined in a dynamical system. The stationary processes that we consider here will be henceforth supposed to be defined on an underlying dynamical system $(X, \mathcal{A}, \mu, T)$, where T is an invertible, bimeasurable, measure-preserving transformation of X. This means that X_0 is defined by a finite measurable partition $\mathcal{P}$ of X, and that the entire process is given by $X_p \stackrel{\text{def}}{=} X_0 \circ T^p$ for each $p \in \mathbb{Z}$. For k and l in $\mathbb{Z}$, $k \leq l$, we note

$$\mathcal{P}_k^l \stackrel{\text{def}}{=} \bigvee_{j=k}^{l} T^{-j}\mathcal{P}$$

the finite partition generated by $X_k, \dots, X_l$. For every partition $\mathcal{Q}$ of X and every $x \in X$, we denote by $\mathcal{Q}(x)$ the atom of $\mathcal{Q}$ containing x.

The process being completely defined by the partition $\mathcal{P}$ and the transformation T, it will also be refered to by the couple $(\mathcal{P}, T)$. In this context, the process $(\mathcal{P}, T)$ is *reliable* if for each $B \in \bigvee_{j=-\infty}^{+\infty} T^{-j}\mathcal{P}$,

$$\mu\left(B \,\middle|\, \mathcal{P}_{-n}^m(x)\right) \xrightarrow[n,m\to\infty]{} \mathbf{1}_B \qquad (\mu\text{-a.e.}).$$

The process $(\mathcal{P}, T)$ is said to be a *generator* of the dynamical system if $\sigma(X_p,\, p \in \mathbb{Z}) = \mathcal{A}$. The following section gives examples of reliable processes among some "natural" generators of classical dynamical systems.

2. Examples of reliable processes

2.1. Bernoulli shifts, Markov processes. Let us start with the simple case where the random variables X_p are independent, identically distributed. This is the case of the natural generator of the system $(X, \mathcal{A}, \mu, T)$ called *Bernoulli shift*, where X is the set of all bi-infinite sequences of letters in the alphabet, $\mathcal{A}$ is the Borel σ-algebra of X, μ is a product probabilty $P^{\otimes\mathbb{Z}}$ (P being a probability defined on the alphabet), and T is the shift of the coordinates. The *past* of the process $\sigma(X_p,\, p < 0)$ is then independent of the *future* $\sigma(X_p,\, p \geq 0)$, so we are precisely in the case studied by Cairoli, for which every martingale which is bounded in L^∞ converges almost surely. As a consequence, an i.i.d. process is always reliable.

More generally, an application of Cairoli's result can also prove the reliability of a k-step Markov process (X_p), because in this case the past of the process is independent of its future conditionally on the variables $X_0, \dots, X_{k-1}$.

2.2. Examples in zero entropy.

Rotations. The following example has been suggested to the author by Benjamin Weiss. Let α be an irrational real number, and T_α be the transformation of the torus $\mathbb{R}/\mathbb{Z}$ defined by $T_\alpha(x) \stackrel{\text{def}}{=} x + \alpha \,(\text{mod } 1)$. The only probability measure on the torus which is preserved by T_α is the Lebesgue leasure λ. If $\mathcal{P}$ is a non-trivial partition of $\mathbb{R}/\mathbb{Z}$ in a finite number of intervals, it is easy to see that for every positive integers m and n, $\mathcal{P}_{-n}^{m}$ is still a partition of $\mathbb{R}/\mathbb{Z}$ in a finite number of intervals. Moreover, as each extremity of the intervals of $\mathcal{P}$ has a dense orbit in the torus, we have for each $x \in \mathbb{R}/\mathbb{Z}$

$$\lambda\left(\mathcal{P}_{-n}^{m}(x)\right) \xrightarrow[n,m\to\infty]{} 0.$$

But for every measurable subset B of the torus, this implies by the Lebesgue density theorem that for λ-almost every $x \in \mathbb{R}/\mathbb{Z}$,

$$\lambda\left(B \,|\, \mathcal{P}_{-n}^{m}(x)\right) \xrightarrow[n,m\to\infty]{} \mathbf{1}_B.$$

Therefore, the process $(\mathcal{P}, T_\alpha)$ is reliable (and it is also a generator for the dynamical system $(\mathbb{R}/\mathbb{Z}, \lambda, T_\alpha)$).

