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Higher Rank Graph C^*-Algebras

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Abstract. Building on recent work of Robertson and Steger, we associate a C^*–algebra to a combinatorial object which may be thought of as a higher rank graph. This C^*–algebra is shown to be isomorphic to that of the associated path groupoid. Various results in this paper give sufficient conditions on the higher rank graph for the associated C^*–algebra to be: simple, purely infinite and AF. Results concerning the structure of crossed products by certain natural actions of discrete groups are obtained; a technique for constructing rank 2 graphs from "commuting" rank 1 graphs is given.

Contents

In this paper we shall introduce the notion of a higher rank graph and associate a C^*–algebra to it in such a way as to generalise the construction of the C^*–algebra of a directed graph as studied in [CK, KPRR, KPR] (amongst others). Graph C^*–algebras include up to strong Morita equivalence Cuntz–Krieger algebras and AF algebras. The motivation for the form of our generalisation comes from the recent work of Robertson and Steger [RS1, RS2, RS3]. In [RS1] the authors study crossed product C^*–algebras arising from certain group actions on $\tilde{A}_2$-buildings and show that they are generated by two families of partial isometries which satisfy certain relations amongst which are Cuntz–Krieger type relations [RS1, Equations (2), (5)] as well as more intriguing commutation relations [RS1, Equation (7)]. In [RS2] they give a more general framework for studying such algebras involving certain families

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of commuting $0-1$ matrices. In particular the associated C^*–algebras are simple, purely infinite and generated by a family of Cuntz–Krieger algebras associated to these matrices. It is this framework which we seek to cast in graphical terms to include a wider class of examples (including graph C^*–algebras).

What follows is a brief outline of the paper. In the first section we introduce the notion of a higher rank graph as a purely combinatorial object: a small category Λ gifted with a degree map $d : \Lambda \to \mathbf{N}^k$ (called shape in [RS2]) playing the role of the length function. No detailed knowledge of category theory is required to read this paper. The associated C^*–algebra $C^*(\Lambda)$ is defined as the universal C^*–algebra generated by a family of partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ satisfying relations similar to those of [KPR]. (Our standing assumption is that our higher rank graphs satisfy conditions analogous to a directed graph being row–finite and having no sinks.) We then describe some basic examples and indicate the relationship between our formalism and that of [RS2].

In the second section we introduce the path groupoid $\mathcal{G}_\Lambda$ associated to a higher rank graph Λ (cf. [R, D, KPRR]). Once the infinite path space Λ^∞ is formed (and a few elementary facts are obtained) the construction is fairly routine. It follows from the gauge-invariant uniqueness theorem (Theorem 3.4) that $C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$. By the universal property $C^*(\Lambda)$ carries a canonical action of $\mathbf{T}^k$ defined by

$$\alpha_t(s_\lambda) = t^{d(\lambda)} s_\lambda \tag{1}$$

called the gauge action. In the third section we prove the gauge–invariant uniqueness theorem, which is the key result for analysing $C^*(\Lambda)$ (cf. [BPRS, aHR], see also [CK, RS2] where similar techniques are used to prove simplicity). It gives conditions under which a homomorphism with domain $C^*(\Lambda)$ is faithful: roughly speaking, if the homomorphism is equivariant for the gauge action and nonzero on the generators then it is faithful. This theorem has a number of interesting consequences, amongst which are the isomorphism mentioned above and the fact that the higher rank Cuntz–Krieger algebras of [RS2] are isomorphic to C^*–algebras associated to suitably chosen higher rank graphs.

In the fourth section we characterise, in terms of an aperiodicity condition on Λ, the circumstances under which the groupoid $\mathcal{G}_\Lambda$ is essentially free. This aperiodicity condition allows us to prove a second uniqueness theorem analogous to the original theorem of [CK]. In 4.8 and 4.9 we obtain conditions under which $C^*(\Lambda)$ is simple and purely infinite respectively which are similar to those in [KPR] but with the aperiodicity condition replacing condition (L).

In the next section we show that, given a functor $c : \Lambda \to G$ where G is a discrete group, then as in [KP] one may construct a skew product $G \times_c \Lambda$ which is also a higher rank graph. If G is abelian then there is a natural action $\alpha^c : \widehat{G} \to \operatorname{Aut} C^*(\Lambda)$ such that

$$\alpha^c_\chi(s_\lambda) = \langle \chi, c(\lambda) \rangle s_\lambda; \tag{2}$$

moreover $C^*(\Lambda) \rtimes_{\alpha^c} \widehat{G} \cong C^*(G \times_c \Lambda)$. Comparing (1) and (2) we see that the gauge action α is of the form α^d and as a consequence we may show that the crossed product of $C^*(\Lambda)$ by the gauge action is isomorphic to $C^*(\mathbf{Z}^k \times_d \Lambda)$; this C^*–algebra is then shown to be AF. By Takai duality $C^*(\Lambda)$ is strongly Morita equivalent to a crossed product of this AF algebra by the dual action of $\mathbf{Z}^k$. Hence $C^*(\Lambda)$ belongs to the bootstrap class $\mathcal{N}$ of C^*–algebras for which the UCT applies (see [RSc]) and is consequently nuclear. If a discrete group G acts freely on a k-graph Λ, then the quotient object Λ/G inherits the structure of a k–graph; moreover (as a generalisation of [GT, Theorem 2.2.2]) there is a functor $c : \Lambda/G \to G$ such that $\Lambda \cong G \times_c (\Lambda/G)$ in an equivariant way. This fact allows us to prove that

$$C^*(\Lambda) \rtimes G \cong C^*(\Lambda/G) \otimes \mathcal{K}\left(\ell^2(G)\right)$$

where the action of G on $C^*(\Lambda)$ is induced from that on Λ. Finally in Section 6, a technique for constructing a 2-graph from "commuting" 1-graphs A, B with the same vertex set is given. The construction depends on the choice of a certain bijection between pairs of composable edges: $\theta : (a, b) \mapsto (b', a')$ where $a, a' \in A^1$ and $b, b' \in B^1$; the resulting 2-graph is denoted $A *_\theta B$. It is not hard to show that every 2-graph is of this form.

Throughout this paper we let $\mathbf{N} = \{0, 1, \dots\}$ denote the monoid of natural numbers under addition. For $k \geq 1$, regard $\mathbf{N}^k$ as an abelian monoid under addition with identity 0 (it will sometimes be useful to regard $\mathbf{N}^k$ as a small category with one object) and canonical generators e_i for $i = 1, \dots, k$. We shall also regard $\mathbf{N}^k$ as the positive cone of $\mathbf{Z}^k$ under the usual coordinatewise partial order: thus $m \leq n$ if and only if $m_i \leq n_i$ for all i, where $m = (m_1, \dots, m_k)$, and $n = (n_1, \dots, n_k)$. (This makes $\mathbf{N}^k$ a lattice.)

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1. Higher rank graph C^*–algebras

In this section we first introduce what we shall call a higher rank graph as a purely combinatorial object. (We do not know whether this concept has been studied before.) Our definition of a higher rank graph is modelled on the path category of a directed graph (see [H], [Mu], [MacL, §II.7] and Example 1.3). Thus a higher rank graph will be defined to be a small category gifted with a degree map (called shape in [RS2]) satisfying a certain factorisation property. We then introduce the associated C^*–algebra whose definition is modelled on that of the C^*–algebra of a graph as well as the definition of [RS2].

Definitions 1.1. A k**-graph** (rank k graph or higher rank graph) (Λ, d) consists of a countable small category Λ (with range and source maps r and s respectively) together with a functor $d : \Lambda \to \mathbf{N}^k$ satisfying the **factorisation property:** for every $\lambda \in \Lambda$ and $m, n \in \mathbf{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$, $d(\nu) = n$. For $n \in \mathbf{N}^k$ we write $\Lambda^n := d^{-1}(n)$. A morphism between k-graphs (Λ_1, d_1) and (Λ_2, d_2) is a functor $f : \Lambda_1 \to \Lambda_2$ compatible with the degree maps.

Remarks 1.2. The factorisation property of 1.1 allows us to identify $\mathrm{Obj}(\Lambda)$, the objects of Λ with Λ^0. Suppose $\lambda\alpha = \mu\alpha$ in Λ then by the the factorisation property $\lambda = \mu$; left cancellation follows similarly. We shall write the objects of Λ as $u, v, w, \dots$ and the morphisms as greek letters $\lambda, \mu, \nu \dots$. We shall frequently refer to Λ as a k-graph without mentioning d explicitly.

It might be interesting to replace $\mathbf{N}^k$ in Definition 1.1 above by a monoid or perhaps the positive cone of an ordered abelian group.

Recall that $\lambda, \mu \in \Lambda$ are composable if and only if $r(\mu) = s(\lambda)$, and then $\lambda\mu \in \Lambda$; on the other hand two finite paths λ, μ in a directed graph may be composed to give the path $\lambda\mu$ provided that $r(\lambda) = s(\mu)$; so in 1.3 below we will need to switch the range and source maps.

Example 1.3. Given a 1-graph Λ, define $E^0 = \Lambda^0$ and $E^1 = \Lambda^1$. If we define $s_E(\lambda) = r(\lambda)$ and $r_E(\lambda) = s(\lambda)$ then the quadruple (E^0, E^1, r_E, s_E) is a directed graph in the sense of [KPR, KP]. On the other hand, given a directed graph $E = (E^0, E^1, r_E, s_E)$, then $E^* = \cup_{n \geq 0} E^n$, the collection of finite paths, may be viewed as small category with range and source maps given by $s(\lambda) = r_E(\lambda)$ and $r(\lambda) = s_E(\lambda)$. If we let $d : E^* \to \mathbf{N}$ be the length function (i.e., $d(\lambda) = n$ iff $\lambda \in E^n$) then (E^*, d) is a 1-graph.

We shall associate a C^*–algebra to a k-graph in such a way that for $k = 1$ the associated C^*–algebra is the same as that of the directed graph. We shall consider other examples later.

