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# Weighted $L_2$ Cohomology of Asymptotically Hyperbolic Manifolds

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Abstract. The main results are summarized by Figure 1. They demonstrate the resiliency of the isomorphism constructed in [Nai99] between weighted cohomology and a variant of weighted $L_2$ cohomology. Our attention is restricted from generic locally symmetric spaces to spaces whose ends are hyperbolic, diffeomorphic to $(0,\infty) \times (S^1)^{n-1}$, and carry exponentially warped product metrics. For weighting functions which are exponential in the Busemann coordinates of these ends, the standard $w$ weighted $L_2$ cohomology will be utilized in lieu of the variant defined in [Fra98]. The resulting standard $w$ weighted $L_2$ cohomology groups may be infinite dimensional vector spaces, but the precise weighting functions at which this undesirable behavior occurs are characterized. For the remaining exponential weights, the $w$ weighted $L_2$ cohomology is again an analogue of weighted cohomology. An immediate consequence of finite dimensionality of the standard $w$ weighted $L_2$ groups is a $w$ weighted Hodge theory summarized by a strong $w$ weighted Kodaira decomposition. This is outlined in the introduction.

After the asymptotically hyperbolic case is complete, the literature on weighted Hardy inequalities on the half line is used to derive certain extensions to some non-hyperbolic end metrics and non-exponential weighting functions. The two most immediate applications are as follows. First, say a function on the half line $k(t)$ satisfies $k'k^{-1} \leq -c$ for $c > 0$. Then one may replace the exponential in the metric of $(0,\infty) \times (S^1)^{n-1}$ by $k(t)$ and weight by powers of $k(t)$ rather than $e^{-t}$, and Figure 1 holds. Second, the analysis allows one to consider weighting functions which on each end are $w(t) = \mathrm{e}^{\alpha t^2}$ for $\alpha \in \mathbb{R}$. These weighting functions compute either de Rham cohomology or compactly supported de Rham cohomology when $\alpha < 0$ or $\alpha > 0$, respectively.

Contents



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## 1. Introduction

Given the abstract, this introduction should begin by specializing the isomorphism theorem of [Nai99] from generic locally symmetric spaces to arithmetic quotients of hyperbolic spaces. First, suppose $\Gamma \subseteq \mathrm{SO}_{n,1}(\mathbb{Q})$ is neat arithmetic, and recall the real hyperbolic $n$ space $\mathbb{R}\mathcal{H}^n = \mathrm{SO}_{n,1}(\mathbb{R})/\mathrm{S}(\mathrm{O}_n(\mathbb{R}) \times \mathrm{O}_1(\mathbb{R}))$. The neatness condition requires the quotient $\Gamma\backslash\mathbb{R}\mathcal{H}^n$ to be a manifold (without singularities) whose ends are diffeomorphic to $(0,\infty)\times(S^1)^{n-1}$ carrying the metric

$$ds^2 = dt^2 + \sum_{\ell=1}^{n-1} \mathrm{e}^{-2t} d\theta_\ell^2.$$

Suppose next $\lambda$ is a real number, identified without comment with $\lambda\alpha$ for $\{\alpha\}$ a basis of the restricted root space decomposition of $\mathfrak{so}_{n,1}(\mathbb{R})$. Each such $\lambda$ describes a weighting function $w_\lambda$ which goes as $\mathrm{e}^{2\lambda t}$ on each end. For such weighting functions on a locally symmetric space, Franke defined the following variant of $w_\lambda$ weighted $L_2$ cohomology in [Fra98].

$$A^q_{\lambda+\log}(\Gamma\backslash\mathbb{R}\mathcal{H}^n) := \Bigg\{\omega \ ; \ \int_{\Gamma\backslash\mathbb{R}\mathcal{H}^n} |\omega|^2 w_\lambda \big((2\lambda)^{-1}\log w_\lambda\big)^j \ \mathrm{dvol} < \infty \quad \text{and similarly for } d\omega \ \forall j \in 0,1,2,\dots \Bigg\}.$$

The second condition makes this a cochain complex, with $q^{\text{th}}$ cohomology denoted $H^q_{\lambda+\log}(\Gamma\backslash\mathbb{R}\mathcal{H}^n)$. On the other hand, for $\lambda$ a half integer, i.e., $\lambda \in \{\dots,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\dots\}$, there is a weighted cohomology theory $W^\lambda H^q(\overline{\Gamma\backslash\mathbb{R}\mathcal{H}^n})$ on the reductive Borel-Serre compactification of the argument, defined in [GHM94]. Nair's theorem now states

$$H^q_{\lambda+\log}(\Gamma\backslash\mathbb{R}\mathcal{H}^n) \cong W^{\lfloor(1-n)/2+\lambda\rfloor+(1/2)} H^q(\overline{\Gamma\backslash\mathbb{R}\mathcal{H}^n}), \qquad 0 \le q \le n.$$

In fact, for the real hyperbolic spaces it would be possible to state this theorem in terms of intersection cohomology, since in this case truncation by weight is essentially truncation by degree. However, this is not true for even slightly more complicated spaces, such as arithmetic quotients of complex hyperbolic space. Anticipating generalizations, weighted cohomology and minor variants will be used.

The standard $w$ weighted $L_2$ cohomology is easier to define, but it is not so well behaved. Switching to the context of this work, suppose $(M^n, g)$ is a complete, finite volume Riemannian manifold with finitely many ends. Suppose further there is an open core whose closure is a compact manifold with boundary, and that this core intersects each end nicely. Finally, for this paper suppose each of the ends is diffeomorphic to $(0,\infty)\times(S^1)^{n-1}$, and that the metric is the metric given above. If there are $m$ such ends, let $(\Xi(1),\dots,\Xi(m))$ be a tuple of real numbers indexing a continuous weighting function $w_\Xi : M \to (0,\infty)$ which restricts to $e^{(2\Xi(i)+(n-1))t}$ on each end. Then define

$$A^q_{(2),\Xi}(M) := \left\{\omega \ ; \int_M |\omega|^2 w_\Xi \ \mathrm{dvol} < \infty \text{ and } \int_M |d\omega|^2 w_\Xi \ \mathrm{dvol} < \infty \right\}.$$

There is no harm in interpreting $\omega$ as either a smooth form or an $L_2$ current here, since mollification arguments of [Che80] show both complexes compute the same cohomology on $M$. Nonetheless, the standard convention will be to suppose $\omega$ is smooth. The $q^{\text{th}}$ cohomology of the resulting complex will be denoted $H^q_{(2),\Xi}(M)$ in this work. If $\Xi \equiv -\frac{1}{2}(n-1)$, $w_\Xi \equiv 1$, and $\Xi$ will be dropped from all notations in this case.

$H^q_{(2),\Xi}(M)$ is often an infinite dimensional vector space, for example when $n = 3$, $q = 1$, and $\Xi \equiv -\frac{1}{2}(n-1)$ as above. For $M^3 = \Gamma\backslash\mathbb{R}\mathcal{H}^3$, this follows from the main result of [BC83]. However, for most $\Xi$, all of the $w_\Xi$ weighted cohomology groups will be finite dimensional. In these nice cases, $H^\bullet_{(2),\Xi}(M)$ computes a topological cohomology theory similar to weighted cohomology, and it also admits a good $w_\Xi$ weighted Hodge theory.