This result can be generalized in dimension $d \geq 1$: let $\alpha = (\alpha_1, \dots, \alpha_d)$ where each α_j is an irrational number, and let T_α be the translation by α modulo 1 on $(\mathbb{R}/\mathbb{Z})^d$. If $\mathcal{P}$ is a finite partition of the d-dimensional torus obtained by a product of partitions $\mathcal{P}_j$, $1 \leq j \leq d$, where each $\mathcal{P}_j$ is a non-trivial partition of $\mathbb{R}/\mathbb{Z}$ in a finite number of intervals, then for each $x \in (\mathbb{R}/\mathbb{Z})^d$, $\mathcal{P}_{-n}^{m}(x)$ is a Cartesian product $I_1 \times \cdots \times I_d$, where each I_j is an interval, and

$$\sup_{i\leq j\leq d} \lambda(I_j) \xrightarrow[n,m\to\infty]{} 0.$$

As above, but using in this case the theorem of Jessen-Marcinkiewicz-Zygmund (see, e.g., [2], p. 50) we can conclude that $(\mathcal{P}, T_\alpha)$ is still a reliable process.

A weakly-mixing example: the natural generator for Chacon's transformation.

Proposition 2.1. *Let $(\mathcal{P}, T)$ be a zero entropy process, and let us assume that there exists $\theta > 0$ satisfying the following property: for each $n \geq 1$, for each atom P of $\mathcal{P}_{-n}^{0}$ or of $\mathcal{P}_{0}^{n}$, every atom A of $\mathcal{P}_{-n}^{n}$ contained in P is such that $\mu\,(A \mid P) \geq \theta$. Then the process $(\mathcal{P}, T)$ is reliable.*

Proof. Let B be a set in the σ-algebra generated by the process $(\mathcal{P}, T)$. Because $(\mathcal{P}, T)$ has zero entropy, B is both in $\bigvee_{-\infty}^{0} T^{-j}\mathcal{P}$ and in $\bigvee_{0}^{+\infty} T^{-j}\mathcal{P}$. For almost every x, we therefore have

$$\mu\left(B \,\middle|\, \mathcal{P}_{-n}^{0}(x)\right) \xrightarrow[n\to\infty]{} \mathbf{1}_B(x), \quad \text{and} \tag{3}$$

$$\mu\left(B \mid \mathcal{P}_{0}^{n}(x)\right) \xrightarrow[n\to\infty]{} \mathbf{1}_B(x). \tag{4}$$

We can write for each $n \geq 1$

$$\mu\left(B \,\middle|\, \mathcal{P}_{-n}^{0}(x)\right) = \sum_{\substack{A \in \bigvee_{-n}^{n} T^{-j}\mathcal{P} \\ A \subset \mathcal{P}_{-n}^{0}(x)}} \mu\left(A \,\middle|\, \mathcal{P}_{-n}^{0}(x)\right) \mu\left(B \mid A\right),$$

hence, by the hypothesis, for each A appearing in the right-hand sum,

$$\left|\mu\left(B \mid A\right) - \mathbf{1}_B(x)\right| \leq \frac{1}{\theta}\left|\mu\left(B \,\middle|\, \mathcal{P}_{-n}^{0}(x)\right) - \mathbf{1}_B(x)\right|. \tag{5}$$

If $1 \leq m \leq n$, we also have

$$\mu\left(B \,\middle|\, \mathcal{P}_{-n}^{m}(x)\right) = \sum_{\substack{A \in \bigvee_{-n}^{n} T^{-j}\mathcal{P} \\ A \subset \mathcal{P}_{-n}^{m}(x)}} \mu\left(A \,\middle|\, \mathcal{P}_{-n}^{m}(x)\right) \mu\left(B \mid A\right),$$

and as $\mathcal{P}_{-n}^{m}(x) \subset \mathcal{P}_{-n}^{0}(x)$, this implies by (5)

$$\left|\mu\left(B \,\middle|\, \mathcal{P}_{-n}^{m}(x)\right) - \mathbf{1}_B(x)\right| \leq \frac{1}{\theta}\left|\mu\left(B \,\middle|\, \mathcal{P}_{-n}^{0}(x)\right) - \mathbf{1}_B(x)\right|.$$

In the case where $1 \leq n < m$, we have by the same argument

$$\left|\mu\left(B \,\middle|\, \mathcal{P}_{-n}^{m}(x)\right) - \mathbf{1}_B(x)\right| \leq \frac{1}{\theta}\left|\mu\left(B \mid \mathcal{P}_{0}^{m}(x)\right) - \mathbf{1}_B(x)\right|,$$

which proves that if x also satisfies (3) and (4),

$$\mu\left(B \,\middle|\, \mathcal{P}_{-n}^{m}(x)\right) \xrightarrow[n,m\to\infty]{} \mathbf{1}_B.$$

□

Now let us show that Proposition 2.1 can be applied to the natural generator of Chacon's transformation. We suppose here that the reader is familiar with this transformation, described for example in [3]. We simply recall without proof the following facts.

