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On Hausdorff Dimension of Random Fractals

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Abstract. We study random recursive constructions with finite "memory" in complete metric spaces and the Hausdorff dimension of the generated random fractals. With each such construction and any positive number β we associate a linear operator $V^{(\beta)}$ in a finite dimensional space. We prove that under some conditions on the random construction the Hausdorff dimension of the fractal coincides with the value of the parameter β for which the spectral radius of $V^{(\beta)}$ equals 1.

Contents

1. Introduction

In this paper we compute the Hausdorff dimension of random fractals in a complete metric space $\mathbb{M}$ which are generated by random recursive constructions. This problem was studied by several authors (see Falconer [1], Graf, Mauldin and Williams [4], Kifer [8], Mauldin and Williams [6], Pesin and Weiss [9], Tempelman [11] and the references therein).

Let us remind here the definition of the Hausdorff measures and the Hausdorff dimension. If $\beta \geq 0$, $\delta > 0$ and A is any subset of $\mathbb{M}$, write

$$H_\delta^{(\beta)}(A) = \inf\Big\{\sum_{i=1}^{\infty}(\mathrm{diam}(U_i))^\beta : A \subset \cup_{i=1}^{\infty} U_i, \quad 0 < \mathrm{diam}(U_i) \leq \delta\Big\}.$$

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Let

$$H^{(\beta)}(A) = \lim_{\delta \to 0} H_\delta^{(\beta)}(A).$$

Then $H^{(\beta)}$ is the β-dimensional Hausdorff outer measure. It is known (see, for example [2]) that there is a number $\dim_H A$, called the *Hausdorff dimension* of A, such that

$$H^{(\beta)}(A) = \infty \quad \text{if} \quad \beta < \dim_H A \quad \text{and} \quad H^{(\beta)}(A) = 0 \quad \text{if} \quad \beta > \dim_H A.$$

We study iterated random constructions with a fixed non-random number of "daughter" sets at each step of the construction. In this case our model generalizes random models studied by Falconer [1] and by Mauldin and Williams [6]. Unlike these authors we do not assume that the random scale coefficients are identically distributed. On the other hand, our model generalizes the deterministic model with "finite memory" studied by Tempelman [11] in which the scale coefficients depend on several previous steps. As in [11], we introduce for each $\beta > 0$ a linear "transition" operator $V^{(\beta)}$ associated with the construction; we denote by $\rho(\beta)$ the spectral radius of this operator. Let K denote the random fractal obtained by the iterating process. We prove that $\dim_H K = \alpha$ almost surely, where α is the unique solution of the equation $\rho(\beta) = 1$.

In Section 2 we define random constructions in complete metric spaces and introduce some additional properties of such constructions.

In Section 3 we define a non-random transition operator and study properties of sequences of random variables associated with random constructions. We show that for the number α defined above $\dim_H K \le \alpha$ almost surely.

In Section 4 we study properties of random variables obtained as limits of some martingales constructed in the previous section. In Section 6 these results are used in the definition and study of a special random probability measure which is the crucial tool in the proof of our main result.

In Section 5 we prove some auxiliary statements related to the metric space of sequences with a metric that meets some restrictions specified below.

In Section 6 we prove the main result: $\dim_H K = \alpha$ a.s. In view of the upper estimate for the dimension obtained in Section 2, we prove here that the lower estimate is valid (this is actually the most difficult part of the proof). This is done on one hand by constructing a random measure analogous to the one studied in [6] and on the other hand by using methods developed in [11], namely, by the study of local "cylinder-wise" dimension of this measure and its relation to the "global" Hausdorff dimension.

2. Random constructions

First of all let us introduce some notation related to finite sequences. Denote by $\mathbb{N}$ the set of all positive integers. Let $\Delta = \{1, \ldots, N\}$, where $N \in \mathbb{N}$. We also consider $\Delta^* = \bigcup_{n=1}^{\infty} \Delta^n$, the set of all finite sequences, and the set $\Delta^{\mathbb{N}}$ of all infinite sequences of elements of Δ. If $\sigma = (\sigma_1, \ldots, \sigma_k)$, then $|\sigma| = k$ is the length and if $\eta = (\eta_1, \ldots, \eta_n)$ then $(\sigma, \eta) = (\sigma_1, \ldots, \sigma_k, \eta_1, \ldots, \eta_n)$ is the concatenated sequence. Δ^0 contains only the empty sequence $\emptyset$ with the following property: $(\emptyset, \eta) = (\eta, \emptyset) = \eta$ for any $\eta \in \Delta^*$. If $\pi \in \Delta^*$ or $\pi \in \Delta^{\mathbb{N}}$ then $\pi|n$ denotes the sequence obtained by restricting π to the first n entries, where $\pi|0 = \emptyset$. In Δ^* we consider a partial order: for $\sigma, \eta \in \Delta^*$ we put $\eta < \sigma$ if and only if $\sigma = (\eta, \xi)$ for some $\xi \in \Delta^*$.

Let $(\mathbb{M}, \lambda)$ be a complete metric space; by $\mathrm{diam}(A)$ we denote the diameter of a set $A \subset \mathbb{M}$; $[A]$ denotes the closure of A; $B(x, r)$ denotes the open ball of radius r centered at x. We consider a probability space $(\Omega, \mathcal{G}, P)$ and for each $\omega \in \Omega$ a countable family of closed nonempty subsets of $\mathbb{M}$:

$$\mathbf{I}(\omega) = \{I_\sigma(\omega) : \sigma \in \Delta^*\}$$

We call the family $\mathbf{I}$ *a random construction* if for almost every $\omega \in \Omega$

$$\lim_{n\to\infty} \max_{\sigma\in\Delta^n} \mathrm{diam}(I_\sigma) = 0 \tag{2.1}$$

and

$$I_\sigma \subset I_\eta, \quad \text{if } \eta < \sigma. \tag{2.2}$$

We also consider a family of positive random variables $\{l_\sigma : \sigma \in \Delta^*\}$. We assume that for almost every ω this family is monotone in the sense $l_{\sigma,p}(\omega) < l_\sigma(\omega)$ for each $\sigma \in \Delta^*$and each $p \in \Delta$, and

$$\lim_{n\to\infty} l_{[\pi|n]}(\omega) = 0, \quad \text{for every } \pi \in \Delta^{\mathbb{N}}.$$

Remark. It can be shown that in this case the convergence is uniform:

$$\lim_{n\to\infty} \max_{\sigma\in\Delta^n} l_\sigma(\omega) = 0. \tag{2.3}$$

We study the properties of the "random fractal"

$$K(\omega) = \bigcap_{n=1}^{\infty} \bigcup_{\sigma\in\Delta^n} I_\sigma(\omega).$$

Remark. There are interesting examples of fractals obtained by constructions without the property (2.2) (see, for example, [11]). But the same fractals can be obtained by modified constructions satisfying this condition.

We recall the notion of the Moran index introduced explicitly in [9],[11] (this characteristic was essentially used in [7]). Let $m \geq 1$ be an integer. Consider a sequence $\pi \in \Delta^{\mathbb{N}}$ and positive numbers r and b. If $l_{[\pi|m]} \geq r$ we define the natural number $k(r, \pi) > m$ as follows: $l_{[\pi|k(r,\pi)+1]} < r \leq l_{[\pi|k(r,\pi)]}$; if $l_{[\pi|m]} < r$ we put $k(r, \pi) = m$.

