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On Commuting Matrix Differential Operators

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Abstract. If the differential expressions P and L are polynomials (over $\mathbb{C}$) of another differential expression they will obviously commute. To have a P which does not arise in this way but satisfies $[P, L] = 0$ is rare. Yet the question of when it happens has received a lot of attention since Lax presented his description of the KdV hierarchy by Lax pairs (P, L). In this paper the question is answered in the case where the given expression L has matrix-valued coefficients which are rational functions bounded at infinity or simply periodic functions bounded at the end of the period strip: if $Ly = zy$ has only meromorphic solutions then there exists a P such that $[P, L] = 0$ while P and L are not both polynomials of any other differential expression. The result is applied to the AKNS hierarchy where $L = JD + Q$ is a first order expression whose coefficients J and Q are 2×2 matrices. It is therefore an elementary exercise to determine whether a given matrix Q with rational or simply periodic coefficients is a stationary solution of an equation in the AKNS hierarchy.

Contents

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1. Introduction

Consider the differential expression

$$L = Q_0 \frac{d^n}{dx^n} + \cdots + Q_n.$$

When does a differential expression P exist which commutes with L? This question has drawn attention for well over one hundred years and its relationship with completely integrable systems of partial differential equations has led to a heightened interest in the past quarter century. A recent survey [10] by F. Gesztesy and myself tries to capture a part of that story and might be consulted for further references.

If $L = d^2/dx^2 + q$ the problem is related to the Korteweg-de Vries (KdV) hierarchy which, according to Lax [17], can be represented as the hierarchy of equations $q_t = [P_{2n+1}, L]$ for $n = 0, 1, \dots,$ where P_{2n+1} is a certain differential expression of order $2n+1$. The stationary solutions of these equations give rise to commuting differential expressions and play an important role in the solution of the Cauchy problem of the famous KdV equation (the case $n = 1$). Relying on a classical result of Picard [21], Gesztesy and myself discovered in [8] that, when q is an elliptic function, the existence of an expression P_{2n+1} which commutes with L is equivalent to the property that for all $z \in \mathbb{C}$ all solutions of the equation $Ly = zy$ are meromorphic functions of the independent variable. This discovery was since generalized to cover certain 2×2 first order systems with elliptic coefficients (see [9]) and scalar n-th order equations with rational and simply periodic coefficients (see [24]).

According to the famous results of Burchnall and Chaundy in [1] and [2] a commuting pair of scalar differential expressions is associated with an algebraic curve and this fact has been one of the main avenues of attack on the problems posed by this kind of integrable systems (Its and Matveev [13], Krichever [14], [15], [16]). For this reason such differential expressions or their coefficients have been called algebro-geometric.

In this paper I will consider the case where the coefficients $Q_0, \dots, Q_n$ of L are $m \times m$ matrices with rational or simply periodic entries. First let us make the following definition.[1]

Definition 1. A pair (P, L) of differential expressions is called a pair of nontrivially commuting differential expressions if $[P, L] = 0$ while there exists no differential expression A such that both P and L are in $\mathbb{C}[A]$.

I will give sufficient conditions for the coefficients Q_j which guarantee the existence of a P such that (P, L) is a nontrivially commuting pair when mn is larger than one.[2] Theorem 1 covers the rational case while Theorem 2 covers the periodic case. These results are then applied to the AKNS hierarchy to obtain a characterization of all rational and simply periodic algebro-geometric AKNS potentials (see Theorem 3).

The main ingredients in the proofs are generalizations of theorems by Halphen [12] and Floquet [6], [7] which determine the structure of the solutions of $Ly = zy$.

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[1] The definition is motivated by the following observation. The expressions P and L commute if they are both polynomials of another differential expression A, i.e., if $P, L \in \mathbb{C}[A]$. Note that this does not happen in the case discussed above, i.e., when $L = d^2/dx^2 + q$ and P is of odd order, unless q is constant.

[2] When $m = n = 1$ and $[P, L] = 0$ then P is necessarily a polynomial of L.

The original theorems cover the scalar case. The generalizations, which are quoted in Appendix A, are proven in [11] and [25], respectively.

Algebro-geometric differential expressions with matrix coefficients have attracted a lot of attention in the past. The papers by Cherednik [3], Dickey [4], Dubrovin [5], van Moerbeke [19], Mumford [20], and Treibich [22] form a (rather incomplete) list of investigations into the subject.

The organization of the paper is as follows: Sections 2 and 3 contain the statements and proofs of Theorems 1 and 2, respectively. Section 4 contains a short description of the AKNS hierarchy as well as Theorem 3 and its proof. The proofs of Theorems 1 and 2 rely on several lemmas which do not specifically refer to one or the other case. These lemmas are stated and proved in Section 5. Finally, for the convenience of the reader, three appendices provide the statements of the theorems of Halphen and Floquet, a few facts about higher order systems of differential equations, and the statement of a theorem of Wasow on the asymptotic behavior of solutions of a system of first order differential equations depending on a parameter.

Before we actually get started let us agree on a few pieces of notation. If $\mathbb{F}$ is a field we denote by $\mathbb{F}[x]$ the ring of polynomials with coefficients in $\mathbb{F}$ and by $\mathbb{F}(x)$ the associated quotient field. The ring of $j \times k$ matrices with entries in $\mathbb{F}$ is denoted by $\mathbb{F}^{j\times k}$. The letter $\mathbb{A}$ represents the field of algebraic functions in one variable over the complex numbers. The symbol $\mathbf{1}$ denotes an identity matrix. Occasionally it is useful to indicate its dimension by a subscript as in $\mathbf{1}_k$. Similarly $\mathbf{0}$ and $\mathbf{0}_{j\times k}$ denote zero matrices. Polynomials are to be regarded as polynomials over $\mathbb{C}$ unless the contrary is explicitly stated.

2. The rational case

Theorem 1. *Let L be the differential expression given by*

$$Ly = Q_0 y^{(n)} + Q_1 y^{(n-1)} + \cdots + Q_n y.$$

Suppose that the following conditions are satisfied:

1. *$Q_0, \dots, Q_n \in \mathbb{C}(x)^{m\times m}$ are bounded at infinity.*
2. *Q_0 is constant and invertible.*
3. *The matrix*

$$B(\lambda) = \lambda^n Q_0 + \lambda^{n-1} Q_1(\infty) + \cdots + Q_n(\infty)$$

   is diagonalizable (as an element of $\mathbb{A}^{m\times m}$).
4. *There are linearly independent eigenvectors $v_1, \dots, v_m \in \mathbb{A}^m$ of B such that $\lim_{\lambda\to\infty} v_j(\lambda)$, $j = 1, \dots, m$, exist and are linearly independent eigenvectors of Q_0. In particular Q_0 is diagonalizable.*

If $mn > 1$ and if, for all $z \in \mathbb{C}$, all solutions of $Ly = zy$ are meromorphic, then there exists a differential expression P with coefficients in $\mathbb{C}(x)^{m\times m}$ such that (P, L) is a pair of nontrivially commuting differential expressions.

Note that Conditions 3 and 4 are automatically satisfied if all eigenvalues of Q_0 are algebraically simple.

Proof. Without loss of generality we will assume that Q_0 is diagonal. Lemma 1 gives a large class of differential expressions P which commute with L. Our goal is therefore to check the hypotheses of Lemma 1. After that we will address the question of finding a P which commutes nontrivially with L.

Let $\mathbb{M} = \mathbb{C}(x)$. For $j = 0, \ldots, n$ let $Q_{\infty,j} = Q_j(\infty)$ and let the function τ be the identity. The eigenvectors of B, which are linearly independent as elements of $\mathbb{A}^m$, become linearly dependent (as elements of $\mathbb{C}^m$) for at most finitely many values of λ, since the determinant of the matrix whose columns are these eigenvectors is an algebraic function. Conditions 1–3 of Lemma 1 are then satisfied. Next we have to construct U such that Conditions 4 and 5 are also satisfied.

Let the characteristic polynomial of B be given by

$$\det(B(\lambda) - z) = \prod_{j=1}^{\nu} f_j(\lambda, z)^{m_j}$$

where the $f_j \in \mathbb{C}[\lambda, z]$ are pairwise relatively prime. Denote the degree of $f_j(\lambda, \cdot)$ (which does not depend on λ) by k_j. According to Lemma 2 we may choose λ among infinitely many values such that:

1. $B(\lambda)$ is diagonalizable.
2. $(f_1 \ldots f_\nu)(\lambda, \cdot)$ has $k_1 + \cdots + k_\nu$ distinct roots.
3. if $z_{j,k}(\lambda)$ is a root of $f_j(\lambda, \cdot)$, then λ is a simple root of $(f_1 \ldots f_\nu)(\cdot, z_{j,k}(\lambda))$.

Until further notice we will think of this value of λ as fixed and, accordingly, we will typically suppress the dependence on λ of the quantities considered.

