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Gaussian Weighted Unreduced L_2 Cohomology of Locally Symmetric Spaces

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Abstract. Let $(\mathcal{M}, g)$ be a complete, noncompact Riemannian manifold of finite volume. For $w : \mathcal{M} \to (0, \infty)$ a weighting function, the w weighted unreduced L_2 cohomology is defined as the usual unreduced L_2 cohomology except that dvol is replaced by the measure w dvol. This paper proves that in the case $\mathcal{M} = \Gamma \backslash G/K$ is a locally symmetric space of nonpositive sectional curvature and arbitrary rank whereupon $w(m) = \mathrm{e}^{-\mathrm{dist}(m,p_0)^2}$ is the Gaussian relative to some basepoint p_0, the w weighted unreduced L_2 cohomology is isomorphic to the usual de Rham cohomology. This isomorphism extends to the standard coefficient bundles.

Note that weights for the de Rham cohomology of exponential decay have already been constructed; see [Bor83], [Bor90] and [Fra98]. The Gaussian weight behaves differently in terms of coefficient bundles.

Contents

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0. Introduction

Let $(\mathcal{M}, g)$ be a locally symmetric finite volume manifold of nonpositive sectional curvature. Thus, the universal cover $X = G/K$ is a globally symmetric space of noncompact type. For convenience, say X is Riemannian irreducible, so G is $\mathbb{R}$ semisimple. In particular, the center of G contains no nonidentity $\mathbb{Q}$ split tori. Let $K \subset G$ be a maximal compact subgroup. Finally, suppose a rational structure $G_{\mathbb{Q}} \subset G$ with arithmetic torsionfree $\Gamma \subset G_{\mathbb{Q}}$ the fundamental group of $\mathcal{M}$. Nonarithmetic lattices occur in $\mathrm{rank}_{\mathbb{R}} G = 1$ and present no difficulties per §3.2.

Let E be a finite dimensional representation of $G_{\mathbb{C}}$. There is as described in [MM63] a coefficient bundle $\mathbb{E} = E \times_{\Gamma} X \cong (\Gamma \backslash G) \times_K E$. Any choice of orthonormal basis of E induces (local) orthonormal sections of a Hermitian metric on $\mathbb{E}$, parallel to a canonical flat connection. This bundle metric on $\mathbb{E}$ may be explicitly described as follows. Fix $\langle -, - \rangle_0$ a metric on the base vector space E; this should be averaged over K so as to be K invariant. Let $x_0 \in X$ be stabilized by K. Then for v, w elements of E, $v \times_{\Gamma} (g \cdot x_0)$ is an element of the fiber of $E \times_{\Gamma} X$ at $(\Gamma g) \cdot x_0$. We define $\langle v \times_{\Gamma} (g \cdot x_0), w \times_{\Gamma} (g \cdot x_0) \rangle_{\mathbb{E}} = \langle g^{-1} \cdot v, g^{-1} \cdot w \rangle_0$.

Remark 0.1. 1) $\mathbb{C} \times \mathcal{M}$ the untwisted coefficient bundle arises from the trivial representation.
2) On Riemannian (M, g), there is no reason to both add a weight in front of dvol and allow the choice of arbitrary bundle metrics on $\mathbb{C} \times \mathcal{M}$. However, in the locally symmetric case the bundle metrics on $\mathbb{C}$ may not be standard.

Franke constructed [Fra98, p. 187] weighting functions $w_\lambda \in O(a^\lambda)$ on each Siegel set in the almost fundamental domain of the fundamental theorem of reduction theory of [BHC62]. In our notation:

- $P = UMA$ is the Langlands decomposition of a minimal rational parabolic, and $\mathfrak{a}$ is the Lie algebra of A with Killing dual $\check{\mathfrak{a}}$. Also, take $\{q_i\}_{i=1}^c$ a set of representatives for $\Gamma \backslash G_{\mathbb{Q}} / P_{\mathbb{Q}}$ indexing the cusps of $\mathcal{M}$.
- $H(-)$ is the associated height function:

$$(H = \log \circ \pi_A) : G = (UMA)K \to \mathfrak{a} \cong \mathbb{R}^\ell$$

  for $\ell = \mathrm{rank}_{\mathbb{Q}} G$.
- $\langle -, - \rangle$ denotes the pairing between $\mathfrak{a}$ and $\check{\mathfrak{a}}$.
- $\chi : \mathbb{R} \to \mathbb{R}$ is a cutoff function which is 1 on $[T, \infty)$ and 0 on $(-\infty, T-1]$ with bounded gradient.
- Breaking with [Fra98], $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ denotes the positive restricted $\mathbb{Q}$ roots of $\mathfrak{g}$ according to P, and $\Sigma^{++}(\mathfrak{g}, \mathfrak{a})$ denotes the simple restricted $\mathbb{Q}$ roots.

Each $\lambda \in \check{\mathfrak{a}}$ produces a weight function on $\mathcal{M}$ of bounded gradient [Fra98, p. 187]:

$$(1) \quad w_\lambda(\Gamma g K) = \sum_{i=1}^{c} \sum_{[\gamma] \in (\Gamma \cap UM) \backslash \Gamma} \left\{ \exp(\langle \lambda, H(\gamma q_i^{-1} g) \rangle) \prod_{\alpha \in \Sigma^{++}(\mathfrak{g}, \mathfrak{a})} \chi(\langle \alpha, H(\gamma q_i^{-1} g) \rangle) \right\}.$$

Remark 0.2. Some care must be taken in choosing T above. The details are omitted to avoid a lengthy review of reduction theory [Fra98, §2.1].

The present argument replaces the linear term λ by a *negative definite* quadratic form $-Q \in \check{\mathfrak{a}} \otimes \check{\mathfrak{a}}$:

$$(2) \quad w_{-Q}(\Gamma g K) = \sum_{i=1}^{c} \sum_{[\gamma] \in (\Gamma \cap UM)\backslash\Gamma} \left\{ \exp(-\langle Q, H(\gamma q_i^{-1} g)\rangle) \prod_{\alpha \in \Sigma^{++}(\mathfrak{g},\mathfrak{a})} \chi(\langle \alpha, H(\gamma q_i^{-1} g)\rangle) \right\}.$$

Here, if $Q = \sum_{i,j} \lambda_i \otimes \lambda_j$, then $\langle Q, H\rangle = \sum_{i,j} \lambda_i(H)\lambda_j(H)$ for $H \in \mathfrak{a}$.

Note that there exists $Q > 0$ which causes the weight above to be $O(\mathrm{e}^{-r^2})$. Indeed, the Killing form is the restriction of the metric of $\mathcal{M}$ to the flat sector corresponding to $\mathfrak{a}^{\geq 0}$, and one may choose Q realizing the Killing form. Thus the fundamental theorem of reduction theory will imply $w_{-Q} \in O(\mathrm{e}^{-r^2})$.

Remark 0.3. Note $\exp(\langle \lambda, H(-)\rangle)$ in these weighting functions may instead be thought of as the exponential of a Busemann function of X. This appears in the literature as a scholium of Lemma 1.3 [Leu95, p. 395].

These weighting functions will be used in the following context. Define

$$(3) \quad \Omega^\bullet_{(2),w}(\mathcal{M},\mathbb{E}) = \left\{ \omega \in \Omega^\bullet(\mathcal{M},\mathbb{E}) \;\middle|\; \int_{\mathcal{M}} |\eta|^2_{\mathbb{E}} w \text{ dvol} < \infty \text{ for } \eta = \omega, d\omega \right\}.$$

This is a cochain complex using the exterior derivative for forms with coefficients in $\mathbb{E}$. The resulting cohomology is denoted $H^\bullet_{(2),w}(\mathcal{M},\mathbb{E})$. In case $w = w_\lambda$ or $w = w_{-Q}$, the subscript of the weight may replace the weight itself in all notations. Finally, w is omitted if $w \equiv 1$.