The Moran index of the construction $\mathbf{I}(\omega)$, corresponding to a constant b, is the minimal number $\gamma_\omega(b)$ with the following property: for any $x \in \mathbb{M}$ and any $\pi \in \Delta^{\mathbb{N}}$ and $n > m$ there exist at most $\gamma_\omega(b)$ pairwise disjoint sets $I_{[\eta^{(t)}|k(l_{[\pi|n]},\eta^{(t)})]}$, $t \in \mathbb{N}$, where $\eta^{(t)} \in \Delta^{\mathbb{N}}$, such that $B(x, bl_{[\pi|n]}) \cap I_{[\eta^{(t)}|k(l_{[\pi|n]},\eta^{(t)})]} \neq \emptyset$; if such a number $\gamma_\omega(b)$ does not exist we put $\gamma_\omega(b) = \infty$.

Let $L_{\sigma,p} = l_{\sigma,p}/l_\sigma$ for $\sigma \in \Delta^*$, $p \in \Delta$. We assume in the sequel that the following conditions are fulfilled:

i. For each $\sigma \in \Delta^*$ and for almost every ω

$$\mathrm{diam}(I_\sigma(\omega)) \leq l_\sigma(\omega). \tag{2.4}$$

ii. The random vectors $(L_{\sigma,1}, \ldots, L_{\sigma,N}), \sigma \in \Delta^*$, are independent.

iii. There exists an integer $m \geq 1$ such that for any $\sigma \in \Delta^*$ and any $\eta \in \Delta^{m-1}$ the random vectors $(L_{\sigma,\eta,1}, \dots, L_{\sigma,\eta,N})$ and $(L_{\eta,1}, \dots, L_{\eta,N})$ have the same distribution.

In case of need we also consider the following restrictions.

iv. If neither $\sigma < \eta$ nor $\eta < \sigma$ then for almost every ω

$$I_\eta(\omega) \cap I_\sigma(\omega) \cap K(\omega) = \emptyset. \tag{2.5}$$

v. For almost every ω

$$\lim_{n\to\infty} \frac{\log l_{[\pi|n+1]}(\omega)}{\log l_{[\pi|n]}(\omega)} = 1 \quad \text{for all } \pi \in \Delta^{\mathbb{N}}. \tag{2.6}$$

vi. For almost every ω there exists $b = b(\omega) > 0$ such that

$$\gamma_\omega(b) < \infty. \tag{2.7}$$

Remarks. 1. The case $m = 1$ when the random vectors $(L_{\sigma,1}, \dots, L_{\sigma,N})$ are identically distributed is covered in [1] and [6].

2. It is obvious that (2.6) is satisfied if there exists a positive random variable $a(\omega)$ such that for each $\sigma \in \Delta^*$and for almost every ω

$$L_\sigma(\omega) \geq a(\omega). \tag{2.8}$$

Note that in [11] conditions (2.6) and (2.8) are referred to as "regularity" and "strong regularity" respectively.

3. It is clear that condition (2.5) is fulfilled if

$$I_\eta(\omega) \cap I_\sigma(\omega) = \emptyset \tag{2.9}$$

as long as neither $\sigma < \eta$ nor $\eta < \sigma$.

4. Condition (2.7) admits stronger but more tractable versions (see [11]). We consider here the simplest one. Let

$$\Lambda(A, B) = \inf\{\lambda(x, y) : x \in A, y \in B\} \text{ for } A, B \in \mathbb{M}.$$

It is easy to see that $\gamma_\omega(b) = 1$ if the following stronger version of condition (2.9) is met:

$$\Lambda(I_\eta(\omega), I_\sigma(\omega)) \geq b \max(l_\eta(\omega), l_\sigma(\omega)) \tag{2.10}$$

if neither $\sigma < \eta$ nor $\eta < \sigma$.

5. While condition (2.4) establishes an upper bound for the diameter of the set $I_\sigma(\omega)$, condition (2.7) implies a lower bound. The stronger condition (2.10) means that in the metric subspace $K(\omega) \subset \mathbb{M}$ the intrinsic diameter of $I_\sigma(\omega) \cap K(\omega)$ cannot be smaller than $bl_\sigma(\omega)$ (see [11] for details).

There are numerous examples of fractals in finite dimensional spaces. We give an example of a non-random fractal in the Hilbert space $\mathbb{H}$ that is not contained in any finite dimensional subspace of $\mathbb{H}$.

Example 2.1. Let $\mathbf{e}_1, \mathbf{e}_2, \dots$ be an orthonormal base in the Hilbert space $\mathbb{H}$. Denote $a_n = 4^{-n}$, $r_n = 4^{-n-1}$; we put here $\Delta = \{0, 1\}$. If $\pi = (p_1, p_2, \dots) \in \Delta^{\mathbb{N}}$ then

$$\mathbf{x}_{\pi|n} = \sum_{k=1}^{n} a_k p_k \mathbf{e}_k, \quad \mathbf{x}_\pi = \sum_{k=1}^{\infty} a_k p_k \mathbf{e}_k$$

and $I_{\pi|n} = [B(\mathbf{x}_{\pi|n}, r_n)]$. The fractal $K = \{\mathbf{x}_\pi : \pi \in \Delta^{\mathbb{N}}\}$. It is easy to check that our construction satisfies conditions (i)-(iii) and (2.10). If $\pi = (1, 1, \dots)$ then the vectors $\mathbf{x}_{\pi|n},\ n = 1, 2, \dots$ are contained in K and are linearly independent.

Following [11] we say that two constructions $I^{(1)}$ and $I^{(2)}$ are *conjunctive* if $I^{(1)}_\sigma(\omega) \cap I^{(2)}_\sigma(\omega) \neq \emptyset$ for almost every ω and for each $\sigma \in \Delta^*$. The following proposition is a simple corollary of this definition.

Proposition 2.1. *For almost every ω conjunctive constructions define the same fractal.*

This gives us the opportunity to study simple conjunctive constructions possessing better properties and defining the same fractal as the given one. We will use this opportunity later. Here we confine ourselves to the following example.

Example 2.2. Assume the random construction $I_\sigma(\omega) = B(x_\sigma(\omega), r_\sigma(\omega)), \sigma \in \Delta^*$, meets condition (2.9); then for any d such that $0 < d < 1$, the conjunctive construction $B(x_\sigma(\omega), dr_\sigma(\omega)), \sigma \in \Delta^*$, enjoys the stronger property (2.10).

3. The transition operator and random sequences related to random constructions

Let $m > 1$. Consider N^{m-1}-dimensional real vector space

$$\Phi = \{u(x_1, \dots, x_{m-1}) : 1 \leq x_1, \dots, x_{m-1} \leq N\}.$$

For any $\beta > 0$ we define a linear operator $V^{(\beta)} : u \longrightarrow uV^{(\beta)} = w$ in Φ by

$$w(x_2, \dots, x_m) = \sum_{x_1 \in \Delta} u(x_1, \dots, x_{m-1})\, \mathbf{E}\, L^\beta_{x_1, \dots, x_m}.$$

Remark. In case $m = 1$ we can also consider an operator $V^{(\beta)}$ given by $N \times N$ matrix with identical rows $(\mathbf{E}\, L^\beta_1, \dots, \mathbf{E}\, L^\beta_N)$.

Let us introduce the random variables

$$\begin{aligned} S^{(p_2, \dots, p_m)}_{\beta, n} &= \sum_{\substack{\sigma \in \Delta^n: \\ \sigma_{n-m+2} = p_2, \dots, \sigma_n = p_m}} l^\beta_\sigma \\ &= \sum_{\sigma \in \Delta^{n-m+1}} l^\beta_{\sigma, p_2, \dots, p_m} \\ &= \sum_{\sigma \in \Delta^{n-m}} \sum_{p_1 \in \Delta^1} l^\beta_{\sigma, p_1, \dots, p_m} \\ &= \sum_{p_1 \in \Delta^1} \sum_{\sigma \in \Delta^{n-m}} l^\beta_{\sigma, p_1, \dots, p_{m-1}} L^\beta_{\sigma, p_1, \dots, p_m}, \end{aligned}$$

where $p_2, \dots, p_m \in \Delta$, $n \geq m$.