Let $z_{j,k}$ be a root of $f_j(\lambda, \cdot)$, i.e., an eigenvalue of $B(\lambda)$ of multiplicity m_j. The equation $Ly = zy$ is equivalent to a first-order system $\psi' = A(z, \cdot)\psi$ where $A(z, x) \in \mathbb{C}^{mn \times mn}$ remains bounded as x tends to infinity. By Lemma 3 the characteristic polynomial of $A(z, \infty)$ is a constant multiple of $\prod_{j=1}^{\nu} f_j(\lambda, z)^{m_j}$ and hence λ is an eigenvalue of $A(z_{j,k}, \infty)$ of algebraic multiplicity m_j. But Lemma 3 implies also that the geometric multiplicity of λ is equal to m_j. Theorem 2.4 of [11] (quoted in Appendix A), which is a generalization of a theorem of Halphen, guarantees then the existence of m_j linearly independent solutions

$$\psi_{j,k,\ell}(x) = R_{j,k,\ell}(x) \exp(\lambda x), \quad \ell = 1, \ldots, m_j$$

of $\psi' = A(z_{j,k}, \cdot)\psi$ where the components of $R_{j,k,\ell}$ are rational functions. The common denominator q of these components is a polynomial in x whose coefficients are independent of λ and $z_{j,k}$ since the poles of the solutions of $Ly = zy$ may occur only at points where one of the coefficient matrices Q_j has a pole. Moreover, q may be chosen such that the entries of qQ_j are polynomials for all $j \in \{1, \ldots, n\}$.

The $R_{j,k,\ell}$, $\ell = 1, \ldots, m_j$, may have poles at infinity whose order can be determined from asymptotic considerations. We denote the largest order of these poles, i.e., the largest degree of the numerators of the components of the $R_{j,k,\ell}$ by s and perform the substitution

$$\psi(x) = \frac{\exp(\lambda x)}{q(x)} \sum_{j=0}^{s} \alpha_j x^{s-j}.$$

This turns the equation $\psi' = A(z, \cdot)\psi$ into the equivalent equation

$$\sum_{\ell=0}^{s+s'} x^{s+s'-\ell} \sum_{j+k=\ell} \{(s-j)q_{k-1} - \Gamma_k(\lambda, z)\}\alpha_j = 0 \tag{1}$$

where $s' = \deg(q)$ and where the Γ_k and q_k are defined respectively by

$$q(x)A(z,x) + (q'(x) - \lambda q(x)) = \sum_{k=0}^{s'} \Gamma_k(\lambda, z)x^{s'-k} \quad \text{and} \quad q(x) = \sum_{k=0}^{s'} q_k x^{s'-k}$$

(quantities whose index is out of range are set equal to zero). Equation (1) represents a system of $(s+s'+1)mn$ linear homogeneous equations for the $(s+1)mn$ unknown components of the coefficients α_j and is thus equivalent to the equation $\widetilde{A}(\lambda,z)\beta = 0$ where $\widetilde{A}$ is an appropriate $(s+s'+1)mn \times (s+1)mn$ matrix and β is a vector with $(s+1)mn$ components comprising all components of all the α_j. Lemma 4 applies to the equation $\widetilde{A}(\lambda,z)\beta = 0$ with $R = \mathbb{C}[\lambda]$ and $g(\lambda,z) = \det(B(\lambda) - z)$. We therefore conclude that there are polynomials $\beta_1, \ldots, \beta_{(s+1)mn}$ in $\mathbb{C}[\lambda,z]^{(s+1)mn}$ (some of which might be zero) such that

$$\beta_1(\lambda, z_{j,k}), \ldots, \beta_{m_j}(\lambda, z_{j,k})$$

are linearly independent solutions of $\widetilde{A}(\lambda, z_{j,k})\beta = 0$ for $k = 1, \ldots, k_j$ and $j = 1, \ldots, \nu$. Hence

$$\psi_{j,k,\ell}(x) = \frac{\exp(\lambda x)}{q(x)} (x^s \mathbf{1}_{mn}, \ldots, x^0 \mathbf{1}_{mn}) \beta_\ell(\lambda, z_{j,k}).$$

Using next that $f_j(\lambda, z_{j,k}) = 0$ and the fact that z^{k_j} has a constant nonvanishing coefficient in $f_j(\lambda,\cdot)$ we obtain that $\psi_{j,k,\ell}$ can be expressed as

$$\psi_{j,k,\ell}(x) = \frac{\exp(\lambda x)}{q(x)} (x^s \mathbf{1}_{mn}, \ldots, x^0 \mathbf{1}_{mn}) \sum_{r=0}^{k_j-1} \widetilde{\beta}_{\ell,j,r}(\lambda) z_{j,k}^r$$

where the $\widetilde{\beta}_{\ell,j,r}$ are elements of $\mathbb{C}[\lambda]^{(s+1)mn}$. (They are independent of the subscript k.) The first m components of each $\psi_{j,k,\ell}$ form a solution $y_{j,k,\ell}$ of $Ly = z_{j,k}y$. One obtains

$$y_{j,k,\ell}(x) = \exp(\lambda x) \sum_{r=0}^{k_j-1} \gamma_{\ell,j,r}(\lambda,x) z_{j,k}^r,$$

where $\gamma_{\ell,j,r}(\lambda,\cdot) \in \mathbb{C}(x)^m$ and $\gamma_{\ell,j,r}(\cdot,x) \in \mathbb{C}[\lambda]^m$.

Now define

$$S_j = (\gamma_{1,j,0}, \ldots, \gamma_{1,j,k_j-1}, \gamma_{2,j,0}, \ldots, \gamma_{m_j,j,k_j-1}),$$

$$V_j = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ z_{j,1}^{k_j-1} & \cdots & z_{j,k_j}^{k_j-1} \end{pmatrix},$$

$$Z_j = \oplus_{r=1}^{m_j} V_j, \quad \text{and} \quad Y_j(\lambda,x) = S_j(\lambda,x) Z_j \exp(\lambda x).$$

The matrix Y_j is a $m \times m_j k_j$ matrix. The m_j columns whose index is equal to k modulo k_j are the linearly independent solutions of $Ly = z_{j,k}y$ whose asymptotic behavior is given by $\exp(\lambda x)$. Finally we define the $m \times m$ matrices

$$S(\lambda,x) = (S_1(\lambda,x), \ldots, S_\nu(\lambda,x)), \quad Z = \oplus_{j=1}^n Z_j, \text{ and}$$

$$Y(\lambda,x) = (Y_1(\lambda,x), \ldots, Y_\nu(\lambda,x)) = S(\lambda,x) Z \exp(\lambda,x).$$

We now study the asymptotic behavior of $Y(\lambda,x)$ as λ tends to infinity. By Lemma 3 the matrix $A(z_{j,k},\infty)$ is diagonalizable and there is a positive integer h such that the eigenvalues of $A(z_{j,k},\infty)$ are given by

$$\mu_{j,k,\ell} = \lambda\left(\sigma_{j,k,\ell,0} + \sum_{r=1}^{\infty} \sigma_{j,k,\ell,r}\lambda^{-r/h}\right), \quad \ell = 1,\dots,mn$$

where the numbers $\sigma_{j,k,\ell,0}$ are different from zero. Define the diagonal matrices $M_r = \operatorname{diag}(\sigma_{j,k,1,r},\dots,\sigma_{j,k,mn,r})$ and order the eigenvalues in such a way that, for $r = 0,\dots,h-1$,

$$M_r = \begin{pmatrix} \sigma_{j,k,1,r}\mathbf{1}_{p_r} & \mathbf{0} \\ \mathbf{0} & \Sigma_{j,k,r} \end{pmatrix},$$

where $p_0 \geq p_1 \geq \cdots \geq p_{h-1}$ and where $\sigma_{j,k,1,r}$ is not an eigenvalue of $\Sigma_{j,k,r}$. Moreover, require that the m_j eigenvalues which are equal to λ are first. Then we have $\sigma_{j,k,1,0} = 1$, $\sigma_{j,k,1,1} = \cdots = \sigma_{j,k,1,h-1} = 0$, and $p_{h-1} \geq m_j$. There are p_0 eigenvalues which are asymptotically equal to λ and there are p_{h-1} eigenvalues which differ from λ by a function which stays bounded as λ tends to infinity.