We consider the two-letter alphabet $\{1, s\}$, and we inductively construct a family of words $(B_k)_{k\geq 1}$ by setting

$$B_1 \overset{\text{def}}{=} 1, \quad \text{and} \quad \forall k \geq 1,\ B_{k+1} \overset{\text{def}}{=} B_k B_k s B_k.$$

The word B_k is also called *k-block*. Let h_k be the length of B_k; we clearly have $h_{k+1} = 3h_k + 1$.

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The word B_k is also called *k-block*. Let h_k be the length of B_k; we clearly have $h_{k+1} = 3h_k + 1$. Let's denote by X the set of all sequences $x = (x_p)_{p\in\mathbb{Z}} \in \{1, s\}^{\mathbb{Z}}$ which for each $k \geq 1$ can be written as a concatenation of an infinite number of k-blocks, possibly separated by isolated symbols s, (called *spacers*). Chacon's transformation (denoted by T) can be defined as the shift of the coordinates on X: $(Tx)_p \overset{\text{def}}{=} x_{p+1}$. The decomposition in k-blocks and spacers is unique for every sequence in X, and there is only one probability measure μ on X which is preserved by T. This probability satisfies for each $k \geq 1$ the following properties:

- Let F be the set of all sequences in X on which we can read B_k in a fixed window of length h_k. Then F is the disjoint union of 3 equiprobable sets F_1, F_2 and F_3, respectively corresponding to the sequences for which this k-block appears in first, second or third position in a $(k+1)$-block.
- This same set F can also be written as the disjoint union of 4 sets denoted by F^{BBB}, F^{BBsB}, F^{BsBB}, F^{BsBsB}, according to the existence of the spacers between this k-block, the preceding and the following ones. We have

$$\mu(F^{BsBB}) = \mu(F^{BBsB}) = \mu(F)/3, \quad \text{and} \tag{6}$$
$$\mu(F^{BBB}) = \mu(F^{BsBsB}) = \mu(F)/6.$$

The process $(\mathcal{P}, T)$, where $\mathcal{P}$ is defined by the 0-coordinate of the sequence $x \in X$ is under the probability μ a stationary process, which is a generator of the dynamical system. It is well-known that this process has zero entropy; now let us check that it satisfies the hypothesis of Proposition 2.1.

Let $n \geq 2$, and let P be a fixed atom of $\mathcal{P}_{-n}^{0}$. Let k be the greatest integer such that we can read a k-block in the word of length $n+1$ characterizing P. we have

$$h_k \leq n+1 \leq 2h_{k+1} + 1 = 6h_k + 3. \tag{7}$$

Let us consider the last occurrence of B_k in this word, and let F be the set of all sequences $x \in X$ for which B_k appears in the same window. We have $P \subset F$, and F is the disjoint union of 9 equiprobable sets $F_{i,j}$, $1 \leq i, j \leq 3$, corresponding to the 9 possible positions of this k-block in the $k+2$-block including it. Then, each $F_{i,j}$ is itself a disjoint union of 4 sets $F_{i,j}^{w}$, $w \in \{BBB, BsBB, BBsB, BsBsB\}$, according to the existence of the spacers before and after this $(k+2)$-block. Because of (6), all these sets satisfy

$$\mu\left(F_{i,j}^{w}\right) \geq \mu(F_{i,j})/6 \geq \mu(F)/54.$$

But (7) implies that each $F_{i,j}^{w}$ is entirely contained in an atom of $\mathcal{P}_{-n}^{n}$. From this, we can deduce that each atom A of $\mathcal{P}_{-n}^{n}$ contained in P satisfies $\mu(A) \geq \mu(P)/54$. A similar argument prove the same result if, at the begining, P is an atom of $\mathcal{P}_{0}^{n}$. By Proposition 2.1, we can then conclude that the natural generator of Chacon's transformation is a reliable process.