The last point will be quickly recalled from the literature. First, let $L^q_{(2),\Xi}(M)$ be the complex of currents defined similarly to $A^q_{(2),\Xi}(M)$ above. Then the standard exterior derivative $d$ has a closure $\bar{d}_\Xi$ defined on a subspace of this complex, and it also has an adjoint $\bar{d}^*_{w_\Xi}$. The somewhat clumsy notation is meant to remind the reader that the adjoint depends on the choice of $w_\Xi$ pointwise, even though $H^q_{(2),\Xi}(M)$ only depends on $w_\Xi$ up to bounded changes. Now the weighted Hodge Laplacian may be defined as

$$\Delta_{w_\Xi} := \bar{d}_\Xi \bar{d}^*_{w_\Xi} + \bar{d}^*_{w_\Xi} \bar{d}_\Xi.$$

In general, the kernel of that operator will not compute $w_\Xi$ weighted $L_2$ cohomology, and the best that can be said is that there is a weak Kodaira decomposition

$$L^\bullet_{(2),\Xi}(M) = \ker \Delta_{w_\Xi} \oplus \overline{\mathrm{im}\bar{d}_\Xi} \oplus \overline{\mathrm{im}\bar{d}^*_{w_\Xi}}.$$

However, results of [BL92], building on [Che80], show that whenever all of the $H^q_{(2),\Xi}(M)$ are finite dimensional there is a strong Kodaira decomposition

$$L^\bullet_{(2),\Xi}(M) = \ker \Delta_{w_\Xi} \oplus \ \mathrm{im}\bar{d}_\Xi \oplus \ \mathrm{im}\bar{d}^*_{w_\Xi}.$$

Then $H^\bullet_{(2),\Xi}(M)$ is canonically a graded Hilbert space via identification with $\ker \Delta_{w_\Xi}$, and so even on complete manifolds a nice Hodge theory results for any topological cohomology theory computed by $H^\bullet_{(2),\Xi}(M)$. There is one caveat, however. As already noted, $\ker \Delta_{w_\Xi}$ depends on $w_\Xi$ pointwise while $H^\bullet_{(2),\Xi}(M)$ does not. Nonetheless, hopefully the reader is convinced that it is worthwhile to pursue involved computations to determine when $H^\bullet_{(2),\Xi}(M)$ actually is finite dimensional.

The present work will use $\overline{M}$ to denote the point end compactification of $M$, and it will show that all of the $\Xi$ with finite dimensional $H^q_{(2),\Xi}(M)$ compute a minor variant of weighted cohomology denoted $\mathcal{W}^\xi H^q(\overline{M})$. In fact, on each end this variant may be thought of as exactly weighted cohomology. As the central point of upcoming proofs will be interpreting $L_2$ norms of forms on each end in the warped product metric in terms of weights, the variant will be referred to as warped cohomology. Then Figure 1 is stated precisely in two results.

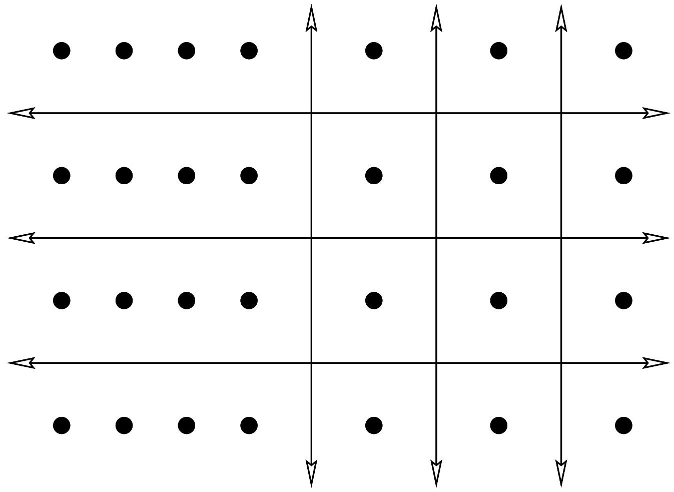

FIGURE 1. This diagram summarizes the main theorems for $M^3$ a three dimensional Riemannian manifold with two hyperbolic ends. Points in the plane parametrize the two exponential coefficients of a weighting function exponential on each end. The weighted $L_2$ cohomology is infinite dimensional on the lines drawn and does not vary within a component in their complement. The large dots represent standard $\xi \in (\mathbb{Z}+\frac{1}{2})^2$, which parametrize weighted cohomology groups that compute the weighted $L_2$ cohomology of their component.

1. All the $H^q_{(2),\Xi}(M)$ are finite dimensional vector spaces unless, for some $\Xi(i)$ for $1 \leq i \leq m$, we have $\Xi(i) \in [-n+1,\dots,0] \cap \mathbb{Z}$. See Proposition 5.1.
2. Moreover, suppose for a tuple of half integers $(\xi(1),\dots,\xi(m))$ and a tuple of real numbers $(\Xi(1),\dots,\Xi(m))$, we have $\xi(i)$ and $\Xi(i)$ in the same component of $\mathbb{R} - ([-n+1,\dots,0] \cap \mathbb{Z})$ for $1 \leq i \leq m$. Then

$$H^q_{(2),\Xi}(M) \cong \mathcal{W}^\xi H^q(M), \qquad 0 \leq q \leq n.$$

   See Theorem 3.2.

For extremely positive $\xi$, the warped cohomology is isomorphic to de Rham cohomology, which in turn inherits a weighted Hodge theory. For extremely negative $\xi$, the warped cohomology is isomorphic to compactly supported de Rham cohomology, etc. The theorem also recovers the fact that the unweighted $L_2$ cohomology groups are finite dimensional if and only if $n$ is even. For $M^3$ a two dimensional example with two ends, the situation is summarized by Figure 1. The vertical and horizontal lines represent the $\Xi$ where $w_\Xi$ is infinite dimensional, and the topological interpretation of the corresponding $H^\bullet_{(2),\Xi}(M^3)$ thus remains a mystery. For the other $\Xi$, all the $H^\bullet_{(2),\Xi}(M^3)$ for the $\Xi$ in the same components are isomorphic. They compute $\mathcal{W}^\xi H^\bullet(\overline{M}^3)$ for any tuple of half integers $\xi$ in said component.

The difficult part of proving a theorem like the above is controlling the behavior of forms on the horospherical directions in a $w_\Xi$ $L_2$ bounded way. Without going into any details, the argument here hinges on carefully defining certain operators on function spaces which are used to integrate out dependence of forms on each of the circles in the end $(0,\infty)\times(S^1)^{n-1}$. This reduces the $w_\Xi$ weighted $L_2$ cohomology of each end to a product of several $e^{\pm ct}$ $L_2$ weighted cohomologies of $(0,\infty)$ for $c\in\mathbb{R}$, and these can be computed using an application of Fubini's theorem. However, the main issue in understanding the weighted $L_2$ cohomology of $(0,\infty)$ for a random $k(t)>0$ is a certain delicate $k(t)$ weighted Hardy inequality. Muckenhoupt's paper [Muc72] provides a precise description of which $k(t)$ allow such a weighted Hardy inequality. However, applying the Muckenhoupt criterion demands checking the suprenum of a product of integrals is finite, and the analysis of this product is quite delicate and often impractical for concrete examples. Moreover, it is not clear to the author that $k(t)$ satisfying the Muckenhoupt criterion would demand $k(t)^2$ or $k(t)^{-2}$ does so. To circumvent these difficulties, $k(t)$ will be taken to be the product of a decreasing function and a decreasing exponential, which is equivalent to $k(t)^{-1}k'(t)\leq -c$ for all $t\in(0,\infty)$ and some $c>0$. The main result of [BH92] then verifies that $k(t)$ satisfies the Muckenhoupt criterion in this circumstance, making the $k(t)$ weighted $L_2$ cohomology of $(0,\infty)$ computable. There are two consequences of applying the refined half line computation to the earlier arguments in this paper.

1. Suppose $k^{-1}k'\leq -c$ for $c>0$ as above. Let $(M,g)$ still have ends diffeomorphic to $(0,\infty)\times(S^1)^{n-1}$, but change the metric to
$$ds^2=dt^2+\sum_{\ell=1}^{n-1}k(t)^2d\theta_\ell^2.$$
Also, weight by $w_\Xi$ restricting to $k(t)^{-2\Xi(i)+(1-n)}$ on each end. The obvious generalizations of the results above hold, providing a precise description of which $w_\Xi$ produce infinite dimensional weighted $L_2$ cohomology and certain strong $w_\Xi$ weighted Kodaira decompositions. See Theorem 6.1.
2. Alternately, revert to the asymptotically hyperbolic metric, but say we choose instead weighting functions $w_\alpha$ which go as $\mathrm{e}^{\alpha t^2}$ on each end. The corresponding $w_\alpha$ weighted $L_2$ cohomology computes compactly supported de Rham cohomology for $\alpha>0$ and de Rham cohomology for $\alpha<0$. See Theorem 6.2.

It is likely the second result generalizes to generic locally symmetric spaces $\Gamma\backslash X$.

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## 2. Asymptotically hyperbolic manifolds

The first step to proving the main theorem is to carefully describe the Riemannian manifolds to which it applies.