To each eigenvalue $\mu_{j,k,\ell}$ we have an eigenvector $u_{j,k,\ell}$ of the form

$$u_{j,k,\ell} = \begin{pmatrix} v_{j,k,\ell} \\ \mu_{j,k,\ell}v_{j,k,\ell} \\ \vdots \\ \mu_{j,k,\ell}^{n-1}v_{j,k,\ell} \end{pmatrix},$$

where $v_{j,k,\ell}$ is an appropriate eigenvector of $B(\mu_{j,k,\ell})$ associated with the eigenvalue $z_{j,k}$, and can, by assumption, be chosen to be holomorphic at infinity. Define $T_{j,k}$ to be $(mn)\times(mn)$ matrix whose columns are the vectors $u_{j,k,1}$, … , $u_{j,k,mn}$. Then $T_{j,k}(\lambda)$ is invertible at and near infinity. Let

$$\breve{A}_{j,k}(\lambda,x) = \lambda^{-1}T_{j,k}^{-1}A(z_{j,k},x)T_{j,k} = \lambda^{-1}\operatorname{diag}(\mu_{j,k,1},\dots,\mu_{j,k,mn}) + \lambda^{-1}X_{j,k}(\lambda,x)$$

where, according to Lemma 5, $X_{j,k}(\lambda,x)$ is bounded as λ tends to infinity. Furthermore, $X_{j,k}(\lambda,x)$ tends to zero as x tends to infinity. Hence, given a $\delta > 0$, there is an $x_0(\delta)$ and a number $r(\delta)$ such that $\|X_{j,k}(\lambda,x)\| < \delta$ whenever $|x-x_0(\delta)| \leq r(\delta)$. The matrix $\breve{A}_{j,k}$ satisfies now the assumptions of Lemma 6 with $\rho = \lambda^{-1/h}$, $\Omega = \{x : |x-x_0(\delta)| < r(\delta)\}$, and $S = \{\rho : 0 < |\rho| < \rho_0\}$ for some suitable constant ρ_0. The matrix Γ is the upper left $p_{h-1}\times p_{h-1}$ block of M_h and hence diagonal. The matrix $\Delta(x)$ is the upper left $p_{h-1}\times p_{h-1}$ block of $X_{j,k}(\infty,x)$. Hence Lemma 6 guarantees the existence of p_{h-1} linearly independent solutions for $\lambda y' = \breve{A}_{j,k}y$ whose asymptotic behavior is given by

$$\begin{pmatrix} \exp(\Gamma(x-x_0))(\mathbf{1}_{p_{h-1}} + \Upsilon(x)) \\ \mathbf{0}_{(mn-p_{h-1})\times p_{h-1}} \end{pmatrix} \exp(\lambda,x). \tag{2}$$

Moreover, given any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\Upsilon(x)\| < \varepsilon$ for all $x \in \{x : |x-x_0(\delta)| < r(\delta)\}$. Since the first m_j entries in the diagonal of Γ are zero we obtain that the asymptotic behavior of the first m_j columns of matrix (2) is given by

$$E_{j,k}(x) = \begin{pmatrix} \mathbf{1}_{m_j} + \Upsilon_{1,1}(x)) \\ \exp(\Gamma_{2,2}(x-x_0))\Upsilon_{2,1}(x) \\ \mathbf{0}_{(mn-p_{h-1})\times m_j} \end{pmatrix} \exp(\lambda,x)$$

where $\Upsilon_{1,1}$ and $\Upsilon_{2,1}$ are the upper left $m_j \times m_j$ block and the lower left $(p_{h-1} - m_j) \times m_j$ block of Υ, respectively, and where $\Gamma_{2,2}$ is the lower right $(p_{h-1} - m_j) \times (p_{h-1} - m_j)$ block of Γ.

We have now arrived at the following result: the m_j columns of Y_j whose index is equal to k modulo k_j have asymptotic behavior whose leading order is given by the first m rows of $T_{j,k}E_{j,k}(x)C$ where C is an appropriate constant and invertible $m_j \times m_j$ matrix. By choosing the eigenvectors $u_{j,k,1}, \ldots, u_{j,k,m_j}$ (which are all associated with the eigenvalue $\mu_{j,k,1} = \lambda$) appropriately we may assume that $C = \mathbf{1}$. Hence, up to terms of a negligible size, the linearly independent eigenvectors of Q_0 are the columns of $Y \exp(-\lambda x) = SZ$ when λ and x are large. This implies that Y is invertible.

Similarly, considering the differential expression L_∞ defined by

$$(L_\infty y)(x) = Q_0 y^{(n)}(x) + Q_1(\infty) y^{(n-1)}(x) + \cdots + Q_n(\infty) y(x)$$

we obtain the invertible matrices

$$S_\infty(\lambda) = (S_{\infty,1}(\lambda), \ldots, S_{\infty,\nu}(\lambda))$$

and

$$Y_\infty(\lambda, x) = (Y_{\infty,1}(\lambda, x), \ldots, Y_{\infty,\nu}(\lambda, x)) = S_\infty(\lambda) Z(\lambda) \exp(\lambda x)$$

where Z is as before. The m_j columns of $Y_{\infty,j}$ whose index is equal to k modulo k_j are those solutions of $L_\infty y = z_{j,k} y$ whose asymptotic behavior is given by $\exp(\lambda x)$. Note that S_∞ is x-independent, since the matrices $A(z_{j,k}, \infty)$ are diagonalizable. Furthermore, since $L_\infty(v \exp(\lambda x)) = (B(\lambda)v) \exp(\lambda x)$, the columns of $S_\infty Z$ are eigenvectors of $B(\lambda)$, which, to leading order as λ tends to infinity, are eigenvectors of Q_0.

Let $d \in \mathbb{C}[\lambda]$ be such that $dS(\cdot, x) S_\infty(\cdot)^{-1}$ becomes a polynomial (at least $d(\lambda) = \det(S_\infty(\lambda))$ will do). Then we may define matrices $U_j \in \mathbb{C}(x)^{m \times m}$ by the equation

$$\sum_{j=0}^{g} \lambda^{g-j} U_j(x) = d(\lambda) S(\lambda, x) S_\infty(\lambda)^{-1}$$

and a differential expression

$$U = \sum_{j=0}^{g} U_j(x) D^{g-j}.$$

Then, obviously,

$$U(S_\infty(\lambda) Z \exp(\lambda x)) = d(\lambda) S(\lambda, x) Z \exp(\lambda x) = d(\lambda) Y(\lambda, x).$$

Since YY_∞^{-1} is close to the identity when λ and x are sufficiently large we obtain that U_0 is invertible and hence that Conditions 4 and 5 of Lemma 1 are satisfied.

Applying Lemma 1 gives now the existence of a nonempty set F of polynomials such that the differential expression P defined by $PU = UDf(L_\infty)$ commutes with L when $f \in F$. Assume that P and L commute trivially. Then, by the first statement of Lemma 7, Q_0 is a multiple of the identity and P and L are polynomials of a unique first order differential expression $G = D + G_1$ where $G_1 \in \mathbb{C}(x)^{m \times m}$ is bounded at infinity and where $G_1(\infty)$ is a multiple of the identity. Let y be a solution of $Gy = gy$ where $g \in \mathbb{C}$ and let φ be the polynomial such that $L = \varphi(G)$. Then y satisfies also $Ly = \varphi(g)y$ and hence every solution of $Gy = gy$ is meromorphic. By applying what we just proved to the expression G rather than

L we know that we also have a differential expression $\widetilde{U}$ and a nonempty set $\widetilde{F}$ of polynomials such that, when $f \in \widetilde{F}$, the differential expression P defined by $PU = UCf(D + G_1(\infty))$ commutes with G and hence with L for all matrices $C \in \mathbb{C}^{m\times m}$. The second statement of Lemma 7 shows that P is not a polynomial of G, if C is not a multiple of the identity. Hence, in this case, (P, L) is a nontrivially commuting pair. □

3. The simply periodic case

If f is an ω-periodic function we will use f^* to denote the one-valued function given by $f^*(t) = f(\frac{\omega}{2\pi i}\log(t))$. Conversely, if a function f^* is given $f(x)$ will refer to $f^*(\exp(2\pi ix/\omega))$. We say that a periodic function f is bounded at the ends of the period strip if f^* is bounded at zero and infinity. A meromorphic periodic function which is bounded at the ends of the period strip can not be doubly periodic unless it is a constant. The function f is a meromorphic periodic function bounded at the ends of the period strips if and only if f^* is a rational function bounded at zero and infinity. For more information on periodic functions see, e.g., Markushevich [18], Chapter III.4.

The field of meromorphic functions with period ω will be denoted by $\mathbb{P}_\omega$.

Theorem 2. *Let L be the differential expression given by*

$$Ly = Q_0 y^{(n)} + Q_1 y^{(n-1)} + \cdots + Q_n y.$$

Suppose that the following conditions are satisfied:

1. $Q_0, \ldots, Q_n \in \mathbb{P}_\omega^{m\times m}$ *are bounded at the ends of the period strip.*
2. Q_0 *is constant and invertible.*
3. *The matrix*

$$B(\lambda) = \lambda^n Q_0 + \lambda^{n-1} Q_1^*(\infty) + \cdots + Q_n^*(\infty)$$

   is diagonalizable (as an element of $\mathbb{A}^{m\times m}$).
4. *There are linearly independent eigenvectors $v_1, \ldots, v_m \in \mathbb{A}^m$ of B such that $\lim_{\lambda\to\infty} v_j(\lambda)$, $j = 1, \ldots, m$, exist and are linearly independent eigenvectors of Q_0. In particular Q_0 is diagonalizable.*

If $mn > 1$ and if, for all $z \in \mathbb{C}$, all solutions of $Ly = zy$ are meromorphic, then there exists a differential expression P with coefficients in $\mathbb{P}_\omega^{m\times m}$ such that (P, L) is a pair of nontrivially commuting differential expressions.