Definitions 1.4. The k-graph Λ is **row finite** if for each $m \in \mathbf{N}^k$ and $v \in \Lambda^0$ the set $\Lambda^m(v) := \{\lambda \in \Lambda^m : r(\lambda) = v\}$ is finite. Similarly Λ has **no sources** if $\Lambda^m(v) \neq \emptyset$ for all $v \in \Lambda^0$ and $m \in \mathbf{N}^k$.

Clearly if E is a directed graph then E is row finite (resp. has no sinks) if and only if E^* is row finite (resp. has no sources). Throughout this paper we will assume (unless otherwise stated) that any k-graph Λ is row finite and has no sources, that is

$$0 < \#\Lambda^n(v) < \infty \text{ for every } v \in \Lambda^0 \text{ and } n \in \mathbf{N}^k. \tag{3}$$

The Cuntz–Krieger relations [CK, p.253] and the relations given in [KPR, §1] may be interpreted as providing a representation of a certain directed graph by partial isometries and orthogonal projections. This view motivates the definition of $C^*(\Lambda)$.

Definitions 1.5. Let Λ be a k-graph (which satisfies the standing hypothesis (3)). Then $C^*(\Lambda)$ is defined to be the universal C^*–algebra generated by a family $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying:

(i) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections,
(ii) $s_{\lambda\mu} = s_\lambda s_\mu$ for all $\lambda, \mu \in \Lambda$ such that $s(\lambda) = r(\mu)$,
(iii) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$,
(iv) for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$ we have $s_v = \sum_{\lambda \in \Lambda^n(v)} s_\lambda s_\lambda^*$.

For $\lambda \in \Lambda$, define $p_\lambda = s_\lambda s_\lambda^*$ (note that $p_v = s_v$ for all $v \in \Lambda^0$). A family of partial isometries satisfying (i)–(iv) above is called a $*$–**representation** of Λ.

Remarks 1.6. (i) If $\{t_\lambda : \lambda \in \Lambda\}$ is a $*$–representation of Λ then the map $s_\lambda \mapsto t_\lambda$ defines a $*$–homomorphism from $C^*(\Lambda)$ to $C^*(\{t_\lambda : \lambda \in \Lambda\})$.

(ii) If E^* is the 1-graph associated to the directed graph E (see 1.3), then by restricting a $*$–representation to E^0 and E^1 one obtains a Cuntz–Krieger family for E in the sense of [KPR, §1]. Conversely every Cuntz–Krieger family for E extends uniquely to a $*$–representation of E^*.

(iii) In fact we only need the relation (iv) above to be satisfied for $n = e_i \in \mathbf{N}^k$ for $i = 1, \ldots, k$, the relations for all n will then follow (cf. [RS2, Lemma 3.2]). Note that the definition of $C^*(\Lambda)$ given in 1.5 may be extended to the case where there are sources by only requiring that relation (iv) hold for $n = e_i$ and then only if $\Lambda^{e_i}(v) \neq \emptyset$ (cf. [KPR, Equation (2)]).

(iv) For $\lambda, \mu \in \Lambda$ if $s(\lambda) \neq s(\mu)$ then $s_\lambda s_\mu^* = 0$. The converse follows from 2.11.

(v) Increasing finite sums of p_v's form an approximate identity for $C^*(\Lambda)$ (if Λ^0 is finite then $\sum_{v\in\Lambda^0} p_v$ is the unit for $C^*(\Lambda)$). It follows from relations (i) and (iv) above that for any $n \in \mathbf{N}^k$, $\{p_\lambda : d(\lambda) = n\}$ forms a collection of orthogonal projections (cf. [RS2, 3.3]); likewise increasing finite sums of these form an approximate identity for $C^*(\Lambda)$ (see 2.5).

(vi) The above definition is not stated most efficiently. Any family of operators $\{s_\lambda : \lambda \in \Lambda\}$ satisfying the above conditions must consist of partial isometries. The first two axioms could also be replaced by:

$$s_\lambda s_\mu = \begin{cases} s_{\lambda\mu} & \text{if } s(\lambda) = r(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

Examples 1.7. (i) If E is a directed graph, then by 1.6 (i) and (ii) we have $C^*(E^*) \cong C^*(E)$ (see 1.3).

(ii) For $k \geq 1$ let $\Omega = \Omega_k$ be the small category with objects $\mathrm{Obj}\,(\Omega) = \mathbf{N}^k$, and morphisms $\Omega = \{(m,n) \in \mathbf{N}^k \times \mathbf{N}^k : m \leq n\}$; the range and source maps are given by $r(m,n) = m$, $s(m,n) = n$. Let $d : \Omega \to \mathbf{N}^k$ be defined by $d(m,n) = n - m$. It is then straightforward to show that Ω_k is a k-graph and $C^*(\Omega_k) \cong \mathcal{K}\left(\ell^2(\mathbf{N}^k)\right)$.

(iii) Let $T = T_k$ be the semigroup $\mathbf{N}^k$ viewed as a small category, then if $d : T \to \mathbf{N}^k$ is the identity map then (T, d) is a k-graph. It is not hard to show that $C^*(T) \cong C(\mathbf{T}^k)$, where s_{e_i} for $1 \leq i \leq k$ are the canonical unitary generators.

(iv) Let $\{M_1, \ldots, M_k\}$ be square $\{0,1\}$ matrices satisfying conditions (H0)–(H3) of [RS2] and let $\mathcal{A}$ be the associated C^*-algebra. For $m \in \mathbf{N}^k$ let W_m be the collection of undecorated words in the finite alphabet A of shape m as defined in [RS2] then let

$$W = \bigcup_{m\in\mathbf{N}^k} W_m.$$

Together with range and source maps $r(\lambda) = o(\lambda)$, $s(\lambda) = t(\lambda)$ and product defined in [RS2, Definition 0.1] W is a small category. If we define $d : W \to \mathbf{N}^k$ by $d(\lambda) = \sigma(\lambda)$, then one checks that d satisfies the factorisation property, and then from the second part of (H2) we see that (W, d) is an irreducible k-graph in the sense that for all $u, v \in W_0$ there is $\lambda \in W$ such that $s(\lambda) = u$ and $r(\lambda) = v$.

We claim that the map $s_\lambda \mapsto s_{\lambda,s(\lambda)}$ for $\lambda \in W$ extends to a *-homomorphism $C^*(W) \to \mathcal{A}$ for which $s_\lambda s_\mu^* \mapsto s_{\lambda,\mu}$ (since these generate $\mathcal{A}$ this will show that the map is onto). It suffices to verify that $\{s_{\lambda,s(\lambda)} : \lambda \in W\}$ constitutes a $*$–representation of W. Conditions (i) and (iii) are easy to check, (iv) follows from [RS2, 0.1c,3.2] with $u = v \in W^0$. We check condition (ii): if $s(\lambda) = r(\mu)$

apply [RS2, 3.2]

$$s_{\lambda,s(\lambda)}s_{\mu,s(\mu)} = \sum_{W^{d(\mu)}(s(\lambda))} s_{\lambda\nu,\nu}s_{\mu,s(\mu)} = s_{\lambda\mu,\mu}s_{\mu,s(\mu)} = s_{\lambda\mu,s(\lambda\mu)}$$

where the sum simplifies using [RS2, 3.1, 3.3] . We shall show below that $C^*(W) \cong \mathcal{A}$.

We may combine higher rank graphs using the following fact, whose proof is straightforward.

Proposition 1.8. *Let (Λ_1, d_1) and (Λ_2, d_2) be rank k_1, k_2 graphs respectively, then $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$ is a rank $k_1 + k_2$ graph where $\Lambda_1 \times \Lambda_2$ is the product category and $d_1 \times d_2 : \Lambda_1 \times \Lambda_2 \to \mathbf{N}^{k_1+k_2}$ is given by $d_1 \times d_2(\lambda_1, \lambda_2) = (d_1(\lambda_1), d_2(\lambda_2)) \in \mathbf{N}^{k_1} \times \mathbf{N}^{k_2}$ for $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$.*

An example of this construction is discussed in [RS2, Remark 3.11]. It is clear that $\Omega_{k+\ell} \cong \Omega_k \times \Omega_\ell$ where $k, \ell > 0$.

Definition 1.9. Let $f : \mathbf{N}^\ell \to \mathbf{N}^k$ be a monoid morphism, then if (Λ, d) is a k-graph we may form the ℓ-graph $f^*(\Lambda)$ as follows: (the objects of $f^*(\Lambda)$ may be identified with those of Λ and) $f^*(\Lambda) = \{(\lambda, n) : d(\lambda) = f(n)\}$ with $d(\lambda, n) = n$, $s(\lambda, n) = s(\lambda)$ and $r(\lambda, n) = r(\lambda)$.

Examples 1.10. (i) Let Λ be a k–graph and put $\ell = 1$, then if we define the morphism $f_i(n) = ne_i$ for $1 \le i \le k$, we obtain the **coordinate graphs** $\Lambda_i := f_i^*(\Lambda)$ of Λ (these are 1–graphs).

(ii) Suppose E is a directed graph and define $f : \mathbf{N}^2 \to \mathbf{N}$ by $(m_1, m_2) \mapsto m_1 + m_2$; then the two coordinate graphs of $f^*(E^*)$ are isomorphic to E^*. We will show below that $C^*(f^*(E^*)) \cong C^*(E^*) \otimes C(\mathbf{T})$.

(iii) Suppose E and F are directed graphs and define $f : \mathbf{N} \to \mathbf{N}^2$ by $f(m) = (m, m)$ then $f^*(E^* \times F^*) = (E \times F)^*$ where $E \times F$ denotes the cartesian product graph (see [KP, Def. 2.1]).