**Definition 2.1** (a.h.)**.** An oriented Riemannian manifold $(M,g)$ is said to be asymptotically hyperbolic when $M=M_0\cup(\cup_{i=1}^m\mathcal{E}_i)$ for an open core $M_0$ with $\overline{M}_0$ a compact manifold with boundary. Also,

1. Each $\mathcal{E}_i$ is diffeomorphic to $(0,\infty)\times(S^1)^{n-1}$, $1\leq i\leq m$. Moreover, let $t$ be the coordinate on $(0,\infty)$ and $\theta_1,\ldots,\theta_{n-1}$ parametrize $(S^1)^{n-1}$ in the usual

way. Under the given diffeomorphism, the metric on each end pulls back to

$$ds^2 = (dt)^2 + \sum_{\ell=1}^{n-1} \mathrm{e}^{-2t}(d\theta_\ell)^2. \tag{1}$$

2. Moreover, under the diffeomorphisms of the last item, $M_0 \cap \mathcal{E}_i$ is identified with $(0,1) \times (S^1)^{n-1}$ for $1 \le i \le m$.
3. For $i \neq j$, $\mathcal{E}_i \cap \mathcal{E}_j = \phi$.

The next step is to define the standard $w$ weighted $L_2$ cohomology presheaf for $w : M \to (0, \infty)$ a continuous weighting function. A quick preliminary is to describe $\overline{M}$.

**Definition 2.2** ($\overline{M}$)**.** The point end compactification $\overline{M}$ of an a.h. manifold $M$ is (as a set) $M \cup \{\infty_1, \ldots, \infty_m\}$, where $m$ is the number of ends of $M$. The topology has as a basis all the open sets of $M$ as well as the unions of the $\{\infty_i\}$ and the subsets of $\mathcal{E}_i$ identified with $(T, \infty) \times (S^1)^{n-1}$, $1 \le i \le m$.

**Definition 2.3** ($A^\bullet_{(2),w}(U)$)**.** Let $U \subseteq \overline{M}$ be open. For $w : M \to (0, \infty)$ any continuous weighting function, the (smooth) $w$ weighted $L_2$ cochains on $U$ are

$$A^q_{(2),w}(U) := \left\{ \omega \text{ smooth };\ \begin{array}{l} \int_{U\cap M} \langle \omega, \omega \rangle w \text{ dvol} < \infty \text{ and} \\ \int_{U \cap M} \langle d\omega, d\omega \rangle w \text{ dvol} < \infty \end{array} \right\}.$$

$H^q_{(2),w}(U)$ is the $q^{\text{th}}$ cohomology of the cochain complex

$$A^0_{(2),w}(U) \xrightarrow{d} A^1_{(2),w}(U) \xrightarrow{d} \ldots \xrightarrow{d} A^{n-1}_{(2),w}(U) \xrightarrow{d} A^n_{(2),w}(U).$$

In the case $w \equiv 1$, it will be dropped from all notations.

A subtle point now arises. For $0 \le q \le n$, the above presheaf $A^q_{(2),w}(-)$ is in general not a sheaf. Upcoming patching arguments will instead use its sheafification, $\mathcal{A}^q_{(2),w}(-)$. Recall

$$\mathcal{A}^q_{(2),w}(U) := \{\omega \text{ smooth on } U\ ;\ \forall p \in U,\ \exists\ V_p \subseteq U,\ p \in V_p,\ \text{s.t. } \omega|_{V_p} \in A^q_{(2),w}(V)\}.$$

In other words, $\mathcal{A}^\bullet_{(2),w}(U)$ is $A^\bullet_{(2),w,\mathrm{loc}}(U)$. Given that, it is perhaps worthwhile to briefly recall why the hypercohomology $\mathbb{H}^q(\overline{M}, \mathcal{A}^\bullet_{(2),w})$ of the differential graded sheaf $\mathcal{A}^\bullet_{(2),w}$ actually computes $H^q_{(2),w}(M)$. Recall that the latter is defined to be $q^{\text{th}}$ cohomology of

$$A^0_{(2),w}(M) \xrightarrow{d} A^1_{(2),w}(M) \xrightarrow{d} \ldots \xrightarrow{d} A^{n-1}_{(2),w}(M) \xrightarrow{d} A^n_{(2),w}(M),$$

although $M$ could also be replaced by $\overline{M}$ in the sequence above. A moment's thought will then show that *since* $\overline{M}$ *is compact,* $A^q_{(2),w}(\overline{M}) = \mathcal{A}^q_{(2),w}(\overline{M})$. Finally, $\mathcal{A}^\bullet_{(2),w}$ is a fine differential graded sheaf by the second item of Definition 2.1 and the argument of Proposition 4.4 of [Zuc82]. Thus each of the grades $\mathcal{A}^q_{(2),w}$ is (stalkwise) acyclic, and the hypercohomology of $\mathcal{A}^\bullet_{(2),w}$ computes the cohomology of its global sections on $\overline{M}$.

In order to complete the definition of the weighted $L_2$ cohomologies described in the introduction, the weighting functions must be defined. For applications to weighted Hodge theory, this choice of $w$ would be significant at each $p \in M$.

However, as the results of the present paper are exclusively weighted $L_2$ cohomology computations, the present choice of $w$ will be a bit careless. To begin, the author wishes to describe a tuple of real numbers $\Xi = (\Xi(1), \ldots, \Xi(m))$ as a warp profile, where $m$ is the number of ends of $M$. The lower case $\xi = (\xi(1), \ldots, \xi(m))$ will only be used when all the $\xi(i)$ are in $\mathbb{Z} + \frac{1}{2}$, and only such $\xi$ will be described as being standard warp profiles.

**Definition 2.4** ($\Xi$)**.** The weighting function $w_\Xi : M \to (0, \infty)$ is defined piecemeal. On each end $\mathcal{E}_i$, it is $w_\Xi(t) = \mathrm{e}^{(2\Xi(i)+(n-1))t}$, and on the complement of all ends it is identically 1. (Thus $w_\Xi$ is continuous but not smooth. A smoothed $w_\Xi$ could be chosen, however.) The abbreviation $\Xi$ may be used in place of $w_\Xi$ in all other notations, e.g., $\mathcal{A}^\bullet_{(2),\Xi}$, etc. In particular, should $\Xi \equiv -\frac{1}{2}(n-1)$ so that $w_\Xi \equiv 1$, the $\Xi$ is dropped in any $w_\Xi$ $L_2$ cohomology notation.

## 3. Warped cohomology of a.h. manifolds

The present discussion would be a good deal more complicated if the ends of $M$ did not have constant sectional curvature. At certain points, brief indications will be made as to how one would extend similar constructions to slightly more complicated geometries, such as ends of a lattice quotient of the complex hyperbolic plane.

Choose a fixed end $\mathcal{E}_i$, and abbreviate by dropping the subscript. Label $\mathbf{T} = \Pi_1^{n-1}\mathrm{SO}_2(\mathbb{R})$, which is the $(S^1)^{n-1}$ factor viewed as a Lie group. Let $\mathfrak{t}$ be the abelian Lie algebra, and let $C^\bullet(\mathfrak{t})$ be the Lie algebra cohomology complex with coefficients in the trivial infinitesimal representation. The exterior derivative vanishes identically since $\mathfrak{t}$ is abelian, and there is canonical identification $H^\bullet(\mathfrak{t}) \cong C^\bullet(\mathfrak{t}) \cong \wedge^\bullet[d\theta_1, \ldots, d\theta_{n-1}]$. For $\alpha$ a half integer, let $C^\bullet(\mathfrak{t})^{>\alpha}$ denote $\oplus_{-j>\alpha} C^j(\mathfrak{t})$. In the present context, degree will form a suitable notion of weight for defining a variant of weighted cohomology. For other slightly more complicated ends, for example the lattice quotients of complex hyperbolic space mentioned in the last paragraph, the warped product metric of the introduction would be multiwarped and the $(S^1)^{n-1}$ would be replaced by an appropriate infranil manifold. This would demand a definition of weight which did not coincide with degree.