We shall consider the following σ-algebras:

$$\mathcal{F}_n = \sigma(\{L_\eta : \eta \in \Delta^* \quad \text{and} \quad |\eta| \leq n\}) \quad \text{for } n = 1, 2, \dots$$

and

$$\mathcal{F} = \bigvee_{n=1}^{\infty} \mathcal{F}_n.$$

Denote by $\mathbf{S}_{\beta,n}$ the random vector $(S_{\beta,n}^{(p_2,\ldots,p_m)} : 1 \le p_2,\ldots,p_m \le N)$. By $\mathbf{E}\,Y$ we denote the expectation of a random variable or a random vector Y.

Lemma 3.1. *For any $\beta > 0$ and $n \ge m$ there exists $\mathbf{E}\,\mathbf{S}_{\beta,n}$ and*

$$\mathbf{E}(\mathbf{S}_{\beta,n}|\mathcal{F}_{n-1}) = \mathbf{S}_{\beta,n-1}V^{(\beta)}.$$

Proof. Using the fact that for any $\sigma \in \Delta^{n-m}, p_1,\ldots,p_m \in \Delta$ the variables $l_{\sigma,p_1,\ldots,p_{m-1}}$ are $\mathcal{F}_{n-1}$-measurable and $L_{\sigma,p_1,\ldots,p_m}$ are independent of $\mathcal{F}_{n-1}$ we have

$$\begin{aligned}\mathbf{E}(S_{\beta,n}^{p_2,\ldots,p_m}|\mathcal{F}_{n-1}) &= \sum_{p_1\in\Delta}\ \sum_{\sigma\in\Delta^{n-m}} l^{\beta}_{\sigma,p_1,\ldots,p_{m-1}}\,\mathbf{E}\,L^{\beta}_{\sigma,p_1,\ldots,p_m}\\ &= \sum_{p_1\in\Delta}\Big(\sum_{\sigma\in\Delta^{n-m}} l^{\beta}_{\sigma,p_1,\ldots,p_{m-1}}\Big)\,\mathbf{E}\,L^{\beta}_{p_1,\ldots,p_m}\\ &= \sum_{p_1\in\Delta} S_{\beta,n-1}^{(p_1,\ldots,p_{m-1})}\,\mathbf{E}\,L^{\beta}_{p_1,\ldots,p_m}\end{aligned}$$

which holds for any $p_2,\ldots,p_m \in \Delta$. □

Denote $v^{(\beta)}(x_1,\ldots,x_m) = \mathbf{E}\,L^{\beta}_{x_1,\ldots,x_m}$ and let $\rho(\beta) = \rho(V^{(\beta)})$ be the spectral radius of the operator $V^{(\beta)}$. Since for every $x_1,\ldots,x_m \in \Delta$ almost surely $0 < L_{x_1,\ldots,x_m} < 1$, it is easy to prove the following statement.

Lemma 3.2. *$\rho(\cdot)$ is a continuous strictly decreasing function such that $\rho(0) > 1$ and $\lim_{\beta\to\infty}\rho(\beta) = 0$. Therefore, there is a unique solution α of the equation $\rho(\beta) = 1$.*

Since $\mathbf{E}\,L^{\beta}_{p_1,\ldots,p_m} > 0$ for any $p_1,\ldots,p_m \in \Delta$, the operators $V^{(\beta)}$ are indecomposable. Therefore by the Perron-Frobenius theorem (see, for example, [3]), $\rho(\beta)$ is an eigenvalue of $V^{(\beta)}$ and there exists a positive right eigenvector $r^{(\beta)} = (r^{(\beta)}(x_1,\ldots,x_{m-1}) : 1 \le x_1,\ldots,x_{m-1} \le N) \in \Phi$, that is, $V^{(\beta)}r^{(\beta)} = \rho(\beta)r^{(\beta)}$.

Perform the following standard transformation:

$$\widetilde{v}^{(\beta)}(x_1,\ldots,x_m) = \frac{v^{(\beta)}(x_1,\ldots,x_m)r^{(\beta)}(x_2,\ldots,x_m)}{\rho(\beta)r^{(\beta)}(x_1,\ldots,x_{m-1})}. \tag{3.1}$$

We notice that the new operator $\widetilde{V}^{(\beta)} = (\widetilde{v}^{(\beta)}(x_1,\ldots,x_m) : 1 \le x_1,\ldots,x_m \le N)$ is stochastic, i.e., for all $x_1,\ldots,x_{m-1} \in \Delta$

$$\sum_{x_m\in\Delta}\widetilde{v}^{(\beta)}(x_1,\ldots,x_m) = 1$$

We can rewrite (3.1) as

$$v^{(\beta)}(x_1,\ldots,x_m) = \frac{\rho(\beta)r^{(\beta)}(x_1,\ldots,x_{m-1})\widetilde{v}^{(\beta)}(x_1,\ldots,x_m)}{r^{(\beta)}(x_2,\ldots,x_m)}.$$

Denote

$$\widetilde{S}_{\beta,n}^{(p_2,\ldots,p_m)} = r^{(\beta)}(p_2,\ldots,p_m)S_{\beta,n}^{(p_2,\ldots,p_m)}/\rho^n(\beta), n \ge m$$

and

$$\widetilde{\mathbf{S}}_{\beta,n} = (\widetilde{S}_{\beta,n}^{(p_2,\ldots,p_m)} : 1 \le p_2,\ldots,p_m \le N), n \ge m.$$

From Lemma 3.1 it follows immediately that

$$\mathbf{E}(\widetilde{S}^{(p_2,\dots,p_m)}_{\beta,n}|\mathcal{F}_{n-1}) = \sum_{p_1\in\Delta} \widetilde{S}^{(p_1,\dots,p_{m-1})}_{\beta,n-1}\widetilde{v}^{(\beta)}(p_1,\dots,p_m), \tag{3.2}$$

for any $p_2,\dots,p_m \in \Delta$, i.e.,

$$\mathbf{E}(\widetilde{\mathbf{S}}_{\beta,n}|\mathcal{F}_{n-1}) = \widetilde{\mathbf{S}}_{\beta,n-1}\widetilde{V}^{(\beta)}, n > m.$$

Define

$$Z_{\beta,n} = \sum_{p_2,\dots,p_m} \widetilde{S}^{(p_2,\dots,p_m)}_{\beta,n}, n \geq m.$$

Lemma 3.3. *The sequence* $(Z_{\beta,n},\ \mathcal{F}_n), n \geq m$, *is a positive martingale.*

Proof. Indeed from (3.2) we have

$$\begin{aligned}\mathbf{E}(Z_{\beta,n}|\mathcal{F}_{n-1}) &= \sum_{p_2,\dots,p_m} \mathbf{E}(\widetilde{S}^{(p_2,\dots,p_m)}_{\beta,n}|\mathcal{F}_{n-1})\\ &= \sum_{p_2,\dots,p_m}\sum_{p_1} \widetilde{S}^{(p_1,\dots,p_{m-1})}_{\beta,n-1}\widetilde{v}^{(\beta)}(p_1,\dots,p_m)\\ &= \sum_{p_1,\dots,p_{m-1}} \widetilde{S}^{(p_1,\dots,p_{m-1})}_{\beta,n-1}\sum_{p_m}\widetilde{v}^{(\beta)}(p_1,\dots,p_m)\\ &= \sum_{p_1,\dots,p_{m-1}} \widetilde{S}^{(p_1,\dots,p_{m-1})}_{\beta,n-1} = Z_{\beta,n-1},\end{aligned}$$

since $\widetilde{V}^{(\beta)}$ is stochastic. □

Therefore, by the martingale convergence theorem, for almost every ω there exists the limit $\lim_{n\to\infty} Z_{\beta,n}$; if $\beta = \alpha$ we denote this limit by X. As an immediate consequence we obtain the following upper estimate for the Hausdorff dimension of the random fractal K.