3. Construction of an unreliable generating process

Now we are going to prove the existence of an unreliable generating process in every aperiodic dynamical system with finite entropy (this last condition being required only because we are dealing with processes taking values in finite alphabets).

Theorem 3.1. *In every aperiodic dynamical system $(X, \mathcal{A}, \mu, T)$ with finite entropy, we can find a finite partition $\mathcal{P}$ and a set B such that:*

- *$(\mathcal{P}, T)$ is a generating process.*
- *$\mu(B) > 0$.*
- *For μ-almost every $x \in B$, $\liminf_{n,m\longrightarrow\infty} \mu\left(B \,\middle|\, \mathcal{P}_{-n}^{m}(x)\right) \leq 1/2$.*

We recall that a *Rokhlin tower* is a finite family $\mathcal{T} = (B, TB, T^2B, \ldots, T^{h-1}B)$ of mutually disjoint sets, which are successive images of a fixed set B, called the *basis* of $\mathcal{T}$. The integer h is the *height* of the tower, each T^jB is a *rung*, and we denote by the same symbol $\mathcal{T}$ the union of the h rungs of the tower. We have

$$\mu(\mathcal{T}) = \mu\left(\bigcup_{j=0}^{h-1} T^jB\right) = h\mu(B).$$

A Rokhlin tower $\mathcal{T}'$ will be called *sub-tower* of $\mathcal{T}$ if it has the same height and if the basis of $\mathcal{T}'$ is contained in the basis of $\mathcal{T}$.

First we establish the following lemma.

Lemma 3.2. *Let (ε_k) be a sequence of positive real numbers, and (l_k) a sequence of positive integers. In every aperiodic dynamical system, we can find a sequence $(\mathcal{T}_k)$ of Rokhlin towers, $\mathcal{T}_k = (B_k, TB_k, \ldots, T^{h_k-1}B_k)$, with the following properties holding for each $k \geq 1$:*

$$\mu(\mathcal{T}_k) \geq 1 - \varepsilon_k \tag{8}$$

$$\mathcal{T}_k \subset \bigcup_{j=l_k}^{h_{k+1}-1-l_k} T^jB_{k+1}. \tag{9}$$

Proof. Given the sequences (ε_k) and (l_k), it is easy to construct a sequence (δ_k) of positive real numbers and a sequence (h_k) of integers with $\delta_1 \leq \varepsilon_1/2$, and such that for each $k \geq 1$ and each $m \leq k+1$,

$$\delta_{k+1} + \frac{2(l_{k+1} + h_k)}{h_{k+1}} < \frac{\varepsilon_m}{2^{k+1}}. \tag{10}$$

The system being aperiodic, we can find for each $k \geq 1$ a Rokhlin tower $\mathcal{T}_k^1$ of height h_k such that $\mu(\mathcal{T}_k^1) \geq 1 - \delta_k$. Let B_k^1 be the basis e of $\mathcal{T}_k^1$.

For each $k \geq 1$, we remove from B_k^1 the points contradicting (9); more precisely, we define

$$B_k^2 \stackrel{\text{def}}{=} B_k^1 \cap \bigcup_{j=l_{k+1}}^{h_{k+1}-1-l_{k+1}-h_k} T^jB_{k+1}.$$

B_k^2 is the basis of a sub-tower $\mathcal{T}_k^2$ of $\mathcal{T}_k^1$. The points which are in $\mathcal{T}_k^1$ but not in $\mathcal{T}_k^2$ are either out of $\mathcal{T}_{k+1}^1$, or in the first or last $(h_k + l_{k+1})$ rungs of $\mathcal{T}_{k+1}^1$. We thus have

$$\mu(\mathcal{T}_k^2) \geq \mu(\mathcal{T}_k^1) - \delta_{k+1} - \frac{2(l_{k+1} + h_k)}{h_{k+1}}. \tag{11}$$

Finally, we define for each $k \geq 1$

$$B_k \stackrel{\text{def}}{=} B_k^2 \setminus \bigcup_{j=1}^{+\infty} \left(\mathcal{T}_{k+j}^1 \setminus \mathcal{T}_{k+j}^2\right).$$

Then B_k is the basis of a Rokhlin tower $\mathcal{T}_k$ with height h_k, which now satisfies the property (9). By (10) and (11), we have

$$\mu(\mathcal{T}_k) \geq \mu(T_k^1) - \sum_{j=1}^{+\infty} \varepsilon_k/2^{k+j} \geq 1 - \varepsilon_k.$$

□

Proof of Theorem 3.1. As T has finite entropy, we can fix a generating process $(\mathcal{Q}, T)$, where $\mathcal{Q}$ is a finite partition indexed by the alphabet $\{1, \ldots, K\}$. We consider the sequence $(\mathcal{T}_k)$ of Rokhlin towers given by Lemma 3.2, for the sequences $\varepsilon_k \stackrel{\text{def}}{=} 2^{-k}$ and $l_k \stackrel{\text{def}}{=} k + 2^k - 1$.