These conventions allow for a restatement of a restatement of prior work. In [Fra98, p. 90], Franke proves a result which rephrases Borel's results [Bor83] on forms of uniform moderate growth. In the present notation,

$$(4) \qquad \lim_{\lambda \to (\infty,\ldots,\infty)} H^\bullet_{(2),-\lambda}(\mathcal{M},\mathbb{E}) \cong H^\bullet(\mathcal{M},\mathbb{E}).$$

The present result is similar. It asserts $H^\bullet_{(2),-Q}(\mathcal{M},\mathbb{E}) \cong H^\bullet(\mathcal{M},\mathbb{E})$. But there is one novelty. Namely, the present result is uniform in $\mathbb{E}$, meaning for fixed Q_0 the current result holds for all $\mathbb{E}$. In contrast, any fixed λ_0 has some $\mathbb{E}(\lambda_0)$ for which $w_{-\lambda_0}$ produces a nontrivial boundary truncation.

The precise statement requires more notation. Let $\overline{\mathcal{M}}$ be the reductive Borel-Serre compactification of $\mathcal{M}$ introduced in [Zuc82]. For the remainder, $j : \mathcal{M} \hookrightarrow \overline{\mathcal{M}}$ is open inclusion so that $j_*\Omega^\bullet(\mathbb{E})$ is an incarnation of $R^\bullet j_*\mathbb{E}$, while $\mathcal{A}^\bullet(E)$ is the presheaf of special forms on $\overline{\mathcal{M}}$ of [GHM94, §14], Equation (11). The sheafification $\mathrm{Sh}(\mathcal{A}^\bullet(E))$ is the weighted cohomology sheaf $W^{-\infty}\mathcal{C}^\bullet(E)$, whose hypercohomology is the usual de Rham cohomology with coefficients in $\mathbb{E}$. Finally, $\mathcal{L}^\bullet_{-Q}(E)$ is the sheafification of the presheaf $L^\bullet_{-Q}(E)$ of w_{-Q} L_2 forms.

Lemma 0.4. *Let $\mathcal{M}$ locally symmetric with $\mathbb{E}$ a standard bundle induced by E a finite dimensional representation. $\mathbb{E}$ carries a Hermitian metric in the standard quasiisometry class. Let $Q > 0$ in $\check{\mathfrak{a}} \otimes \check{\mathfrak{a}}$. Then there is a quasiisomorphism on $\overline{\mathcal{M}}$*

$$(5) \qquad j_*\Omega^\bullet(\mathbb{E}) \leftarrow \mathrm{Sh}(\mathcal{A}^\bullet(E)) \rightarrow \mathcal{L}^\bullet_{-Q}(E).$$

The following consequence is immediate, since quasiisomorphisms imply isomorphic hypercohomology.

Theorem 0.5. *Suppose the hypotheses of Lemma* 0.4*. Then one has*

$$H^\bullet_{(2),-Q}(\mathcal{M},\mathbb{E}) \cong H^\bullet(\mathcal{M},\mathbb{E}).$$

Corollary 0.6 follows. In fact, there now exists a strong weighted Kodaira decomposition* given the finite dimensional answer above, while the Gaussian $w = \mathrm{e}^{-r^2}$ results via choice $\mathbb{E} = \mathbb{C} \times \mathcal{M}$ for $\mathbb{C}$ trivial and $-Q$ realizing the Killing form.

Corollary 0.6. *Suppose* $(\mathcal{M}, g)$ *is a locally symmetric nonpositively curved finite volume Riemannian manifold. Let* r *be distance to a basepoint and* $w = e^{-r^2}$ *be the Gaussian weight on* $\mathcal{M}$*. Then* $H^\bullet_{(2),w}(\mathcal{M}) \cong H^\bullet(\mathcal{M})$*, where the latter is the usual de Rham cohomology.*

The above should be compared to the following conjecture by Edward Bueler. To the author's knowledge, this remains open even in the *rank one* finite volume locally symmetric case.

Bueler's Conjecture (1.1 of [Bue99])**.** *Suppose* (M^n, g) *is complete,* $Ric \geq -c^2$*. Let* $w(m) = \rho_t(m, p_0)$ *for* $p_0 \in M$ *fixed and* t *fixed, where* $\rho_t(x, y)$ *is the heat kernel of the metric Laplacian on functions associated to* g*. Then for* $\beta^p = \dim H^p(M)$ *the usual Betti number of* M*,*

$$\dim \ker \Delta_w = \beta^p, \quad 0 \leq p \leq n. \tag{6}$$

Also, the complex dom d *is Fredholm, implying a strong Kodaira decomposition and finite dimensional* $H^\bullet_{(2),w}(\mathcal{M})$*.*

Finally, the result has certain apparently unexciting implications for Lie algebra cohomology. Nonetheless, errors might be made in deriving them from the literature. A short appendix is included.

Remark 0.7. The author thanks Gopal Prasad for comments on an early manuscript and the preprint and thanks the referee for enumerating several improvements and a few errata.

1. Sheaves on $\overline{\mathcal{M}}$

1.1. Topology of $\overline{\mathcal{M}}$. The introduction mentioned in passing the reductive Borel-Serre compactification of $\mathcal{M}$, denoted here $\overline{\mathcal{M}}$. This compactification originated in [Zuc82] and at its creation was optimized for computing L_2 cohomology sheaves. It also plays a central role in [GHM94], being the compactification on which the various incarnations of weighted cohomology are constructed. Besides [Zuc82], the compactification was also used in Theorem A of [Nai99]. The present argument is similar but less delicate due to rapid decay of the weight.

To facilitate computations of stalk cohomologies of differential graded sheaves, say at $y \in Y$, a sequence $\{U_i\}_{i=1}^\infty$ of open neighborhoods of y with $\{y\} = \cap_{i=1}^\infty U_i$ is useful. Such a system of neighborhoods will be called *fundamental.* The main issue with a compactification $\overline{\mathcal{M}}$ is to build such fundamental neighborhoods about $n \in \partial\overline{\mathcal{M}}$ rather than $m \in \mathcal{M}$.

By definition, $\overline{X}$ is naturally a quotient of the Borel-Serre enlargement $\widetilde{X}$ of [BS73], with $\Gamma\backslash\widetilde{X} = \widetilde{\mathcal{M}}$ a manifold with corners. This is described in [Zuc82], pp.

*See [BL92] for an explanation and references.

184 and 190, where the projections $p_P : e'(P) \to \hat{e}(P)$ of individual boundary strata patch to a global projection.

Thus, say $P = UMA$ is the rational Langlands decomposition of the real points of an arbitrary rank fixed parabolic $\mathbb{Q}$ subgroup P. Here, U is the unipotent radical of P. Recall $x_0 \in X$ corresponding to K is a fixed choice of basepoint corresponding to a fixed Gaussian. Then A is identified with a maximal $\mathbb{Q}$ split torus in the center of P/U, so that A is moreover invariant under the Cartan involution corresponding to x_0. Then MA is similarly identified to P/U, while M is identified with intersection of the squares of all $\mathbb{Q}$ characters [GHM94, p. 150].

Since U is the exponential of a space of positive $\mathfrak{a}$ weights, it is a simply connected nilpotent group. On the other hand, M is reductive with the Cartan involution of G having a well-defined restriction to M, so that $X_M = M/(K \cap M)$ is a globally symmetric space of rank strictly less than X for P nontrivial. We finally fix notations $\Gamma_U = \Gamma \cap U$ and Γ_M for the projection of Γ onto $M \cong U\backslash P/A$. Then Γ_U is always a lattice subgroup in U given that P is rational, so that via a standard property of simply connected nilpotent groups $\Gamma_U\backslash U$ is compact. On the other hand, $\mathcal{M}_M = \Gamma_M\backslash X_M$ is a lower rank finite volume locally symmetric space.