Theorem 3.1. *For almost every* ω

$$\dim_H K(\omega) \leq \alpha,$$

where α *is defined in Lemma* 3.2*.*

Proof. Let $\beta > \alpha$. So by Lemma 3.2 $\rho(\beta) < 1$. Along with $Z_{\beta,n}$ we consider also

$$\overline{Z}_{\beta,n} = \sum_{p_2,\dots,p_m} S^{(p_2,\dots,p_m)}_{\beta,n}.$$

Denote here

$$\frac{1}{\xi} = \min_{x_1,\dots,x_{m-1}} r^{(\beta)}(x_1,\dots,x_{m-1}) > 0.$$

Then it is easy to see that for any n

$$0 \leq \overline{Z}_{\beta,n} \leq \xi\rho^n(\beta)Z_{\beta,n}.$$

Since $\rho(\beta) < 1$ and the sequence $Z_{\beta,n}$ is convergent a.s., this implies that for almost every ω

$$\overline{Z}_{\beta,n} \to 0 \text{ as } n \to \infty.$$

Now, for any n we have $K \subset \bigcup_{\sigma \in \Delta^n} I_\sigma$. By (2.3) for any small $\delta > 0$ we can find $n = n(\delta)$ such that $l_\sigma \le \delta$ for every $\sigma \in \Delta^n$. Hence

$$H_\delta^{(\beta)}(K) \le \sum_{\sigma \in \Delta^n} l_\sigma^\beta = \overline{Z}_{\beta,n}.$$

Let $\delta \to 0$, then $n(\delta) \to \infty$ and, therefore, $H^{(\beta)}(K) = 0$ for any $\beta > \alpha$. This proves the theorem. □

Fix $\sigma \in \Delta^*$. Assume $|\sigma| = k$. We define the random variables

$$S_{\sigma;\beta,n}^{(p_2,\dots,p_m)} = \Big(\sum_{\eta \in \Delta^{n-m+1}} l^\beta_{\sigma,\eta,p_2,\dots,p_m}\Big)/l_\sigma^\beta$$
$$= \sum_{\eta \in \Delta^{n-m+1}} \prod_{t=1}^{n} L^\beta_{\sigma,[\eta,p_2,\dots,p_m|t]},$$

where $\beta > 0$, $p_2, \dots, p_m \in \Delta$, $n \ge m$. We consider also the random vector

$$\mathbf{S}_{\sigma;\beta,n} = \big(S_{\sigma;\beta,n}^{(p_2,\dots,p_m)} : 1 \le p_2, \dots, p_m \le N\big).$$

The following lemma generalizes Lemma 3.1.

Lemma 3.4. *For any $\beta > 0$ and $n \ge m$*

$$\mathbf{E}(\mathbf{S}_{\sigma;\beta,n}|\mathcal{F}_{k+n-1}) = \mathbf{S}_{\sigma;\beta,n-1}V^{(\beta)}.$$

Proof. We have

$$S_{\sigma;\beta,n}^{(p_2,\dots,p_m)} = \Big(\sum_{p_1 \in \Delta} \sum_{\eta \in \Delta^{n-m}} l^\beta_{\sigma,\eta,p_1,\dots,p_m}\Big)/l_\sigma^\beta$$
$$= \Big(\sum_{p_1 \in \Delta} \sum_{\eta \in \Delta^{n-m}} l^\beta_{\sigma,\eta,p_1,\dots,p_{m-1}} L^\beta_{\sigma,\eta,p_1,\dots,p_m}\Big)/l_\sigma^\beta.$$

Now, since for each $\sigma \in \Delta^k$ and $\eta \in \Delta^{n-m}$ the random variables l_σ and $l_{\sigma,\eta,p_1,\dots,p_{m-1}}$ are $\mathcal{F}_{k+n-1}$-measurable, $L_{\sigma,\eta,p_1,\dots,p_m}$ is independent of $\mathcal{F}_{k+n-1}$ and $\mathbf{E}\, L^\beta_{\sigma,\eta,p_1,\dots,p_m} = \mathbf{E}\, L^\beta_{p_1,\dots,p_m}$, we get

$$\mathbf{E}(S_{\sigma;\beta,n}^{(p_2,\dots,p_m)}|\mathcal{F}_{k+n-1}) = \sum_{p_1 \in \Delta} \Big(\frac{1}{l_\sigma^\beta} \sum_{\eta \in \Delta^{n-m}} l^\beta_{\sigma,\eta,p_1,\dots,p_{m-1}}\Big) v^{(\beta)}(p_1, \dots, p_m)$$
$$= \sum_{p_1 \in \Delta} S_{\sigma;\beta,n-1}^{(p_1,\dots,p_{m-1})} v^{(\beta)}(p_1, \dots, p_m),$$

which finishes the proof. □

Define

$$Z_{\sigma;\beta,n} = \Big(\sum_{p_2,\dots,p_m} r^{(\beta)}(p_2, \dots, p_m) S_{\sigma;\beta,n}^{(p_2,\dots,p_m)}\Big)/\rho^n(\beta).$$

The same way as before it can be shown that $(Z_{\sigma;\beta,n}, \mathcal{F}_{k+n})$, $n \ge m$, form a positive martingale. Therefore by the martingale convergence theorem for almost every ω there exists $\lim_n Z_{\sigma;\beta,n}$. When $\beta = \alpha$ we denote this limit by X_σ. These limit random variables X_σ play a significant role in constructing a special random measure in Section 6 that will be an essential tool in the prove that $\dim_H K(\omega) \ge \alpha$.

4. Properties of the random variables X and X_σ

Note that for any k the family $\{X_\sigma : \sigma \in \Delta^k\}$ consists of independent variables and independent of $\mathcal{F}_k$. There are the following relations between the limit random variables X and X_σ, $\sigma \in \Delta^*$.

Lemma 4.1.

(a) $X_\emptyset = X/l_\emptyset^\alpha$.
(b) *For any* k $l_\sigma^\alpha X_\sigma = \sum_{\eta\in\Delta^k} l_{\sigma,\eta}^\alpha X_{\sigma,\eta}$.
(c) $X = \sum_{\eta\in\Delta^k} l_\eta^\alpha X_\eta$.

Proof. (a) For any $n \geq m$, $p_2,\dots,p_m \in \Delta$ we have

$$S_{\emptyset;n}^{(p_2,\dots,p_m)} = \Big(\sum_{\eta\in\Delta^{n-m+1}} l^\alpha_{\eta,p_2,\dots,p_m}\Big)/l_\emptyset^\alpha = S_n^{(p_2,\dots,p_m)}/l_\emptyset^\alpha,$$

therefore, $Z_{\emptyset;n} = Z_n/l_\emptyset^\alpha$ for $n = m, m+1,\dots$ and by taking limit as $n\to\infty$ we get $X_\emptyset = X/l_\emptyset^\alpha$.

