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Stationary Measures for Random Walks in a Random Environment with Random Scenery

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Abstract. Let Γ act on a countable set V with only finitely many orbits. Given a Γ-invariant random environment for a Markov chain on V and a random scenery, we exhibit, under certain conditions, an equivalent stationary measure for the environment and scenery from the viewpoint of the random walker. Such theorems have been very useful in investigations of percolation on quasi-transitive graphs.

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1. Introduction

Given a state space for a Markov chain, one might assign transition probabilities randomly in order to finish specifying the Markov chain. In such a case, one speaks about *random walk in a random environment*, or RWRE for short. If we do not condition on the transition probabilities, such a stochastic process is usually no longer a Markov chain. The first investigation of RWREs is due to Solomon [9]. Their properties are often surprising.

Alternatively, given a completely specified Markov chain, which we shall refer to as a random walk, there might be a random field on the state space, i.e., a collection of random variables indexed by the state space. This random field is called a *random scenery*. As the random walker moves, he observes the scenery at his location. Perhaps the first explicit investigation of random walks in random scenery was Lang and Nguyen [6].

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Of course, one may combine these processes to obtain a *random walk in a random environment with random scenery*, or RWRERS for short. This has not been looked at much except in the case where the scenery arises from percolation on a graph and determines the environment (Häggström [4], Häggström and Peres [5], Lyons and Schramm [8]). In fact, the purpose of those investigations was to find out information about the scenery; the corresponding RWRE was used as a tool to probe the scenery.

In general, one would like a stationary probability measure on the trajectories of an RWRERS that is equivalent to (mutually absolutely continuous with) the natural probability measure giving the environment, the scenery, and the trajectory of the Markov chain given the environment. Here, stationarity means that when looked at from the viewpoint of the random walker, one should see a stationary environment and a stationary scenery. In order to make sense of this, one needs to be able to compare the environment and scenery at one state to those at another. The simplest assumption is that there is a group Γ of "symmetries" of the state space V that acts transitively on V. Then Γ induces an action on functions on V, in particular, on environments and sceneries. Restricting one's attention to the σ-field $\mathcal{I}$ of Γ-invariant events, one can ask whether there is a stationary probability measure on $\mathcal{I}$ that is equivalent to the natural one.

In many cases of interest, there is such a stationary probability measure that one can explicitly give. We present some general theorems of this sort. These are "soft" theorems, in contrast to most theorems in the literature that describe more quantitative behavior of the processes. There are some surprising phenomena even with such soft theorems. Compare the following two examples:

Example 1. Consider a regular tree $T = (V, E)$ of degree 3 and fix $o \in V$. Let Γ be the group of automorphisms of T. Declare each edge in E "open" with probability $2/3$ independently. Let ω consist of the subgraph formed by the open edges. Consider simple random walk starting at o on the connected component $C(o)$ of o in ω. This has an equivalent stationary initial probability measure, namely, the law of ω (product measure) biased by the degree of o in ω.

Example 2. With notation as above, let ζ be a fixed end of T. Let Γ_ζ be the group of automorphisms on T that fix ζ. This subgroup is also transitive on V. However, in this case, simple random walk on $C(o)$ does not have any stationary probability measure equivalent to the natural probability measure: Let $Y(x)$ be the vertex in $C(x)$ that is closest to ζ. Let $w(n)$ denote the location of the walker at time n. Let $\mathcal{A}_n$ be the event that $C\big(w(n)\big)$ is infinite and $w(n) = Y\big(w(n)\big)$; this event is Γ_ζ-invariant. Note that when the walker starts at o, we have $C\big(w(n)\big) = C(o)$ and $Y\big(w(n)\big) = Y(o)$. As time evolves, the probability of $\mathcal{A}_n$ tends to 0, yet the probability of $\mathcal{A}_0$ is positive.

It turns out that an important issue for finding a stationary measure is whether Γ is unimodular or not (see Section 2 for the definition). The group Γ of Example 1 is unimodular, but the group Γ_ζ of Example 2 is not. In many applications, V is a countable group Γ such as $\mathbb{Z}^d$, in which case Γ acts on itself by multiplication; since Γ is countable, it is unimodular.

In order to state one of our theorems, we need some notation. The space of trajectories of the walk is $V^{\mathbb{N}}$. Let $(\Xi, \mathcal{F})$ be a measurable space which will be used to define the environment and the scenery.