Proof. The proof of this theorem is very close to that of Theorem 1. We record the few points where more significant deviations exist. For notational simplicity we will assume that $\omega = 2\pi$.

Lemma 1 is now used with $\mathbb{M} = \mathbb{P}_\omega$, $Q_{\infty,j} = Q_j^*(\infty)$, and $\tau(x) = e^{ix}$. As before we have to construct the expression U: The role of Halphen's theorem (or, more precisely, Theorem 2.4 of [11]) is now played by Theorem 1 in [25] (quoted in Appendix A), which is a variant Floquet's theorem. We have therefore the existence of m_j linearly independent functions

$$\psi_{j,k,\ell}(x) = R_{j,k,\ell}^*(e^{ix}) \exp(\lambda x), \quad \ell = 1, \ldots, m_j$$

where the components of $R^*_{j,k,\ell}$ are rational functions. The substitution

$$y(x) = \frac{\exp(\lambda x)}{q(\mathrm{e}^{ix})} \sum_{j=0}^{s} \alpha_j \mathrm{e}^{ix(s-j)}$$

turns the equation $y' = A(z_{j,k}, \cdot)y$ into a system of linear algebraic equation with m_j linearly independent solutions. This way one shows as before that

$$\psi_{j,k,\ell}(x) = \frac{\exp(\lambda x)}{q(\mathrm{e}^{ix})} (\mathrm{e}^{six}\mathbf{1}_{mn}, \dots, \mathrm{e}^{ix}\mathbf{1}_{mn}, \mathbf{1}_{mn}) \sum_{r=0}^{k_j-1} \widetilde{\beta}_{\ell,j,r}(\lambda) z^r_{j,k}$$

where the $\widetilde{\beta}_{\ell,j,r}$ are elements of $\mathbb{C}[\lambda]^{(s+1)mn}$. Doing this for $k = 1, \dots, k_j$ and for $j = 1, \dots, \nu$ and selecting the first m components of all the resulting vectors provides once more an $m \times m$ matrices S, Z, and $Y = SZ\exp(\lambda x)$. Again the entries of S are polynomials with respect to λ but now they are rational functions with respect to e^{ix}. By considering the constant coefficient expression

$$L_\infty = Q_0 \frac{d^n}{dx^n} + \dots + Q^*_n(\infty)$$

one obtains also matrices S_∞ and $Y_\infty = S_\infty Z \exp(\lambda x)$ and U is defined as before through a multiple of $S(\lambda, x) S_\infty(\lambda)^{-1}$. The investigation of the asymptotic behavior of Y and Y_∞ as λ tends to infinity, which leads to proving the invertibility of U_0, is unchanged as it did not use the special structure of the Q_j, except that one should choose $\exp(ix_0)$ large rather than x_0 large.

Finally, the argument that it is possible to pick, among all expressions commuting with L, an expression which does not commute trivially remains unchanged. □

4. Application to the AKNS system

Let $L = Jd/dx + Q(x)$, where

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad Q(x) = \begin{pmatrix} 0 & -iq(x) \\ ip(x) & 0 \end{pmatrix}.$$

Note that $J^2 = -\mathbf{1}_2$ and that $JQ + QJ = 0$.

The AKNS hierarchy is then a sequence of equations of the form

$$Q_t = [P_{n+1}, L], \quad n = 0, 1, 2, \dots$$

where P_{n+1} is a differential expression of order $n + 1$ such that $[P_{n+1}, L]$ is a multiplication operator. For this to happen P_{n+1} has to be very special. It can be recursively computed in the following way: Let

$$P_{n+1} = \sum_{\ell=0}^{n+1} (k_{n+1-\ell}(x) + v_{n+1-\ell}(x)J + W_{n+1-\ell}(x))L^\ell,$$

where the k_j and v_j are scalar-valued and where the W_j are 2×2 matrices with vanishing diagonal elements. Requiring that $[P_{n+1}, L]$ is a multiplication operator yields $k'_j = 0$ for $j = 0, \dots, n + 1$ and the recursion relations

$$W_0 = 0$$

$$v'_j \mathbf{1}_2 = W_j Q + Q W_j, \quad W_{j+1} = \frac{1}{2} J(W'_j - 2 v_j Q_j), \quad j = 0, \dots, n + 1.$$

This gives finally

$$[P_{n+1}, L] = 2v_{n+1}JQ - JW'_{n+1}.$$

The first few AKNS equations are

$$Q_t = -c_0 Q' + 2c_1 JQ,$$
$$Q_t = -\frac{c_0}{2} J(Q'' - 2Q^3) - c_1 Q' + 2c_2 JQ,$$
$$Q_t = \frac{c_0}{4}(Q''' - 6Q^2 Q') - c_1 J(Q'' - 2Q^3) - c_2 Q' + 2c_3 JQ.$$

Here we are interested in the stationary solutions of AKNS equations. Therefore we make the following definition.

Definition 2. Suppose p and q are meromorphic functions. Then Q is called an *algebro-geometric* AKNS *potential* (or simply *algebro-geometric*) if Q is a stationary solution of some AKNS equation.

The goal of this section is to prove the following theorem.

Theorem 3. *Let* $Q = \begin{pmatrix} 0 & -iq \\ ip & 0 \end{pmatrix}$ *and assume either that* p, q *are rational functions bounded at infinity or else that* p, q *are meromorphic* ω*-periodic functions bounded at the ends of the period strip. Then* Q *is an algebro-geometric* AKNS *potential if and only if for all* $z \in \mathbb{C}$ *all solutions of the equation* $Jy' + Qy = zy$ *are meromorphic with respect to the independent variable.*

Before we begin the proof of this result let us recall the following two results which were proven by Gesztesy and myself in [9]. The first one (Theorem 4 below) asks that p and q are meromorphic and provides one direction in the proof of Theorem 3. The second one (Theorem 5 below) is the analogue of Theorem 3 for the case of elliptic coefficients and is stated here for comparison purposes.

Theorem 4. *Let* $Q = \begin{pmatrix} 0 & -iq \\ ip & 0 \end{pmatrix}$ *where* p, q *are meromorphic functions. If* Q *is an algebro-geometric* AKNS *potential then for all* $z \in \mathbb{C}$ *all solutions of the equation* $Jy' + Qy = zy$ *are meromorphic with respect to the independent variable.*

Theorem 5. *Let* $Q = \begin{pmatrix} 0 & -iq \\ ip & 0 \end{pmatrix}$ *with* p, q *elliptic functions with a common period lattice. Then* Q *is an elliptic algebro-geometric* AKNS *potential if and only if for all* $z \in \mathbb{C}$ *all solutions of the equation* $Jy' + Qy = zy$ *are meromorphic with respect to the independent variable.*

Now we are ready to prove Theorem 3:

Proof of Theorem 3. We only need to prove that Q is algebro-geometric if all solutions of $Ly = zy$ are meromorphic since the converse follows from Theorem 4. Suppose Q is periodic. The desired conclusion follows from Theorem 2 once we have checked its hypotheses. But Conditions 1 and 2 are satisfied by our assumptions while Conditions 3 and 4 hold automatically when the eigenvalues of Q_0 $(= J)$ are distinct. For convenience, however, let us mention that the eigenvalues of

$$B(\lambda) = \begin{pmatrix} i\lambda & -iq^*(\infty) \\ ip^*(\infty) & -i\lambda \end{pmatrix}$$

are $\pm\sqrt{q^*(\infty)p^*(\infty)-\lambda^2}$ and that these are distinct for all but two values of λ. The eigenvectors may be chosen as

$$v_1 = \frac{1}{2\lambda}\begin{pmatrix} i\lambda + z_1(\lambda) \\ ip^*(\infty) \end{pmatrix} \quad \text{and} \quad v_2 = \frac{1}{2\lambda}\begin{pmatrix} iq^*(\infty) \\ i\lambda + z_1(\lambda) \end{pmatrix}$$

where $z_1(\lambda)$ is the branch of $\sqrt{q^*(\infty)p^*(\infty)-\lambda^2}$ which is asymptotically equal to $i\lambda$.

The proof for the rational case is virtually the same. □

The solutions of $Jy' + Qy = zy$ are analytic at every point which is neither a pole of p nor of q. Since it is a matter of routine calculations to check whether a solution of $Jy' + Qy = zy$ is meromorphic at a pole of p or q and since there are only finitely many poles of Q modulo periodicity, Theorem 3 provides an easy method which allows one to determine whether a rational function Q bounded at infinity or a meromorphic simply periodic function Q bounded at the ends of the period strip is a stationary solution of an equation in the AKNS hierarchy.