(b) Fix $p_2,\dots,p_m \in \Delta$. Denote here

$$\widetilde{\Delta}_{n-m+1} = \{\gamma \in \Delta^n : \gamma_{n-m+2} = p_2,\dots,\gamma_n = p_m\}.$$

Then

$$S_{\sigma;n}^{(p_2,\dots,p_m)} = \Big(\sum_{\gamma\in\widetilde{\Delta}_{n-m+1}} l^\alpha_{\sigma,\gamma}\Big)/l_\sigma^\alpha = \sum_{\gamma\in\widetilde{\Delta}_{n-m+1}} \prod_{t=1}^n L^\alpha_{\sigma,[\gamma|t]}. \tag{4.1}$$

Let $\eta\in\Delta^k$. Notice that

$$l_{\sigma,\eta} = l_\sigma \prod_{t=1}^k L_{\sigma,[\eta|t]}. \tag{4.2}$$

Also, if $\gamma\in\widetilde{\Delta}_{n-m+1}$ we have the following obvious identities for finite sequences in Δ^*:

$$\begin{aligned} (\sigma,[\eta|t]) &= (\sigma,[\eta,\gamma|t]), \quad \text{if} \quad 1\leq t\leq k,\\ (\sigma,\eta,[\gamma|t-k]) &= (\sigma,[\eta,\gamma|t]), \quad \text{if} \quad k+1\leq t\leq k+n. \end{aligned} \tag{4.3}$$

From (4.1), (4.2) and (4.3) it follows that

$$\begin{aligned} \sum_{\eta\in\Delta^k} l^\alpha_{\sigma,\eta} S^{(p_2,\dots,p_m)}_{\sigma,\eta;n} &= \sum_{\eta\in\Delta^k} l^\alpha_{\sigma,\eta}\Big(\sum_{\gamma\in\widetilde{\Delta}_{n-m+1}}\prod_{t=1}^n L^\alpha_{\sigma,\eta,[\gamma|t]}\Big)\\ &= l_\sigma^\alpha \sum_{\eta\in\Delta^k}\Big(\prod_{t=1}^k L^\alpha_{\sigma,[\eta,\gamma|t]} \sum_{\gamma\in\widetilde{\Delta}_{n-m+1}} \prod_{t=k+1}^{k+n} L^\alpha_{\sigma,[\eta,\gamma|t]}\Big)\\ &= l_\sigma^\alpha \sum_{\eta\in\widetilde{\Delta}_{n+k-m+1}} \prod_{t=1}^{n+k} L^\alpha_{\sigma,[\eta|t]} = l_\sigma^\alpha S^{(p_2,\dots,p_m)}_{\sigma;n+k}. \end{aligned}$$

This implies that for any $\sigma\in\Delta^*, n\geq m$ and k

$$l_\sigma^\alpha Z_{\sigma;\alpha,n+k} = \sum_{\eta\in\Delta^k} l^\alpha_{\sigma,\eta} Z_{\sigma,\eta;\alpha,n}. \tag{4.4}$$

Now, by taking limit of both sides of (4.4) as $n \to \infty$ we obtain (b).

(c) This follows from (a) and (b). □

Obviously, for almost every ω $X_\sigma(\omega) \geq 0$ for each $\sigma \in \Delta^*$. We are going to prove that each of X_σ is positive with probability 1. The key to this is the following theorem about the moments of X_σ which is rather close to the one considered in [6].

Theorem 4.1. *For each $\sigma \in \Delta^*$ the random sequence $\{Z_{\sigma;n}, n \geq m\}$ is L^k-bounded for any $k \in \mathbb{N}$.*

Proof. For simplicity, we will prove the theorem for the special case $\sigma = \emptyset$; the general case of $\sigma \neq \emptyset$ can be handled analogously. Denote here

$$\widetilde{Z}_{\beta,n} = Z_{\beta,n}\rho^n(\beta) \quad \text{for } \beta > 0 \text{ and } n = m, m+1, \ldots$$

Notice that

$$\widetilde{Z}_{\beta,n} = \sum_{\sigma \in \Delta^{n-m+1}} \sum_{p_2,\ldots,p_m} l^\beta_{\sigma,p_2,\ldots,p_m} r^{(\beta)}(p_2,\ldots,p_m).$$

Therefore $\widetilde{Z}_{\beta,n} < N^n \max_{p_2,\ldots,p_m} r^{(\beta)}(p_2,\ldots,p_m)$ and thus $\mathbf{E}\,\widetilde{Z}^t_{\beta,n} < \infty$ for any $\beta > 0$, $t > 0$ and n. Using induction on t we shall show that for any $\beta \geq \alpha$, $t \in \mathbb{N}$

$$\mathbf{E}\,\widetilde{Z}^t_{\beta,n} \leq c\gamma^n, n = m, m+1, \ldots \tag{4.5}$$

for some constants $c = c(\beta,t) > 0$ and $\gamma = \gamma(\beta,t) \in (0,1]$ such that $\gamma(\beta,t) < 1$ if $\beta > \alpha$. When $t = 1$ we have

$$\mathbf{E}\,\widetilde{Z}_{\beta,n} = \rho^n(\beta)\,\mathbf{E}\,Z_{\beta,n}.$$

Since $\rho(\beta) \in (0,1)$ for $\beta > \alpha$ and $(Z_{\beta,n}, \mathcal{F}_n)$ forms a martingale, (4.5) holds in this case.

Let $k > 1$. Assume that (4.5) holds for any $t \in [1, k-1]$. We have

$$\widetilde{Z}_{\beta,n+1} = \sum_{\eta=(\sigma,p_1,\ldots,p_{m-1})} l^\beta_\eta \Big(\sum_{p_m} L^\beta_{\eta,p_m} r^{(\beta)}(p_2,\ldots,p_m)\Big).$$

Thus

$$\widetilde{Z}^k_{\beta,n+1} = \sum_{h=1}^{k} \sum_{\substack{j_1 \geq \cdots \geq j_h \geq 1 \\ j_1+\cdots+j_h=k}} \sum_{\substack{\eta^{(1)},\ldots,\eta^{(h)} \in \Delta^n \\ \eta^{(s)} \neq \eta^{(t)}, s \neq t}} \prod_{i=1}^{h} l^{j_i\beta}_{\eta^{(i)}} \Big(\sum_{p_m} L^\beta_{\eta^{(i)},p_m} r^{(i)}(p^{(i)}_2,\ldots,p^{(i)}_{m-1},p_m)\Big)^{j_i},$$

where $\eta^{(i)} = (\sigma^{(i)}, p^{(1)}_1, \ldots, p^{(i)}_{m-1}) \in \Delta^n$.

Since for every $\eta \in \Delta^n$ the random variable l_η is $\mathcal{F}_n$-measurable and the family

$$\Big\{\sum_{p_m} L^\beta_{\eta,p_m} r^{(\beta)}(p_2,\ldots,p_m) : \eta \in \Delta^n\Big\}$$

consists of independent random variables and does not depend on $\mathcal{F}_n$, we find

$$(4.6)\quad \mathbf{E}(\widetilde{Z}^k_{\beta,n+1}|\mathcal{F}_n) = \sum_{h=1}^{k} \sum_{\substack{j_1\geq\cdots\geq j_h\geq 1\\ j_1+\cdots+j_h=k}} \sum_{\substack{\eta^{(1)},\ldots,\eta^{(h)}\in\Delta^n\\ \eta^{(s)}\neq\eta^{(t)},s\neq t}} \prod_{i=1}^{h} l^{j_i\beta}_{\eta^{(i)}}\, \mathbf{E}\Big[\Big(\sum_{p_m} L^{\beta}_{p_1^{(i)},\ldots,p_{m-1}^{(i)},p_m} r^{(\beta)}(p_2^{(i)},\ldots,p_{m-1}^{(i)},p_m)\Big)^{j_i}\Big].$$

For the term in (4.6) with $h = k$ the only choice is $j_1 = \cdots = j_k = 1$. Recall that for any $p_1,\ldots,p_{m-1}\in\Delta$

$$\sum_{p_m}\mathbf{E}\, L^{\beta}_{p_1,\ldots,p_{m-1},p_m} r^{(\beta)}(p_2,\ldots,p_m) = \rho(\beta) r^{(\beta)}(p_1,\ldots,p_{m-1}).$$

Hence this term is

$$(4.7)\qquad \rho^k(\beta) \sum_{\substack{\eta^{(1)},\ldots,\eta^{(k)}\in\Delta^n\\ \eta^{(s)}\neq\eta^{(t)},s\neq t}} \prod_{i=1}^{k} l^{\beta}_{\eta^{(i)}} r^{(\beta)}(p_1^{(i)},\ldots,p_{m-1}^{(i)}) \leq \rho^k(\beta)\widetilde{Z}^k_{\beta,n}.$$