Construction of the partition $\mathcal{P}$. The partition $\mathcal{P}$ will be indexed by the alphabet $\{1, \ldots, K\} \times \{b, c, d, e\}$, the first coordinate of $\mathcal{P}(x)$ being $\mathcal{Q}(x)$, which will ensure that $(\mathcal{P}, T)$ is still a generating process.

It remains to define the second coordinate of $\mathcal{P}(x)$, denoted by $\mathcal{R}(x)$. Let us begin by defining $\mathcal{R}(x) \stackrel{\text{def}}{=} a$ on the first tower $\mathcal{T}_1$. Next, let $k \geq 2$ be such that $\mathcal{R}(x)$ is defined on $\mathcal{T}_{k-1}$. We want to define $\mathcal{R}(x)$ on $\mathcal{T}_k \setminus \mathcal{T}_{k-1}$:

- On the first $(2^k - 1)$ and last $(2^k - 1)$ rungs of $\mathcal{T}_k$, $\mathcal{R}(x)$ is going to be b or c (to be specified later).
- On the k rungs from 2^k to $2^k + k - 1$, we define $\mathcal{R}(x) \stackrel{\text{def}}{=} d$.
- On the k rungs from $h_k - (2^k - 1) - k$ to $h_k - 2^k$, we define $\mathcal{R}(x) \stackrel{\text{def}}{=} e$.
- For x in intermediate rungs but not in $\mathcal{T}_{k-1}$, we define $\mathcal{R}(x) \stackrel{\text{def}}{=} a$.

Here we can point out that the observation of exactly k consecutive identical symbols d or e in the $\mathcal{R}$-name of a point x enables us to situate x in the rungs of $\mathcal{T}_k$.

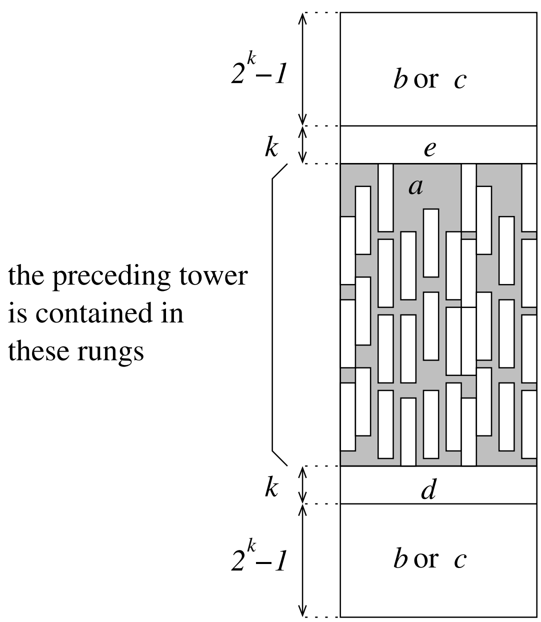

FIGURE 1. Definition of $\mathcal{R}(x)$ on $\mathcal{T}_k \setminus \mathcal{T}_{k-1}$

Now let us see how to choose between b and c on the extremities of $\mathcal{T}_k$. We temporarily consider a partition denoted by $\mathcal{P}^*$ on the tower $\mathcal{T}_k$, which coincides with $\mathcal{P}$ except on the $2 \times (2^k - 1)$ rungs where $\mathcal{R}(x)$ is still not defined, and where we set the second coordinate of $\mathcal{P}^*(x)$ to be the symbol $*$. We cut $\mathcal{T}_k$ into sub-towers, each one corresponding to a fixed $\mathcal{P}^*$-name. In other words, we consider the sub-towers of $\mathcal{T}_k$ whose basis are the atoms of the partition $\bigvee_{j=0}^{h_k-1} T^{-j}\mathcal{P}^*$ restricted to B_k. These sub-towers are called *$\mathcal{P}^*$-k-columns*.