Now choose $\eta_1 \times \eta_2 \subset UM$ open with η_1 so large as to include a fundamental domain for the nilpotent lattice $\Gamma_U \subset U$ and $\eta_2 \subset M$ being $K \cap M$ invariant with compact closure. Let $\{H_i\}_{i=1}^r$ be dual to $\{\alpha_i\}_{i=1}^r = \Sigma^{++}(\mathfrak{p}, \mathfrak{a})$ for r the rank of P. We define the group isomorphism $\psi : (0,\infty)^{\dim A} \to A$ as $\psi(t_1, \ldots, t_r) = \exp(\sum_{i=1}^r t_i H_i)$. Note ψ is *not* an isometry for the usual flat metric of $(0,\infty)^r$. Henceforth, $d\vec{t}^2$ denotes not $\sum_{i=1}^r dt_i^2$ but rather the flat pullback metric from $\mathfrak{a}^{>0}$.

For $T > 0$, the Siegel set $\mathfrak{S}(T, \eta_1 \times \eta_2)$ is $\eta_1\eta_2\psi([T,\infty)^{\dim A}) \subset G$. The term also refers to $\mathfrak{S}(T, \eta_1 \times \eta_2) \cdot x_0$. The fundamental neighborhoods of $n \in \partial\overline{\mathcal{M}}$ are minor modifications of such Siegel sets.

Describing these requires a review of the construction of $\overline{\mathcal{M}}$. To do so, let $\widetilde{X}$ denote the Borel-Serre enlargement of the globally symmetric space $X = G/K$ and $\overline{X}$ be the reductive Borel-Serre enlargement of [Zuc82]. Then the added boundary strata $e'(P)$ of $\widetilde{M} = \Gamma\backslash\widetilde{X}$ for P a $\mathbb{Q}$ parabolic admits a natural fibration

$$\Gamma_U\backslash U \longrightarrow e'(P) \xrightarrow{p_P} (\Gamma_M\backslash X_M = \hat{e}(P)). \tag{7}$$

Zucker verifies that these projections p_P patch in a well defined way given the incidence of $e'(P)$ and $e'(R)$ for $P \subset R$ parabolic, so that $\overline{\mathcal{M}}$ is the singular image under the patched projection of the manifold with corners $\widetilde{\mathcal{M}}$. Thus, each point in a boundary strata of $\overline{\mathcal{M}}$ is associated to a point in some $\Gamma_M\backslash X_M$.

We finally describe how to modify a Siegel set to form a fundamental neighborhood to $n \in \partial\overline{\mathcal{M}}$. By construction, there is a rational parabolic $P = UMA$ with $n \in \hat{e}(P)$. Choose $\eta_2(i)$ for $i = 1, 2, \ldots$ a sequence of fundamental neighborhoods about n in $\Gamma_M\backslash X_M$, and choose $\eta_1 \subset U$ as above. Then for $\pi_\Gamma : X \to \mathcal{M}$ projection,

$$\widetilde{U}_i = \pi_\Gamma[\mathfrak{S}(i, \eta_1 \times \eta_2(i))] \sqcup \eta_2(i) \quad \subset \quad \Gamma\backslash X \sqcup \hat{e}(P) \tag{8}$$

is the closure of the interior of $\pi_\Gamma[\mathfrak{S}(i, \eta_1 \times \eta_2(i))]$ in $\overline{\mathcal{M}}$ and forms a fundamental system of neighborhoods about $n \in \partial\overline{\mathcal{M}}$ as $i \to \infty$.

1.2. L_2 sheaves and metrics. We briefly recall the two sheaves on $\overline{\mathcal{M}}$ which will be shown to be quasiisomorphic. The first $j_*\Omega^\bullet(\mathbb{E})$ is

$$\Gamma(U, j_*\Omega^\bullet(\mathbb{E})) = \Omega^\bullet(U \cap \mathcal{M}, \mathbb{E}). \tag{9}$$

Here, Γ denotes sections rather than a group. For the second, consider this presheaf:

$$\Gamma(U, L^\bullet_{-Q}(E)) = \left\{ \omega \in \Gamma(j_*\Omega^\bullet(\mathbb{E})) \;\Big|\; \int_{U\cap\mathcal{M}} (|\omega|^2 + |d\omega|^2) w_{-Q} \mathrm{dvol} < \infty \right\}. \tag{10}$$

Then $\mathcal{L}^\bullet_{-Q}(E)$ is the sheafification, i.e., the differential graded sheaf implementing the corresponding local integrability condition.

One may now check that the sections on the compact space $\overline{\mathcal{M}}$ of $\mathcal{L}^\bullet_{-Q}(E)$ are exactly the globally weighted L^2 forms, given that sheafification replaces the global square integrability with the local version. These global sections also compute the hypercohomology, since the argument for fineness on p. 191 of [Zuc82] generalizes to the present weighted L_2 sheaf.

Finally, the differential graded presheaf $\mathcal{A}^\bullet(E)$ of special forms is constructed in [GHM94, §14]. It fits into the d.g.s. in the middle of the quasiisomorphism of Equation (5):

$$j_*\Omega^\bullet(\mathbb{E}) \leftarrow \mathrm{Sh}(\mathcal{A}^\bullet(E)) \rightarrow \mathcal{L}^\bullet_{-Q}(E).$$

To describe these, let U_i correspond to a fundamental system about $n \in \partial\overline{\mathcal{M}}$.

$$\Gamma(U_i, \mathcal{A}^\bullet(E)|_{\hat{e}(P)}) = \left\{ \sum \alpha_j \otimes \omega_j \otimes \phi_j \;\Big|\; \alpha_j \in \mathbb{C}, \omega_j \in \Omega^\bullet(\overline{\eta}_2(i), \mathbb{E}), \phi_j \in C^\bullet(\mathfrak{u}, E) \right\}. \tag{11}$$

Here, $C^\bullet(\mathfrak{u}, E) = \wedge^\bullet\mathfrak{u}^* \otimes E$ is identified with $\Omega^\bullet(U, \mathbb{E})^U \subset \Omega^\bullet((\Gamma \cap U)\backslash U, \mathbb{E})$, and $\eta_2(i) \subset X_M$ is an open set with compact closure.

The arrow pointing left in Equation (5) is inclusion. It is true but not obvious that the arrow which points right is also inclusion. To see this, recall $U_i \cong (\Gamma_U\backslash U) \times \overline{\eta}(i) \times (0, \infty)^r$ via a diffeomorphic decomposition. The $\overline{\eta}(i)$ is a Riemannian factor, say with coordinates carrying a local metric dx_M^2 with $\nabla R \equiv 0$ on X_M. The $\Gamma_U\backslash U$ factor collapses; say β runs over $\Sigma^+(\mathfrak{p}, \mathfrak{a}_P)$ for $\mathfrak{u} = \oplus_\beta \mathfrak{u}_\beta$ with du_β^2 an inner product. Then taking ψ as our coordinates on the last factor, [Bor74, §4] shows that the pullback of the metric on $\mathcal{M}$ via the Langlands decomposition is

$$ds^2 = d\vec{t}^{\,2} + dx_M^2 + \sum_{\beta>0} \exp(-2\langle\beta, H \circ \psi(\vec{t})\rangle) du_\beta^2. \tag{12}$$

Here, $d\vec{t}^{\,2}$ is the pullback under ψ of the flat metric on $\mathfrak{a}^{>0}$ rather than $\sum_{i=1}^r dt_i^2$. The above formula then demands every special form is square integrable versus the weights of Equation (2).

