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The Normal Symbol on Riemannian Manifolds

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Abstract. For an arbitrary Riemannian manifold X and Hermitian vector bundles E and F over X we define the notion of the normal symbol of a pseudodifferential operator P from E to F. The normal symbol of P is a certain smooth function from the cotangent bundle T^*X to the homomorphism bundle $\mathrm{Hom}(E, F)$ and depends on the metric structures, resp. the corresponding connections on X, E and F. It is shown that by a natural integral formula the pseudodifferential operator P can be recovered from its symbol. Thus, modulo smoothing operators, resp. smoothing symbols, we receive a linear bijective correspondence between the space of symbols and the space of pseudodifferential operators on X. This correspondence comprises a natural transformation between appropriate functors. A formula for the asymptotic expansion of the product symbol of two pseudodifferential operators in terms of the symbols of its factors is given. Furthermore an expression for the symbol of the adjoint is derived. Finally the question of invertibility of pseudodifferential operators is considered. For that we use the normal symbol to establish a new and general notion of elliptic pseudodifferential operators on manifolds.

Contents

1. Introduction

It is a well-known fact that on Euclidean space one can construct a canonical linear isomorphisms between symbol spaces and corresponding spaces of pseudodifferential operators. Furthermore one has natural formulas which represent a

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pseudodifferential operator in terms of its symbol, resp. which give an expression for the symbol of a pseudodifferential operator. By using symbols one gets much insight in the structure of pseudodifferential operators on Euclidean space. In particular they give the means to construct a (pseudo) inverse of an elliptic differential operator on $\mathbb{R}^d$.

Compared to the symbol calculus for pseudodifferential operators on $\mathbb{R}^n$ it seems that the symbol calculus for pseudodifferential operators on manifolds is not that well-established. But as the pure consideration of symbols and operators in local coordinates does not reveal the geometry and topology of the manifold one is working on, it is very desirable to build up a general theory of symbols for pseudodifferential operators on manifolds. In his articles [13, 14] Widom gave a proposal for a symbol calculus on manifolds. By using a rather general notion of a phasefunction, Widom constructs a map from the space of pseudodifferential operators on manifolds to the space of symbols and shows by an abstract argument for the case of scalar symbols that this map is bijective modulo smoothing operators, resp. symbols. In our paper we introduce a symbol calculus for pseudodifferential operators between vector bundles having the feature that both the symbol map and its inverse have a concrete representation. In particular we thus succeed in giving an integral representation for the inverse of our symbol map or in other words for the operator map. Essential for our approach is an appropriate notion of a phasefunction. Because of our special choice of a phasefunction the resulting symbol calculus is natural in a category theoretical sense.

Using the integral representation for the operator map, it is possible to write down a formula for the symbol of the adjoint of a pseudodifferential operator and for the symbol of the product of two pseudodifferential operators.

The normal symbol calculus will furthermore give us the means to build up a natural notion of elliptic symbols, respectively elliptic pseudodifferential operators on manifolds. It generalizes the classical notion of ellipticity as defined for example in Hörmander [7] and also the concept of ellipticity introduced by Douglis, Nirenberg [2]. Our framework of ellipticity allows the construction of parametrices of nonclassical, respectively nonhomogeneous, elliptic pseudodifferential operators. Moreover we do not need principal symbols for defining ellipticity. Instead we define elliptic operators in terms of their normal symbol, as the globally defined normal symbol of a pseudodifferential operator carries more information than its principal symbol. Moreover, the normal symbol of a pseudodifferential operator shows immediately whether the corresponding operator is invertible modulo smoothing operators or not.

Let us also mention that another application of the normal symbol calculus on manifolds lies in quantization theory. There one is interested in a quantization map associating pseudodifferential operators to certain functions on a cotangent bundle in a way that Dirac's quantization condition is fulfilled. But the inverse of a symbol map does exactly this, so it is a quantization map. See Pflaum [8, 9] for details.

Note that our work is also related to the recent papers of Yu. Safarov [11, 10].

2. Symbols

First we will define the notion of a symbol on a Riemannian vector bundle. Let us assume once and for all in this article that $\mu, \rho, \delta \in \mathbb{R}$ are real numbers such that $0 \leq \delta < \rho \leq 1$ and $1 \leq \rho + \delta$. The same shall hold for triples $\tilde{\mu}, \tilde{\rho}, \tilde{\delta} \in \mathbb{R}$.

We sometimes use properties of symbols on $\mathbb{R}^d \times \mathbb{R}^N$. As these are well-described in the mathematics literature we only refer the interested reader to Hörmander [6, 7], Shubin [12] or Grigis, Sjöstrand [5] for a general introduction to symbol theory on $\mathbb{R}^d \times \mathbb{R}^N$ and for proofs.