Define the shift $\mathcal{S} : V^{\mathbb{N}} \to V^{\mathbb{N}}$ by

$$(\mathcal{S}w)(n) := w(n+1)\,,$$

and let

$$\mathcal{S}(\xi, w) := (\xi, \mathcal{S}w) \qquad \forall(\xi, w) \in \Xi \times V^{\mathbb{N}}\,.$$

For $\gamma \in \Gamma$, we set

$$\gamma(\xi, w) := (\gamma\xi, \gamma w)\,,$$

where $(\gamma w)(n) := \gamma\big(w(n)\big)$.

A quadruple $(\Xi, \mathcal{F}, \mathbf{P}, \Gamma)$ is called a *measure-preserving dynamical system* if Γ acts measurably on the measure space $(\Xi, \mathcal{F}, \mathbf{P})$ preserving the measure $\mathbf{P}$. We call a measurable function $p : \Xi \times V \times V \to [0, 1]$, written $p : (\xi, x, y) \mapsto p_\xi(x, y)$, a *random environment* (from Ξ) if for all $\xi \in \Xi$ and all $x \in V$, we have $\sum_{y\in V} p_\xi(x, y) = 1$. The natural action of Γ on p is the one induced by the diagonal one, $(\gamma p)(\xi, x, y) := p(\gamma^{-1}\xi, \gamma^{-1}x, \gamma^{-1}y)$. Unless otherwise stated, we shall use such actions implicitly. Given $x \in V$ and a measurable map $\xi \mapsto \nu_\xi(x)$ from $\Xi \to [0, \infty)$, let $\widehat{\mathbf{P}}_x$ denote the joint distribution on $\Xi \times V^{\mathbb{N}}$ of ξ biased by $\nu_\xi(x)$ and the trajectory of the Markov chain determined by p_ξ starting at x. That is, if θ_ξ^x denotes the probability measure on $V^{\mathbb{N}}$ determined by p_ξ with $w_0 = x$, then for all events $\mathcal{A}$, we have

$$\widehat{\mathbf{P}}_x[\mathcal{A}] := \int_\Xi d\mathbf{P}(\xi)\,\nu_\xi(x) \int_{(\xi,w)\in\mathcal{A}} d\theta_\xi^x(w)\,.$$

Let $\mathcal{I}$ be the σ-field of Γ-invariant events in $\Xi \times V^{\mathbb{N}}$. *We assume throughout this note that Γ is a locally compact group acting on V and that all stabilizers of elements of V have finite Haar measure.*

The following theorem generalizes similar results in Häggström [4], Häggström and Peres [5], Lyons [7], and Lyons and Schramm [8].

Theorem 1. *Let V be a countable set with a transitive action by a unimodular group Γ. Let $(\Xi, \mathcal{F}, \mathbf{P}, \Gamma)$ be a measure-preserving dynamical system and p be a Γ-invariant random environment from Ξ. Suppose that $\nu : (\xi, x) \mapsto \nu_\xi(x)$ is a Γ-invariant measurable mapping from $\Xi \times V \to [0, \infty)$ such that for each $\xi \in \Xi$, ν_ξ is a stationary distribution for the Markov chain determined by p_ξ. Then for any $o \in V$, the restriction of $\widehat{\mathbf{P}}_o$ to the Γ-invariant σ-field is an $\mathcal{S}$-invariant measure; that is, $(\Xi \times V^{\mathbb{N}}, \mathcal{I}, \widehat{\mathbf{P}}_o, \mathcal{S})$ is a measure-preserving dynamical system. If $\mathbf{E}[\nu_\bullet(o)] = 1$, then $\widehat{\mathbf{P}}_o$ is a probability measure.*

As an example of an $\mathcal{I}$-measurable function, we offer $p_\xi(w(0), \bullet)$, the environment at the location of the walker. A function $\Upsilon : \Xi \times V \to \mathbb{R}$ can be regarded as a random real-valued scenery, where $\Upsilon(\xi, x)$ is the scenery at x given by the outcome ξ. If Υ is a Γ-invariant measurable function, then $\Upsilon\big(\xi, w(0)\big)$ is $\mathcal{I}$-measurable. Thus, the theorem implies that the walker will see a stationary scenery.

Example 3 (Alili [1])**.** Let $V := \Gamma := \mathbb{Z}$, $\Xi := (0,1)^{\mathbb{Z}}$, $\mathbf{P}$ be any $\mathbb{Z}$-invariant measure on Ξ, and for all $\xi \in \Xi$,

$$p_\xi(x,y) := \begin{cases} \xi(x) & \text{if } y = x+1, \\ 1-\xi(x) & \text{if } y = x-1, \\ 0 & \text{otherwise.} \end{cases}$$

Write $\rho(x) := \xi(x-1)/\xi(x)$. Suppose that $A(x) := \sum_{n \ge x} \prod_{k=x+1}^{n} \rho(x) < \infty$ a.s. Then $\nu_\xi(x) := \big(1+\rho(x)\big)A(x)$ is a stationary measure with $(\xi, x) \mapsto \nu_\xi(x)$ being $\mathbb{Z}$-invariant.