2. Proof of quasiisomorphism

2.1. Contraction onto U invariant forms. A direct limit of cochain homotopies is a cochain homotopy. Thus, the quasiisomorphism will be implied by checking for the appropriate presheaves that on each fundamental neighborhood U_i of a generic boundary point $n \in \partial\overline{\mathcal{M}}$ we may construct cochain homotopy equivalences of each of $\Gamma(U_i, L^\bullet_{-Q}(E))$ and $\Gamma(U_i, j_*\Omega^\bullet(\mathbb{E}))$ onto the space of special forms $\Gamma(U_i, \mathcal{A}^\bullet(E))$.

The first step of the argument described in §2.1 is known. It reproves the Nomizu/van Est theorem of [Nom54] and [vE58] while checking L_2 boundedness conditions on boundary neighborhoods. See [Zuc82, pp. 195–201]. However, the argument has already been cited multiple times without discussion by other authors. A mild variant reappears here.

2.1.1. $\Gamma_U\backslash U$ as a tower. Let β vary over $\Sigma^+(\mathfrak{p}, \mathfrak{a}_P)$ for $\mathfrak{u} = \oplus_\beta \mathfrak{u}_\beta$. Since $\Sigma^+(\mathfrak{p}, \mathfrak{a}_P)$ is a set of restricted roots, perhaps dim $\mathfrak{u}_\beta > 1$. Nonetheless, we may choose a basis $\{X_i\}_{i=1}^{n-1} \subset \mathfrak{u}$ so that each $X_i \in \mathfrak{u}_{\beta(i)}$ for $i = 1 \ldots n-1$ and moreover $i > j$ demands $\beta(i) \le \beta(j)$. In particular, these conventions demand $X_1 \in \mathfrak{z}(\mathfrak{u})$.

Definition 2.1. For $0 \le k \le n-1$, $\mathfrak{u}_{(k)} = \langle X_1, \ldots, X_k\rangle$ with the brackets denoting the real span. $U_{(k)} = \exp \mathfrak{u}_k$ and $\Gamma_{U,(k)} = U_{(k)} \cap \Gamma_U$. As $\mathfrak{u}_{(k)}$ is an ideal in $\mathfrak{u}$, set $\mathfrak{u}^{(k)} = \mathfrak{u}/\mathfrak{u}_{(k)}$ a Lie algebra with $U^{(k)} = \exp \mathfrak{u}^{(k)}$ and $\Gamma_U^{(k)} = \Gamma_U/\Gamma_{U,(k)}$. Finally, put $T^{(k)} = (\Gamma_U^{(k)}/\Gamma_U^{(k-1)})\backslash(U^{(k)}/U^{(k-1)})$.

Here, $T^{(k)}$ is a torus since $T^{(k)} \cong \{\exp n\ell X_k\}_{n\in\mathbb{Z}}\backslash\{\exp tX_k\}_{t\in\mathbb{R}}$ for some length ℓ. Thus, $T^{(k)} \cong \mathrm{SO}(2)$. This allows $\Gamma_U\backslash U$ to be written as a tower of SO(2) principal bundles. The tower written horizontally is

(13)

$$\begin{array}{ccccccc} T^{(1)} & & T^{(i)} & & T^{(i+1)} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ (\Gamma_U\backslash U) & \to \cdots & \Gamma_U^{(i)}\backslash U^{(i)} & \to & \Gamma_U^{(i+1)}\backslash U^{(i+1)} & \to \cdots \to & \Gamma_U^{(n-1)}\backslash U^{(n-1)} = \{e\}. \end{array}$$

Fix a fundamental neighborhood $V = U_k$ for some k, i.e., $V = (\Gamma_U\backslash U) \times \eta(k) \times (0,\infty)^r$ for $\eta(k) \subset X_M$ an open set with compact closure. For $V^{(i)} = (\Gamma_U^{(i)}\backslash U^{(i)}) \times \eta(k)\times(0,\infty)^r$, extend the notation $\Pi^{(i)}$ to also denote $\Pi^{(i)} : V \to V^{(i)}$ the composite projection onto $V^{(i)}$. Finally, define $C^\infty(V)^i = \{f \in C^\infty(V) | Xf \equiv 0 \text{ if } X \in \mathfrak{u}_{(i)}\}$. The following lemma follows via an omitted induction.

Lemma 2.2. *There is a bijective correspondence $\bar{f} \leftrightarrow f$ between $C^\infty(V^{(i)})$ and $C^\infty(V)^i$ which respects $\Pi^{(i)}$ in the sense that $\bar{f} \circ \Pi^{(i)} = f$.*

2.1.2. Antiderivative operators of SO(2) actions. For the remainder of this section, we fix a principal SO(2) bundle $\mathrm{SO}(2) \to \mathcal{E} \overset{\Pi}{\to} B$. In the application, $\mathcal{E} = V^{(i)}$ while $B = V^{(i+1)}$. We use Z to denote the vector field which traverses the SO(2) fiber in time 2π. In the application, this is some renormalization of X_i dependent on the circumference of the circular fibers.

Definition 2.3. Let $f : \mathcal{E} \to \mathbb{C}$. Then $a(f)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x e^{2\pi s Z})ds$. We put $f_0 = f - a(f)$.

Definition 2.4. Pick any constant c, and let $N \subset \mathcal{E}$ be an open set with a SO(2) principal bundle chart $\varphi : N \to W \times \mathrm{SO}(2;\mathbb{R})$. Then letting θ stand for $\exp(\theta(E_{21} - E_{12}))$ in the argument of $\varphi(w, -)$,

$$[\bar{s}f]\varphi(w,\theta) = \int_c^\theta f_0(\varphi(w,\psi))d\psi - \frac{1}{2\pi}\int_0^{2\pi}\int_c^\theta f_0(\varphi(w,\psi))d\psi\, d\theta \tag{14}$$

is readily verified to be independent of c and moreover the local choice of N. The associated globally defined operator on $\mathcal{E}$ will be denoted $s : C_c^\infty(\mathcal{E}) \to C_c^\infty(\mathcal{E})$.

These two operators on functions allow us to redefine Zucker's cochain homotopy operators without recourse to $d^*_{\mathrm{SO}(2)}$ the exterior derivative adjoint in an SO(2) direction. Certain boundedness properties and commutation properties of each are required in the computations that complete the argument. These are collected in the following lemma, which is stated without proof. Each part follows from either i) the definitions, ii) the Schwarz inequality, or iii) dominated convergence.

Lemma 2.5. i) *Say μ is some measure on $\mathcal{E}$ which restricts to a product measure on each* SO(2) *principal bundle chart. Then $a : C_c^\infty(\mathcal{E}) \to C_c^\infty(\mathcal{E})$ is $w\ \mu\ L_2$ bounded for any $w > 0$ constant on every* SO(2) *fiber.*

ii) *The same holds for $s : C_c^\infty(\mathcal{E}) \to C_c^\infty(\mathcal{E})$.*

iii) $Zaf = aZf \equiv 0$.

iv) $Zsf = sZf = f_0$.

v) *Suppose that in every bundle chart $\varphi(W \times \mathrm{SO}(2))$, the vector field X has no component in the vertical* SO(2) *direction. Then $[X, s] = \mathbf{0}$ and $[X, a] = \mathbf{0}$.*