Each $\mathcal{P}^*$-k-column is then cut into 2^k+1 sub-towers of equal measure, which will bear different $\mathcal{P}$-names: these sub-columns will be called *$\mathcal{P}$-k-columns*. Let us number from 0 to 2^k the 2^k+1 $\mathcal{P}$-k-columns obtained with a single $\mathcal{P}^*$-k-column. On the $\mathcal{P}$-k-column numbered by 0, we define $\mathcal{R}(x)$ by putting the symbol c in place of $*$ on each rung. Now consider the $\mathcal{P}$-k-column numbered by j, for $1 \leq j \leq 2^k$. On this column, we replace the symbols $*$ on the 2^k-1 first rungs with $(j-1)$ b followed by (2^k-j) c, and the symbols $*$ on the 2^k-1 last rungs with $(j-1)$ c followed by (2^k-j) b (see Figure 2).

Construction and property of the set B. We denote by C_k the union of all the $\mathcal{P}$-k-columns numbered by 0, i.e., the set of all points in $\mathcal{T}_k$ with $\mathcal{R}$-name beginning and ending with (2^k-1) consecutive c. We have $0<\mu(C_k)<2^{-k}$. Then we define

$$C \stackrel{\text{def}}{=} \bigcup_{k=2}^{+\infty} C_k, \quad \text{and } B \stackrel{\text{def}}{=} X \setminus C.$$

We have $1/2 < \mu(B) < 1$.

To complete the proof of the theorem, it remains to verify that for μ-almost every $x \in B$, $\liminf_{n,m\longrightarrow\infty} \mu\left(B \,\middle|\, \mathcal{P}_{-n}^m(x)\right) \leq 1/2$. But for almost every $x \in B$, we can find an integer k_0 such that $x \in \bigcap_{k\geq k_0} \mathcal{T}_k$. Then by the definition of B, for each $k \geq k_0$, x is in a $\mathcal{P}$-k-column numbered by $j_k \neq 0$. Let i_k be the height of x in the tower $\mathcal{T}_k$, i.e., the only integer in $\{0,\dots,h_k-1\}$ such that $x \in T^{i_k}B_k$, and let us define

$$n_k \stackrel{\text{def}}{=} i_k - (j_k-1) \quad \text{and } m_k \stackrel{\text{def}}{=} h_k - 1 - i_k - (2^k - j_k).$$

Because $x \in \mathcal{T}_{k_0}$, we have $2^k-1+k \leq i_k < h_k-(k+2^k-1)$ as soon as $k>k_0$, which ensures that n_k and m_k are greater than k. Now let us consider the atom $\mathcal{P}_{-n_k}^{m_k}(x)$. As shown on Figure 3, the $\mathcal{P}$-name characterizing this atom stops just before seeing the symbols b written in the $\mathcal{P}$-k-column containing x. The k consecutive symbols d and e in this $\mathcal{P}$-name tell us that $\mathcal{P}_{-n_k}^{m_k}(x)$ is contained in $T^{i_k}B_k$. But then we see that $\mathcal{P}_{-n_k}^{m_k}(x)$ is the disjoint union of two sets of equal measure, one of which is the union of rungs of height i_k in some $\mathcal{P}_k$-columns numbered by 0, hence contained in C. We have therefore

$$\mu\left(B \,\middle|\, \mathcal{P}_{-n_k}^{m_k}(x)\right) \leq \frac{1}{2}.$$

□

4. Comments and questions

Reliable generators. Section 2 shows that in various classical examples of dynamical systems it is possible to find a reliable generating process. Does the existence of such a reliable generator hold in every finite entropy dynamical system?

The answer to this question may be all the less obvious because the examples of processes given in this work seem to be reliable for very different reasons.