Example 4. Suppose that $G = (V, E)$ is a graph and Γ is a closed transitive (on vertices) group of automorphisms of G. Let $\mathbf{P}$ be a Γ-invariant probability measure on 2^E. That is, we choose a random subgraph of G. The case that $\mathbf{P}$ is product measure, as in Example 1, is called *Bernoulli percolation*. The random subgraph has connected components, often called "percolation clusters". These clusters are of great interest. One method that has recently proven quite powerful for studying the clusters is to use them for a random environment (and/or scenery). Namely, let D be the degree of vertices in G. Denote the subgraph by ω. An RWRE called *delayed simple random walk* is defined via the transition probabilities $p_\omega(x,y) := 1/D$ if $[x,y] \in \omega$ and $p_\omega(x,x) = d_\omega(x)/D$, where $d_\omega(x)$ is the degree of x in ω. This was introduced by Häggström [4] and used also by Häggström and Peres [5], Benjamini, Lyons, and Schramm [3], and Lyons and Schramm [8]. If Γ is unimodular, we take $\Xi := 2^E$ and $\nu \equiv 1$ in Theorem 1.

Example 5. In the same setting as Example 4, consider the transition probabilities $p_\omega(x,y) := 1/d_\omega(x)$ if $[x,y] \in \omega$ and $d_\omega(x) \neq 0$, with $p_\omega(x,x) = 1$ if $d_\omega(x) = 0$. This is called *simple random walk on percolation clusters*. In this case, we take $\nu_\omega(x) := d_\omega(x)$ if $d_\omega(x) \neq 0$ and $\nu_\omega(x) := 1$ if $d_\omega(x) = 0$. The paper by Benjamini, Lyons, and Schramm [3] gives a number of potential-theoretic properties of simple random walk on percolation clusters.

2. Definitions

Let V be a countable set. If Γ acts on V (on the left), we say that Γ is *transitive* if for every $x, y \in V$, there is a $\gamma \in \Gamma$ with $\gamma x = y$. If the orbit space $\Gamma \backslash V$ is finite, then Γ is *quasi-transitive*.

Recall that on every locally compact group Γ, there is a unique (up to a constant scaling factor) Borel measure $|\bullet|$ that, for every $\gamma \in \Gamma$, is invariant under left multiplication by γ; this measure is called (left) *Haar measure*. The group is *unimodular* if Haar measure is also invariant under right multiplication. For example, when Γ is countable, the Haar measure is (a constant times) counting measure, so Γ is unimodular.

When Γ acts on V, let

$$S(x) := \{\gamma \in \Gamma : \gamma x = x\}$$

denote the stabilizer of x. We shall write

$$m(x) := |S(x)| .$$

As stated prior to Theorem 1, we make the standing assumption that Γ is a locally compact group acting on V and that all stabilizers of elements of V have finite

Haar measure. Under this assumption, it is not hard to show that the following conditions are equivalent:

1. Γ is unimodular.
2. $m(\bullet)$ is Γ-invariant.
3. For all x and y in the same orbit, $|S(x)y| = |S(y)x|$.

(See Trofimov [10].)

Häggström [4] introduced the Mass-Transport Principle in studying percolation on regular trees. Following is a generalization.

Lemma 1. *Let Γ act quasi-transitively on V and $f : V \times V \to [0, \infty]$ be invariant under the diagonal action of Γ. Choose a complete set $\{o_1, \ldots, o_L\}$ of representatives in V of the orbits of Γ and write $m_i := m(o_i)$. Then*

$$\sum_{i=1}^{L} \sum_{z \in V} f(o_i, z) = \sum_{j=1}^{L} 1/m_j \sum_{y \in V} f(y, o_j) m(y) .$$

See Cor. 3.7 of Benjamini, Lyons, Peres, and Schramm [2].

3. Proofs

Theorem 1 generalizes as follows to quasi-transitive actions:

Theorem 2. *Let V be a countable set with a quasi-transitive action by a unimodular group Γ. Let $\{o_1, \ldots, o_L\}$ be a complete set of representatives of $\Gamma \backslash V$ and write $m_i := m(o_i)$. Let $(\Xi, \mathcal{F}, \mathbf{P}, \Gamma)$ be a measure-preserving dynamical system and p be a Γ-invariant random environment from Ξ. Suppose that $\nu : (\xi, x) \mapsto \nu_\xi(x)$ is a Γ-invariant measurable mapping from $\Xi \times V \to [0, \infty)$ such that for each $\xi \in \Xi$, ν_ξ is a stationary distribution for the Markov chain determined by p_ξ. Write*

$$\widehat{\mathbf{P}} := \sum_{i=1}^{L} m_i^{-1} \widehat{\mathbf{P}}_{o_i} .$$

Then the restriction of $\widehat{\mathbf{P}}$ to the Γ-invariant σ-field is an $\mathcal{S}$-invariant measure. If

$$\sum_i m_i^{-1} \mathbf{E}[\nu_\bullet(o_i)] = 1 ,$$

then $\widehat{\mathbf{P}}$ is a probability measure.