Proposition 1.11. *Let Λ be a k-graph and $f : \mathbf{N}^\ell \to \mathbf{N}^k$ a monoid morphism, then there is a $*$–homomorphism $\pi_f : C^*(f^*(\Lambda)) \to C^*(\Lambda)$ such that $s_{(\lambda,n)} \mapsto s_\lambda$; moreover if f is surjective, then π_f is too.*

Proof. By 1.6(i) it suffices to show that this is a $*$–representation of $f^*(\Lambda)$. Properties (i)–(iii) are straightforward to verify and property (iv) follows by observing that for fixed $n \in \mathbf{N}^\ell$ and $v \in \Lambda^0$ the map $f^*(\Lambda)^n(v) \to \Lambda^{f(n)}(v)$ given by $(\lambda, n) \mapsto \lambda$ is a bijection. If f is surjective, then it is clear that every generator s_λ of $C^*(\Lambda)$ is in the range of π_f. □

Later in 3.5 we will also show that π_f is injective if f is injective.

2. The path groupoid

In this section we construct the path groupoid $\mathcal{G}_\Lambda$ associated to a higher rank graph (Λ, d) along the lines of [KPRR, §2]. Because some of the details are not quite the same as those in [KPRR, §2] we feel it is useful to sketch the construction. First we introduce the following analog of an infinite path in a higher rank graph:

Definitions 2.1. Let Λ be a k-graph, then

$$\Lambda^\infty = \{x : \Omega_k \to \Lambda : x \text{ is a } k\text{-graph morphism}\},$$

is the infinite path space of Λ. For $v \in \Lambda^0$ let $\Lambda^\infty(v) = \{x \in \Lambda^\infty : x(0) = v\}$. For each $p \in \mathbf{N}^k$ define $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma^p(x)(m,n) = x(m+p, n+p)$ for $x \in \Lambda^\infty$ and $(m,n) \in \Omega$. (Note that $\sigma^{p+q} = \sigma^p \circ \sigma^q$).

By our standing assumption (3) one can show that for every $v \in \Lambda^0$ we have $\Lambda^\infty(v) \neq \emptyset$. Our definition of Λ^∞ is related to the definition of W_∞, the space of infinite words, given in the proof of [RS2, Lemma 3.8]. If E^* is the 1-graph associated to the directed graph E then $(E^*)^\infty$ may be identified with E^∞.

Remarks 2.2. By the factorisation property the values of $x(0,m)$ for $m \in \mathbf{N}^k$ completely determine $x \in \Lambda^\infty$. To see this, suppose that $x(0,m)$ is given for all $m \in \mathbf{N}^k$ then for $(m,n) \in \Omega$, $x(m,n)$ is the unique element $\lambda \in \Lambda$ such that $x(0,n) = x(0,m)\lambda$.

More generally, let $\{n_j : j \geq 0\}$ be an increasing cofinal sequence in $\mathbf{N}^k$ with $n_0 = 0$ (for example one could take $n_j = jp$ where $p = (1, \dots, 1) \in \mathbf{N}^k$); then $x \in \Lambda^\infty$ is completely determined by the values of $x(0, n_j)$. Moreover, given a sequence $\{\lambda_j : j \geq 1\}$ in Λ such that $s(\lambda_j) = r(\lambda_{j+1})$ and $d(\lambda_j) = n_j - n_{j-1}$ there is a unique $x \in \Lambda^\infty$ such that $x(n_{j-1}, n_j) = \lambda_j$. For $(m,n) \in \Omega$ we define $x(m,n)$ by the factorisation property as follows: let j be the smallest index such that $n \leq n_j$. Then $x(m,n)$ is the unique element of degree $n-m$ such that $\lambda_1 \cdots \lambda_j = \mu x(m,n)\nu$ where $d(\mu) = m$ and $d(\nu) = n_j - n$. It is straightforward to show that x has the desired properties.

We now establish a factorisation property for Λ^∞ which is an easy consequence of the above remarks.

Proposition 2.3. *Let Λ be a rank k graph. For all $\lambda \in \Lambda$ and $x \in \Lambda^\infty$ with $x(0) = s(\lambda)$, there is a unique $y \in \Lambda^\infty$ such that $x = \sigma^{d(\lambda)}y$ and $\lambda = y(0, d(\lambda))$; we write $y = \lambda x$. Note that for every $x \in \Lambda^\infty$ and $p \in \mathbf{N}^k$ we have $x = x(0,p)\sigma^p x$.*

Proof. Fix $\lambda \in \Lambda$ and $x \in \Lambda^\infty$ with $x(0) = s(\lambda)$. The sequence $\{n_j : j \geq 0\}$ defined by $n_0 = 0$ and $n_j = (j-1)p + d(\lambda)$ for $j \geq 1$ is cofinal. Set $\lambda_1 = \lambda$ and $\lambda_j = x((j-2)p, (j-1)p)$ for $j \geq 2$ and let $y \in \Lambda^\infty$ be defined by the method given in 2.2. Then y has the desired properties. □

Next we construct a basis of compact open sets for the topology on Λ^∞ indexed by Λ.

Definitions 2.4. Let Λ be a rank k graph. For $\lambda \in \Lambda$ define

$$Z(\lambda) = \{\lambda x \in \Lambda^\infty : s(\lambda) = x(0)\} = \{x : x(0, d(\lambda)) = \lambda\}.$$

Remarks 2.5. Note that $Z(v) = \Lambda^\infty(v)$ for all $v \in \Lambda^0$. For fixed $n \in \mathbf{N}^k$ the sets $\{Z(\lambda) : d(\lambda) = n\}$ form a partition of Λ^∞ (see 1.6(v)); moreover for every $\lambda \in \Lambda$ we have

$$Z(\lambda) = \bigcup_{\substack{d(\mu)=n \\ r(\mu)=s(\lambda)}} Z(\lambda\mu). \tag{4}$$

We endow Λ^∞ with the topology generated by the collection $\{Z(\lambda) : \lambda \in \Lambda\}$. Note that the map given by $\lambda x \mapsto x$ induces a homeomorphism between $Z(\lambda)$ and

$Z(s(\lambda))$ for all $\lambda \in \Lambda$. Hence, for every $p \in \mathbf{N}^k$ the map $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ is a local homeomorphism.

Lemma 2.6. *For each $\lambda \in \Lambda$, $Z(\lambda)$ is compact.*

Proof. By 2.5 it suffices to show that $Z(v)$ is compact for all $v \in \Lambda^0$. Fix $v \in \Lambda^0$ and let $\{x_n\}_{n\geq 1}$ be a sequence in $Z(v)$. For every m, $x_n(0,m)$ may take only finitely many values (by (3)). Hence there is a $\lambda \in \Lambda^m$ such that $x_n(0,m) = \lambda$ for infinitely many n. We may therefore inductively construct a sequence $\{\lambda_j : j \geq 1\}$ in Λ^p such that $s(\lambda_j) = r(\lambda_{j+1})$ and $x_n(0,jp) = \lambda_1 \cdots \lambda_j$ for infinitely many n (recall $p = (1,\ldots,1) \in \mathbf{N}^k$). Choose a subsequence $\{x_{n_j}\}$ such that $x_{n_j}(0,jp) = \lambda_1 \cdots \lambda_j$. Since $\{jp\}$ is cofinal, there is a unique $y \in \Lambda^\infty(v)$ such that $y((j-1)p, jp) = \lambda_j$ for $j \geq 1$; then $x_{n_j} \to y$ and hence $Z(v)$ is compact. □

Note that Λ^∞ is compact if and only if Λ^0 is finite.

Definition 2.7. If Λ is k-graph then let

$$\mathcal{G}_\Lambda = \{(x,n,y) \in \Lambda^\infty \times \mathbf{Z}^k \times \Lambda^\infty : \sigma^\ell x = \sigma^m y, n = \ell - m\}.$$

Define range and source maps $r, s : \mathcal{G}_\Lambda \to \Lambda^\infty$ by $r(x,n,y) = x$, $s(x,n,y) = y$. For $(x,n,y), (y,\ell,z) \in \mathcal{G}_\Lambda$ set $(x,n,y)(y,\ell,z) = (x,n{+}\ell,z)$, and $(x,n,y)^{-1} = (y,-n,x)$; $\mathcal{G}_\Lambda$ is called the path groupoid of Λ (cf. [R, D, KPRR]).

One may check that $\mathcal{G}_\Lambda$ is a groupoid with $\Lambda^\infty = \mathcal{G}_\Lambda^0$ under the identification $x \mapsto (x,0,x)$. For λ, $\mu \in \Lambda$ such that $s(\lambda) = s(\mu)$ define

$$Z(\lambda,\mu) = \{(\lambda z, d(\lambda) - d(\mu), \mu z) : z \in \Lambda^\infty(s(\lambda))\}.$$

We collect certain standard facts about $\mathcal{G}_\Lambda$ in the following result.

Proposition 2.8. *Let Λ be a k–graph. The sets $\{Z(\lambda,\mu) : \lambda,\mu \in \Lambda, s(\lambda) = s(\mu)\}$ form a basis for a locally compact Hausdorff topology on $\mathcal{G}_\Lambda$. With this topology $\mathcal{G}_\Lambda$ is a second countable, r–discrete locally compact groupoid in which each $Z(\lambda,\mu)$ is a compact open bisection. The topology on Λ^∞ agrees with the relative topology under the identification of Λ^∞ with the subset $\mathcal{G}_\Lambda^0$ of $\mathcal{G}_\Lambda$.*

Proof. One may check that the sets $Z(\lambda,\mu)$ form a basis for a topology on $\mathcal{G}_\Lambda$. To see that multiplication is continuous, suppose that $(x,n,y)(y,\ell,z) = (x,n+\ell,z) \in Z(\gamma,\delta)$. Since $(x,n,y),(y,\ell,z)$ are composable in $\mathcal{G}_\Lambda$ there are $\kappa,\nu \in \Lambda$ and $t \in \Lambda^\infty$ such that $x = \gamma\kappa t$, $y = \nu t$ and $z = \delta\kappa t$. Hence $(x,k,y) \in Z(\gamma\kappa,\nu)$ and $(y,\ell,z) \in Z(\nu,\delta\kappa)$ and the product maps the open set $\mathcal{G}_\Lambda^2 \cap (Z(\gamma\kappa,\nu) \times Z(\nu,\delta\kappa))$ into $Z(\gamma,\delta)$. The remaining parts of the proof are similar to those given in [KPRR, Proposition 2.6]. □

Note that $Z(\lambda,\mu) \cong Z(s(\lambda))$, via the map $(\lambda z, d(\lambda) - d(\mu), \mu z) \mapsto z$. Again we note that in the case $k = 1$ we have $\Lambda = E^*$ for some directed graph E and the groupoid $\mathcal{G}_{E^*} \cong \mathcal{G}_E$, the graph groupoid of E which is described in detail in [KPRR, §2].