5. The lemmas

Lemma 1. *Let $\mathbb{M}$ be a field of meromorphic functions on $\mathbb{C}$ and consider the differential expression $L = \sum_{j=0}^n Q_j D^{n-j}$ where $Q_0 \in \mathbb{C}^{m\times m}$ is invertible and $Q_j \in \mathbb{M}^{m\times m}$ for $j = 1, \dots, n$. Suppose that there exist differential expressions*

$$L_\infty = \sum_{j=0}^{n} Q_{\infty,j}\ D^{n-j} \quad \textit{and} \quad U = \sum_{j=0}^{g} U_j(\tau(x))D^{g-j}$$

with the following properties:

1. *$Q_{\infty,0}, \dots, Q_{\infty,n}$ are in $\mathbb{C}^{m\times m}$ and $Q_{\infty,0} = Q_0$.*
2. *τ is a meromorphic function on $\mathbb{C}$.*
3. *There is a set $\Lambda \subset \mathbb{C}$ with at least $g + n + 1$ distinct elements such that, for each $\lambda \in \Lambda$, the matrix $B(\lambda) = \sum_{j=0}^n \lambda^{n-j} Q_{\infty,j}$ has m linearly independent eigenvectors $v_1(\lambda), \dots, v_m(\lambda) \in \mathbb{C}^m$ respectively associated with the (possibly degenerate) eigenvalues $z_1(\lambda), \dots, z_m(\lambda)$.*
4. *$U_0, \dots, U_g \in \mathbb{M}^{m\times m}$ and U_0 is invertible.*
5. *$U(v_j(\lambda)\exp(\lambda x))$ is a solution of $Ly = z_j(\lambda)y$ for $j = 1, \dots, m$.*

Finally, define the algebra

$$\mathcal{C} = \{C \in \mathbb{C}^{m\times m} : [Q_{\infty,0}, C] = \cdots = [Q_{\infty,n}, C] = 0\}.$$

Then there exists a nonempty set $F \subset \mathbb{C}[u]$ with the following property: for each polynomial $f \in F$ and each polynomial $h \in \mathcal{C}[u]$ there exists a differential expression P with coefficients in $\mathbb{M}^{m\times m}$ such that $[P, L] = 0$. In fact, P is given by $PU = Uh(D)f(L_\infty)$.

Proof. Consider the differential expression $V = LU - UL_\infty$ and fix $\lambda \in \Lambda$. Since

$$L_\infty(v_j(\lambda)\exp(\lambda x)) = z_j v_j(\lambda)\exp(\lambda x)$$

we obtain

$$V(v_j(\lambda)\exp(\lambda x)) = (L - z_j)U(v_j(\lambda)\exp(\lambda x)) = 0.$$

V is a differential expression of order $g+n$ at most, i.e., $V = \sum_{k=0}^{g+n} V_k(x) D^k$ for suitable matrices V_k. Hence

$$0 = \exp(-\lambda x) V(v_j(\lambda) \exp(\lambda x)) = \left(\sum_{k=0}^{g+n} V_k(x) \lambda^k \right) v_j(\lambda).$$

For fixed x and λ we now have an $m \times m$ matrix $\widetilde{V}(\lambda, x) = \sum_{k=0}^{g+n} V_k(x) \lambda^k$ whose kernel contains all eigenvectors of $B(\lambda)$ and is therefore m-dimensional. This means that $\widetilde{V}(\lambda, x) = 0$. Since this is the case for at least $g+n+1$ different values of λ we conclude that $V_0 = \cdots = V_{g+n} = 0$ and hence that

$$LU = UL_\infty.$$

Since U_0 is invertible $Uy = 0$ has mg linearly independent solutions. Let $\{y_1, \dots, y_{mg}\}$ be a basis of $\ker(U)$. With each element y_ℓ of this basis we may associate a differential expression H_ℓ with coefficients in $\mathbb{C}^{m \times m}$ in the following way. Since $y_\ell \in \ker(U)$, so is $L_\infty y_\ell$ and, in fact, $L_\infty^j y_\ell$ for every $j \in \mathbb{N}$. Since $\ker(U)$ is finite-dimensional there exists a $k \in \mathbb{N}$ and complex numbers $\alpha_0, \dots, \alpha_k$ such that $\alpha_0 \neq 0$ and

$$\sum_{j=0}^{k} \alpha_{k-j} L_\infty^j y_\ell = 0.$$

Then define $H_\ell = \sum_{j=0}^{k} \alpha_{k-j} L_\infty^j$. Since the expressions H_ℓ commute among themselves we obtain that

$$\ker(U) \subset \ker \left(\prod_{\ell=1}^{mg} H_\ell \right).$$

Hence the set

$$F = \{ f \in \mathbb{C}[u] : \ker(U) \subset \ker(f(L_\infty)) \}$$

is not empty.

Note that $[L_\infty, D] = 0$ and $[L_\infty, C] = 0$ if $C \in \mathcal{C}$. For any $h \in \mathcal{C}[u]$ and any $f \in F$ let $P_\infty = h(D) f(L_\infty)$. Then $[P_\infty, L_\infty] = 0$ and $\ker(U) \subset \ker(P_\infty) \subset \ker(UP_\infty)$. Corollary 1 in Appendix B shows that there is an expression P such that $PU = UP_\infty$. Hence $[P, L]U = PLU - LPU = UP_\infty L_\infty - UL_\infty P_\infty = U[P_\infty, L_\infty] = 0$ and thus, recalling that U_0 is invertible, $[P, L] = 0$. □

Lemma 2. *Let*

$$B(\lambda) = \sum_{j=0}^{n} \lambda^{n-j} B_j$$

where $B_0, \dots, B_n \in \mathbb{C}^{m \times m}$ and where B_0 is invertible. Suppose the characteristic polynomial of B has the prime factorization $\prod_{j=1}^{\nu} f_j(\lambda, z)^{m_j}$. If weight $nr + s$ is assigned to the monomial $\lambda^s z^r$, then the weight of the heaviest monomial in f_j is a multiple of n, say nk_j and the coefficients of z^{k_j} and λ^{nk_j} in f_j are nonzero.

Let Λ be the set of all complex numbers λ satisfying the following two conditions:

1. *$(f_1 \dots f_\nu)(\lambda, \cdot)$ has $k_1 + \cdots + k_\nu$ distinct roots.*
2. *If $z_{j,k}$ is a root of $f_j(\lambda, \cdot)$, then λ is a simple root of $(f_1 \dots f_\nu)(\cdot, z_{j,k})$.*

Then the complement of Λ *is finite.*

Moreover, there is an integer h *and there are complex numbers* $\rho_{j,k,r}$ *such that, for sufficiently large* λ*, the roots of* $f_j(\lambda,\cdot)$*,* $j = 1, \ldots, \nu$*, are given by*

$$z_{j,k} = \lambda^n \left(\rho_{j,k,0} + \sum_{r=1}^{\infty} \rho_{j,k,r} \lambda^{-r/h} \right), \quad k = 1, \ldots, k_j$$

where the numbers $\rho_{j,k,0}$ *are different from zero.*

Proof. First we agree, as usual, that the weight of a polynomial is equal to the weight of its heaviest monomial. It is then easy to see that the characteristic polynomial $f(\lambda, z) = \det(B(\lambda) - z)$ has weight mn. Suppose $f = g_1 g_2$ and let $f = \sum_{j=0}^{mn} \alpha_j w_j$ where w_j is a polynomial all of whose terms have weight j. Doing the same with g_1 and g_2 one can show that any factor of f has a weight which is a multiple of n, say kn, and that the coefficients of z^k and λ^{kn} in that factor are nonzero. In particular then, this is true for the prime factors.

Therefore $f_j(\lambda, \cdot)$ has k_j distinct roots for all but finitely many values of λ. Moreover, by Bezout's theorem, the curves defined by f_j and f_ℓ intersect only in finitely many points if j is different from ℓ. Hence the first condition is satisfied for all but finitely many values of λ.

The discriminant of $(f_1 \ldots f_\nu)(\cdot, z)$ is a polynomial in z. Hence there are at most finitely many values of z for which $(f_1 \ldots f_\nu)(\cdot, z)$ has multiple roots. For each of these exceptional z-values there are only finitely many of the multiple roots. Hence there are only finitely many values of λ such that there is a z for which $(f_1 \ldots f_\nu)(\cdot, z)$ has a multiple root.