2.1.3. Central case of cochain homotopy. Recall the conventions $\Sigma^{++}(\mathfrak{p},\mathfrak{a}) = \{\alpha_i\}_{i=1}^r$ and $\lambda = \sum c_i\alpha_i$. Let $\mathcal{N}$ the set of $\mathfrak{a}$ weights of $C^\bullet(\mathfrak{u}, E)$ induced by $\mathfrak{u} = \oplus_{(\mathfrak{p},\mathfrak{a})}\mathfrak{u}_\alpha$, and say $2\rho = \sum_{\Sigma^+(\mathfrak{p},\mathfrak{a})} \dim \mathfrak{u}_\beta \beta$ is the exponential decay rate of the volume form. Change conventions slightly so that $L^\bullet_{(2)}((0,\infty)^r, k(\vec{t}))$ may denote $k(\vec{t})$ weighted L^2 forms on the sector. Finally, V is some fixed fundamental neighborhood U_i. Then we have the decomposition $V \cong_{\mathrm{dif}} (\Gamma_U\backslash U) \times \eta_2 \times (0,\infty)^r$. Thus

$$(15)\quad \Gamma(A^\bullet(\mathbb{E})) = \mathbb{C} \otimes \Omega^\bullet(\overline{\eta}_2, \mathbb{E}) \otimes C^\bullet(\mathfrak{u}, E)$$

$$\Gamma(L^\bullet_{-Q}(E)) = \oplus_{\nu\in\mathcal{N}}\Big\{\big[L^\bullet_{(2)}((0,\infty)^r, \exp(-\langle Q, H\circ\psi(\vec{t})\rangle - 2\langle \nu+\rho, H\circ\psi(\vec{t})\rangle)) \hat{\otimes} L_2((\Gamma\cap U)\backslash U))\big]\hat{\otimes} L^\bullet_2(\overline{\eta}_2, \mathbb{E})\otimes C^\bullet(\mathfrak{u}, E)_\nu\Big\}.$$

Before describing the generic case of a cochain homotopy which projects down the tower of Equation (13), we begin with the case of $Z = X_1$ central. Several simplifications arise.

Thus, say L_Z is the Lie derivative of Z. Since Z is central, in terms of the decompositions the Lie derivative is given by $L_Z(f \otimes \phi_1 \otimes \phi_2) = Zf \otimes \phi_1 \otimes \phi_2$. Moreover, a computation using Z central also verifies $[d, s\otimes\mathbf{1}\otimes\mathbf{1}] = \mathbf{0}$ for s the antiderivative with respect to Z. Neither fact is true for noncentral directions.

Definition 2.6. The linear operators P, B are $P = a\otimes\mathbf{1}\otimes\mathbf{1}$ and $B = (s\otimes\mathbf{1}\otimes\mathbf{1})\circ\iota_Z$ according to Equation (15).

Using Cartan's magic formula $d\iota_Z + \iota_Z d = L_Z$, we see that (B, P) forms a cochain homotopy of $\Omega^\bullet_c(V, \mathbb{E})$:

$$(16)\qquad dB + Bd = s(d\iota_Z + \iota_Z d) = s(L_Z) = \mathbf{1} - P.$$

Lemma 2.5 and Equation (12) will moreover imply that P and B are bounded via the weighted L_2 conditions. Thus, (B, P) is a well-defined cochain homotopy on each of the three spaces of sections. Note that the boundedness check for B uses ι_Z bounded, which follows since $|Z| \to 0$ as $m \to \partial\overline{\mathcal{M}}$. As the forms in im P for each of the three d.g.s. are invariant under the SO(2) action given by integrating Z, they may now be viewed as forms on $V^{(2)} = (\Gamma_U^{(2)}\backslash U^{(2)}) \times \eta(i) \times (0,\infty)^r$ rather than on

$V = V^{(1)}$. The next section describes the induction argument that continues this process until the output forms are invariant under $\mathfrak{u}$ and hence U.

2.1.4. Generic cochain homotopy. Recall that for $0 \le \ell \le n-1$, $C^\infty(V)^\ell = \{f \in C^\infty(V) \mid Xf \equiv 0 \text{ if } X \in \mathfrak{u}_{(\ell)}\}$. Via the Langlands decomposition, we may view $\{\frac{\partial}{\partial t_i}\}_{i=1}^r \sqcup \{Y_j\} \sqcup \{X_k\}$ as a frame on V. Here the coordinate fields span $(0,\infty)^r$, Y_j span the patch η_2 of X_M, and $X_k \in \mathfrak{u}_{\beta(k)}$ so that $k_1 \le k_2$ demands $\beta(k_1) \ge \beta(k_2)$. Then using coefficient functions $C^\infty(V)^\ell$ in the dual coframe produces a filtration $\Omega^\bullet(V,\mathbb{E}) \supset \Omega^\bullet(V,\mathbb{E})^1 \supset \dots \Omega^\bullet(V,\mathbb{E})^{n-1}$. Each space of forms of Equation (15) intersects the subspaces of this filtration nontrivially, and the central case of cochain homotopy of Subsection 2.1.3 contracts each onto its $\Omega^\bullet(V,\mathbb{E})^1$ subspace. The special forms are $\Omega^\bullet(V,\mathbb{E})^{n-1}$.

Induction hypothesis: Each space of forms of Equation (15) has been contracted onto its $(\ell-1)^{\text{st}}$ subspace.

Definition 2.7. Extend the definition of $s(f)$ and $a(f)$ to act on $\Omega^\bullet(V,\mathbb{E})$ by acting on the coefficient functions of the above coframe. Fix throughout the subsection the convention that Z is the positive multiple of X_ℓ with flow traversing the SO(2) fiber of $V^{(\ell-1)}$ in time 2π. Then $P = a$ and $\widetilde{B} = s \circ \iota_Z$.

$(P, \widetilde{B})$ is *not* a cochain homotopy equivalence, but the argument may be salvaged. To begin, suppose throughout $f\tau\otimes\eta\otimes\phi$ is decomposed according to Equation (15), with η a form on $\eta_2 \subset X_M$. Then for $\Theta_j : C^\bullet(\mathfrak{u},E) \to C^{\bullet+1}(\mathfrak{u},E)$ left exterior multiplication by X_j, ι_j left interior multiplication, and $\Theta_{t,i}$, $\iota_{t,i}$ similar for $\tau \in \wedge^\bullet[dt_i]_{i=1}^r$, we have the following expression for the exterior derivative. It is intended that this formula extend linearly:

$$\begin{aligned} (17)\qquad d(f\tau\otimes\eta\otimes\phi) &= \sum_{i=1}^r \frac{\partial f}{\partial t_i}\Theta_{t,i}\tau\otimes\eta\otimes\phi + (-1)^{\deg\tau\wedge\eta}(Zf)\tau\otimes\eta\otimes\Theta_Z\psi \\ &+ \sum_{j\neq\ell}(-1)^{\deg\tau\wedge\eta}X_j f\tau\otimes\eta\otimes\Theta_j\phi + (-1)^{\deg\eta}f\tau\otimes d\eta\otimes\phi \\ &+ (-1)^{\deg\tau\wedge\eta}f\tau\otimes\eta\otimes d\phi. \end{aligned}$$

Recalling $\widetilde{B}(f\tau\otimes\eta\otimes\phi) = (-1)^{\deg\tau\wedge\eta}sf\tau\otimes\eta\otimes\phi$, we compute $d\widetilde{B}(f\tau\otimes\eta\otimes\phi)$ and $\widetilde{B}d(f\tau\otimes\eta\otimes\phi)$.

$$\begin{aligned} (18)\qquad \widetilde{B}d(f\otimes\eta\otimes\phi) &= (-1)^{\deg\tau\wedge\eta}\sum_{i=1}^r\left(s\frac{\partial}{\partial t_i}\right)\Theta_{t,i}\tau\otimes\eta\otimes\iota_Z\phi \\ &+ s(Zf)\tau\otimes\eta\otimes\iota_Z\Theta_Z\phi + \sum_{j\neq\ell}s(X_jf)\tau\otimes\eta\otimes\iota_Z\Theta_j\phi \\ &+ (-1)^{\deg\tau+1}(sf)\tau\otimes d\eta\otimes\iota_Z\phi + (sf)\tau\otimes\eta\otimes\iota_Z d\phi \\ d\widetilde{B}(f\otimes\eta\otimes\phi) &= (-1)^{\deg\tau\wedge\eta}\sum_{i=1}^r\frac{\partial}{\partial t_i}(sf)\Theta_{t,i}\tau\otimes\eta\otimes\iota_Z\phi \\ &+ (Zs)f\tau\otimes\eta\otimes\Theta_Z\iota_Z\phi + \sum_{j\neq\ell}(X_js)f\tau\otimes\eta\otimes\Theta_j\iota_Z\phi \\ &+ (-1)^{\deg\tau}(sf)\tau\otimes d\eta\otimes\phi + (sf)\tau\otimes\eta\otimes d\iota_Z\phi. \end{aligned}$$