Proposition 2.9. *Let Λ be a k-graph and let $f : \mathbf{N}^\ell \to \mathbf{N}^k$ be a morphism. The map $x \mapsto f^*(x)$ given by $f^*(x)(m,n) = (x(f(m),f(n)), n-m)$ defines a continuous surjective map $f^* : \Lambda^\infty \to f^*(\Lambda)^\infty$. Moreover, if the image of f is cofinal* (*equivalently $f(p)$ is strictly positive in the sense that all of its coordinates are nonzero*) *then f^* is a homeomorphism.*

Proof. Given $x \in f^*(\Lambda)^\infty$ choose a sequence $\{m_i\}$ such that $n_j = \sum_{i=1}^j m_i$ is cofinal in $\mathbf{N}^\ell$. Set $n_0 = 0$ and let $\lambda_j \in \Lambda^{f(m_j)}$ be defined by the condition that $x(n_{j-1}, n_j) = (\lambda_j, m_j)$. We must show that there is an $x' \in \Lambda^\infty$ such that $x'(f(n_{j-1}), f(n_j)) = \lambda_j$. It suffices to show that the the intersection $\cap_j Z(\lambda_1 \cdots \lambda_j) \neq \emptyset$. But this follows by the finite intersection property. One checks that $x = f^*(x')$. Furthermore the inverse image of $Z(\lambda, n)$ is $Z(\lambda)$ and hence f^* is continuous.

Now suppose that the image of f is cofinal, then the procedure defined above gives a continuous inverse for f^*. Given $x \in f^*(\Lambda)^\infty$, then since $f(n_j)$ is cofinal, the intersection $\cap_j Z(\lambda_1 \cdots \lambda_j)$ contains a single point x'. Note that x' depends on x continuously. □

For higher rank graphs of the form $f^*(\Lambda)$ with f surjective (see 1.9), the associated groupoid $\mathcal{G}_{f^*(\Lambda)}$ decomposes as a direct product as follows.

Proposition 2.10. *Let Λ be a k-graph and let $f : \mathbf{N}^\ell \to \mathbf{N}^k$ be a surjective morphism. Then*

$$\mathcal{G}_{f^*(\Lambda)} \cong \mathcal{G}_\Lambda \times \mathbf{Z}^{\ell-k}.$$

Proof. Since f is surjective, the map $f^* : \Lambda^\infty \to f^*(\Lambda)^\infty$ is a homeomorphism (see 2.9). The map f extends to a surjective morphism $f : \mathbf{Z}^\ell \to \mathbf{Z}^k$. Let $j : \mathbf{Z}^k \to \mathbf{Z}^\ell$ be a section for f and let $i : \mathbf{Z}^{\ell-k} \to \mathbf{Z}^\ell$ be an identification of $\mathbf{Z}^{\ell-k}$ with $\ker f$. Then we get a groupoid isomorphism by the map

$$((x, n, y), m) \mapsto (f^*x, i(m) + j(n), f^*y),$$

where $((x, n, y), m) \in \mathcal{G}_\Lambda \times \mathbf{Z}^{\ell-k}$. □

Finally, as in [RS2, Lemma 3.8] we demonstrate that there is a nontrivial $*$–representation of (Λ, d).

Proposition 2.11. *Let (Λ, d) be a k-graph. Then there exists a representation $\{S_\lambda : \lambda \in \Lambda\}$ of Λ on a Hilbert space with all partial isometries S_λ nonzero.*

Proof. Let $\mathcal{H} = \ell^2(\Lambda^\infty)$, then for $\lambda \in \Lambda$ define $S_\lambda \in \mathcal{B}(\mathcal{H})$ by

$$S_\lambda e_y = \begin{cases} e_{\lambda y} & \text{if } s(\lambda) = y(0), \\ 0 & \text{otherwise,} \end{cases}$$

where $\{e_y : y \in \Lambda^\infty\}$ is the canonical basis for $\mathcal{H}$. Notice that S_λ is nonzero since $\Lambda^\infty(s(\lambda)) \neq \emptyset$; one then checks that the family $\{S_\lambda : \lambda \in \Lambda\}$ satisfies conditions 1.5(i)–(iv). □

3. The gauge invariant uniqueness theorem

By the universal property of $C^*(\Lambda)$ there is a canonical action of the k-torus $\mathbf{T}^k$, called the **gauge action**: $\alpha : \mathbf{T}^k \to \operatorname{Aut} C^*(\Lambda)$ defined for $t = (t_1, \ldots, t_k) \in \mathbf{T}^k$ and $s_\lambda \in C^*(\Lambda)$ by

$$\alpha_t(s_\lambda) = t^{d(\lambda)} s_\lambda \tag{5}$$

where $t^m = t_1^{m_1} \cdots t_k^{m_k}$ for $m = (m_1, \ldots, m_k) \in \mathbf{N}^k$. It is straightforward to show that α is strongly continuous. As in [CK, Lemma 2.2] and [RS2, Lemma 3.6] we shall need the following.

Lemma 3.1. *Let Λ be a k-graph. Then for $\lambda, \mu \in \Lambda$ and $q \in \mathbf{N}^k$ with $d(\lambda)$, $d(\mu) \le q$ we have*

$$s_\lambda^* s_\mu = \sum_{\substack{\lambda\alpha=\mu\beta \\ d(\lambda\alpha)=q}} s_\alpha s_\beta^*. \tag{6}$$

Hence every nonzero word in s_λ, s_μ^* may be written as a finite sum of partial isometries of the form $s_\alpha s_\beta^*$ where $s(\alpha) = s(\beta)$; their linear span then forms a dense $*$–subalgebra of $C^*(\Lambda)$.

Proof. Applying 1.5(iv) to $s(\lambda)$ with $n = q - d(\lambda)$, to $s(\mu)$ with $n = q - d(\mu)$ and using 1.5 (ii) we get

$$\begin{aligned} s_\lambda^* s_\mu &= p_{s(\lambda)} s_\lambda^* s_\mu p_{s(\mu)} = \left(\sum_{\Lambda^{q-d(\lambda)}(s(\lambda))} s_\alpha s_\alpha^* \right) s_\lambda^* s_\mu \left(\sum_{\Lambda^{q-d(\mu)}(s(\mu))} s_\beta s_\beta^* \right) \\ &= \left(\sum_{\Lambda^{q-d(\lambda)}(s(\lambda))} s_\alpha s_{\lambda\alpha}^* \right) \left(\sum_{\Lambda^{q-d(\mu)}(s(\mu))} s_{\mu\beta} s_\beta^* \right). \end{aligned} \tag{7}$$

By 1.6(iv) if $d(\lambda\alpha) = d(\mu\beta)$ but $\lambda\alpha \neq \mu\beta$, then the range projections $p_{\lambda\alpha}$, $p_{\mu\beta}$ are orthogonal and hence one has $s_{\lambda\alpha}^* s_{\mu\beta} = 0$. If $\lambda\alpha = \mu\beta$ then $s_{\lambda\alpha}^* s_{\mu\beta} = p_v$ where $v = s(\alpha)$ and so $s_\alpha s_{\lambda\alpha}^* s_{\mu\beta} s_\beta^* = s_\alpha p_v s_\beta^* = s_\alpha s_\beta^*$; formula (6) then follows from formula (7). The rest of the proof is now routine. □

Following [RS2, §4]: for $m \in \mathbf{N}^k$ let $\mathcal{F}_m$ denote the C^*–subalgebra of $C^*(\Lambda)$ generated by the elements $s_\lambda s_\mu^*$ for $\lambda, \mu \in \Lambda^m$ where $s(\lambda) = s(\mu)$, and for $v \in \Lambda^0$ denote $\mathcal{F}_m(v)$ the C^*–subalgebra generated by $s_\lambda s_\mu^*$ where $s(\lambda) = v$.

Lemma 3.2. *For $m \in \mathbf{N}^k$, $v \in \Lambda^0$ there exist isomorphisms*

$$\mathcal{F}_m(v) \cong \mathcal{K}\left(\ell^2(\{\lambda \in \Lambda^m : s(\lambda) = v\})\right)$$

and $\mathcal{F}_m \cong \bigoplus_{v \in \Lambda^0} \mathcal{F}_m(v)$. Moreover, the C^*–algebras $\mathcal{F}_m$, $m \in \mathbf{N}^k$, form a directed system under inclusion, and $\mathcal{F}_\Lambda = \overline{\cup \mathcal{F}_m}$ is an AF C^*–algebra.

Proof. Fix $v \in \Lambda^0$ and let λ, μ, α, $\beta \in \Lambda^m$ be such that $s(\lambda) = s(\mu)$ and $s(\alpha) = s(\beta)$, then by 1.6(v) we have

$$\left(s_\lambda s_\mu^*\right)\left(s_\alpha s_\beta^*\right) = \delta_{\mu,\alpha} s_\lambda s_\beta^*, \tag{8}$$

so that the map which sends $s_\lambda s_\mu^* \in \mathcal{F}_m(v)$ to the matrix unit

$$e_{\lambda,\mu}^v \in \mathcal{K}\left(\ell^2(\{\lambda \in \Lambda^m : s(\lambda) = v\})\right)$$

for all λ, $\mu \in \Lambda^m$ with $s(\lambda) = s(\mu) = v$ extends to an isomorphism. The second isomorphism also follows from (8) (since $s(\mu) \neq s(\alpha)$ implies $\mu \neq \alpha$). We claim that $\mathcal{F}_m$ is contained in $\mathcal{F}_n$ whenever $m \le n$. To see this we apply 1.5(iv) to give

$$s_\lambda s_\mu^* = s_\lambda p_{s(\lambda)} s_\mu^* = \sum_{\Lambda^\ell(s(\lambda))} s_\lambda s_\gamma s_\gamma^* s_\mu^* = \sum_{\Lambda^\ell(s(\lambda))} s_{\lambda\gamma} s_{\mu\gamma}^* \tag{9}$$

where $\ell = n - m$. Hence the C^*–algebras $\mathcal{F}_m$, $m \in \mathbf{N}^k$, form a directed system as required. □

Note that $\mathcal{F}_\Lambda$ may also be expressed as the closure of $\cup_{j=1}^{\infty}\mathcal{F}_{jp}$ where $p = (1,\ldots,1) \in \mathbf{N}^k$.