The last statement follows from standard considerations of the behavior of algebraic functions near a point. In particular, the power n on λ is determined by an inspection of the Newton polygon associated with f_j. □

Lemma 3. *Let*

$$B(\lambda) = \sum_{j=0}^{n} \lambda^{n-j} B_j$$

where $B_0, \ldots, B_n \in \mathbb{C}^{m \times m}$ *and where* B_0 *is invertible. Define*

$$A(z) = \begin{pmatrix} \mathbf{0} & \mathbf{1}_m & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & & & \cdots & \mathbf{1}_m \\ B_0^{-1}(z - B_n) & -B_0^{-1} B_{n-1} & -B_0^{-1} B_{n-2} & \cdots & -B_0^{-1} B_1 \end{pmatrix},$$

a matrix whose n^2 *entries are* $m \times m$ *blocks.*

The vector $v \in \mathbb{C}^m$ *is an eigenvector of* $B(\lambda)$ *associated with the eigenvalue* z *if and only if*

$$u = \begin{pmatrix} v \\ \lambda v \\ \vdots \\ \lambda^{n-1} v \end{pmatrix}$$

is an eigenvector of $A(z)$ *associated with the eigenvalue* λ*. In particular,* z *has geometric multiplicity* k *as an eigenvalue of* $B(\lambda)$ *if and only if* λ *has geometric* *multiplicity k as an eigenvalue of $A(z)$. Also,*

$$\det(A(z)-\lambda)=(-1)^{nm}\det(B_0^{-1})\det(B(\lambda)-z). \tag{3}$$

If B is diagonalizable (as an element of $\mathbb{A}^{m\times m}$), then $A(z)$ is diagonalizable for all but finitely many values of z.

Moreover, if $z_{j,k}$ is a zero of $f_j(\lambda,\cdot)$, then there are complex numbers $\sigma_{j,k,\ell,r}$ and an integer h such that the eigenvalues $\mu_{j,k,1}, \dots, \mu_{j,k,mn}$ of $A(z_{j,k})$ are given by

$$\mu_{j,k,\ell}=\lambda\left(\sigma_{j,k,\ell,0}+\sum_{r=1}^{\infty}\sigma_{j,k,\ell,r}\lambda^{-r/h}\right),\quad \ell=1,\dots,mn$$

where the numbers $\sigma_{j,k,\ell,0}$ are different from zero.

Proof. That $B(\lambda)v=zv$ if and only if $A(z)u=\lambda u$ follows immediately from direct computation. The validity of (3) is proven by blockwise Gaussian elimination.

Assume now that B is diagonalizable and let $T\in\mathbb{A}^{m\times m}$ be an invertible matrix whose columns are eigenvectors of B. The determinant of T is an algebraic function in λ which is zero or infinity only for finitely many distinct values of λ and $B(\lambda)$ is diagonalizable for all λ but these. From Lemma 2 we know also that there are only finitely many values of λ for which $\prod_{j=1}^{\nu}f_j(\cdot,z)$ has repeated zeros. To all these exceptional values of λ correspond finitely many eigenvalues z of $B(\lambda)$. We assume now that z is a complex number distinct from all those values. If μ is now an eigenvalue of $A(z)$ then it is a zero of $f_j(\cdot,z)$ for some j but not a zero of $f_\ell(\cdot,z)$, if $\ell\neq j$. Hence its algebraic multiplicity is m_j. Additionally, z is an eigenvalue of geometric multiplicity m_j of $B(\mu)$, since $B(\mu)$ is diagonalizable. The previous argument shows that μ has geometric multiplicity m_j as eigenvalue of $A(z)$. Since this is true for any eigenvalue of $A(z)$, the matrix $A(z)$ must be diagonalizable. The last statement follows again from standard considerations of the behavior of algebraic functions near a point, using that $z_{j,k}$ is an algebraic function of λ (whose behavior near infinity is of the form given in Lemma 2) and that $\mu_{j,k,r}$ are algebraic functions of $z_{j,k}$. □

Lemma 4. *Let R be an integral domain, Q its fraction field, g an element of $R[z]$, and K a field extension of Q in which g splits into linear factors. Suppose A is a matrix in $R[z]^{j\times k}$. Then there exist k vectors $v_1,\dots,v_k\in R[z]^k$ with the following property: if $z_0\in K$ is any of the roots of g and if the dimension of $\ker(A(z_0))$ is μ, then $v_1(z_0), \dots, v_\mu(z_0)$ are linearly independent solutions of $A(z_0)x=0$.*

Proof. Suppose g has the prime factorization $g_1^{m_1}\dots g_\nu^{m_\nu}$. If $g(z_0)=0$ then precisely one of the prime factors of g, say g_ℓ, satisfies $g_\ell(z_0)=0$. Note that $F_\ell=Q[z]/\langle g_\ell\rangle$ is a field and that we may view A as an element of $F_\ell^{j\times k}$. Since F_ℓ is isomorphic to a subfield of K any K^k-solution of $A(z_0)x=0$ is a scalar multiple of a representative of an F_ℓ^k-solution of $Ax=0$ (evaluated at z_0) and vice versa. Therefore there is a basis $\{x_{\ell,1}(z_0),\dots,x_{\ell,\mu_\ell}(z_0)\}$ of $\ker(A(z_0))$ where the $x_{\ell,r}$ are in $Q[z]^k$. By choosing appropriate multiples in R we may even assume that the $x_{\ell,r}$ are in $R[z]^k$. Notice that if z_0' is another root of g_ℓ then $\{x_{\ell,1}(z_0'),\dots,x_{\ell,\mu_\ell}(z_0')\}$ is a basis of $\ker(A(z_0'))$. We define also $x_{\ell,r}=0$ for $r=\mu_\ell+1,\dots,k$.

For $r = 1, \dots, k$ we now let

$$v_r = \sum_{\ell=1}^{\nu} \left(\prod_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{\nu} g_{\ell'} \right) x_{\ell,r}.$$

This proves the lemma once we recall that $g_\ell(z_0) = 0 = g_{\ell'}(z_0)$ implies that $\ell = \ell'$. □

Lemma 5. *Suppose $A \in \mathbb{C}^{mn \times mn}$ and $T \in \mathbb{A}^{mn \times mn}$ have the following properties:*

1. *The first $(n-1)m$ rows of A are zero.*
2. *T is invertible at and near infinity and its columns $T_{1:mn,j}$ have the form*

$$T_{1:mn,j} = \begin{pmatrix} v_j \\ \mu_j v_j \\ \vdots \\ \mu_j^{n-1} v_j \end{pmatrix},$$

where the μ_j are complex-valued algebraic functions of λ with the asymptotic behavior $\mu_j(\lambda) = \lambda(\sigma_j + o(1))$ as λ tends to infinity and where the v_j are $\mathbb{C}^m$-valued algebraic functions of λ which are holomorphic at infinity.

Then $(T^{-1}AT)(\lambda)$ is bounded as λ tends to infinity.

Proof. The first $(n-1)m$ rows of AT are zero. Consequently we need to consider only the last m columns of T^{-1}. Let $B_n, \dots, B_1$ denote the $m \times m$ matrices which occupy the last m rows of A (with decreasing index as one moves from left to right) and let τ_ℓ^* denote the row-vector in the last m columns of row ℓ in T^{-1} (note that $\tau_\ell \in \mathbb{A}^m$). Then

$$(T^{-1}AT)_{\ell,k} = \sum_{j=1}^{n} \mu_k^{n-j} \tau_\ell^* B_j v_k.$$

We will show below that τ_ℓ has the asymptotic behavior $\tau_\ell = \lambda^{1-n}(\tau_{0,\ell} + o(1))$ with $\tau_{0,\ell} \in \mathbb{C}^m$ as λ tends to infinity. Hence

$$(T^{-1}AT)_{\ell,k} = \sum_{j=1}^{n} \lambda^{1-j}(\sigma_k^{n-j} \tau_{0,\ell}^* B_j v_k(\infty) + o(1)) = \sigma_k^{n-1} \tau_{0,\ell}^* B_1 v_k(\infty) + o(1)$$

as λ tends to infinity and this will prove the claim.

The minor of T which arises when one deletes row r and columns s of T will be denoted by $M_{s,r}$. We have then that

$$(T^{-1})_{r,s} = \frac{(-1)^{r+s}}{\det(T)} \det(M_{s,r}).$$

The k-th entry in row $m\alpha + \beta$, where $\beta \in \{1, \dots, m\}$ and $\alpha \in \{0, \dots, n-1\}$, equals λ^α times a function which is bounded as λ tends to infinity. Hence $\det(T) = \lambda^N(t_0 + o(1))$ for some nonzero complex number t_0 and for $N = mn(n-1)/2$. By the same argument we have that $\det(M_{m\alpha+\beta,r}) = \lambda^{N'}(m_{m\alpha+\beta,r} + o(1))$ where $N' = N - \alpha$ and $m_{m\alpha+\beta,r} \in \mathbb{C}$. Hence

$$(T^{-1})_{r,s} = (-1)^{r+s} \lambda^{-\alpha} \frac{m_{m\alpha+\beta,r} + o(1)}{t_0}.$$

For the first part of the proof we need only the case $\alpha = n-1$. $\square$

Lemma 6. *Let $\Omega \subset \mathbb{C}$ be an open simply connected set containing x_0 and $S \subset \mathbb{C}$ a sector centered at zero. Suppose that $A : S \times \Omega \to \mathbb{C}^{n\times n}$ is holomorphic on $S \times \Omega$ and admits a uniform asymptotic expansion*

$$A(\rho, x) \sim \sum_{r=0}^{\infty} A_r(x)\rho^r$$

as ρ tends to zero. Suppose that, for $r = 0, \ldots, h-1$, the matrices A_r are constant and have the block-diagonal form

$$A_r = \begin{pmatrix} \sigma_r \mathbf{1}_{p_r} & 0 \\ 0 & \Sigma_r \end{pmatrix}$$

where $p_0 \geq p_1 \geq \cdots \geq p_{h-1} = p$ and where σ_r is not an eigenvalue of Σ_r. Denote the upper left $p \times p$ block of A_h by $A_{h;1,1}$ and assume that $A_{h;1,1}(x) = \Gamma + \Delta(x)$ where $\Gamma \in \mathbb{C}^{p\times p}$ and $\Delta : \Omega \to \mathbb{C}^{p\times p}$. Let $\alpha = \sum_{r=0}^{h-1} \sigma_r \rho^{r-h}$.