Definition 2.1. Let $E \to X$ be a Riemannian or Hermitian vector bundle over the smooth manifold X, $\varrho : TX \to X$ its tangent bundle and $\pi : T^*X \to X$ its cotangent bundle. Then an element $a \in \mathcal{C}^\infty(\pi^*(E|_U))$ with $U \subset X$ open is called a symbol over $U \subset X$ with values in E of order μ and type (ρ, δ) if for every trivialization $(x, \zeta) : T^*X|_V \to \mathbb{R}^{2d}$ and every trivialization $\Psi : E|V \to V \times \mathbb{R}^N$ with $V \subset U$ open the following condition is satisfied:

(Sy) For every $\alpha, \beta \in \mathbb{N}^d$ and every $K \subset V$ compact there exists $C = C_{K,\alpha,\beta} > 0$ such that for every $\xi \in T^*X|_K$ the inequality

$$\left|\left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^{|\beta|}}{\partial \zeta^\beta} \Psi(a(\xi)) \right|\right| \leq C \, (1 + ||\xi||)^{\mu + \delta|\alpha| - \rho|\beta|} \tag{1}$$

is valid.

The space of these symbols is denoted by $\mathrm{S}^\mu_{\rho,\delta}(U, T^*X, E)$ or shortly $\mathrm{S}^\mu_{\rho,\delta}(U, E)$. It gives rise to the sheaf $\mathrm{S}^\mu_{\rho,\delta}(\cdot, T^*X, E)$ of symbols on X with values in E of order μ and type (ρ, δ).

By defining $\mathrm{S}^{-\infty}_{\rho,\delta}(U, E) = \bigcap_{\mu \in \mathbb{N}} \mathrm{S}^\mu_{\rho,\delta}(U, E)$ and $\mathrm{S}^{\infty}_{\rho,\delta}(U, E) = \bigcup_{\mu \in \mathbb{N}} \mathrm{S}^\mu_{\rho,\delta}(U, E)$ we get the sheaf $\mathrm{S}^{-\infty}_{\rho,\delta}(\cdot, E)$ of smoothing symbols resp. the sheaf $\mathrm{S}^{\infty}_{\rho,\delta}(\cdot, E)$ of symbols of type (ρ, δ).

In case E is the trivial bundle $X \times \mathbb{C}$ of a Riemannian manifold X we write $\mathrm{S}^\mu_{\rho,\delta}$ with $\mu \in \mathbb{R} \cup \{\infty, -\infty\}$ for the corresponding symbol sheaves $\mathrm{S}^\mu_{\rho,\delta}(\cdot, X \times \mathbb{C})$.

The space of symbols $\mathrm{S}^\mu_{\rho,\delta}(U, TX, E)$ consists of functions $a \in \mathcal{C}^\infty(\varrho^*(E|_U))$ such that a fulfills condition (Sy). It gives rise to sheaves $\mathrm{S}^\mu_{\rho,\delta}(\cdot, TX, E)$, $\mathrm{S}^\infty_{\rho,\delta}(\cdot, TX, E)$ and $\mathrm{S}^{-\infty}_{\rho,\delta}(\cdot, TX, E)$.

It is possible to extend this definition to one of symbols over conic manifolds but we do not need a definition in this generality and refer the reader to Duistermaat [3] and Pflaum [8].

We want to give the symbol spaces $\mathrm{S}^\mu_{\rho,\delta}(U, E)$ a topological structure. So first choose a compact set $K \subset U$, a (not necessarily disjunct) partition $K = \bigcup\limits_{\iota \in J} K_\iota$ of K into compact subsets together with local trivializations $(x_\iota, \zeta_\iota) : T^*X|_{V_\iota} \to \mathbb{R}^{2d}$ and trivializations $\Psi_\iota : E|V_\iota \to V_\iota \times \mathbb{R}^N$ such that $K_\iota \subset V_\iota \subset U$ and V_ι open. Then we can attach for any $\alpha, \beta \in \mathbb{R}^d$ a seminorm $p = p_{(K_\iota, (x_\iota, \zeta_\iota)), \alpha, \beta} : \mathrm{S}^\mu_{\rho,\delta}(U, E) \to \mathbb{R}^+ \cup \{0\}$ to the symbol spaces $\mathrm{S}^\mu_{\rho,\delta}(U, E)$ by

$$p(a) = \sup_{\iota \in J} \left\{ \frac{\left|\left| \frac{\partial^{|\alpha|}}{\partial {x_\iota}^\alpha} \frac{\partial^{|\beta|}}{\partial {\zeta_\iota}^\beta} \Psi_\iota(a(\xi)) \right|\right|}{(1 + |\xi|)^{\mu + |\alpha|\delta - |\beta|\rho}} \; : \; \xi \in T^*X|_{K_\iota} \right\}. \tag{2}$$