Thus using $\Theta_Z\iota_Z+\iota_Z\Theta_Z=\mathbf{1}$, $\Theta_j\iota_Z+\iota_Z\Theta_j=\mathbf{0}$, $Zs=sZ=\mathbf{1}-a$, and the obvious cancellations, one obtains

$$
\begin{aligned}
(19)\qquad (d\widetilde{B}+\widetilde{B}d)(f\tau\otimes\eta\otimes\phi) &= (Zs)f\otimes\eta\otimes\phi+(sf)\tau\otimes\eta\otimes(\iota_Z d+d\iota_Z)\phi \\
&= (1-P)(f\tau\otimes\eta\otimes\phi)-Q(f\tau\otimes\eta\otimes\phi)
\end{aligned}
$$

for $Q(f\tau\otimes\eta\otimes\phi)=-(sf)\tau\otimes\eta\otimes(d\iota_Z+\iota_Z d)\phi$ and extended linearly. Some preliminary facts about Q must be argued before constructing the genuine cochain homotopy.

Lemma 2.8. i) *Let $-\lambda_0$ be the lowest $\mathfrak{a}$ weight of $C^\bullet(\mathfrak{u},E)$, while $Z\in\mathfrak{u}_\beta$. Then for N so large that $C^\bullet(\mathfrak{u},E)_{-\lambda_0+n\beta}=\{0\}$ for $n\geq N$, one has $Q^N=\mathbf{0}$.*
ii) $[d,P]=\mathbf{0}$.
iii) $[d,Q]=\mathbf{0}$.
iv) $QP=PQ=\mathbf{0}$.
v) *Q is bounded in the weighted L_2 norm on forms.*

Sketch. i) follows from the similar statement for $\iota_Z d+d\iota_Z$, since $d: C^\bullet(\mathfrak{u},E)\to C^\bullet(\mathfrak{u},E)$ preserves the weight space decomposition. ii) is a computation, and iii) follows from ii) since $[d,d\widetilde{B}+\widetilde{B}d]=\mathbf{0}$. iv) is a computation using $saf=asf\equiv 0$. v) follows from the similar statement for s and the metric Equation (12). □

Now label $B=\widetilde{B}(\sum_{i=0}^{N-1}Q^i)$ for N per the lemma. A computation checks that (B,P) is a cochain homotopy, using i)-iv) of the lemma. A check using v) shows that the cochain homotopy respects the complexes of weighted L_2 forms and special forms. Thus each space of forms in decomposition Equation (15) may be contracted onto its U invariant subspace.

Next, apply the Poincaré lemma to η_2 so that $Y_*f\equiv 0$ for the coefficient functions in the coframe $\{\frac{\partial}{\partial t_i}\}\sqcup\{Y_j\}\sqcup\{X_k\}$. This is standard. The final cochain homotopies compute the weighted L_2 cohomology of sectors $(0,\infty)^r$.

2.2. Contraction on $\mathfrak{a}^{>0}$.

2.2.1. Cochain homotopies on sectors. The goal of this subsection is to generalize a result first quoted in [Bul01] from half lines to sectors. The original result is that for $k(t)$ a weight on $(0,\infty)$ satisfying the differential inequality $\frac{d}{dt}\log k(t)<-\epsilon^2<\infty$, one has the following inequality:

$$
(20)\qquad \int_0^\infty\left\|\int_0^t f(s)ds\right\|^2 k(t)dt<\text{const}\int_0^\infty\|f(t)\|^2k(t)dt.
$$

The present argument is thus quite indebted to [Muc72] and [BH92].

The setting throughout is $(0,\infty)\cdot C$, where $C\subset S^{n-1}$ is a set of vectors on the sphere which is open with compact closure so that $(0,\infty)\cdot C$ is starlike about $\vec{0}$. The scalar multiplication will also be thought of as providing coordinates, so that $\vec{x}$ in the cone may also be written as $(t,\vec{\theta})$ for $t\in(0,\infty)$ and $\vec{\theta}\in C$. We also use t rather than the more typical r for the radial coordinate, given the previous paragraph.

Proposition 2.9. *In the setting above, let $k(\vec{x}\,) > 0$ be a smooth weighting function on $(0,\infty)\cdot C$ so that raywise within the cone the following differential inequality holds:*

$$\frac{\partial}{\partial t}\log\, k(t,\vec{\theta}\,) \leq -2\epsilon < 0, \text{ uniformly over all } \vec{\theta} \in C. \tag{21}$$

Abbreviate $X = \frac{\partial}{\partial t}$, and define the integral operator by

$$I(f_I dt\; d\theta_I + g_I d\theta_I) = \left(\int_X f_I\right) dt\; d\theta_I + \left(\int_X g_I\right) d\theta_I. \tag{22}$$

Here, the $d\theta_I$ are wedges of duals of coordinate fields of S^{n-1} which expand linearly away from $\vec{0}$ when lifted to the cone, and $(\int_X f)(\vec{x}\,) = \int_0^{|\vec{x}\,|} f(s\vec{x}/|\vec{x}\,|)ds$. Finally, define $B_\bullet = \iota_X \circ I$ for $\bullet \geq 1$ and $B_0 \equiv \mathbf{0}$, where ι_X is contraction by X. Then each $B_\bullet$ is bounded in the $k(\vec{x}\,)$ weighted L_2 norm on compactly supported smooth forms on the cone, and thus each extends to the L_2 space of forms $L_2((0,\infty)\cdot C, k(\vec{x}\,)\mathrm{dvol})$.

Remark 2.10. The most natural of these operators is B_1, which is just the line integral $B_1\phi = \int_{\vec{0}}^{\vec{x}} \phi$.

Corollary 2.11. *Given the raywise differential inequality* (21)*, the weighted L_2 cohomology of the sector is*

$$H^\bullet_{(2)}((0,\infty)\cdot C, k(\vec{x}\,)) \cong \mathbb{C} \oplus 0 \cdots \oplus 0. \tag{23}$$

In fact, the cochain homotopy onto the right-hand side is given by taking $P_0 f = f(\vec{0}\,)$, $P_\bullet \equiv \mathbf{0}$ else, and $B_\bullet$ per Proposition 2.9*.*

Remark 2.12. This would imply that in Equation (15),

$$H^\bullet_{(2)}((0,\infty)^r, \exp(-\langle Q, H\cdot\psi(\vec{t}\,)\rangle - 2\langle \nu+\rho, H\circ\psi(\vec{t}\,)\rangle) \cong \mathbb{C}\oplus 0 \oplus 0 \cdots. \tag{24}$$

That in turn ends the final cochain homotopy on the differential presheaf sections, proving Lemma 0.4 hence Theorem 0.5 hence 0.6.