Then there exists a subsector S' of S and an $n \times p$ matrix $Y(\rho, x)$ whose columns are linearly independent solutions of $\rho^h y' = Ay$ and for which

$$R(\rho, x) = Y(\rho, x)\exp(-\alpha x)$$

has in S' an asymptotic expansion of the form

$$R(\rho, x) \sim \sum_{j=0}^{\infty} R_j(x)\rho^j$$

as ρ tends to zero. Moreover, for every positive ε there exists a positive δ such that $\|\Delta(x)\| < \delta$ for all $x \in \Omega$ implies

$$R_0(x) = \begin{pmatrix} \exp(\Gamma(x-x_0))(\mathbf{1}_p + \Upsilon(x)) \\ \mathbf{0}_{(n-p)\times p} \end{pmatrix}$$

with $\|\Upsilon(x)\| < \varepsilon$ for all $x \in \Omega$.

Proof. The key to the proof of this lemma is Theorem 26.2 in Wasow [23] which we have (essentially) quoted in Appendix C and which implies immediately Corollary 2. A repeated application of this corollary shows that there are p linearly independent solutions y_j of $\rho^h y' = Ay$ of the form

$$y_j = P_0 Q_0 \ldots P_{h-1} Q_{h-1} w_j \exp(\alpha x), \quad j = 1, \ldots, p$$

where the P_k and Q_k are matrices and where the w_j are vectors whose properties are described presently. Let $p_{-1} = n$. Then P_k is an $p_{k-1} \times p_{k-1}$ matrix which is asymptotically equal to $\mathbf{1}_{p_{k-1}}$. The matrix Q_k is a constant $p_{k-1} \times p_k$ matrix whose upper block is equal to $\mathbf{1}_{p_k}$ and whose lower block is a zero matrix. Finally, the w_j are linearly independent solutions of the $p \times p$-system $w' = B(\rho, x)w$ where

$$B(\rho, x) = \rho^{-h} Q^* \left(A(\rho, x) - \sum_{r=0}^{h-1} A_r(x)\rho^r \right) Q.$$

Note that $B(\rho, x)$ has the asymptotic behavior

$$B(\rho, x) \sim \sum_{r=0}^{\infty} B_r(x)\rho^r$$

as ρ tends to zero where $B_0(x) = A_{h;1,1}(x)$.

The equation $w' = B(\rho, x)w$ has a fundamental matrix W whose asymptotic behavior is given by

$$W(\rho, x) \sim \sum_{r=0}^{\infty} W_r(x)\rho^r$$

where

$$W_0(x) = \exp\left(\int_{x_0}^{x} A_{h;1,1}(t)dt\right) = \exp\left(\Gamma(x - x_0) + \int_{x_0}^{x} \Delta(t)dt\right)$$

is an invertible matrix. Since $\| \exp(T_1 + T_2) - \exp(T_1)\| \leq \|T_2\| \exp(\|T_1\| + \|T_2\|)$ we have that

$$W_0(x) = \exp(\Gamma(x - x_0))(\mathbf{1}_p + \Upsilon(x))$$

where the norm of Υ becomes small if the norm of Δ becomes small. The fact that the matrices P_k are asymptotically equal to identity matrices and that the upper blocks of the Q_k are equal to identity matrices gives now the desired conclusion. □

Lemma 7. *Suppose that* $\mathbb{M}$*,* L*,* L_∞*,* U*,* $\mathcal{C}$*, and* F *are as in Lemma* 1*. Given an expression* P_∞ *let* P *be defined by* $PU = UP_\infty$*. Then the following two statements hold:*

1. *Let* $P_\infty = Df(L_\infty)$*, where* f *is a monic polynomial in* F*. If* (P, L) *is a trivially commuting pair, then* Q_0 *and* $Q_{\infty,1}$ *are multiples of the identity. Moreover, there exists a first order differential expression* $G = D + Q_0^{-1}(Q_1 - \eta_1)/n$ (*where* η_1 *is a suitable constant*) *such that both* P *and* L *are polynomials of* G*.*
2. *Let* $P_\infty = Cf(L_\infty)$*, where* $C \in \mathcal{C}$ *and where* $f \in F$ *is monic. If both* P *and* L *are polynomials of an expression* $D + G_1$ *then* C *is a multiple of the identity matrix.*

Proof. Assume that

$$L = \sum_{j=0}^{n} Q_j D^{n-j} = \sum_{j=0}^{n'} \eta_j G^{n'-j} \quad \text{and} \quad P = \sum_{j=0}^{r} P_j D^{r-j} = \sum_{j=0}^{r'} \gamma_j G^{r'-j},$$

where G is a differential expression of order k and the coefficients η_j and γ_j are complex numbers.

To prove the first statement assume also that $P_\infty = Df(L_\infty)$ where $f \in F$ has degree s. Since the order of L is equal to $n = kn'$ and the order of P is equal to $r = sn + 1 = kr'$ we have necessarily $k = 1$, $n' = n$, and $r' = r = sn + 1$. Therefore we assume now that $G = G_0 D + G_1$.

Note that $LU = UL_\infty$, $PU = UP_\infty$, and $Q_{\infty,0} = Q_0$ imply that

$$U_0 Q_0 = Q_0 U_0, \quad U_0 P_{\infty,0} = P_0 U_0, \tag{4}$$

$$U_1 Q_0 + U_0 Q_{\infty,1} = Q_0 U_1 + Q_1 U_0 + n Q_0 U_0', \tag{5}$$

and that

$$U_1 P_0 + U_0 P_{\infty,1} = P_0 U_1 + P_1 U_0 + r P_0 U_0', \tag{6}$$

where $P_{\infty,j}$ is the coefficient of D^{r-j} in P_∞. Since $P_{\infty,0} = Q_0^s$ we find firstly that $P_0 = U_0 Q_0^s U_0^{-1} = Q_0^s$. Next, since $Q_0 = \eta_0 G_0^n$ and $P_0 = \gamma_0 G_0^{sn+1}$, we have that $G_0 = \eta_0^s \gamma_0^{-1} \mathbf{1}$. Hence G_0, Q_0, and P_0 are all multiples of the identity matrix. Therefore we can (and will) assume from now that $G_0 = \mathbf{1}$ by changing the coefficients η_j and γ_j appropriately. In particular, $Q_0 = \eta_0 \mathbf{1}$.

We find next that

$$Q_1 = (n\eta_0 G_1 + \eta_1 \mathbf{1}), \quad P_1 = (r\gamma_0 G_1 + \gamma_1 \mathbf{1}),$$

and that

$$P_{\infty,1} = (sQ_{\infty,1} + \kappa)Q_0^{s-1}$$

where κ is the coefficient of u^{s-1} in $f(u)$ if $n = 1$ and $\kappa = 0$ if $n > 1$. Inserting these expressions into (5) and (6) and eliminating the terms with U_0' gives

$$Q_{\infty,1} = [n\kappa + \eta_0^{-s}(r\eta_1\gamma_0 - n\gamma_1\eta_0)]\mathbf{1}.$$

We also obtain that $G_1 = Q_0^{-1}(Q_1 - \eta_1)/n$. This proves the first statement of the lemma.

To prove the second statement, let $G = D + G_1$. This implies, as before, that $Q_0 = \eta_0 \mathbf{1}$ and $P_{\infty,0} = P_0 = \gamma_0 \mathbf{1}$. On the other hand, since $P_\infty = Cf(L_\infty)$, we have that $P_{\infty,0} = C\eta_0^s$. Thus C is a multiple of the identity. □

Appendix A. The theorems of Halphen and Floquet

The proofs of Theorems 1 and 2 rely on results of Halphen [12] and Floquet [6], [7], or rather on generalizations to systems of their results. These generalizations were proven in [11] and [25], respectively, and are repeated here for the convenience of the reader.