The system of these seminorms gives $\mathrm{S}^{\mu}_{\rho,\delta}(U,E)$ the structure of a Fréchet space such that the restriction morphisms $\mathrm{S}^{\mu}_{\rho,\delta}(U,E) \to \mathrm{S}^{\mu}_{\rho,\delta}(V,E)$ for $V \subset U$ open are continuous. Additionally we have natural and continuous inclusions $\mathrm{S}^{\mu}_{\rho,\delta}(U,E) \subset \mathrm{S}^{\tilde{\mu}}_{\tilde{\rho},\tilde{\delta}}(U,E)$ for $\tilde{\mu} \geq \mu$, $\tilde{\rho} \leq \rho$ and $\tilde{\delta} \geq \delta$. Like in the case of symbols on $\mathbb{R}^d \times \mathbb{R}^N$ one can show that pointwise multiplication of symbols is continuous with respect to the Fréchet topology on $\mathrm{S}^{\mu}_{\rho,\delta}(U,E)$.

Proposition 2.2. *Let $a \in \mathcal{C}^{\infty}(\pi^*(E|_U))$. If there exists an open covering of U by patches V_ι with local trivializations $(x_\iota, \zeta_\iota) : T^*X|_{V_\iota} \to \mathbb{R}^{2d}$ and trivializations $\Psi_\iota : E|V_\iota \to V_\iota \times \mathbb{R}^N$ such that for every (x_ι,ζ_ι) the above condition (Sy) holds, then a is a symbol of order μ and type (ρ,δ).*

Proof. Let $(x,\zeta) : T^*X|_V \to \mathbb{R}^{2d}$, $V \subset U$ open be a trivialization. Then we can write on $V_\iota \cap V$

$$\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \frac{\partial^{|\beta|}}{\partial \zeta^{\beta}} = \sum_{\substack{\tilde{\alpha}+\alpha' \leq \alpha \\ \iota}} (f_{\tilde{\alpha},\alpha',\beta} \circ \pi)\, \zeta_\iota^{\alpha'} \frac{\partial^{|\tilde{\alpha}|}}{\partial {x_\iota}^{\tilde{\alpha}}} \frac{\partial^{|\beta+\alpha'|}}{\partial {\zeta_\iota}^{\beta+\alpha'}}\,, \tag{3}$$

where $f_{\tilde{\alpha},\alpha',\beta} \in \mathcal{C}^{\infty}(V_\iota \cap V)$. As $\rho + \delta \geq 1$ holds and (Sy) is true for (x_ι,ζ_ι), the claim now follows. $\square$

Example 2.3. (*i*) Let X be a Riemannian manifold. Then every smooth function $a : T^*U \to \mathbb{C}$ which is a polynomial function of order $\leq \mu$ on every fiber lies in $\mathrm{S}^{\mu}_{1,0}(U)$. In particular we have an embedding $\mathcal{D}_0 \to \mathrm{S}^{\infty}_{\rho,\delta}$, where $\mathcal{D}_0$ is the sheaf of polynomial symbols on X, i.e., for every $U \subset X$ open $\mathcal{D}_0(U) = \{f : T^*U \to \mathbb{C} :\ f|_{T^*_xX} \text{ is a polynomial for every } x \in U\}$.

(*ii*) Again let X be Riemannian. Then the mapping $l : T^*X \to \mathbb{C}$, $\xi \mapsto ||\xi||^2$ is a symbol of order 2 and type $(1,0)$ on X, but not one of order $\mu < 2$. Next regard the function $a : T^*X \to \mathbb{C}$, $\xi \mapsto \frac{1}{1+||\xi||^2}$. This function is a symbol of order -2 and type $(1,0)$ on X, but not one of order $\mu < -2$.

(*iii*) Assume $\varphi : X \to \mathbb{R}$ to be smooth and bounded. Then $a_\varphi : T^*X \to \mathbb{R}$, $\xi \mapsto (1+||\xi||^2)^{\varphi(\pi(\xi))}$ comprises a symbol of order $\mu = \sup_{x\in X}\varphi(x)$ and type $(1,\delta)$, where $0 < \delta < 1$ is arbitrary.

The following theorem is an essential tool for the use of symbols in the theory of partial differential equations and extends a well-known result for the case of $E = \mathbb{R}^d \times \mathbb{R}^N$ to arbitrary Riemannian vector bundles.

Theorem 2.4. *Let $a_j \in \mathrm{S}^{\mu_j}_{\rho,\delta}(U,E)$, $j \in \mathbb{N}$ be symbols such that $(\mu_j)_{j\in\mathbb{N}}$ is a decreasing sequence with $\lim\limits_{j\to\infty} \mu_j = -\infty$. Then there exists a symbol $a \in \mathrm{S}^{\mu_0}_{\rho,\delta}(U,E)$ unique up to smoothing symbols, such that $a - \sum_{j=0}^k a_j \in \mathrm{S}^{\mu_k}_{\rho,\delta}(U,E)$ for every $k \in \mathbb{N}$. This induces a locally convex Hausdorff topology on the quotient vector space $\mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(U,E)$, which is called the* topology of asymptotic convergence.