Let $\Omega^\bullet(\mathcal{E})$ be the de Rham complex on $\mathcal{E}$. Then there is a canonical identification

$$\Omega^q(\mathcal{E}) \cong \oplus_{r=0,1} C^\infty(\mathcal{E}) \otimes \wedge^r[dt] \otimes C^{q-r}(\mathfrak{t}). \tag{2}$$

For later work, it will be useful to describe the de Rham exterior derivative under this decomposition. First, let $\Theta_\ell : C^q(\mathfrak{t}) \to C^{q+1}(\mathfrak{t})$ be left exterior multiplication by $d\theta_\ell$ and $\iota_\ell : C^q(\mathfrak{t}) \to C^{q-1}(\mathfrak{t})$ be left interior multiplication by $\frac{\partial}{\partial\theta_\ell}$. Then the following formula holds:

$$\iota_\ell\Theta_k + \Theta_k\iota_\ell = \delta_{k,\ell}\mathbf{1}, \tag{3}$$

where $\delta_{k,\ell}$ is the Kronecker delta. Define operators $\Theta_t : \wedge^q[dt] \to \wedge^{q+1}[dt]$ and $\iota_t : \wedge^q[dt] \to \wedge^{q-1}[dt]$ similarly. Then the de Rham exterior derivative is identified with

$$d(f \otimes \tau \otimes \phi) = \frac{\partial f}{\partial t} \otimes (\Theta_t\tau) \otimes \phi + (-1)^{\deg\tau} \sum_{\ell=1}^{n-1} \frac{\partial f}{\partial\theta_\ell} \otimes \tau \otimes (\Theta_\ell\phi), \tag{4}$$

where $\phi \in C^\bullet(\mathfrak{t})$ and $\tau \in \wedge^\bullet[dt]$.

The crux of coming arguments is the fact that the weight of a form in $C^\bullet(\mathfrak{t})$, simply the negative of the degree on real hyperbolic quotients, encodes the $L_2$ growth rates in the (multi)warped end metric. In the present circumstance, this may be stated by noting $\phi \in C^j(\mathfrak{t})$ satisfies

$$\langle 1\otimes 1\otimes \phi, 1\otimes 1\otimes\phi\rangle_1 \;=\; C^2 \mathrm{e}^{2jt} \text{ on } \mathcal{E} \text{ for some } C\in\mathbb{R}. \tag{5}$$

In particular, the fact that $|\mathrm{dvol}|^2 \equiv 1$ demands that for appropriate $\Phi \in C^{n-1}(\mathfrak{t})$,

$$\mathrm{dvol}_{\mathcal{E}} = \mathrm{e}^{-(n-1)t}\otimes (dt)\otimes(\Phi). \tag{6}$$

Recall the convention of using $\xi$ to denote $(\xi(1),\ldots,\xi(m))$ a tuple in $(\mathbb{Z}+\frac{1}{2})^m$. The next definition follows the preface of [GHM94].

**Definition 3.1** ($\mathcal{W}^\xi C^\bullet$)**.** Let $i$ be for the moment fixed, abbreviating $\mathcal{E}_i$ by $\mathcal{E}$. Let $\mathcal{E}(T)$ for $T >> 0$ be the subset identified with $(T,\infty)\times(S^1)^{n-1}$. Let $V \subseteq M$ be open with $\omega$ a smooth form on $V$. We say that $\omega$ is $\xi$ *warped at* $\infty_i$ if there is some $T >> 0$ so that $\omega|_{\mathcal{E}(T)}$ may be written in terms of (2) as $1\otimes 1\otimes\phi$ for $\phi \in C^\bullet(\mathfrak{t})^{>\xi(i)}$. Now let $i$ be again variable. For $U\subseteq \overline{M}$ be open, the $\xi$ warped cohomology presheaf is now defined by

$$\mathcal{W}^\xi C^q(U) := \{\omega \text{ smooth on } U\cap M \;;\; \omega \text{ is } \xi \text{ warped at } \infty_i \;\forall\, \infty_i \in U\}.$$

The exterior derivative formula (4) makes this a differential graded presheaf, which the reader may check is a differential graded sheaf. By definition, $\mathcal{W}^\xi H^q(\overline{M}) := \mathbb{H}^q(\overline{M}, \mathcal{W}^\xi C^\bullet)$.

Note $\mathcal{W}^\xi C^\bullet$ is a fine differential graded sheaf, since $\mathcal{W}^\xi C^0$ is fine by the previous argument and each higher grade is a $\mathcal{W}^\xi C^0$ module. Thus $\mathcal{W}^\xi H^q(\overline{M})$ is computed by the $q^{\text{th}}$ cohomology of the complex

$$\mathcal{W}^\xi C^0(\overline{M}) \xrightarrow{d} \mathcal{W}^\xi C^1(\overline{M}) \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{W}^\xi C^{n-1}(\overline{M}) \xrightarrow{d} \mathcal{W}^\xi C^n(\overline{M}).$$

However, the typical computation would construct a Mayer Vietoris sequence subordinate to the cover of Definition 2.1 and use $\mathcal{W}^\xi C(\mathcal{E}_i(T)\cup\{\infty_i\}) \cong C^\bullet(\mathfrak{t})^{>\xi(i)}$. The next section of this work may be modified to prove the previous isomorphism.

The isomorphism theorem of the introduction may now be stated explicitly.

**Theorem 3.2.** *Let $(M,g)$ be a.h. per Definition 2.1, let $\Xi$ be a tuple of $m$ real numbers, and let $\xi$ be a tuple of $m$ half integers. Suppose that for $1\le i\le m$, $\xi(i)$ and $\Xi(i)$ are in the same component of $\mathbb{R}-(\mathbb{Z}\cap[-n+1,0])$. Recall the $w_\Xi$ of Definition 2.4. Then*

$$H^q_{(2),\Xi}(M) \cong \mathcal{W}^\xi H^q(\overline{M}), \qquad 0\le q\le n.$$

The technique will be to establish a quasiisomorphism between $\mathcal{W}^\xi C^\bullet$ and $\mathcal{A}^\bullet_{(2),\Xi}$. The differential graded sheaf in the middle of the quasiisomorphism is the intersection of the two sheaves, which is the sheafification of the intersection of the two presheaves in $i^*\Omega^\bullet$, for $i: M\to\overline{M}$ and $\Omega^\bullet$ the de Rham differential graded sheaf on $M$. Fixing the notation, for $U\subseteq\overline{M}$ open,

$$\mathcal{W}^\xi C^q_{(2),\Xi}(U) := \mathcal{W}^\xi C^q(U)\cap A^q_{(2),\Xi}(U)$$

with sheafification $\mathcal{W}^\xi \mathcal{C}^\bullet_{(2),\Xi}$.

**Lemma 3.3.** *The following is a quasiisomorphism, where both arrows are inclusions of differential graded sheaves.*

$$\mathcal{W}^\xi C^\bullet \leftarrow \mathcal{W}^\xi \mathcal{C}^\bullet_{(2),\Xi} \rightarrow \mathcal{A}^\bullet_{(2),\Xi},$$

To recall why Lemma 3.3 implies Theorem 3.2, let $H^q(\mathcal{S}^\bullet)_p$ denote the stalk of the cohomology sheaf of the differential graded sheaf $\mathcal{S}^\bullet$ for $p \in \overline{M}$. Then the fundamental theorem on page 202 of [Bre97] shows that it suffices to check the inclusion pullback mappings $H^q(\mathcal{W}^\xi \mathcal{C}^\bullet_{(2),\Xi})_p \rightarrow H^q(\mathcal{W}^\xi C^\bullet)_p$ and $H^q(\mathcal{W}^\xi \mathcal{C}^\bullet_{(2),\Xi})_p \rightarrow H^q(\mathcal{A}^\bullet_{(2),\Xi})_p$ are isomorphisms at all $p \in \overline{M}$.

As an aside, the isomorphism of stalk cohomologies is quickly verified for all $p$ except $p = \infty_1, \ldots, \infty_m$. The next section will thus construct appropriate cochain homotopies on $\mathcal{E}_i$ to verify the isomorphism on the $\infty_i$ stalks, $1 \leq i \leq m$.