Proof of Corollary 2.11. Via dominated convergence, one has that $[d, I] \equiv \mathbf{0}$. Now since the coordinate coframe in (22) was dual to a coordinate frame, the Lie derivative of the radial direction L_X in fact just differentiates coefficient functions in terms of this coframe. Thus, $L_X I\phi = \phi$. Now for ϕ a form of degree at least one, so that neither B term is a zero operator, Cartan's formula shows

$$(dB + Bd)\phi = d\iota_X I\phi + \iota_X I d\phi = (d\iota_X + \iota_X d)I\phi = L_X I\phi = \phi = (\mathbf{1} - P)\phi.$$

On the other hand, for functions $(dB_0 + B_1 d)f = \int_{\vec{0}}^{\vec{x}} df = f(\vec{x}\,) - f(\vec{0}\,)$. Finally, the critical boundedness check for $B_\bullet$ follows from 2.9. □

Proof of Proposition 2.9. The proposition follows given that one can use the uniform raywise estimate (21) to show that for some finite constant $C(I, 2\epsilon)$ and any multiindex I fixed one has

$$\int_{(0,\infty)\cdot C} \left\|\int_X f_I\right\|^2 t^{-2\#I} k(\vec{x}\,)\,\mathrm{dvol} \leq C(I,2\epsilon)\int_{(0,\infty)\cdot C} \|f_I\|^2 t^{-2\#I} k(\vec{x}\,)\,\mathrm{dvol}. \tag{25}$$

Note that $\mathrm{dvol} = t\; dt\; d\vec{\theta}$ simply adds another factor of t. The lemmas of the next subsection, namely 2.13, 2.14, and 2.15, prove the estimate (25) above. □

2.2.2. Lemmas and estimates. As stated in the last subsection and the introduction, the present argument follows [Muc72] and [BH92]. Also, the lemmas below check that the constant which bounds the growth rate of the exterior derivative is traced explicitly through the present computation and found to be $4/\epsilon^2$ for 2ϵ per (21).

Lemma 2.13. *Let $n \in \mathbb{Z}$. Consider the inequality*

$$\int_C \int_{(0,\infty)} \left\| \int_X f \right\|^2 t^n k(t,\vec{\theta}\,)\, dt\, d\vec{\theta} \le C(\lambda) \int_C \int_{(0,\infty)} \|f\|^2 t^n k(t,\vec{\theta}\,)\, dt\, d\vec{\theta} \tag{26}$$

for f varying over those square integrable functions of $k(\vec{x}\,)t^n\, dt\, d\vec{\theta}$. If there exists a function $\lambda(\vec{x}\,)$ so that

$$\left\{ k(s,\vec{\theta}\,)^{-1} t^{-n} \lambda(s,\vec{\theta}\,) \int_s^\infty k(t,\vec{\theta}\,) t^n \left(\int_0^t \lambda(u,\vec{\theta}\,)^{-1} du \right)\, dt \right\} \le C_\lambda < \infty \tag{27}$$

uniformly over $\vec{\theta} \in C$ and $s > 0$, then (26) *holds with $C(\lambda) = C_\lambda < \infty$.*

Lemma 2.14. *Suppose we define a possibly infinite constant B as follows:*

$$B = \sup_C \sup_{r>0} \sqrt{\int_0^r k(t,\vec{\theta}\,)^{-1} t^{-n} dt \int_r^\infty k(t,\vec{\theta}\,) t^n dt}\,. \tag{28}$$

Then we may choose $\lambda(t,\vec{\theta}\,) = t^n k(t,\vec{\theta}\,)\sqrt{\int_0^t s^{-n} k(t,\vec{\theta}\,) ds}$ which guarantees the expression of (27) *of Lemma* 2.13 *is bounded above by $C_\lambda \le 4B^2$.*

Lemma 2.15. *Suppose we label $M = \sup_{t>0} e^{2\epsilon t} (\int_t^\infty e^{-\epsilon t})^{-2} = \epsilon^{-2}$. Then given the uniform raywise bound of* (21)*, one has $\sqrt{M} \ge B$ for B the constant of* (28) *of Lemma* 2.14*. Thus* (26) *of Lemma* 2.13 *holds with*

$$C_\lambda \le 4B^2 \le 4M = 4\epsilon^{-2} < \infty. \tag{29}$$

Proof of Lemma 2.13. Using the Schwarz inequality for $\lambda(s,\vec{\theta}\,)ds$,

$$\left\| \int_X f \right\|^2 \le \int_0^t \|f(s,\vec{\theta}\,)\|^2 \lambda(s,\vec{\theta}\,) ds \left(\int_0^t \lambda(u,\vec{\theta}\,)^{-1} du \right).$$

Now one substitutes into the weighted L_2 integral and applies Fubini's theorem.

$$\begin{aligned}
&\int_C \int_0^\infty \left\| \int_0^t f(s,\vec{\theta}\,) ds \right\|^2 k(\vec{x}\,) t^n dt\, d\vec{\theta} \\
&\qquad \le \int_C \int_0^\infty \int_0^t \|f(s,\vec{\theta}\,)\|^2 \lambda(s,\vec{\theta}\,) ds \left(\int_0^t \lambda(u,\vec{\theta}\,)^{-1} du \right) k(t,\vec{\theta}\,) t^n dt\, d\vec{\theta} \\
&\qquad = \int_C \int_0^\infty \int_s^\infty \|f\|^2 \lambda(s,\vec{\theta}\,) \left(\int_0^t \lambda(u,\vec{\theta}\,)^{-1} du \right) k(t,\vec{\theta}\,) t^n dt\, ds\, d\vec{\theta} \\
&\qquad = \int_C \int_0^\infty \{---\} \|f\|^2 k(s,\vec{\theta}\,) s^n ds\, d\vec{\theta} \\
&\qquad \le C_\lambda \int_C \int_0^\infty \|f\|^2 k(s,\vec{\theta}\,) s^n ds\, d\vec{\theta}
\end{aligned}$$

with $\{---\}$ per the statement of the lemma. The lemma follows. $\square$

Proof of Lemma 2.14. Recall $\lambda(s,\vec{\theta}) = s^n k(s,\vec{\theta})\sqrt{\int_0^s u^{-n}k(u,\vec{\theta})^{-1}du}$. For this choice of λ, let us note as a preliminary that for each $\vec{\theta}$

$$\int_0^t \lambda(u,\vec{\theta})^{-1}du = \int_0^t 2\frac{d}{du}\sqrt{\int_0^u w^{-n}k(w,\vec{\theta})^{-1}dw}\,du = 2\sqrt{\int_0^t u^{-n}k(u,\vec{\theta})^{-1}du}.$$

Thus one obtains the following estimates, beginning by plugging $\lambda(t,\vec{\theta})$ above into the expression of Lemma 2.13 to be bounded.

$$\begin{aligned}
&k(s,\vec{\theta})^{-1}s^{-n}s^n k(s,\vec{\theta})\left(\int_0^s u^{-n}k(u,\vec{\theta})^{-1}du\right)^{1/2}\int_s^\infty k(t,\vec{\theta})\left[\int_0^t \lambda(u,\vec{\theta})^{-1}du\right]dt\\
&\quad= \left(\int_0^s u^{-n}k(u,\vec{\theta})^{-1}du\right)^{1/2}\int_s^\infty k(t,\vec{\theta})t^n\left[2\left(\int_0^t u^{-n}k(u,\vec{\theta})^{-1}du\right)^{1/2}\right]dt\\
&\quad\le (2B)\left(\int_0^s u^{-n}k(u,\vec{\theta})^{-1}du\right)^{1/2}\int_s^\infty k(t,\vec{\theta})t^n\left(\int_t^\infty u^n k(u,\vec{\theta})du\right)^{-1/2}dt\\
&\quad= (2B)\left(\int_0^s u^{-n}k(u,\vec{\theta})^{-1}du\right)^{1/2} 2\left(\int_s^\infty t^n k(t,\vec{\theta})dt\right)^{1/2}\\
&\quad\le 4B^2.
\end{aligned}$$