Now, if $h < k$ in (4.6), then $j_1 \geq 2$. Since

$$\max_{1\leq j\leq k}\ \max_{p_1,\ldots,p_{m-1}\in\Delta} \mathbf{E}\Big[\Big(\sum_{p_m} L^{\beta}_{p_1,\ldots,p_m} r^{(\beta)}(p_2,\ldots,p_m)\Big)^{j}\Big] < \infty$$

and

$$\min_{1\leq j\leq k}\ \min_{p_1,\ldots,p_{m-1}\in\Delta} r^{(j\beta)}(p_1,\ldots,p_{m-1}) > 0,$$

there is a constant $c_1 = c_1(\beta,k)$ such that

$$(4.8)\qquad \sum_{\substack{\eta^{(1)},\ldots,\eta^{(h)}\in\Delta^n\\ \eta^{(s)}\neq\eta^{(t)},s\neq t}} \prod_{i=1}^{h} l^{j_i\beta}_{\eta^{(i)}}\, \mathbf{E}\Big[\Big(\sum_{p_m} L^{\beta}_{p_1^{(i)},\ldots,p_{m-1}^{(i)},p_m} r^{(\beta)}(p_2^{(i)},\ldots,p_{m-1}^{(i)},p_m)\Big)^{j_i}\Big]$$

$$\leq c_1 \sum_{\substack{\eta^{(1)},\ldots,\eta^{(h)}\in\Delta^n\\ \eta^{(s)}\neq\eta^{(t)},s\neq t}} \prod_{i=1}^{h} l^{j_i\beta}_{\eta^{(i)}} r^{(j_i\beta)}(p_1^{(i)},\ldots,p_{m-1}^{(i)}) \leq c_1 \prod_{i=1}^{h}\widetilde{Z}_{j_i\beta,n}.$$

Thus, from (4.6), (4.7) and (4.8) it follows that

$$(4.9)\qquad \mathbf{E}(\widetilde{Z}^k_{\beta,n+1}|\mathcal{F}_n) \leq \rho^k(\beta)\widetilde{Z}^k_{\beta,n} + c_1 \sum_{h=1}^{k-1} \sum_{\substack{j_1\geq\cdots\geq j_h\geq 1\\ j_1+\cdots+j_h=k}} \prod_{i=1}^{h}\widetilde{Z}_{j_i\beta,n}.$$

By the Hölder inequality

$$\mathbf{E}\Big(\prod_{i=1}^{h}\widetilde{Z}_{j_i\beta,n}\Big) \leq \prod_{i=1}^{h}\|\widetilde{Z}_{j_i\beta,n}\|_h.$$

Hence by taking expectations in (4.9) we can get

$$(4.10)\qquad \mathbf{E}\,\widetilde{Z}^k_{\beta,n+1} \leq \rho^k(\beta)\,\mathbf{E}\,\widetilde{Z}^k_{\beta,n} + c_1 M_n,$$

where

$$M_n = \sum_{h=1}^{k-1} \sum_{\substack{j_1 \geq \cdots \geq j_h \geq 1 \\ j_1+\cdots+j_h=k}} \prod_{i=1}^{h} \|\widetilde{Z}_{j_i\beta,n}\|_h.$$

Iterating (4.10) backward we find

$$\mathbf{E}\,\widetilde{Z}^k_{\beta,n+1} \leq (\rho(\beta))^{(n-m+1)k}\,\mathbf{E}\,\widetilde{Z}^k_{\beta,m} + c_1 \sum_{t=m}^{n} (\rho(\beta))^{(n-t)k} M_t. \tag{4.11}$$

Now, $h \leq k-1$, and by our induction hypothesis if $j_i \geq 1$

$$\sup_n \|\widetilde{Z}_{j_i\beta,n}\|_h < \infty.$$

Since $j_1 \geq 2$ and $\beta \geq \alpha$ then $j_1\beta > \alpha$. Again by the induction hypothesis we have for any n

$$\mathbf{E}\,\widetilde{Z}^h_{j_1\beta,n} = \|\widetilde{Z}_{j_i\beta,n}\|^h_h \leq c(j_1\beta,h)\gamma(j_1\beta,h)^n,$$

where $c(j_1\beta,h) > 0$ and $\gamma(j_1\beta,h) \in (0,1)$. Thus, from (4.11) it is clear that (4.5) holds for $t=k$, which completes the proof. □

Corollary 4.1. *Each of the random variables X_σ, $\sigma \in \Delta^*$, has finite moments of any order $k \in \mathbb{N}$.*

We also have the following important property of the random variables X_σ.

Corollary 4.2.

$$P(X_\sigma > 0) = 1 \quad \textit{for each } \sigma \in \Delta^*. \tag{4.12}$$

Proof. First, since for each $\sigma \in \Delta^*$ the expectation $\mathbf{E}\,X_\sigma = \mathbf{E}\,Z_{\sigma;\alpha,m} > 0$ we find that $P(X_\sigma > 0) > 0$ or $P(X_\sigma = 0) < 1$. Consider here the following σ-algebras

$$\mathcal{F}^n_n = \sigma\{L_\eta : \eta \in \Delta^n\}, \quad \mathcal{F}^n = \sigma\{L_\eta : \eta \in \Delta^* \quad \text{and} \quad |\eta| \geq n\}, \quad n \in \mathbb{N}$$

Since $\mathcal{F}^n_n$, $n \in \mathbb{N}$ are independent and, by construction of random variables X_σ the event $\{X_\sigma = 0\}$ belongs to the "tail" σ-algebra $\mathcal{F}^n$ for any $\sigma \in \Delta^n$ and for any $n > 0$, the relation (4.12) follows now by the Kolmogorov's Zero-or-One Law (see, for example, [10]). □

Now, we shall prove that in fact there are only finitely many different distributions of X_σ, $\sigma \in \Delta^*$. More precisely the following holds.

Theorem 4.2. *Let $|\sigma| = k \geq m-1$ and $\sigma_{k-m+2} = p_1, \ldots, \sigma_k = p_{m-1}$ for some $p_1, \ldots, p_{m-1} \in \Delta$. Then X_σ and $X_{p_1,\ldots,p_{m-1}}$ are identically distributed.*

In order to prove Theorem 4.2 we establish the following lemma. Denote here

$$\Delta^{(n)} = \bigcup_{t=1}^{n} \Delta^t, \quad \Delta^{(n)}_0 = \bigcup_{t=0}^{n} \Delta^t.$$

Lemma 4.2. *For any n the random vectors*

$$(L_{\sigma,\eta}; \eta \in \Delta^{(n)}) \quad \textit{and} \quad (L_{p_1,\ldots,p_{m-1},\eta}; \eta \in \Delta^{(n)})$$

have the same distribution, provided that $|\sigma| = k \geq m-1$ and $\sigma_{k-m+2} = p_1, \ldots, \sigma_k = p_{m-1}$.