## 4. Proof of the isomorphism

Again fix $\mathcal{E}_i$, abbreviated $\mathcal{E}$. Label $\bar{\mathcal{E}} = \mathcal{E} \cup \{\infty_i\}$. For $T >> 0$, let $\bar{\mathcal{E}}(T)$ denote $\mathcal{E}(T) \cup \{\infty_i\}$. It suffices to show the identities on $A^\bullet_{(2),\Xi}(\mathcal{E}(T))$ and $\mathcal{W}^\xi C^\bullet(\mathcal{E}(T))$ are cochain homotopy equivalent to a composition of projections to the subcomplex $1 \otimes 1 \otimes C^\bullet(\mathfrak{t})^{>\xi(i)} \subseteq \mathcal{W}^\xi C^\bullet_{(2),\Xi}(\bar{\mathcal{E}}(T))$. (Presheaves may be used since the direct limit of cochain homotopies is a cochain homotopy.) Note there is no harm in shifting the $(0, \infty)$ factor so that $T = 0$. That said, the cochain homotopies are achieved in $n$ steps. The first $n-1$ steps project all complexes to appropriate subcomplexes of $C^\infty(0,\infty) \otimes \wedge^\bullet[dt] \otimes C^\bullet(\mathfrak{t})$, following (2). Specifically, each of these $n-1$ steps removes dependence on one of the $\theta_\ell$ directions of the $(S^1)^{n-1}$ factor. The final step converts the remaining coefficient function on $(0,\infty)$ to a constant and contracts away $dt$'s.

First, certain operators on coefficient function spaces need to be defined. This will take some time, but it will expedite the actual computation of the first $n-1$ cochain homotopies.

**Definition 4.1.** For $1 \leq \ell \leq n$, define $C^\infty(\mathcal{E})^\ell$ as follows.

$$C^\infty(\mathcal{E})^\ell := \{f \in C^\infty(\mathcal{E}) \; ; \; \frac{\partial}{\partial \theta_j} f \equiv 0 \; \forall j < \ell\}.$$

Note in particular that $C^\infty(\mathcal{E})^1 := C^\infty(\mathcal{E})$, while $C^\infty(\mathcal{E})^n$ may be canonically identified with $C^\infty(0,\infty)$. Also, there is a sequence of inclusions

$$C^\infty(0,\infty) = C^\infty(\mathcal{E})^n \hookrightarrow \cdots \hookrightarrow C^\infty(\mathcal{E})^2 \hookrightarrow C^\infty(\mathcal{E})^1 = C^\infty(\mathcal{E}).$$

Under the identification (2), define

$$\Omega^\bullet(\mathcal{E})^\ell := C^\infty(\mathcal{E})^\ell \otimes \wedge^\bullet[dt] \otimes C^\bullet(\mathfrak{t}).$$

$\Omega^\bullet(\mathcal{E})^\ell$ will is a cochain complex, due to the exterior derivative formula (4).

In interpreting the following proposition, recall $\Omega^\bullet(\mathcal{E})^1 = \Omega^\bullet(\mathcal{E})$ and $\Omega^\bullet(\mathcal{E})^n = C^\infty(0,\infty) \otimes \wedge^\bullet[dt] \otimes C^\bullet(\mathfrak{t})$.

**Proposition 4.2.** *Consider the following inclusions.*

$$\Omega^\bullet(\mathcal{E})^n \hookrightarrow \Omega^\bullet(\mathcal{E})^{n-1} \hookrightarrow \cdots \hookrightarrow \Omega^\bullet(\mathcal{E})^1 = \Omega^\bullet(\mathcal{E}).$$

*There linear $B_\ell : \Omega^\bullet(\mathcal{E})^\ell \rightarrow \Omega^{\bullet-1}(\mathcal{E})^\ell$ of degree $-1$ and $P_\ell : \Omega^\bullet(\mathcal{E})^\ell \rightarrow \Omega^\bullet(\mathcal{E})^{\ell+1}$ of degree $0$ so that the following hold for $1 \leq \ell \leq n-1$.*

1. *$(B_\ell, P_\ell)$ is a cochain homotopy, i.e.,*

$$dB_\ell + B_\ell d = \mathbf{1} - P_\ell. \tag{7}$$

2. *Both $B_\ell$ and $P_\ell$ are $w_\Xi$ $L_2$ bounded, and each moreover fixes the subcomplex $1 \otimes 1 \otimes C^\bullet(\mathfrak{t}) \subseteq \Omega^\bullet(\mathcal{E})^\ell$.*
3. *For any $\alpha \in \mathbb{Z} + \frac{1}{2}$, $B_\ell$ and $P_\ell$ preserve $\oplus_{-j>\alpha} 1 \otimes 1 \otimes C^j(\mathfrak{t})$.*

**Definition 4.3.** Subcomplexes of $\mathcal{W}^\xi C^\bullet(\bar{\mathcal{E}})$, $A^\bullet_{(2),\Xi}(\mathcal{E})$, and $\mathcal{W}^\xi C^\bullet_{(2),\Xi}(\bar{\mathcal{E}})$ are defined as follows, in each case for $0 \le \ell \le n$.

$$\mathcal{W}^\xi C^\bullet(\bar{\mathcal{E}})^\ell := \mathcal{W}^\xi C^\bullet(\bar{\mathcal{E}}) \cap \Omega^\bullet(\mathcal{E})^\ell.$$
$$A^\bullet_{(2),\Xi}(\mathcal{E})^\ell := A^\bullet_{(2),\Xi}(\mathcal{E}) \cap \Omega^\bullet(\mathcal{E})^\ell.$$
$$\mathcal{W}^\xi C^\bullet_{(2),\Xi}(\bar{\mathcal{E}})^\ell := \mathcal{W}^\xi C^\bullet_{(2),\Xi}(\bar{\mathcal{E}}) \cap \Omega^\bullet(\mathcal{E})^\ell.$$

**Corollary 4.4.** *The composition $P_{n-1}P_{n-2}\dots P_1$ is cochain homotopic to the identities of $\mathcal{W}^\xi C^\bullet(\bar{\mathcal{E}})$ and $A^\bullet_{(2),\Xi}(\mathcal{E})$, and also to the identity map of their intersection $\mathcal{W}^\xi C^\bullet_{(2),\Xi}(\bar{\mathcal{E}})$. The images of the composition of projections are $\mathcal{W}^\xi C^\bullet(\bar{\mathcal{E}})^n$, $A^\bullet_{(2),\Xi}(\mathcal{E})^n$, and $\mathcal{W}^\xi C^\bullet_{(2),\Xi}(\bar{\mathcal{E}})^n$ respectively.*

In order to construct $B_\ell$ and $P_\ell$, consider $\ell$ as fixed. The definition depends on operators defined on $\mathcal{E}$ in the coordinates $(t, \theta_1, \dots, \theta_{n-1})$.

**Definition 4.5** ($\mathrm{Av}_\ell$)**.** $\mathrm{Av}_\ell : C^\infty(\mathcal{E})^\ell \to C^0(\mathcal{E})$ is defined by

$$[Av_\ell(f)](t, \theta_1, \dots, \theta_{n-1}) := \frac{1}{2\pi}\int_0^{2\pi} f(t, \theta_1, \dots, \theta_{\ell-1}, \psi, \theta_{\ell+1}, \dots, \theta_{n-1})d\psi.$$

For $f \in C^\infty(\mathcal{E})^\ell$, $f_0$ will be used as an abbreviation for $f - \mathrm{Av}_\ell(f)$.

It will be checked in due course that for any suitable $f$, $\mathrm{Av}_\ell(f) \in C^\infty(\mathcal{E})^{\ell+1}$. In particular, $\mathrm{Av}_\ell(f)$ is smooth.

The operator required to define $B_\ell$ is the series operator, denoted $S_\ell$. $S_\ell$ will produce an antiderivative with respect to $\frac{\partial}{\partial\theta_\ell}$ of the function $f$, so that $\mathrm{Av}_\ell$ vanishes on this antiderivative. The operator is so named due to its expression in terms of Fourier coefficients, following the appendix of [Zuc82]. Reiterating this motivation, suppose we had an expression for $f \in C^\infty(\mathcal{E})^\ell$ as

$$f(t, \theta_\ell, \dots, \theta_{n-1}) = \sum_{j=0}^{\infty} a_j(t, \theta_{\ell+1}, \dots \theta_{n-1})\cos(j\theta_\ell) + b_j(t, \theta_{\ell+1}, \dots \theta_{n-1})\sin(j\theta_\ell).$$

Then the motivation is that $S_\ell(f)$ has the following expansion:

$$(S_\ell f)(t, \theta_\ell, \dots, \theta_{n-1}) = \sum_{j=1}^{\infty} j^{-1} b_j(t, \theta_{\ell+1}, \dots \theta_{n-1})\cos(j\theta_\ell) - j^{-1} a_j(t, \theta_{\ell+1}, \dots \theta_{n-1})\sin(j\theta_\ell).$$

With this motivation given, the precise definition will now be made in terms of coordinates.