Still more generally, we may remove the hypothesis that Γ be unimodular by means of the following modification:

Theorem 3. *Let V be a countable set with a quasi-transitive action by a group Γ. Let $\{o_1, \ldots, o_L\}$ be a complete set of representatives of $\Gamma \backslash V$ and write $m_i := m(o_i)$. Let $(\Xi, \mathcal{F}, \mathbf{P}, \Gamma)$ be a measure-preserving dynamical system and p be a Γ-invariant random environment from Ξ. Suppose that $\nu : (\xi, x) \mapsto \nu_\xi(x)$ is a Γ-invariant measurable mapping from $\Xi \times V \to [0, \infty)$ such that for each $\xi \in \Xi$, $x \mapsto m(x)\nu_\xi(x)$ is a stationary distribution for the Markov chain determined by p_ξ. Write*

$$\widehat{\mathbf{P}} := \sum_{i=1}^{L} \widehat{\mathbf{P}}_{o_i} .$$

Then the restriction of $\widehat{\mathbf{P}}$ *to the* Γ*-invariant* σ*-field is an* $\mathcal{S}$*-invariant measure. If*

$$\sum_i \mathbf{E}[\nu_\bullet(o_i)] = 1\,,$$

then $\widehat{\mathbf{P}}$ *is a probability measure.*

Note that this incorporates Theorem 2 because when Γ is unimodular, the function $(\xi, x) \mapsto m(x)\nu_\xi(x)$ is Γ-invariant.

Proof. Let F be a Γ-invariant function on $\Xi \times V^{\mathbb{N}}$. We must show that $\int d\widehat{\mathbf{P}}\, F \circ \mathcal{S} = \int d\widehat{\mathbf{P}}\, F$.

Set

$$f(x, y; \xi) := \nu_\xi(x) p_\xi(x, y) \int d\theta_\xi^y(w)\, F(\xi, w)\,.$$

Thus, we have

$$\int d\widehat{\mathbf{P}}\, F \circ \mathcal{S} = \sum_{i=1}^{L} \sum_{y \in V} \int d\mathbf{P}(\xi)\, f(o_i, y; \xi)\,.$$

Our assumptions imply that f, and hence $\mathbf{E}[f(x, y; \bullet)]$, is Γ-invariant. Consequently, Lemma 1 gives

$$\begin{aligned}
\int d\widehat{\mathbf{P}}\, F \circ \mathcal{S} &= \sum_{j=1}^{L} \sum_{y \in V} \int d\mathbf{P}(\xi)\, m(y) f(y, o_j; \xi)/m_j \\
&= \sum_{j=1}^{L} \int d\mathbf{P}(\xi) \sum_{y \in V} \nu_\xi(y) m(y) p_\xi(y, o_j)/m_j \int d\theta_\xi^{o_j}(w)\, F(\xi, w) \\
&= \sum_{j=1}^{L} \int d\mathbf{P}(\xi)\, \nu_\xi(o_j) \int d\theta_\xi^{o_j}(w)\, F(\xi, w) = \int d\widehat{\mathbf{P}}\, F\,.
\end{aligned}$$

□

Example 6. Suppose that $G = (V, E)$ is a graph and Γ is a closed quasi-transitive group of automorphisms of G. Let $\mathbf{P}$ be a Γ-invariant probability measure on 2^E. Write

$$\alpha(x) := \sum_{[x,y] \in E} \sqrt{m(y)/m(x)}\,.$$

Given $\omega \in 2^E$, consider the transition probabilities $p_\omega(x, y) := \alpha(x)^{-1} \sqrt{m(y)/m(x)}$ for $[x, y] \in \omega$ and $p_\omega(x, x) := 1 - \sum_{[x,y] \in \omega} p_\omega(x, y)$. The resulting Markov chain on ω is reversible with stationary measure $x \mapsto m(x)\alpha(x)$. In the unimodular transitive case, this Markov chain is delayed simple random walk. Whether Γ is unimodular or not, we may take $\Xi := 2^E$ and $\nu_\omega(x) := \alpha(x)$ in Theorem 3.

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