Proof. It's enough to prove that for any family of Borel sets $B_\eta,\ \eta \in \Delta^{(n)}$

$$P\big(L_{\sigma,\eta} \in B_\eta, \eta \in \Delta^{(n)}\big) = P\big(L_{p_1,\dots,p_{m-1},\eta} \in B_\eta, \eta \in \Delta^{(n)}\big).$$

First of all we have

$$P(L_{\sigma,\eta} \in B_\eta : \eta \in \Delta^{(n)}) = P(L_{\sigma,\eta,p} \in B_{\eta,p} : \eta \in \Delta_0^{(n-1)}, p \in \Delta). \tag{4.13}$$

Recall that $(L_{\sigma,\eta,1}, L_{\sigma,\eta,2},\dots),\ \eta \in \Delta_0^{(n-1)}$ are independent and distributed identically to

$$(L_{p_1,\dots,p_{m-1},\eta,1}, L_{p_1,\dots,p_{m-1},\eta,2},\dots).$$

Therefore each part in (4.13) is equal to

$$\prod_{\eta\in\Delta_0^{(n-1)}} P(L_{\sigma,\eta,p} \in B_{\eta,p} : p \in \Delta) = \prod_{\eta\in\Delta_0^{(n-1)}} P(L_{p_1,\dots,p_{m-1},\eta,p} \in B_{\eta,p} : p \in \Delta)$$

$$= P(L_{p_1,\dots,p_{m-1},\eta} \in B_\eta : \eta \in \Delta^{(n)}).$$

□

Since by the definition

$$S_{\sigma;n}^{(q_1,\dots,q_{m-1})} = \sum_{\gamma\in\Delta^{n-m+1}} \prod_{t=1}^{n} L_{\sigma,[\gamma,q_1,\dots,q_{m-1}|t]},\ n \geq m-1,$$

we immediately get that $S_{\sigma;n}^{(q_1,\dots,q_{m-1})}$ is distributed identically to $S_{p_1,\dots,p_{m-1};n}^{(q_1,\dots,q_{m-1})}$ provided $|\sigma| = k$ and $\sigma_{k-m+2} = p_1,\dots,\sigma_k = p_{m-1}$. This implies that $Z_{\sigma;n}$ has the same distribution as $Z_{p_1,\dots,p_{m-1};n}$ and since for each $\sigma \in \Delta^*$

$$X_\sigma = \lim_n Z_{\sigma;n} \quad \text{for almost every } \omega\ ,$$

this establishes Theorem 4.2.

5. Dimension of the sequence space

In this section we prove some auxiliary statements related to the metric space $(\Delta^{\mathbb{N}}, \lambda^*)$, where λ^* is a metric that meets some restrictions specified below. We define *cylinders* by

$$C_n(x) = \{y \in \Delta^{\mathbb{N}} : [y|n] = [x|n]\}, \quad \text{for } x \in \Delta^{\mathbb{N}} \text{ and } n \in \mathbb{N}\ .$$

Let us consider a family of positive numbers $\{l_y : y \in \Delta^*\}$ with properties (2.3) and (2.6). We assume that the metric λ^* satisfies the following condition:

$$\operatorname{diam}(C_n(x)) \leq l_{[x|n]}.$$

Then the family $I_{[x|n]} = C_n(x), x \in \Delta^{\mathbb{N}}, n \in \mathbb{N}$ is a (non-random) construction. It is clear that the generated fractal $K = \Delta^{\mathbb{N}}$. The construction $C_n(x), x \in \Delta^{\mathbb{N}}, n \in \mathbb{N}$ meets condition (2.5) and it is monotone.

Note that, since the construction is specified, the condition (vi) of Section 2 presents some restriction on the metric λ^*.

Let us denote by $\mathcal{A}$ the σ-algebra generated by the cylinders. It is easy to check (see, for example, Proposition 1.1. in [11]) that any λ^*-Borel set is contained in $\mathcal{A}$; moreover, the λ^*-topology in $\Delta^{\mathbb{N}}$ coincides with the product topology; in particular, $\Delta^{\mathbb{N}}$ is λ^*-compact.

Let μ be a finite measure on $\mathcal{A}$. *The ball-wise local dimension of* μ *at a point* $x \in \Delta^{\mathbb{N}}$ is defined as follows:

$$d_{\mathcal{B},\mu}(x) \stackrel{\text{def}}{=} \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

We consider also the *the cylinder-wise local dimension of* μ *at* $x \in \Delta^{\mathbb{N}}$:

$$d_{\mathcal{C},\mu}(x) \stackrel{\text{def}}{=} \liminf_{n \to \infty} \frac{\log \mu(C_n(x))}{\log[l_{[x|n]}]}. \tag{5.1}$$

The following theorem states an important relation between these notions of local dimension.

Theorem 5.1. *Under condition* (vi) *of Section* 2*,*

$$d_{\mathcal{B},\mu}(x) \geq d_{\mathcal{C},\mu}(x), x \in \Delta^{\mathbb{N}}.$$

The proof of this statement is contained in the proof of Theorem 1.4 in [11].

Theorem 5.2. *Assume* $d_{\mathcal{B},\mu}(x) \geq d$ *on a set* F *with* $\mu(F) > 0$ *where* d *is a positive constant. Then* $\dim_H F \geq d$*.*

Proof. [1] Fix $0 < \epsilon < d$ and $s > 0$. Let us consider the set $F_s^\epsilon \subset \Delta^{\mathbb{N}}$, where $\frac{\log[\mu(B(x,r))]}{\log r} \geq d - \epsilon$ for any $r \leq s$. If $\delta < s$ then for any δ-cover of F_s^ϵ by balls $\{B(x_i, r_i)\}$ with $x_i \in F_s^\epsilon$ we have $\sum_i r_i^{d-\epsilon} \geq \sum_i \mu(B(x_i, r_i)) \geq \mu(F_s^\epsilon) > 0$ for sufficiently small s. This implies that $\dim_H F_s^\epsilon \geq d - \epsilon$ for such s. Obviously, $F = \uparrow \lim_{s \to 0} F_s^\epsilon$, and therefore $\dim_H \Delta^{\mathbb{N}} \geq \dim_H F = \lim_{s \to 0} \dim_H(F_s^\epsilon) \geq d - \epsilon$. Since ϵ was chosen arbitrarily the theorem is proved. ☐

Corollary 5.1. *Let condition* (vi) *be fulfilled. Assume* $d_{\mathcal{C},\mu}(x) \geq d$ *on a set* F *with* $\mu(F) > 0$ *where* d *is a positive constant. Then* $\dim_H \Delta^{\mathbb{N}} \geq d$*.*

6. Main results

In this section we prove the main theorem that under restrictions on the random construction $\mathbf{I}$ introduced earlier we have $\dim_H K = \alpha$ almost surely. We are going to do this using cylinder-wise local dimension of a special measure μ on the space $\Delta^{\mathbb{N}}$ and its relation to the "global" Hausdorff dimension as established in the previous section.

Let $\mathcal{A}$ denote the σ-algebra in $\Delta^{\mathbb{N}}$ generated by the cylinders. If $\sigma = [\pi|n]$, where $\sigma \in \Delta^n, \pi \in \Delta^{\mathbb{N}}$, we denote $C_n(\sigma) = C_n(\pi)$.

Lemma 6.1. *For almost every* ω *the relations*

$$\mu_\omega(C_n(\sigma)) = l_\sigma^\alpha(\omega) X_\sigma(\omega)/X(\omega), \quad \pi \in \Delta^{\mathbb{N}},\ n \in \mathbb{N},$$

define a probability measure on $\mathcal{A}$*.*

Proof. Lemma 4.1 shows that for almost every ω the measure μ_ω is a consistent on the algebra of the cylindrical sets. Moreover for almost every ω

$$\mu_\omega(\Delta^{\mathbb{N}}) = \sum_{\sigma \in \Delta^n} \mu_\omega(C_n(\sigma)) = \frac{1}{X(\omega)} \sum_{\sigma \in \Delta^n} l_\sigma^\alpha X_\sigma(\omega) = 1.$$

☐

[1] This is an adaptation of the proof of Theorem 1.6 in [11].

The proof of the following lemma is a modification of the proof of Theorem 3.5 in [6].