**Definition 4.6** ($S_\ell$)**.** $S_\ell : C^\infty(\mathcal{E})^\ell \to C^0(\mathcal{E})$ is defined by

$$[S_\ell f](t, \theta_1, \dots, \theta_{n-1}) = \int_0^{\theta_\ell} f_0(t, \theta_1, \dots, \theta_{\ell-1}, \psi, \theta_{\ell+1}, \dots, \theta_{n-1})d\psi.$$

This is well defined regardless of choice of $\theta_\ell$. This amounts to checking

$$(8) \qquad \int_{2\pi n}^{2\pi m} f_0(t, \theta_1, \ldots, \theta_{\ell-1}, \psi, \theta_{\ell+1}, \ldots, \theta_{n-1}) d\psi = 0, \qquad m, n \in \mathbb{Z}.$$

This integral and an application of Fubini's theorem will also show $\mathrm{Av}_\ell[S_\ell(f)] \equiv 0$. The required properties of $\mathrm{Av}_\ell$ and $S_\ell$ are listed below.

**Lemma 4.7.** *The following properties hold true.*

1. *$Av_\ell$ and $S_\ell$ are $w_\Xi$ $L_2$ bounded on $C^\infty(\mathcal{E})^\ell$.*
2. *Using the Lie bracket to denote the commutator of the two operators on $C^\infty(\mathcal{E})^\ell$, one has $[\frac{\partial}{\partial t}, Av_\ell] = 0$, $[\frac{\partial}{\partial t}, S_\ell] = 0$, $[\frac{\partial}{\partial \theta_j}, Av_\ell] = 0$, and $[\frac{\partial}{\partial \theta_j}, S_\ell] = 0$ for $1 \leq j \leq n-1$.*
3. *$Av_\ell : C^\infty(\mathcal{E})^\ell \to C^\infty(\mathcal{E})^{\ell+1}$ and $S_\ell : C^\infty(\mathcal{E})^\ell \to C^\infty(\mathcal{E})^\ell$.*
4. *$\frac{\partial}{\partial \theta_\ell}(S_\ell f) = f - Av_\ell f = f_0$.*

**Proof.** To prove the first item, for $\mathrm{Av}_\ell$ this is an application of the Schwarz inequality. For $S_\ell$, use linearity to consider positive and negative parts of $f_0$ separately. Then choose (without loss of generality) the $\theta_\ell$ in $0 \leq \theta_\ell < 2\pi$ for all appropriate points in $\mathcal{E}$, and replace $\theta_\ell$ by $2\pi$. Now apply $w_\Xi$ $L_2$ boundedness of $\mathrm{Av}_\ell$.

The second item is a standard statement that taking derivatives commutes with integration, and it is proven via dominated convergence in the normal way. For the third item, smoothness follows from the fact that $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_{n-1}}\}$ is a frame of vector fields on $\mathcal{E}$ and the second item. All of the vanishings for the range to be as stated follow from the second item as well, with the exception of $\frac{\partial}{\partial \theta_\ell}\mathrm{Av}_\ell(f) \equiv 0$. This is a consequence of the definition. Finally, for the fourth item, use the integral in the definition of $S_\ell(f)$ and apply the fundamental theorem of calculus. □

We are now in a position to define and utilize the cochain homotopy $(B_\ell, P_\ell)$. The reader is requested to recall the identification $\Omega^\bullet(\mathcal{E}) \cong C^\infty(\mathcal{E}) \otimes \wedge^\bullet[dt] \otimes C^\bullet(\mathfrak{t})$ of (2) is being kept implicit, as is the similar identification with $\ell$ superscripts.

**Definition 4.8** ($(B_\ell, P_\ell)$)**.** $B_\ell : \Omega^\bullet(\mathcal{E})^\ell \to \Omega^{\bullet-1}(\mathcal{E})^\ell$ and $P_\ell : \Omega^\bullet(\mathcal{E})^\ell \to \Omega^\bullet(\mathcal{E})^{\ell+1}$ of the $\ell^{\text{th}}$ cochain homotopy of Proposition 4.2 are defined by

$$\begin{cases} P_\ell(f \otimes \tau \otimes \phi) := (\mathrm{Av}_\ell f) \otimes \tau \otimes \phi \\ B_\ell(f \otimes \tau \otimes \phi) := (-1)^{\deg \tau}(S_\ell f) \otimes \tau \otimes (\iota_\ell \phi). \end{cases}$$

Recall $\iota_\ell : C^\bullet(\mathfrak{t}) \to C^{\bullet-1}(\mathfrak{t})$ is left interior multiplication by $\frac{\partial}{\partial \theta_\ell}$.

With these definitions, Proposition 4.2 reduces to a computation verifying the cochain homotopy formula $dB_\ell + B_\ell d = \mathbf{1} - P_\ell$. Using the formula (4) for $d$ under the identification $\Omega^\bullet(\mathcal{E}) \cong C^\infty(\mathcal{E}) \otimes \wedge^\bullet[dt] \otimes C^\bullet(\mathfrak{t})$ of (2),

$$\begin{aligned}(B_\ell d)(f \otimes \tau \otimes \phi) &= (-1)^{\deg \tau + 1}(S_\ell \frac{\partial}{\partial t} f) \otimes (\Theta_t \tau) \otimes (\iota_\ell \phi) \\ &\quad + \sum_{j=1}^{n-1} (S_\ell \frac{\partial}{\partial \theta_j} f) \otimes \tau(\iota_\ell \Theta_j \phi), \text{ and} \\ (dB_\ell)(f \otimes \tau \otimes \phi) &= (-1)^{\deg \tau}(\frac{\partial}{\partial t} S_\ell f) \otimes (\Theta_t \tau) \otimes (\iota_\ell \phi) \\ &\quad + \sum_{j=1}^{n-1} (\frac{\partial}{\partial \theta_j} S_\ell f) \otimes \tau \otimes (\Theta_j \iota_\ell \phi).\end{aligned}$$

Adding the two terms, the desired formula will follow.

$$\begin{aligned}(dB_\ell + B_\ell d)(f \otimes \tau \otimes \phi) &= \sum_{j=1}^{n-1} (\frac{\partial}{\partial \theta_j} S_\ell f) \otimes \tau \otimes [(\iota_\ell \Theta_j + \Theta_j \iota_\ell)\phi] \\ &= \sum_{j=1}^{n-1} (\frac{\partial}{\partial \theta_j} S_\ell f) \otimes \tau \otimes [(\delta_{j,\ell})\phi] \\ &= (\frac{\partial}{\partial \theta_\ell} S_\ell f) \otimes \tau \otimes \phi \\ &= (f_0) \otimes \tau \otimes \phi \\ &= (\mathbf{1} - P_\ell)(f \otimes \tau \otimes \phi).\end{aligned}$$

This concludes the proof of Proposition 4.2.

The final step is to integrate out the dependence of $f(t) \otimes \tau \otimes \phi$ on $(0, \infty)$. This step breaks down into a case study depending on which of the two inclusions one is working with. First, the output of Proposition 4.2 on each of the complexes $A^\bullet_{(2),\Xi}(\mathcal{E})$, $\mathcal{W}^\xi C^\bullet(\bar{\mathcal{E}})$, and $\mathcal{W}^\xi C^\bullet_{(2),\Xi}(\bar{\mathcal{E}})$ will be described explicitly in terms of the following subcomplexes of the de Rham complex on the half line $(0, \infty)$. Also, $T$ here will again be $T >> 0$, but it is not the same $T$ that was renormed to 0 in the first paragraph of the section.