This concludes the proof of the lemma. $\square$

Proof of Lemma 2.15. For each fixed $\vec{\theta}_0$, choose a decreasing function $d_{\vec{\theta}_0}$ so that $k(t,\vec{\theta}_0) = d_{\vec{\theta}_0}(t)\mathrm{e}^{-\epsilon t}$. Now label $\widetilde{M}$ as the following supremum:

$$\widetilde{M} = \sup_C \sup_t \left(\int_t^\infty s^n k(s,\vec{\theta})ds\right)^2 t^{-2n}k(t,\vec{\theta})^{-2}.$$

Then in fact since all the $d_{\vec{\theta}}$ decrease, one has

$$\widetilde{M} \le \sup_C \sup_t \frac{d_{\vec{\theta}}(t)^2(\int_t^\infty \mathrm{e}^{-\epsilon t}dt)^2}{d_{\vec{\theta}}(t)^2\mathrm{e}^{-2\epsilon t}} = M < \infty.$$

Now flipping the definition about, $k(s,\vec{\theta})^{-1}s^{-n} \le Mk(s,\vec{\theta})s^n(\int_s^\infty t^n k(t,\vec{\theta})dt)^{-2}$. Whence substituting within the integral,

$$\begin{aligned}
&\left(\int_0^t k(s,\vec{\theta})^{-1}s^{-n}ds\right)^{1/2}\\
&\quad\le M^{1/2}\left[\int_0^t k(s,\vec{\theta})s^n\left(\int_s^\infty u^n k(u,\vec{\theta})du\right)^{-2}ds\right]^{1/2}\\
&\quad= M^{1/2}\left[\left(\int_t^\infty s^n k(s,\vec{\theta})ds\right)^{-1} - \left(\int_0^\infty s^n k(s,\vec{\theta})ds\right)^{-1}\right]^{1/2}\\
&\quad\le M^{1/2}\left(\int_t^\infty s^n k(s,\vec{\theta})ds\right)^{-1/2}.
\end{aligned}$$

Thus $M^{1/2} \ge (\int_t^\infty s^n k(s,\vec{\theta})ds)^{1/2}(\int_0^t k(s,\vec{\theta})^{-1}s^{-n}ds)^{1/2}$ uniformly over t and $\vec{\theta} \in C$, as we wished to prove. $\square$

Remark 2.16. Reading through the argument, one notices that $d\log t^n \to 0$ as $t \to \infty$ for any $n \in \mathbb{Z}$. As logarithmic derivatives are multiplicative, one notes after the proof that one might instead use the estimate $4\epsilon^{-2}$ directly without the extra factors of t^n by renaming $\widetilde{k}(t, \vec{\theta}) = t^n k(t, \vec{\theta})$.

3. Appendices

3.1. Appendix: Regularity for locally symmetric spaces. The decay of the weights given by formula (2) is extremely rapid. Informal discussions lead the present author to thus believe there are no applications of the $(\mathfrak{g}, K)$ cohomology of the right regular representation of w_{-Q} or the multiplicative inverse. Nonetheless, references are included here to make these well-defined concepts.

We begin by recalling two regularity results. (N, g) is any complete manifold. Then a regularization result, due to Cheeger [Che80] shows that the (unreduced) L_2 cohomologies computed with either smooth square integrable forms or with the Hilbert complex of L_2 forms coincide. The proof uses Friedrich's mollifiers. Similar results arise as scholia in the weighted case; see [BL92]. This justifies Equation (3) stating that $H^\bullet_{(2),w}(N)$ is the cohomology of the complex

$$\left\{ \omega \mid \int_M |\omega|^2 \; w \, \text{dvol} < \infty \text{ and } \int_M |d\omega|^2 \; w \, \text{dvol} < \infty \right\}.$$

In the locally symmetric case, a *stronger* regularity holds. Retaining $\mathcal{M} = \Gamma\backslash G/K$ as in the body, let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$, i.e., the algebra of left G invariant differential operators on G. We wish to express the w weighted cohomology as the Lie algebra cohomology of a $U(\mathfrak{g})$ module. Stated simply, we want the same cohomology to be produced by an even smaller complex:

$$\left\{ \omega \mid \int_{\Gamma\backslash X} |D\omega|^2 w \, \text{dvol} < \infty \; \forall D \in U(\mathfrak{g}) \right\}. \tag{30}$$

Borel proved this stronger regularity result in the unweighted case [Bor83]. Specifically, he proved a cochain homotopy from the cohomology of Equation (3) onto that of Equation (30). For the argument, mollification takes places via a *convolution in G*.

Definition 3.1 ([Fra98, p. 191])**.** A weight w is admissible if $|\nabla \log w|$ is bounded.

Franke proved the stronger regularity in the weighted case[†] , given that w is admissible. Of course, Gaussians are *inadmissible.* Nonetheless, Borel's argument generalizes immediately to the case at hand. The proof will not be repeated, but a new statement is in order.

Theorem 3.2 ([Bor83], 2.7, p. 618)**.** *Let $w : \Gamma\backslash X \to (0, \infty)$ be a Borel measurable weighting function on a locally symmetric space, with $\widetilde{w}$ the right K invariant lift. We do not suppose $|d\log w|$ is bounded. Put $L^\infty_{2,\widetilde{w}}(\Gamma\backslash G) = V$ be the Harish-Chandra module of smooth vectors within the regular representation $L_{2,\widetilde{w}}(\Gamma\backslash G)$. Then there is an isomorphism $H^\bullet_{(2),w}(\Gamma\backslash X, \mathbb{E}) \cong H^\bullet(\mathfrak{g}, K, V \otimes E)$.*

[†]In fact, he proves both regularities directly with weighted Sobolev inequalities.

Remark 3.3. 1) For most w, $\widetilde{w}dg$ is not G invariant. This does not destroy the statement, but it should be read carefully. For the $(\mathfrak{g}, K)$ cohomology actually depends on $V^\infty = \{f|Df \in L_{2,\widetilde{w}}(\Gamma\backslash G)\forall D \in U(\mathfrak{g})\}$.

2) The argument runs roughly as follows. First, lift the question to $C^0(\Gamma\backslash G) \otimes \wedge^\bullet \mathfrak{g}^* \otimes E$. Here, one may produce a cochain homotopy (B, P) whose projection operator P convolves with an approximate convolution identity supported on a neighborhood of the identity in G. That clearly is $\widetilde{w}dg$ bounded and maps the coefficient functions into C^∞, but unfortunately the associated B does not preserve the subspace $\cap_{X\in\mathfrak{k}}\ker \iota_X$ associated to the $(\mathfrak{g}, K)$ cohomology. The argument circumvents that problem via explicit analysis of an appropriate spectral sequence.

3.2. Appendix: Extensions to nonarithmetic lattices. There are no great difficulties in generalizing these results to the nonarithmetic lattices. Such lattices only exist in case rank$G_{\mathbb{R}} = 1$. Below appears a short description of the minor modifications of the previous argument which apply to the nonarithmetic case.

- There is strictly speaking no $\mathbb{Q}$ rank. However, since we suppose $\mathcal{M}$ noncompact, the case at hand mimics the case where both ranks are one. In particular, $\lambda \in \check{\mathfrak{a}}^{>0}$ should be read as λ nonzero and positive according to a parabolic preserving some line. The correct definition of $-Q < 0$ follows.
- Rational parabolics are generally replaced by parabolics stabilizing a line in the language of [GR70], i.e., the stabilizer of the boundary point the lift of an embedded ray of $\mathcal{M}$. Equivalently, any parabolic P with $P = UA$ and $(\Gamma \cap U)\backslash U$ compact may replace a rational parabolic in the body.
- Given these conventions, $\overline{\mathcal{M}}$ is point-end.
- The special form presheaf is again just U invariant forms in some neighborhood of a boundary point of the point-end compactification. Weighted L_2 cohomology is defined in the usual way [Zuc82, p. 175].

Then one can readily repeat all arguments, using Theorem 1.2 of [GR70] in place of the reduction theory of [BHC62].

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