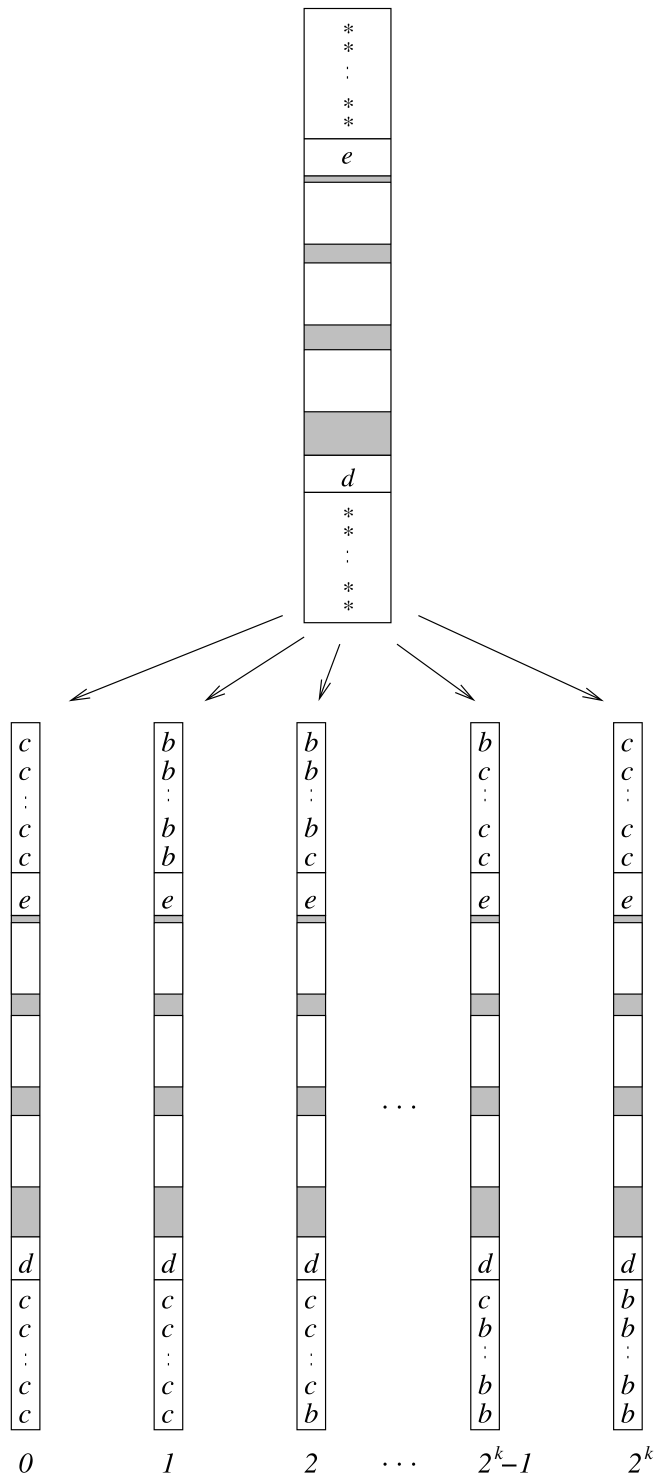

FIGURE 2. Definition of symbols b and c on a $\mathcal{P}^*$-k-column

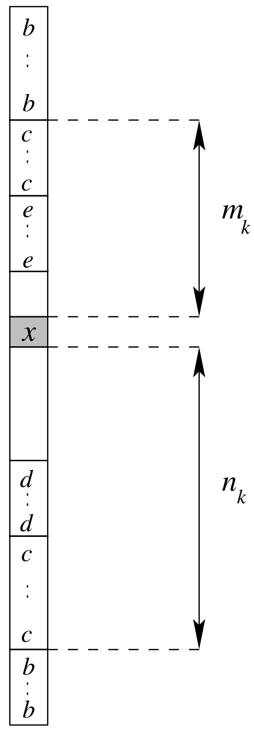

FIGURE 3. Definition of n_k and m_k according to the position of x in its $\mathcal{P}$-k-column

Reliability and operations on processes. In the proof of Theorem 3.1, we see that given a process $(\mathcal{Q}, T)$ in an aperiodic dynamical system, it is always possible to find a finite partition $\mathcal{P}$ which is *finer* than $\mathcal{Q}$ (i.e., $\mathcal{P} = \mathcal{Q} \vee \mathcal{R}$, where $\mathcal{R}$ is another finite partition), such that the process $(\mathcal{P}, T)$ is unreliable. Now what can be said about the reliability of a process $(\mathcal{P}, T)$ when $\mathcal{P} = \mathcal{Q} \vee \mathcal{R}$, with both $(\mathcal{Q}, T)$ and $(\mathcal{R}, T)$ reliable processes?

Conversely, let $(\mathcal{P}, T)$ be a reliable process, and $\mathcal{Q}$ be a finite partition which is coarser than $\mathcal{P}$: is the process $(\mathcal{Q}, T)$ always reliable?

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