Lemma 6.2. *With probability* 1

$$d_{\mathcal{C},\mu}(\pi) \geq \alpha \quad \textit{for every } \pi \in \Delta^{\mathbb{N}}.$$

Proof. Recall that by Theorem 4.1 we have $\mathbf{E}(X_\sigma^t) < \infty$ for any $t > 0$ and for each $\sigma \in \Delta^*$. Fix $k > 0$ and $\beta < \alpha$. Then, since l_σ and X_σ are independent,

$$P(l_\sigma^\alpha X_\sigma > k l_\sigma^\beta) \leq \frac{\mathbf{E}([l_\sigma^{\alpha-\beta}]^t X_\sigma^t)}{k^t} = \frac{\mathbf{E}\, l_\sigma^{(\alpha-\beta)t}\, \mathbf{E}\, X_\sigma^t}{k^t} \leq \frac{c}{k^t}\, \mathbf{E}\, l_\sigma^{(\alpha-\beta)t},$$

where $c = \max_{q_1,\ldots,q_{m-1}}(\mathbf{E}\, X^t_{q_1,\ldots,q_{m-1}}) < \infty$ by Theorem 4.2. Recall the notation

$$\overline{Z}_{\gamma,n} = \sum_{\sigma \in \Delta^n} l_\sigma^\gamma,$$

and that

$$\overline{Z}_{\gamma,n} \leq d\rho^n(\gamma) Z_{\gamma,n} \quad \text{for some constant } d > 0.$$

Thus

$$\sum_{\sigma \in \Delta^n} P(l_\sigma^\alpha X_\sigma > k l_\sigma^\beta) \leq \frac{c}{k^t}\, \mathbf{E}\, \overline{Z}_{(\alpha-\beta)t} \leq \frac{cd}{k^t} \rho^n((\alpha-\beta)t)\, \mathbf{E}\, Z_{(\alpha-\beta)t,m}. \tag{6.1}$$

Choose $t_0 > 0$ such that $\rho((\alpha-\beta)t_0) < 1$. Then from (6.1) it follows

$$\sum_{n=0}^{\infty} P(\exists \sigma \in \Delta^n : l_\sigma^\alpha X_\sigma > k l_\sigma^\beta) < \infty.$$

By the Borel-Cantelli lemma we find that

$$P(\exists n_0 : \forall n \geq n_0\ \forall \pi \in \Delta^{\mathbb{N}},\ l_{[\pi|n]}^\alpha X_{[\pi|n]} \leq k l_{[\pi|n]}^\beta) = 1.$$

Fix ω such that μ_ω is well defined and there is $n_0 = n_0(\omega) \in \mathbb{N}$ such that for all $n \geq n_0$ we have $l_{[\pi|n]}^\alpha(\omega) X_{[\pi|n]}(\omega) \leq k l_{[\pi|n]}^\beta(\omega)$. For each such ω

$$\frac{\log \mu_\omega(C_n(\pi))}{\log l_{[\pi|n]}(\omega)} \geq \beta + \frac{\log k}{\log l_{[\pi|n]}(\omega)}. \tag{6.2}$$

Letting $n \to \infty$ in (6.2) we find

$$\liminf_{n\to\infty} \frac{\log \mu_\omega(C_n(\pi))}{\log l_{[\pi|n]}(\omega)} \geq \beta,$$

which holds for any $\beta < \alpha$ and hence for $\beta = \alpha$. □

The random fractal $K(\omega)$ corresponds to the whole space $\Delta^{\mathbb{N}}$ endowed with some random metric. This correspondence is established as follows. Let $\pi \in \Delta^{\mathbb{N}}$. Since the space $(\mathbb{M}, \lambda)$ is complete, each of the sets I_σ is closed and $\operatorname{diam}(I_\sigma) \to 0$ as $|\sigma| \to \infty$ for almost every ω , the intersection $\cap_{n=1}^{\infty} I_{[\pi|n]}$ is nonempty and consists of a single point which we denote x_π. It is easy to see that

$$K(\omega) = \{x_\pi(\omega) : \pi \in \Delta^{\mathbb{N}}\}.$$

For each such ω we consider the "coding" map $\phi : K \mapsto \Delta^{\mathbb{N}}$ defined by $\phi(x_\pi) = \pi$. The following lemma is proved in [11].

We assume that in the following three statements the condition (2.5) is fulfilled.

Lemma 6.3. *The map ϕ is one-to-one and $\phi(I_{[\pi|n]} \cap K) = C_n(\pi)$ for each $\pi \in \Delta^{\mathbb{N}}$ and $n \in \mathbb{N}$.*

Now fix ω and let λ_K be the restriction of the metric λ to K. Then $\lambda^*_\omega = \lambda_K \circ \phi^{-1}$ is a metric in $\Delta^{\mathbb{N}}$ and for almost every ω we have that ϕ is an isometry between metric spaces $(K(\omega), \lambda_K)$ and $(\Delta^{\mathbb{N}}, \lambda^*_\omega)$. It is obvious that for almost every ω the metric λ^*_ω satisfies the conditions stated in Section 5. Therefore the following statement holds.

Proposition 6.1. *The set $K(\omega)$ is compact for almost every ω.*

The following lemma allows us to use the results of the previous section to compute the dimension of the fractal $K(\omega)$.

Lemma 6.4. *For almost every ω*

$$\dim_H K = \dim_H \Delta^{\mathbb{N}}.$$

Theorem 6.1 (Main)**.** *Assume that conditions* (iv)*,* (v) *and* (vi) *of Section* 2 *are met. Then for almost every ω*

$$\dim_H K(\omega) = \alpha.$$

Proof. We fix $\omega \in \Omega$ for which conditions (iv), (v) and (vi) of Section 2 are fulfilled, the measure μ_ω exists and $d_{\mathcal{C},\mu}(\pi, \omega) \geq \alpha$ for all $\pi \in \Delta^{\mathbb{N}}$. Then by Corollary 5.1 we immediately obtain that for almost every ω

$$\dim_H K(\omega) = \dim_H \Delta^{\mathbb{N}} \geq \alpha,$$

which along with the upper estimate, Theorem 3.1, finishes the proof. □

Remark. As it is mentioned at the beginning, the condition (2.7), finiteness of Moran index of the construction, is not very visual. We can replace (2.7) and (2.5) with a stronger but more tractable condition (2.10).

In the following two corollaries we assume that condition (v) of Section 2 is fulfilled.

Corollary 6.1. [2] *Let $\mathbf{J}$ be a construction. Assume that for almost every ω there is a conjunctive construction $\mathbf{D} = \{D_\sigma : \sigma \in \Delta^*\}$ consisting of closed balls $D_\sigma = [B(x_\sigma, bl_\sigma)]$, $b = b(\omega) > 0$ so that the balls D_σ and D_η are disjoint if neither $\sigma < \eta$ nor $\eta < \sigma$.Then for almost every ω*

$$\dim_H K(\omega) = \alpha.$$

Proof. For each ω we consider another construction $\widetilde{\mathbf{D}} = \{B(x_\sigma, dl_\sigma) : \sigma \in \Delta^*\}$, where $d = d(\omega) < b(\omega)$. It is clear that for almost every ω the construction $\widetilde{\mathbf{D}}(\omega)$ defines the same fractal $K(\omega)$ as the constructions $\mathbf{D}(\omega)$ and $\mathbf{J}(\omega)$. Now as it is noticed in Example (2.2), this construction $\mathbf{D}(\omega)$ satisfies the conditions (2.10) and (2.4). It remains to apply Theorem 6.1. □

Let us note also the following simple particular case.

[2]See also [11].

Corollary 6.2. *Let* **J** *be a random construction. Assume that for almost every* ω *there is* $b = b(\omega) > 0$ *such that* $[B(x_\sigma, bl_\sigma)] \subset I_\sigma$ *where the balls* $[B(x_\sigma, bl_\sigma)]$ *and* $[B(x_\eta, bl_\eta)]$ *are disjoint if neither* $\sigma < \eta$ *nor* $\eta < \sigma$.[3] *Then for almost every* ω

$$\dim_H K(\omega) = \alpha.$$

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[3] A similar condition was considered in [9]