$$\begin{aligned}A^\bullet_{(2),c}((0,\infty)) &:= \left\{ \omega \in \Omega^\bullet((0,\infty))\ ;\ \begin{array}{l} \int_0^\infty |\omega(t)|^2 e^{ct} dt < \infty \text{ and} \\ \int_0^\infty |d\omega(t)|^2 e^{ct} dt < \infty \end{array} \right\} \\ \Omega^\bullet((0,\infty))^{T,>} &:= \{\omega \in \Omega^\bullet((0,\infty)); \omega|_{(T,\infty)} \equiv C \text{ for a } C \in \mathbb{R}\} \\ \Omega^\bullet((0,\infty))^{T,<} &:= \{\omega \in \Omega^\bullet((0,\infty)); \omega|_{(T,\infty)} \equiv 0\} \\ \mathcal{W}^> C^\bullet((0,\infty)) &:= \varinjlim_T \Omega^\bullet((0,\infty))^T \\ \mathcal{W}^< C^\bullet((0,\infty)) &:= \varinjlim_T \Omega^\bullet((0,\infty))^T \\ A^\bullet_{(2),c}((0,\infty))^{T,>} &:= \Omega^\bullet((0,\infty))^{T,>} \cap A^\bullet_{(2),c}((0,\infty)) \\ A^\bullet_{(2),c}((0,\infty))^{T,<} &:= \Omega^\bullet((0,\infty))^{T,<} \cap A^\bullet_{(2),c}((0,\infty)) \\ \mathcal{W}^> C^\bullet_{(2),c}((0,\infty)) &:= \mathcal{W}^> C^\bullet((0,\infty)) \cap A^\bullet_{(2),c}((0,\infty)) \\ \mathcal{W}^< C^\bullet_{(2),c}((0,\infty)) &:= \mathcal{W}^< C^\bullet((0,\infty)) \cap A^\bullet_{(2),c}((0,\infty))\end{aligned}$$

The images of the projections of Proposition 4.2 will now be listed. Here, the first two terms of the standard isomorphism (2), $\Omega^\bullet(\mathcal{E}_i) \cong C^\infty(\mathcal{E}_i)\otimes\wedge^\bullet[dt^i]\otimes C^\bullet(\mathfrak{t})$, are collected.

$$\oplus_{j=1}^{n-1} A^\bullet_{(2),2\Xi(i)+2j}((0,\infty))\otimes C^j(\mathfrak{t}) \hookrightarrow A^\bullet_{(2),\Xi}(\mathcal{E})$$

$$\begin{matrix}(\oplus_{-j<\xi(i)}\mathcal{W}^< C((0,\infty))\otimes C^j(\mathfrak{t}))\\ \oplus \\ (\oplus_{-j>\xi(i)}\mathcal{W}^> C((0,\infty))\otimes C^j(\mathfrak{t}))\end{matrix} \hookrightarrow \mathcal{W}^\xi C^\bullet(\bar{\mathcal{E}})$$

$$\begin{matrix}(\oplus_{-j<\xi(i)}\mathcal{W}^< C_{(2),2\Xi(i)+2j}((0,\infty))\otimes C^\bullet(\mathfrak{t}))\\ \oplus \\ (\oplus_{-j>\xi(i)}\mathcal{W}^> C_{(2),2\Xi(i)+2j}((0,\infty))\otimes C^j(\mathfrak{t}))\end{matrix} \hookrightarrow \mathcal{W}^\xi C^\bullet_{(2),\Xi}(\bar{\mathcal{E}})$$

Recall $\xi(i)$ and $\Xi(i)$ are in the same unbounded component of $\mathbb{R}-(\mathbb{Z}\cap[-n+1,0])$. Thus it suffices construct cochain homotopies projecting each side of the following inclusions onto the same subcomplex. The subcomplex will be either the constant zero forms or the zero complex.

$$\begin{array}{lcll}
\mathcal{W}^> C^\bullet_{(2),c}((0,\infty)) & \hookrightarrow & A^\bullet_{(2),c}((0,\infty)) & \text{when } c<0.\\
\mathcal{W}^< C^\bullet_{(2),c}((0,\infty)) & \hookrightarrow & A^\bullet_{(2),c}((0,\infty) & \text{when } c>0.\\
\mathcal{W}^> C^\bullet_{(2),c}((0,\infty)) & \hookrightarrow & \mathcal{W}^> C^\bullet((0,\infty) & \text{when } c<0.\\
\mathcal{W}^< C^\bullet_{(2),c}((0,\infty)) & \hookrightarrow & \mathcal{W}^< C^\bullet((0,\infty) & \text{when } c>0.
\end{array}$$

The required cochain homotopy operators will now be denoted $(B_k, P_k)$ for $1\le k\le 4$, where the order is taken from the list above. The definition of $(B_1,P_1)$ and $(B_2,P_2)$ is then given below. Here, $\tau\in\wedge^\bullet[dt]$.

$$\begin{cases} B_1(f(t)\wedge\tau) &=& (\int_0^t f(s)\ ds)\wedge(\iota_t\tau)\\ P_1(f(t)\wedge\tau) &=& \{f(1)-\int_0^1 \frac{\partial f}{\partial t}(u)\ du\}\wedge(\Theta_t\iota_t\tau)\end{cases}$$
$$\begin{cases} B_2(f(t)\wedge\tau) &=& (\int_t^\infty f(s)\ ds)\wedge(\iota_{\frac{\partial}{\partial t}}\tau)\\ P_2(f(t)\wedge\tau) &=& 0\end{cases}$$

The other two cochain homotopies are not arrived at directly, but are rather constructed as a direct limit of cochain homotopies $(B_3^T,P_3^T)$ and $(B_4^T,P_4^T)$. The latter will preserve both sides of the following inclusions.

$$\begin{array}{lcll}
A^\bullet_{(2),c}((0,\infty))^{T,>} & \hookrightarrow & \Omega^\bullet((0,\infty)^{T,>} & \text{when } c<0.\\
A^\bullet_{(2),c}((0,\infty))^{T,<} & \hookrightarrow & \Omega^\bullet((0,\infty)^{T,<} & \text{when } c>0.
\end{array}$$

In fact, $(B_3^T,P_3^T)$ and $(B_4^T,P_4^T)$ can be given by the same formula.

$$\begin{cases} P^T_{3,4}(f(t)\wedge\tau) &=& f(T+1)\wedge(\Theta_t\iota_t\tau)\\ B^T_{3,4}(f(t)\wedge\tau) &=& (\int_{T+1}^t f(s)\ ds)\wedge(\iota_t\tau).\end{cases}$$

There remains the issue of checking $(B_1,P_1)$ and $(B_2,P_2)$ are $\mathrm{e}^{\pm ct}$ $L_2$ bounded for $c\neq 0$. This can be seen to be equivalent to the weighted Hardy inequality of Section 6, for $k(t)=\mathrm{e}^{\pm ct}$. When the weighting function is either a strictly increasing or decreasing exponential function, the argument of Proposition 2.39 of [Zuc82] with $a=\infty$ suffices. The appropriate $\lambda(t)$ is also exponential for this case. This concludes the argument for Lemma 3.3 and thus the proof of Theorem 3.2.

The next section will treat the infinite dimensionality of $H^*_{(2),\Xi}(M)$ in those cases not covered by Theorem 3.2. Following this, the extra references for analyzing the $k(t)$ weighted $L_2$ cohomology of the half line will appear in the final section.

## 5. Infinite dimensionality of $H^*_{(2),\Xi}(M)$

The goal of this section is to prove the following proposition.

**Proposition 5.1.** *Let $(M, g)$ be a.h. per Definition* 2.1*, $\Xi = (\Xi(1), \ldots, \Xi(m)) \in \mathbb{R}^m$, $w_\Xi$ per Definition* 2.4*, etc. Label $H^*_{(2),\Xi}(M) := \oplus_{q=0}^n H^q_{(2),\Xi}(M)$. Then the vector space $H^*_{(2),\Xi}(M)$ is infinite dimensional if and only if $\exists\, \Xi(i) \in \mathbb{Z}\cap[-n+1, 0]$.*

**Proof.** $\Longrightarrow$) Given the isomorphism of Theorem 3.2, it suffices to know each $\mathcal{W}^\xi H^q(\overline{M})$ is finite dimensional for any $\xi$. This follows from Proposition 3.10 on page 77 of [Bor84].