Clearly for $t \in \mathbf{T}^k$ the gauge automorphism α_t defined in (5) fixes those elements $s_\lambda s_\mu^* \in C^*(\Lambda)$ with $d(\lambda) = d(\mu)$ (since $\alpha_t(s_\lambda s_\mu^*) = t^{d(\lambda)-d(\mu)} s_\lambda s_\mu^*$) and hence $\mathcal{F}_\Lambda$ is contained in the fixed point algebra $C^*(\Lambda)^\alpha$. Consider the linear map on $C^*(\Lambda)$ defined by

$$\Phi(x) \;=\; \int_{\mathbf{T}^k} \alpha_t(x)\,dt$$

where dt denotes normalised Haar measure on $\mathbf{T}^k$ and note that $\Phi(x) \in C^*(\Lambda)^\alpha$ for all $x \in C^*(\Lambda)$. As the proof of the following result is now standard, we omit it (see [CK, Proposition 2.11], [RS2, Lemma 3.3], [BPRS, Lemma 2.2]).

Lemma 3.3. *Let Φ, $\mathcal{F}_\Lambda$ be as described above.*

(i) *The map Φ is a faithful conditional expectation from $C^*(\Lambda)$ onto $C^*(\Lambda)^\alpha$.*
(ii) *$\mathcal{F}_\Lambda = C^*(\Lambda)^\alpha$.*

Hence the fixed point algebra $C^*(\Lambda)^\alpha$ is an AF algebra. This fact is key to the proof of the gauge–invariant uniqueness theorem for $C^*(\Lambda)$ (see [BPRS, Theorem 2.1], [aHR, Theorem 2.3], see also [CK, RS2] where a similar technique is used in the proof of simplicity).

Theorem 3.4. *Let B be a C^*–algebra, $\pi : C^*(\Lambda) \to B$ be a homomorphism and let $\beta : \mathbf{T}^k \to Aut(B)$ be an action such that $\pi \circ \alpha_t = \beta_t \circ \pi$ for all $t \in \mathbf{T}^k$. Then π is faithful if and only if $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$.*

Proof. If $\pi(p_v) = 0$ for some $v \in \Lambda^0$ then clearly π is not faithful. Conversely, suppose that π is equivariant and that $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$. We first show that π is faithful on $C^*(\Lambda)^\alpha = \overline{\bigcup_{j\geq 0}\mathcal{F}_{jp}}$. For any ideal I in $C^*(\Lambda)^\alpha$, we have $I = \overline{\bigcup_{j\geq 0}(I \cap \mathcal{F}_{jp})}$ (see [B, Lemma 3.1], [ALNR, Lemma 1.3]). Thus it is enough to prove that π is faithful on each $\mathcal{F}_n$. But by 3.2 it suffices to show that it is faithful on $\mathcal{F}_n(v)$, for all $v \in \Lambda^0$. Fix $v \in \Lambda^0$ and $\lambda, \mu \in \Lambda^n$ with $s(\lambda) = s(\mu) = v$ we need only show that $\pi(s_\lambda s_\mu^*) \neq 0$. Since $\pi(p_v) \neq 0$ we have

$$0 \neq \pi(p_v^2) = \pi(s_\lambda^* s_\lambda s_\mu^* s_\mu) = \pi(s_\lambda^*)\pi(s_\lambda s_\mu^*)\pi(s_\mu).$$

Hence $\pi(s_\lambda s_\mu^*) \neq 0$ and π is faithful on $C^*(\Lambda)^\alpha$. Let $a \in C^*(\Lambda)$ be a nonzero positive element; then since Φ is faithful $\Phi(a) \neq 0$ and as π is faithful on $C^*(\Lambda^\alpha)$ we have

$$0 \neq \pi(\Phi(a)) = \pi\left(\int_{\mathbf{T}^k} \alpha_t(a)\,dt\right) = \int_{\mathbf{T}^k} \beta_t(\pi(a))\,dt,$$

hence $\pi(a) \neq 0$ and π is faithful on $C^*(\Lambda)$ as required. □

Corollary 3.5.

(i) *Let (Λ, d) be a k-graph and let $\mathcal{G}_\Lambda$ be its associated groupoid. Then there is an isomorphism $C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$ such that $s_\lambda \mapsto 1_{Z(\lambda,s(\lambda))}$ for $\lambda \in \Lambda$. Moreover, the canonical map $C^*(\mathcal{G}_\Lambda) \to C_r^*(\mathcal{G}_\Lambda)$ is an isomorphism.*
(ii) *Let $\{M_1, \ldots, M_k\}$ be a collection of matrices satisfying* (H0)–(H3) *of* [RS2] *and W the k-graph defined in* 1.7(iv). *Then $C^*(W) \cong \mathcal{A}$, via the map $s_\lambda \mapsto s_{\lambda,s(\lambda)}$ for $\lambda \in W$.*

(iii) *If Λ is a k-graph and $f : \mathbf{N}^\ell \to \mathbf{N}^k$ is injective, then the $*$-homomorphism $\pi_f : C^*(f^*(\Lambda)) \to C^*(\Lambda)$ (see 1.11) is injective. In particular the C^*–algebras of the coordinate graphs Λ_i for $1 \le i \le k$ form a generating family of subalgebras of $C^*(\Lambda)$. Moreover, if f is surjective then $C^*(f^*(\Lambda)) \cong C^*(\Lambda) \otimes C(\mathbf{T}^{\ell-k})$.*

(iv) *Let (Λ_i, d_i) be k_i-graphs for $i = 1, 2$, then $C^*(\Lambda_1 \times \Lambda_2) \cong C^*(\Lambda_1) \otimes C^*(\Lambda_2)$ via the map $s_{(\lambda_1,\lambda_2)} \mapsto s_{\lambda_1} \otimes s_{\lambda_2}$ for $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$.*

Proof. For (i) we note that $s_\lambda \mapsto 1_{Z(\lambda,s(\lambda))}$ for $\lambda \in \Lambda$ is a $*$-representation of Λ; hence there is a $*$-homomorphism $\pi : C^*(\Lambda) \to C^*(\mathcal{G}_\Lambda)$ such that $\pi(s_\lambda) = 1_{Z(\lambda,s(\lambda))}$ for $\lambda \in \Lambda$ (see 1.6(i)). Let β denote the $\mathbf{T}^k$-action on $C^*(\mathcal{G}_\Lambda)$ induced by the $\mathbf{Z}^k$-valued 1–cocycle defined on $\mathcal{G}_\Lambda$ by $(x, k, y) \mapsto k$ (see [R, II.5.1]); one checks that $\pi \circ \alpha_t = \beta_t \circ \pi$ for all $t \in \mathbf{T}^k$. Clearly for $v \in \Lambda^0$ we have $1_{Z(v,v)} \ne 0$, since $\Lambda^\infty(v) \ne \emptyset$ and π is injective. Surjectivity follows from the fact that $\pi(s_\lambda s_\mu^*) = 1_{Z(\lambda,\mu)}$ together with the observation that $C^*(\mathcal{G}_\Lambda) = \overline{\operatorname{span}}\{1_{Z(\lambda,\mu)}\}$. The same argument shows that $C_r^*(\mathcal{G}_\Lambda) \cong C^*(\Lambda)$ and so $C_r^*(\mathcal{G}_\Lambda) \cong C^*(\mathcal{G}_\Lambda)$[1] .

For (ii) we note that there is a surjective $*$-homomorphism $\pi : C^*(W) \to \mathcal{A}$ such that $\pi(s_\lambda) = s_{\lambda,s(\lambda)}$ for $\lambda \in W$ (see 1.7(iv)) which is clearly equivariant for the respective $\mathbf{T}^k$–actions. Moreover by [RS2, Lemma 2.9] we have $s_{v,v} \ne 0$ for all $v \in W_0 = A$ and so the result follows.

For (iii) note that the injection $f : \mathbf{N}^\ell \to \mathbf{N}^k$ extends naturally to a homomorphism $f : \mathbf{Z}^\ell \to \mathbf{Z}^k$ which in turn induces a map $\hat{f} : \mathbf{T}^k \to \mathbf{T}^\ell$ characterised by $\hat{f}(t)^p = t^{f(p)}$ for $p \in \mathbf{N}^\ell$. Let B be the fixed point algebra of the gauge action of $\mathbf{T}^k$ on $C^*(\Lambda)$ restricted to the kernel of $\hat{f}$. The gauge action restricted to B descends to an action of $\mathbf{T}^\ell = \mathbf{T}^k/\operatorname{Ker}\hat{f}$ on B which we denote $\overline{\alpha}$. Observe that for $t \in \mathbf{T}^k$ and $(\lambda, n) \in f^*(\Lambda)$ we have

$$\alpha_t(\pi_f(s_{(\lambda,n)})) = t^{f(n)} s_\lambda = \hat{f}(t)^n s_\lambda;$$

hence $\operatorname{Im}\pi_f \subseteq B$ (if $t \in \operatorname{Ker}\hat{f}$, then $\hat{f}(t)^n = 1$). By the same formula we see that $\pi_f \circ \alpha = \overline{\alpha} \circ \pi_f$ and the result now follows by 3.4. The last assertion follows from part (i) together with the fact that $\mathcal{G}_{f^*(\Lambda)} \cong \mathcal{G}_\Lambda \times \mathbf{Z}^{\ell-k}$ (see 2.10).