Theorem 6. *Let $A \in \mathbb{C}(x)^{n\times n}$ with entries bounded at infinity and suppose that the first-order system $y' = Ay$ has a meromorphic fundamental system of solutions. Then $y' = Ay$ has a fundamental matrix of the type*

$$Y(x) = R(x)\exp(\operatorname{diag}(\lambda_1 x, \dots, \lambda_n x)),$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $A(\infty)$ and $R \in \mathbb{C}(x)^{n\times n}$.

Theorem 7. *Suppose that A is an $n \times n$-matrix whose entries are meromorphic, ω-periodic functions which are bounded at the ends of the period strip. If the first-order system $y' = Ay$ has only meromorphic solutions, then there exists a constant $n \times n$-matrix J in Jordan normal form and an $n \times n$-matrix R^* whose entries are rational functions over $\mathbb{C}$ such that the following statements hold:*

1. *The eigenvalues of $A^*(0)$ and J are the same modulo $i\mathbb{Z}$ if multiplicities are properly taken into account. More precisely, suppose that there are nonnegative integers $\nu_1, \dots, \nu_{r-1}$ such that $\lambda, \lambda + i\nu_1, \dots, \lambda + i\nu_{r-1}$ are all the eigenvalues of $A^*(0)$ which are equal to λ modulo $i\mathbb{Z}$. Then λ is an eigenvalue of J with algebraic multiplicity r.*
2. *The equation $y' = Ay$ has a fundamental matrix Y given by*

$$Y(x) = R^*(\mathrm{e}^{2\pi i x/\omega})\exp(Jx).$$

In particular every entry of Y has the form $f(\mathrm{e}^{2\pi i x/\omega}, x)\mathrm{e}^{\lambda x}$, where $\lambda + i\nu$ is an eigenvalue of $A^*(0)$ for some nonnegative integer ν and where f is rational function in its first argument and a polynomial in its second argument.

Appendix B. Higher order systems of differential equations

In this section we recall two basic facts about systems of linear differential equations of order higher than one.

Consider the system

$$Ty = T_0(x)y^{(n)} + T_1(x)y^{(n-1)} + \cdots + T_n(x)y = 0 \tag{7}$$

where the T_j are $m \times m$ matrices whose entries are continuous functions on some real interval or complex domain Ω and where $T_0(x)$ is invertible for every $x \in \Omega$. Using the analogue of the standard transformation which turns a higher order scalar equation into a first order system, one finds that the system (7) is equivalent to the first order system $u' = Au$ where A is the $mn \times mn$ matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{1}_m & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & & & \cdots & \mathbf{1}_m \\ -T_0^{-1}T_n & -T_0^{-1}T_{n-1} & -T_0^{-1}T_{n-2} & \cdots & -T_0^{-1}T_1 \end{pmatrix}$$

in which all entries represent $m \times m$ blocks. From this it follows immediately that a fundamental system of solutions of $Ty = 0$ has mn elements.

The other property we need is about the existence of a factorization of an n-th order expression into n first order factors.

Theorem 8. *Let T be the differential expression defined in* (7)*. Suppose that F_1, ... , F_{n-1} are $m \times m$ matrices whose entries are continuous functions on Ω and which are invertible for every $x \in \Omega$. Define $F_n = T_0 F_1^{-1} \dots F_{n-1}^{-1}$. Then there exist $m \times m$ matrices Φ_1, ... , Φ_n such that*

$$T = (F_n D - \Phi_n) \dots (F_1 D - \Phi_1).$$

Proof. Denote the elements of a fundamental system of solutions by $y_1, \dots, y_{mn}$ and define, for $j = 1, \dots, n$, the $m \times m$ matrices

$$Y_j = (y_{m(j-1)+1}, \dots, y_{mj}).$$

Next define $W_1 = Y_1$ and $\Phi_1 = F_1 Y_1' Y_1^{-1}$ and suppose we have determined matrices Φ_1, ... , Φ_{j-1}. We will show below that

$$W_j = (F_{j-1} D - \Phi_{j-1}) \dots (F_1 D - \Phi_1) Y_j$$

is invertible so that we can define

$$\Phi_j = F_j W_j' W_j^{-1}.$$

Now let

$$S = (F_n D - \Phi_n) \dots (F_1 D - \Phi_1).$$

Then $S(Y_j) = (F_n D - \Phi_n) \dots (F_j D - \Phi_j) W_j = 0$, i.e., S and T have the same solutions. S and T are therefore equivalent to the same first order system. Since they have the same leading coefficient we finally obtain $S = T$.

We complete the proof by showing that the matrices W_j are invertible, i.e., that their columns $W_{j,1}$, ... , $W_{j,m}$ are linearly independent. This is true for $j = 1$ since the columns of W_1 are the solutions $y_1, \dots, y_m$ which are linearly independent. Assume that $W_1, \dots, W_{j-1}$ are invertible and that

$$0 = \sum_{k=1}^{m} \alpha_k W_{j,k}.$$

Then

$$0 = (F_{j-1}D - \Phi_{j-1}) \dots (F_1 D - \Phi_1) \sum_{k=1}^{m} \alpha_k y_{m(j-1)+k}.$$

Since the space of solutions of $(F_{j-1}D - \Phi_{j-1}) \dots (F_1 D - \Phi_1)y = 0$ is spanned by $y_1, \dots, y_{m(j-1)}$ we obtain that

$$\sum_{k=1}^{m} \alpha_k y_{m(j-1)+k} = \sum_{\ell=1}^{m(j-1)} \beta_\ell y_\ell.$$

But since $y_1, \dots, y_{mj}$ are linearly independent it follows that all $\alpha_1 = \dots = \alpha_n = 0$ (and $\beta_1 = \dots = \beta_{m(j-1)} = 0$). Hence the columns of W_j are linearly independent and W_j is invertible. □

Corollary 1. *Let S and T be differential expressions with matrix coefficients and invertible leading coefficients. If $\ker S \subset \ker T$ then there exists a differential expression R such that $RS = T$.*

Appendix C. Wasow's theorem

For the reader's convenience we provide here a slightly adapted version of Theorem 26.2 in Wasow [23]. The adaptation makes use of formulas (25.19) and (25.20) in [23].

Theorem 9. *Let $\Omega \subset \mathbb{C}$ be an open simply connected set containing the point x_0 and let S be a sector $\{\rho : 0 < |\rho| < \rho_0, \alpha_0 < \arg(\rho) < \beta_0\}$. Suppose that $A : S \times \Omega \to \mathbb{C}^{n\times n}$ is holomorphic and admits a uniform asymptotic expansion*

$$A(\rho, x) \sim \sum_{r=0}^{\infty} A_r(x)\rho^r$$

on $S \times \Omega$. Furthermore suppose that A_0 is diagonal, i.e., $A_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and that the sets $\{\lambda_1(x_0), \dots, \lambda_p(x_0)\}$ and $\{\lambda_{p+1}(x_0), \dots, \lambda_n(x_0)\}$ are disjoint. Then there exists a subsector S^* of S and a subregion Ω^* of Ω and $\mathbb{C}^{n\times n}$-valued functions P and B with the following properties:

1. *P and B are holomorphic in $S^* \times \Omega^*$.*
2. *$P - \mathbf{1}$ and B have the block forms*

$$P(\rho, x) - \mathbf{1} = \begin{pmatrix} 0 & P_{1,2}(\rho, x) \\ P_{2,1}(\rho, x) & 0 \end{pmatrix}$$

and

$$B(\rho, x) = \begin{pmatrix} B_{1,1}(\rho, x) & 0 \\ 0 & B_{2,2}(\rho, x) \end{pmatrix},$$

and the blocks have asymptotic expansion $P_{j,k}(\rho, x) \sim \sum_{r=1}^{\infty} P_{r;j,k}(x)\rho^r$ and $B_{j,j}(\rho, x) \sim \sum_{r=0}^{\infty} B_{r;j,j}(x)\rho^r$, as ρ tends to zero.

3. *$B_0 = A_0$ and $A_0 P_1 - P_1 A_0 = B_1 - A_1$.*

4. *the transformation* $y = Pz$ *takes the differential equation* $\rho^h y' = Ay$ *into* $\rho^h z' = By$.

Corollary 2. *If* $\lambda_1(x) = \cdots = \lambda_p(x) = \sigma$ *for all* $x \in \Omega$ *and*

$$\widetilde{B}(\rho, x) = \frac{1}{\rho}(B_{1,1} - B_{0;1,1}(x)) \sim \sum_{r=1}^{\infty} B_{r;j,j}(x)\rho^{r-1},$$

then the equation $\rho^h y' = Ay$ *has* p *linearly independent solutions of the form* $y(x) = PQw(x)\exp(\sigma x \rho^{-h})$*, where*

$$Q = \begin{pmatrix} \mathbf{1}_{p\times p} \\ \mathbf{0}_{(n-p)\times p} \end{pmatrix}$$

and w *is a solution of the* $p \times p$ *system*

$$\rho^{h-1} w' = \widetilde{B}(\rho, x) w.$$

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