$\Longleftarrow$) The author is unaware of any analogues of Wilder's theorem which seek to show $\mathbb{H}^*(\overline{M}, \mathcal{S}^\bullet)$ is not finitely generated for certain differential graded sheaves. Thus an *ad hoc* Mayer Vietoris argument, which makes use of $\partial M$ discrete, will be given instead. Recall that in the case at hand,

$$\dim \oplus_{j=1}^m H^*_{(2),\Xi}(\mathcal{E}_j) = \infty,$$

since the $\mathcal{E}_i$ in the hypothesis has a copy of $A^\bullet_{(2),0}(0,\infty)$ embedded within the output of the projections of Proposition 4.2. ($H^1_{(2),0}(0,\infty)$ is infinite dimensional, as is outlined in [Zuc82, pp. 183–184].) Now as outlined in [Zuc82, p. 175] one may build a Mayer Vietoris sequence for $H^\bullet_{(2),\Xi}(-)$ subordinate to $M = M_0 \cup (\bigsqcup_{j=1}^m \mathcal{E}_j)$ provided one chooses the cutoff functions of [BT82, pp. 17–25] with bounded gradient. This is possible by the second item of Definition 2.1. Thus the assertion is reduced to the following (co)homological algebra lemma. ☐

**Lemma 5.2.** *Let $A^\bullet$, $B^\bullet$, and $C^\bullet$ be cochain complexes vanishing in negative degree. Suppose there are short exact sequences*

$$0 \to A^q \longrightarrow B^q \longrightarrow C^q \to 0, \quad q \geq 0.$$

*Finally, fix a choice of $q_0 \geq 0$, and suppose $\sum_{q=0}^{q_0} \dim H^q(C^\bullet) < \infty$ and also that $\sum_{q=0}^{q_0-1} \dim H^q(A^\bullet) + \dim H^q(B^\bullet) < \infty$. Then*

$$(\dim H^{q_0}(A^\bullet) < \infty) \Longleftrightarrow (\dim H^{q_0}(B^\bullet) < \infty).$$

**Proof.** The long exact sequence provided by the snake lemma may be truncated as follows.

$$0 \to H^0(A^\bullet) \to \cdots \to H^{q_0}(A^\bullet) \to H^{q_0}(B^\bullet) \to \operatorname{im}(H^{q_0}(B^\bullet) \to H^{q_0}(C^\bullet)) \to 0.$$

Now the dimension of $\operatorname{im}(H^{q_0}(B^\bullet) \to H^{q_0}(C^\bullet))$ must be finite as a subspace of $H^{q_0}(C^\bullet)$. Considering the alternating sum of the dimensions of the long exact sequence provides the result. ☐

## 6. $H^{\bullet}_{(2),k(t)}(0,\infty)$ and applications

Let $k(t)$ be a positive function on $(0,\infty)$, which for convenience is chosen to be smooth. Put $L^{\bullet}_{(2),k(t)}(0,\infty)$ to be those currents $f(t)+g(t)dt$ with $\int_0^\infty [f(t)^2 + g(t)^2]k(t)dt < \infty$. Then $H^0_{(2),k(t)}(0,\infty)$ is trivial or one dimensional as $k(t)$ is integrable. However, $H^1_{(2),k(t)}(0,\infty)$ is often an infinite dimensional vector space, e.g., if $k(t)=t^\alpha$ for $\alpha\in\mathbb{R}$. This happens if and only if $\bar{d}^0$ has closed image in $L^1_{(2),k(t)}(0,\infty)$. That in turn is equivalent to having one of the two $k(t)$ weighted Hardy inequalities hold, in cases $k(t)$ is $L^1$ or $k(t)^{-1}$ is $L^1$ respectively. These are

$$\int_0^\infty \left(\int_0^t f(s)ds\right)^2 k(t)dt < C\int_0^\infty f(t)^2 k(t)dt$$

or

$$\int_0^\infty \left(\int_t^\infty f(s)ds\right)^2 k(t)dt < C\int_0^\infty f(t)^2 k(t)dt.$$

These two cases are dual, and the present discussion will for the remainder suppose $k(t)$ is integrable.

In the case at hand, the weighted Hardy inequality holds when $k'(t)k(t)^{-1}\le -c$ uniformly for all $0<t<\infty$ and for some fixed $c>0$. This is a specialization of the main result of [BH92] to the present context. Rather than merely cite it, the argument will be briefly sketched in three steps.

1. It suffices to choose any function $\lambda(s)$ so that $k(s)^{-1}\lambda(s)\int_s^\infty k(t)(\int_0^t \lambda(u)du)dt$ is bounded. See [Zuc82, p. 183].
2. The Muckenhoupt criterion of [Muc72] states that for the minimum $C$ allowed in the weighted Hardy inequality, $B^2\le C\le 4B^2$ where
$$B := \sup_{0<r<\infty}\sqrt{\int_0^r k(s)^{-1}ds}\sqrt{\int_r^\infty k(s)ds}.$$
For the inequality that provides $C<\infty$, take $\lambda(s)=k(s)\sqrt{\int_0^s k(u)^{-1}du}$.
3. Write $k(t)=\mu(t)\mathrm{e}^{-ct}$ for $\mu(t)$ a decreasing function. The argument of Lemma 2.1 of [BH92] will now verify the Muckenhoupt criterion for $k(t)$.

Thus we now know that the $B_1$ and $B_2$ of the isomorphism proof are bounded for many more $k(t)$ than previously. The other cochain homotopies constructed thus far provide the following theorem.

**Theorem 6.1.** *Suppose a smooth function $k(t)$ on $[0,\infty)$ has $k'(t)k(t)^{-1}\le -c$ for $c>0$. Let $(M,g)$ satisfy Definition 2.1, except that the $e^{-t}$ in Equation (1) is replaced by $k(t)$. Let $\Xi\in\mathbb{R}^m$, and let $w_\Xi$ be defined as in Definition 2.4 except that $w_\Xi$ restricts to $k(t)^{(-2\Xi(i)-n+1)}$ on each end. Let $H^q_{(2),\Xi}(M)$ now refer to this $w_\Xi$ weighted $L_2$ cohomology.*

1. *Let $H^*_{(2),\Xi}(M) := \oplus_{q=0}^n H^q_{(2),\Xi}(M)$. Then $H^*_{(2),\Xi}(M)$ is infinite dimensional if and only if $\exists\ \Xi(i)\in\mathbb{Z}\cap[-n+1,0]$.*
2. *Suppose $\Xi(i)\in\mathbb{R}-(\mathbb{Z}\cap[-n+1,0])$ for all $i$ (i.e., $H^*_{(2),\Xi}(M)$ is finite dimensional). Let $\xi$ be a tuple of integers in $(\mathbb{Z}+\frac{1}{2})^m$ so that $\Xi(i)$ and $\xi(i)$ are in the same component of $\mathbb{R}-(\mathbb{Z}\cap[-n+1,0])$ for $1\le i\le m$. Let $\mathcal{W}^\xi H^q(\overline{M})$*

*be the standard extension of Definition* 3.1. *Then*

$$H^q_{(2),\Xi}(M) \cong \mathcal{W}^\xi H^q(\overline{M}), \quad 0 \le q \le n.$$

**Sketch of proof.** Let $k(t)$ be as in the theorem, and let $g(t) := k(t)^\alpha$ for $\alpha \neq 0$ in $\mathbb{R}$. Then $g'(t)g(t)^{-1} \le \alpha^{-1}(-c)$, and $g(t)$ satisfies the Muckenhoupt criterion as well. Thus in the case of the second item, all of the $A_{(2),k(t)^\alpha}(0,\infty)$ output by Proposition 4.2 are computable. In the case of the first item, the old argument applies since $k(t)^0 \equiv 1$. □

A second interesting application will be stated briefly.

**Theorem 6.2.** *Let $\alpha$ be a nonzero real number, and let $(M,g)$ be a.h. Let $w_\alpha$ be a continuous positive weighting function on $M$ restricting to $e^{\alpha t^2}$ on each of the end coordinates of Definition* 2.1. *Then for $0 \le q \le n$, $H^q_{(2),w_\alpha}(M)$ is isomorphic to either $H^q_c(M)$ or $H^q(M)$ for $\alpha > 0$ or $\alpha < 0$, respectively.*

**Sketch of proof.** Let $\beta$ be any real number, including possibly zero. Then

$$\frac{d}{dt}(\mathrm{e}^{\alpha t^2}\mathrm{e}^{\beta t}) = (2\alpha t + \beta)\mathrm{e}^{\alpha t^2}\mathrm{e}^{\beta t}.$$

If $\alpha$ is negative, a suitable $c$ exists for the derivative-inverse inequality when $t >> 0$. If $\alpha$ is positive, the dual derivative-inverse inequality is satisfied. □

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