For (iv), define a map $\pi : C^*(\Lambda_1 \times \Lambda_2) \to C^*(\Lambda_1) \otimes C^*(\Lambda_2)$ given by $s_{(\lambda_1,\lambda_2)} \mapsto s_{\lambda_1} \otimes s_{\lambda_2}$; this is surjective as these elements generate $C^*(\Lambda_1) \otimes C^*(\Lambda_2)$. We note that $C^*(\Lambda_1) \otimes C^*(\Lambda_2)$ carries a $\mathbf{T}^{k_1+k_2}$ action β defined for $(t_1, t_2) \in \mathbf{T}^{k_1+k_2}$ and $(\lambda_0, \lambda_1) \in \Lambda_1 \times \Lambda_2$ by $\beta_{(t_1,t_2)}(s_{\lambda_1} \otimes s_{\lambda_2}) = \alpha_{t_1} s_{\lambda_1} \otimes \alpha_{t_2} s_{\lambda_2}$. Injectivity then follows by 3.4, since π is equivariant and for $(v, w) \in (\Lambda_1 \times \Lambda_2)^0$ we have $p_v \otimes p_w \ne 0$. □

Henceforth we shall tacitly identify $C^*(\Lambda)$ with $C^*(\mathcal{G}_\Lambda)$.

Remark 3.6. Let Λ be a k-graph and suppose that $f : \mathbf{N}^\ell \to \mathbf{N}^k$ is an injective morphism for which H, the image of f, is cofinal. Then π_f induces an isomorphism of $C^*(f^*(\Lambda))$ with its range, the fixed point algebra of the restriction of the gauge action to $H^\perp$.

4. Aperiodicity and its consequences

The aperiodicity condition we study in this section is an analog of condition (L) used in [KPR]. We first define what it means for an infinite path to be periodic or aperiodic.

[1]This can be also deduced from the amenability of $\mathcal{G}_\Lambda$ (see 5.5).

Definitions 4.1. For $x \in \Lambda^\infty$ and $p \in \mathbf{Z}^k$ we say that p is **a period** of x if for every $(m,n) \in \Omega$ with $m+p \geq 0$ we have $x(m+p, n+p) = x(m,n)$. We say that x is **periodic** if it has a nonzero period. We say that x is **eventually periodic** if $\sigma^n x$ is periodic for some $n \in \mathbf{N}^k$, otherwise x is said to be **aperiodic**.

Remarks 4.2. For $x \in \Lambda^\infty$ and $p \in \mathbf{Z}^k$, p is **a period** of x if and only if $\sigma^m x = \sigma^n x$ for all $m, n \in \mathbf{N}^k$ such that $p = m - n$. Similarly x is eventually periodic, with eventual period $p \neq 0$ if and only if $\sigma^m x = \sigma^n x$ for some $m, n \in \mathbf{N}^k$ such that $p = m - n$.

Definition 4.3. The k-graph Λ is said to satisfy the **aperiodicity condition** (A) if for every $v \in \Lambda^0$ there is an aperiodic path $x \in \Lambda^\infty(v)$.

Remark 4.4. Let E be a directed graph which is row finite and has no sinks. Then the associated 1-graph E^* satisfies the aperiodicity condition if and only if every loop in E has an exit (i.e., satisfies condition (L) of [KPR]). However, if we consider the 2-graph $f^*(E^*)$ where $f : \mathbf{N}^2 \to \mathbf{N}$ is given by $f(m_1, m_2) = m_1 + m_2$ then $p = (1, -1)$ is a period for every point in $f^*(E^*)^\infty$ (even if E has no loops).

Proposition 4.5. *The groupoid $\mathcal{G}_\Lambda$ is essentially free (i.e., the points with trivial isotropy are dense in $\mathcal{G}_\Lambda^0$) if and only if Λ satisfies the aperiodicity condition.*

Proof. Observe that if $x \in \Lambda^\infty$ is aperiodic then $\sigma^m x = \sigma^n x$ implies that $m = n$ and hence $x \in \Lambda^\infty = \mathcal{G}_\Lambda^0$ has trivial isotropy, and conversely. Hence $\mathcal{G}_\Lambda$ is essentially free if and only if aperiodic points are dense in Λ^∞. If aperiodic points are dense in Λ^∞ then Λ clearly satisfies the aperiodicity condition, for $Z(v) = \Lambda^\infty(v)$ must then contain aperiodic points for every $v \in \Lambda^0$. Conversely, suppose that Λ satisfies the aperiodicity condition, then for every $\lambda \in \Lambda$ there is $x \in \Lambda^\infty(s(\lambda))$ which is aperiodic. Then $\lambda x \in Z(\lambda)$ is aperiodic. Hence the aperiodic points are dense in Λ^∞. □

The isotropy group of an element $x \in \Lambda^\infty$ is equal to the subgroup of its eventual periods (including 0).

Theorem 4.6. *Let $\pi : C^*(\Lambda) \to B$ be a $*$–homomorphism and suppose that Λ satisfies the aperiodicity condition. Then π is faithful if and only if $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$.*

Proof. If $\pi(p_v) = 0$ for some $v \in \Lambda^0$ then clearly π is not faithful. Conversely, suppose $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$; then by 3.5(i) we have $C^*(\Lambda) = C_r^*(\mathcal{G}_\Lambda)$ and hence from [KPR, Corollary 3.6] it suffices to show that π is faithful on $C_0(\mathcal{G}_\Lambda^0)$. If the kernel of the restriction of π to $C_0(\mathcal{G}_\Lambda^0)$ is nonzero, it must contain the characteristic function $1_{Z(\lambda)}$ for some $\lambda \in \Lambda$. It follows that $\pi(s_\lambda s_\lambda^*) = 0$ and hence $\pi(s_\lambda) = 0$; in which case $\pi(p_{s(\lambda)}) = \pi(s_\lambda^* s_\lambda) = 0$, a contradiction. □

Definition 4.7. We say that Λ is **cofinal** if for every $x \in \Lambda^\infty$ and $v \in \Lambda^0$ there is $\lambda \in \Lambda$ and $n \in \mathbf{N}^k$ such that $s(\lambda) = x(n)$ and $r(\lambda) = v$.

Proposition 4.8. *Suppose Λ satisfies the aperiodicity condition, then $C^*(\Lambda)$ is simple if and only if Λ is cofinal.*

Proof. By 3.5(i) $C^*(\Lambda) = C_r^*(\mathcal{G}_\Lambda)$; since $\mathcal{G}_\Lambda$ is essentially free, $C^*(\Lambda)$ is simple if and only if $\mathcal{G}_\Lambda$ is minimal. Suppose that Λ is cofinal and fix $x \in \Lambda^\infty$ and $\lambda \in \Lambda$;

then by cofinality there is a $\mu \in \Lambda$ and $n \in \mathbf{N}^k$ so that $s(\mu) = x(n)$ and $r(\mu) = s(\lambda)$. Then $y = \lambda\mu\sigma^n x \in Z(\lambda)$ and y is in the same orbit as x; hence all orbits are dense and $\mathcal{G}_\Lambda$ is minimal.

Conversely, suppose that $\mathcal{G}_\Lambda$ is minimal and that $x \in \Lambda^\infty$ and $v \in \Lambda^0$. Then there is $y \in Z(v)$ such that x, y are in the same orbit. Hence there exist $m, n \in \mathbf{N}^k$ such that $\sigma^n x = \sigma^m y$; then it is easy to check that $\lambda = y(0, m)$ and n have the desired properties. □

Notice that second hypothesis used in the following corollary is the analog of the condition that every vertex connects to a loop and it is equivalent to requiring that for every $v \in \Lambda^0$, there is an eventually periodic $x \in \Lambda^\infty(v)$ with positive eventual period (i.e., the eventual period lies in $\mathbf{N}^k \backslash \{0\}$). The proof follows the same lines as [KPR, Theorem 3.9].

Proposition 4.9. *Let Λ satisfy the aperiodicity condition. Suppose that for every $v \in \Lambda^0$ there are λ, $\mu \in \Lambda$ with $d(\mu) \neq 0$ such that $r(\lambda) = v$ and $s(\lambda) = r(\mu) = s(\mu)$. Then $C^*(\Lambda)$ is purely infinite in the sense that every hereditary subalgebra contains an infinite projection.*

Proof. Arguing as in [KPR, Lemma 3.8] one shows that $\mathcal{G}_\Lambda$ is locally contracting. The aperiodicity condition guarantees that $\mathcal{G}_\Lambda$ is essentially free, hence by [A-D, Proposition 2.4] (see also [LS]) we have $C^*(\Lambda) = C^*_r(\mathcal{G}_\Lambda)$ is purely infinite. □

5. Skew products and group actions

Let G be a discrete group, Λ a k-graph and $c : \Lambda \to G$ a functor. We introduce an analog of the skew product graph considered in [KP, §2] (see also [GT]); the resulting object, which we denote $G \times_c \Lambda$, is also a k-graph. As in [KP] if G is abelian the associated C^*–algebra is isomorphic to a crossed product of $C^*(\Lambda)$ by the natural action of $\widehat{G}$ induced by c (more generally it is a crossed product by a coaction — see [Ma, KQR]). As a corollary we show that the crossed product of $C^*(\Lambda)$ by the gauge action, $C^*(\Lambda) \rtimes_\alpha \mathbf{T}^k$, is isomorphic to $C^*(\mathbf{Z}^k \times_d \Lambda)$, the C^*–algebra of the skew-product k-graph arising from the degree map. It will then follow that $C^*(\Lambda) \rtimes_\alpha \mathbf{T}^k$ is AF and that $\mathcal{G}_\Lambda$ is amenable.

Definition 5.1. Let G be a discrete group, (Λ, d) a k-graph. Given $c : \Lambda \to G$ a functor then define the **skew product** $G \times_c \Lambda$ as follows: the objects are identified with $G \times \Lambda^0$ and the morphisms are identified with $G \times \Lambda$ with the following structure maps

$$s(g, \lambda) = (gc(\lambda), s(\lambda)) \quad \text{and} \quad r(g, \lambda) = (g, r(\lambda)).$$

If $s(\lambda) = r(\mu)$ then (g, λ) and $(gc(\lambda), \mu)$ are composable in $G \times_c \Lambda$ and

$$(g, \lambda)(gc(\lambda), \mu) = (g, \lambda\mu).$$

The degree map is given by $d(g, \lambda) = d(\lambda)$.

One must check that $G \times_c \Lambda$ is a k-graph. If $k = 1$ then any function $c : E^1 \to G$ extends to a unique functor $c : E^* \to G$ (as in [KP, §2]). The skew product graph $E(c)$ of [KP] is related to our skew product in a simple way: $G \times_c E^* = E(c)^*$. A key example of this construction arises by regarding the degree map d as a functor with values in $\mathbf{Z}^k$.

The functor c induces a cocycle $\tilde{c} : \mathcal{G}_\Lambda \to G$ as follows: given $(x, \ell - m, y) \in \mathcal{G}_\Lambda$ so that $\sigma^\ell x = \sigma^m y$ then set

$$\tilde{c}(x, \ell - m, y) = c(x(0, \ell))c(y(0, m))^{-1}.$$

As in [KP] one checks that this is well-defined and that $\tilde{c}$ is a (continuous) cocycle; regarding the degree map d as a functor with values in $\mathbf{Z}^k$, we have $\tilde{d}(x, n, y) = n$ for $(x, n, y) \in \mathcal{G}_\Lambda$. In the following we show that the skew product groupoid obtained from $\tilde{c}$ (as defined in [R]) is the same as the path groupoid of the skew product (cf. [KP, Theorem 2.4]).

Theorem 5.2. *Let G be a discrete group, Λ a k-graph and $c : \Lambda \to G$ a functor. Then $\mathcal{G}_{G \times_c \Lambda} \cong \mathcal{G}_\Lambda(\tilde{c})$ where $\tilde{c} : \mathcal{G}_\Lambda \to G$ is defined as above.*

Proof. We first identify $G \times \Lambda^\infty$ with $(G \times_c \Lambda)^\infty$ as follows: for $(g, x) \in G \times \Lambda^\infty$ define $(g, x) : \Omega \to G \times_c \Lambda$ by

$$(g, x)(m, n) = (gc(x(0, m)), x(m, n));$$

it is straightforward to check that this defines a degree–preserving functor and thus an element of $(G \times_c \Lambda)^\infty$. Under this identification $\sigma^n(g, x) = (gc(x(0, n)), \sigma^n x)$ for all $n \in \mathbf{N}^k$, $(g, x) \in (G \times_c \Lambda)^\infty$. As in the proof of [KP, Theorem 2.4] define a map $\phi : \mathcal{G}_\Lambda(\tilde{c}) \to \mathcal{G}_{G \times_c \Lambda}$ as follows: for $x, y \in \Lambda^\infty$ with $\sigma^\ell x = \sigma^m y$ set $\phi([x, \ell - m, y], g) = (x', \ell - m, y')$ where $x' = (g, x)$ and $y' = (g\tilde{c}(x, \ell - m, y), y)$. Note that

$$\begin{aligned} \sigma^m y' &= \sigma^m(g\tilde{c}(x, \ell - m, y), y) = \sigma^m(gc(x(0, \ell))c(y(0, m))^{-1}, y) \\ &= (gc(x(0, \ell)), \sigma^m y) = (gc(x(0, \ell)), \sigma^\ell x) = \sigma^\ell(g, x) \\ &= \sigma^\ell x', \end{aligned}$$

and hence $(x', \ell - m, y') \in \mathcal{G}_{G \times_c \Lambda}$. The rest of the proof proceeds as in [KP, Theorem 2.4] *mutatis mutandis.* □

Corollary 5.3. *Let G be a discrete abelian group, Λ a k-graph and $c : \Lambda \to G$ a functor. There is an action $\alpha^c : \widehat{G} \to \operatorname{Aut} C^*(\Lambda)$ such that for $\chi \in \widehat{G}$ and $\lambda \in \Lambda$*

$$\alpha^c_\chi(s_\lambda) = \langle \chi, c(\lambda) \rangle s_\lambda.$$

Moreover $C^*(\Lambda) \rtimes_{\alpha^c} \widehat{G} \cong C^*(G \times_c \Lambda)$. In particular the gauge action is of the form, $\alpha = \alpha^d$, and so $C^*(\Lambda) \rtimes_\alpha \mathbf{T}^k \cong C^*(\mathbf{Z}^k \times_d \Lambda)$.

Proof. Since $C^*(\Lambda)$ is defined to be the universal C^*–algebra generated by the s_λ's subject to the relations (1.5) and α^c preserves these relations it is clear that it defines an action of $\widehat{G}$ on $C^*(\Lambda)$. The rest of the proof follows in the same manner as that of [KP, Corollary 2.5] (see [R, II.5.7]). □

In order to show that $C^*(\Lambda) \rtimes_\alpha \mathbf{T}^k$ is AF, we need the following lemma.

Lemma 5.4. *Let Λ be a k-graph and suppose there is a map $b : \Lambda^0 \to \mathbf{Z}^k$ such that $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in \Lambda$, then $C^*(\Lambda)$ is AF.*

Proof. For every $n \in \mathbf{Z}^k$ let A_n be the closed linear span of elements of the form $s_\lambda s_\mu^*$ with $b(s(\lambda)) = n$. Fix λ, $\mu \in \Lambda$ with $b(s(\lambda)) = b(s(\mu)) = n$. We claim that $s_\lambda^* s_\mu = 0$ if $\lambda \neq \mu$. If $s_\lambda^* s_\mu \neq 0$ then by 3.1 there are α, $\beta \in \Lambda$ with $s(\lambda) = r(\alpha)$ and $s(\mu) = r(\beta)$ such that $\lambda\alpha = \mu\beta$; but then we have

$$d(\alpha) + n = d(\alpha) + b(s(\lambda)) = b(s(\lambda\alpha)) = b(s(\mu\beta)) = d(\beta) + b(s(\mu)) = d(\beta) + n.$$

Thus $d(\alpha) = d(\beta)$ and hence by the factorisation property $\alpha = \beta$. Consequently $\lambda = \mu$ by cancellation and the claim is established. It follows that for each v with $b(v) = n$ the elements $s_\lambda s_\mu^*$ with $s(\lambda) = s(\mu) = v$ form a system of matrix units and two systems associated to distinct v's are orthogonal (see 3.2). Hence we have

$$A_n \cong \bigoplus_{b(v)=n} \mathcal{K}\left(\ell^2(s^{-1}(v)\right).$$

By an argument similar to that in the proof of Lemma 3.2, if $n \le m$ then $A_n \subseteq A_m$ (see equation (9)); our conclusion now follows. □

Note that A_n in the above proof is the C^*–algebra of a subgroupoid of $\mathcal{G}_\Lambda$ which is isomorphic to the disjoint union

$$\bigsqcup_{b(v)=n} R_v \times \Lambda^\infty(v)$$

where R_v is the transitive principal groupoid on $s^{-1}(v)$. Since $\mathcal{G}_\Lambda$ is the increasing union of these elementary groupoids, it is an AF-groupoid and hence amenable (see [R, III.1.1]). The existence of such a function $b : \Lambda^0 \to \mathbf{Z}^k$ is not necessary for $C^*(\Lambda)$ to be AF since there are 1–graphs with no loops which do not have this property (see [KPR, Theorem 2.4]).

Theorem 5.5. *Let Λ be a k-graph, then $C^*(\Lambda) \rtimes_\alpha \mathbf{T}^k$ is AF and the groupoid $\mathcal{G}_\Lambda$ is amenable. Moreover, $C^*(\Lambda)$ falls in the bootstrap class $\mathcal{N}$ of* [RSc] *and is therefore nuclear. Hence, if $C^*(\Lambda)$ is simple and purely infinite* (*see Proposition* 4.9), *then it may be classified by its K-theory.*

Proof. Observe that the map $b : (\mathbf{Z}^k \times_d \Lambda)^0 \to \mathbf{Z}^k$ given by $b(n, v) = n$ satisfies

$$b(s(n,\lambda)) - b(r(n,\lambda)) = b(n + d(\lambda), \lambda) - b(n, r(\lambda)) = n + d(\lambda) - n = d(n,\lambda)$$

The first part of the result then follows from 5.4 and 5.3. To show that $\mathcal{G}_\Lambda$ is amenable we first observe that $\mathcal{G}_\Lambda(\tilde{d}) \cong \mathcal{G}_{\mathbf{Z}^k \times_d \Lambda}$ is amenable. Since $\mathbf{Z}^k$ is amenable, we may apply [R, Proposition II.3.8] to deduce that $\mathcal{G}_\Lambda$ is amenable. Since $C^*(\Lambda)$ is strongly Morita equivalent to the crossed product of an AF algebra by a $\mathbf{Z}^k$–action, it falls in the bootstrap class $\mathcal{N}$ of [RSc]. The final assertion follows from the Kirchberg-Phillips classification theorem (see [K, P]). □

We now consider free actions of groups on k-graphs (cf. [KP, §3]). Let Λ be a k-graph and G a countable group, then G **acts on** Λ if there is a group homomorphism $G \to \operatorname{Aut}\Lambda$ (automorphisms are compatible with all structure maps, including the degree): write $(g, \lambda) \mapsto g\lambda$. The action of G on Λ is said to be **free** if it is free on Λ^0. By the universality of $C^*(\Lambda)$ an action of G on Λ induces an action β on $C^*(\Lambda)$ such that $\beta_g s_\lambda = s_{g\lambda}$.

Given a free action of a group G on a k-graph Λ one forms the **quotient** Λ/G by the equivalence relation $\lambda \sim \mu$ if $\lambda = g\nu$ for some $g \in G$. One checks that all structure maps are compatible with $\sim$ and so Λ/G is also a k-graph.

Remark 5.6. Let G be a countable group and $c : \Lambda \to G$ a functor, then G acts freely on $G \times_c \Lambda$ by $g(h, \lambda) = (gh, \lambda)$; furthermore $(G \times_c \Lambda)/G \cong \Lambda$.

Suppose now that G acts freely on Λ with quotient Λ/G; we claim that Λ is isomorphic, in an equivariant way, to a skew product of Λ/G for some suitably chosen c (see [GT, Theorem 2.2.2]). Let q denote the quotient map. For every

$v \in (\Lambda/G)^0$ choose $v' \in \Lambda^0$ with $q(v') = v$ and for every $\lambda \in \Lambda/G$ let λ' denote the unique element in Λ such that $q(\lambda') = \lambda$ and $r(\lambda') = r(\lambda)'$. Now let $c : \Lambda/G \to G$ be defined by the formula

$$s(\lambda') = c(\lambda)s(\lambda)'.$$

We claim that $c(\lambda\mu) = c(\lambda)c(\mu)$ for all λ, $\mu \in \Lambda$ with $s(\lambda) = r(\mu)$. Note that

$$r(c(\lambda)\mu') = c(\lambda)r(\mu') = c(\lambda)r(\mu)' = c(\lambda)s(\lambda)' = s(\lambda');$$

hence, we have $(\lambda\mu)' = \lambda'(c(\lambda)\mu')$ (since the image of both sides agree under q and r). Thus

$$c(\lambda\mu)s(\mu)' = c(\lambda\mu)s(\lambda\mu)' = s[(\lambda\mu)'] = s(c(\lambda)\mu') = c(\lambda)s(\mu') = c(\lambda)c(\mu)s(\mu)'$$

which establishes the desired identity (since G acts freely on Λ). The map $(g, \lambda) \mapsto g\lambda'$ defines an equivariant isomorphism between $G \times_c (\Lambda/G)$ and Λ as required.

The following is a generalization of [KPR, 3.9, 3.10] and is proved similarly.

Theorem 5.7. *Let Λ be a k-graph and suppose that the countable group G acts freely on Λ, then*

$$C^*(\Lambda) \rtimes_\beta G \cong C^*(\Lambda/G) \otimes \mathcal{K}\left(\ell^2(G)\right).$$

Equivalently, if $c : \Lambda' \to G$ is a functor, then

$$C^*(G \times_c \Lambda') \rtimes_\beta G \cong C^*(\Lambda') \otimes \mathcal{K}\left(\ell^2(G)\right)$$

where β, the action of G on $C^*(G \times_c \Lambda')$, is induced by the natural action on $G \times_c \Lambda'$. If G is abelian this action is dual to α^c under the identification of 5.3.

Proof. The first statement follows from the second with $\Lambda' = \Lambda/G$; indeed, by 5.6 there is a functor $c : \Lambda/G \to G$ such that $\Lambda \cong G \times_c (\Lambda/G)$ in an equivariant way. The second statement follows from applying [KP, Proposition 3.7] to the natural G-action on $\mathcal{G}_{G \times_c \Lambda'} \cong \mathcal{G}_{\Lambda'}(\tilde{c})$. The final statement follows from the identifications

$$C^*(\Lambda) \rtimes_{\alpha^c} \widehat{G} \cong C^*(G \times_c \Lambda) \cong C^*(\mathcal{G}_\Lambda(\tilde{c}))$$

and [R, II.2.7]. □

6. 2-graphs

Given a k-graph Λ one obtains for each $n \in \mathbf{N}^k$ a matrix

$$M_\Lambda^n(u, v) = \#\{\lambda \in \Lambda^n : r(\lambda) = u, s(\lambda) = v\}.$$

By our standing assumption the entries are all finite and there are no zero rows. Note that for any $m, n \in \mathbf{N}^k$ we have $M_\Lambda^{m+n} = M_\Lambda^m M_\Lambda^n$ (by the factorisation property); consequently, the matrices M_Λ^m and M_Λ^n commute for all $m, n \in \mathbf{N}^k$. If W is the k-graph associated to the commuting matrices $\{M_1, \ldots, M_k\}$ satisfying conditions (H0)–(H3) of [RS2] which was considered in Example 1.7(iv), then one checks that $M_W^{e_i} = M_i^t$. Further, if $\Lambda = E^*$ is a 1-graph derived from the directed graph E, then M_Λ^1 is the vertex matrix of E.

Now suppose that A and B are 1-graphs with $A^0 = B^0 = V$ such the associated vertex matrices commute. Set $A^1 * B^1 = \{(\alpha, \beta) \in A^1 \times B^1 : s(\alpha) = r(\beta)\}$ and $B^1 * A^1 = \{(\beta, \alpha) \in B^1 \times A^1 : s(\beta) = r(\alpha)\}$; since the associated vertex matrices commute there is a bijection $\theta : (\alpha, \beta) \mapsto (\beta', \alpha')$ from $A^1 * B^1$ to $B^1 * A^1$ such that $r(\alpha) = r(\beta')$ and $s(\beta) = s(\alpha')$. We construct a 2-graph Λ from A, B and θ. This construction is very much in the spirit of [RS2]; roughly speaking an element in Λ of degree $(m,n) \in \mathbf{N}^2$ will consist of a rectangular grid of size (m,n) with edges of A horizontally, edges of B vertically and nodes in V arranged compatibly. First identify $\Lambda^0 = V$. For $(m,n) \in \mathbf{N}^2$ set $W(m,n) = \{(i,j) \in \mathbf{N}^2 : (i,j) \le (m,n)\}$. An element in $\Lambda^{(m,n)}$ is given by $v(i,j) \in V$ for $(i,j) \in W(m,n)$, $\alpha(i,j) \in A^1$ for $(i,j) \in W(m-1,n)$ and $\beta(i,j) \in B^1$ for $(i,j) \in W(m,n-1)$ (set $W(m,n) = \emptyset$ if m or n is negative) satisfying the following compatibility conditions wherever they make sense:

i. $r(\alpha(i,j)) = v(i,j)$ and $r(\beta(i,j)) = v(i,j)$
ii. $s(\alpha(i,j)) = v(i+1,j)$ and $s(\beta(i,j)) = v(i,j+1)$
iii. $\theta(\alpha(i,j), \beta(i+1,j)) = (\beta(i,j), \alpha(i,j+1))$;

for brevity and with a slight abuse of notation we regard this element as a triple (v,α,β) (note that α disappears if $m = 0$ and β disappears if $n = 0$ and v is determined by α and/or β if $mn \neq 0$). Set

$$\Lambda = \bigcup_{(m,n)} \Lambda^{(m,n)}$$

and define $s(v,\alpha,\beta) = v(m,n)$ and $r(v,\alpha,\beta) = v(0,0)$.

Note that if $\lambda \in A^m$ and $\mu \in B^n$ with $m,n > 0$ such that $s(\lambda) = r(\mu)$ there is a unique element $(v,\alpha,\beta) \in \Lambda^{(m,n)}$ such that $\lambda = \alpha(0,0)\alpha(1,0)\cdots\alpha(m-1,0)$ and $\mu = \beta(m,0)\beta(m,1)\cdots\beta(m,n-1)$; denote this element $\lambda\mu$. Further if $\lambda \in A^m$ and $\mu \in B^n$ with $m,n > 0$ such that $r(\lambda) = s(\mu)$ there is a unique element (v,α,β) in $\Lambda^{(m,n)}$ such that $\lambda = \alpha(0,n)\alpha(1,n)\cdots\alpha(m-1,n)$ and $\mu = \beta(0,0)\beta(0,1)\cdots\beta(0,n-1)$; denote this element $\mu\lambda$. Using these two facts it is not difficult to verify that given elements $(v,\alpha,\beta) \in \Lambda^{(m,n)}$ and $(v',\alpha',\beta') \in \Lambda^{(m',n')}$ with $v(m,n) = v'(0,0)$ there is a unique element $(v'',\alpha'',\beta'') \in \Lambda^{(m+m',n+n')}$ such that $v''(i,j) = v(i,j)$, $\alpha''(i,j) = \alpha(i,j)$, $\beta''(i,j) = \beta(i,j)$, $v''(m+i,n+j) = v'(i,j)$, $\alpha''(m+i,n+j) = \alpha'(i,j)$ and $\beta''(m+i,n+j) = \beta'(i,j)$ wherever these formulas make sense. Write $(v'',\alpha'',\beta'') = (v,\alpha,\beta)(v',\alpha',\beta')$. This defines composition in Λ; note that associativity and the factorisation property are built into the construction (as in [RS2]). Finally, we write $\Lambda = A *_\theta B$. It is straightforward to verify that up to isomorphism any 2-graph may be obtained from its constituent 1-graphs in this way.

If $A = B$, then we may take $\theta = \iota$ the identity map. In that case one has $A *_\iota A \cong f^*(A)$ where $f : \mathbf{N}^2 \to \mathbf{N}$ is given by $f(m,n) = m+n$. Hence, by Corollary 3.5(iii) we have $C^*(A *_\iota A) \cong C^*(A) \otimes C(\mathbf{T})$.

To further emphasise the dependence of the product $A *_\theta B$ on the bijection $\theta : A^1 * B^1 \to B^1 * A^1$ consider the following example.

Example 6.1. Let $A = B$ be the 1-graph derived from the directed graph which consists of one vertex and two edges, say $A^1 = \{e,f\}$ (note $C^*(A) \cong \mathcal{O}_2$). Then $A^1 * A^1 = \{(e,e),(e,f),(f,e),(f,f)\}$, and we define the bijection θ to be the flip. It is easy to show that $A *_\theta A \cong A \times A$; hence,

$$C^*(A *_\theta A) \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$$

where the first isomorphism follows from Corollary 3.5(iv) and the second from the Kirchberg-Phillips classification theorem (see [K, P]). But

$$C^*(A *_\iota A) \cong \mathcal{O}_2 \otimes C(\mathbf{T});$$

hence, $A *_\theta A \ncong A *_\iota A$.

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