Proof. It is a well-known fact that the claim holds for $U \subset \mathbb{R}^d$ and $E = \mathbb{R}^d \times \mathbb{R}^N$ (see [7, 12, 5]). Covering U by trivializations $(z_\iota,\zeta_\iota) : T^*X|_{V_\iota} \to \mathbb{R}^{2d}$ and $\Psi_\iota : E|V_\iota \to V_\iota \times \mathbb{R}^N$ we can find a partition (φ_ι) of unity subordinate to the covering (V_ι) and for every index ι a symbol $b_{\iota,k} \in \mathrm{S}^{\mu}_{\rho,\delta}(V_\iota,\mathbb{R}^N)$ such that $b_{\iota,k} = \sum\limits_{j=0}^{k} \Psi_\iota \circ a_j|_{V_\iota} \in$

$\mathrm{S}^{\mu_k}_{\rho,\delta}(V_\iota,\mathbb{R}^N)$. Now define $a = \sum_\iota (\varphi_\iota \circ \pi)\, \Psi_\iota^{-1} \circ b_{\iota,k}$ and check that this a satisfies the claim. Uniqueness of a up to smoothing symbols is clear again from a local consideration. □

Polynomial symbols are not affected by smoothing symbols. The following proposition gives the precise statement.

Proposition 2.5. *Let X be a smooth manifold, and $\mathcal{D}_0$ the sheaf of polynomial symbols on X. Then by composition with the projection $\mathrm{S}^\infty_{\rho,\delta} \to \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}$ the canonical embedding*

$$\begin{array}{rcll} \mathcal{D}_0 & \to & \mathrm{S}^\infty_{\rho,\delta} & \\ \mathcal{D}_0(U) \ni f & \mapsto & f \in \mathrm{S}^\infty_{\rho,\delta}(U), & \quad U \subset X \text{ open} \end{array} \tag{4}$$

gives rise to a monomorphism $\delta : \mathcal{D}_0 \to \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}$ of sheaves of algebras.

Proof. If $f, g \in \mathcal{D}_0(U)$ are polynomial symbols such that $f - g \in \mathrm{S}^{-\infty}(U)$, then $f - g$ is a bounded polynomial function on each fiber of T^*U. Therefore $f - g$ is constant on each fiber, hence $f - g = 0$ by $f - g \in \mathrm{S}^{-\infty}(U)$. □

3. Pseudodifferential operators on manifolds

Let a be a polynomial symbol defined on the (trivial) cotangent bundle $T^*U = U \times \mathbb{R}^d$ of an open set $U \subset \mathbb{R}^d$ of Euclidean space. Then one can write $a = \sum_\alpha (a_\alpha \circ \pi)\zeta^\alpha$, where $\zeta : T^*U \to \mathbb{R}^d$ is the projection on the "cotangent vectors" and the a_α are smooth functions on U. According to standard results of partial differential equations one knows that the symbol a defines the differential operator $A = \mathrm{Op}\,(a)$:

$$\mathcal{C}^\infty(U) \ni f \mapsto \sum_\alpha a_\alpha\, (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial z^\alpha} f \in \mathcal{C}^\infty(U). \tag{5}$$

Sometimes A is called the quantization of a. In case $f \in \mathcal{D}(U)$, the Fourier transform of f is well-defined and provides the following integral representation for Af:

$$Af \;=\; \mathrm{Op}\,(a)\,(f) \;=\; \int_{\mathbb{R}^d} e^{i<\xi,\cdot>}\, a(\xi)\, \hat{f}(\xi)\, d\xi. \tag{6}$$

It would be very helpful for structural and calculational considerations to extend this formula to Riemannian manifolds. To achieve this it is necessary to have an appropriate notion of Fourier transform on manifolds. In the following we are going to define such a Fourier transform and will later get back to the problem of an integral representation for the "quantization map" Op on Riemannian manifolds.

Assume X to be Riemannian of dimension d and consider the exponential function exp with respect to the Levi-Civita connection on X. Furthermore let $E \to X$ be a Riemannian or Hermitian vector bundle. Choose an open neighborhood $W \subset TX$ of the zero section in TX such that $(\varrho, \exp) : W \to X \times X$ maps W diffeomorphically onto an open neighborhood of the diagonal Δ of $X \times X$. Then there exists a smooth function $\psi : TX \to [0,1]$ called a cut-off function such that $\psi|_{\tilde{W}} = 1$ and $\operatorname{supp}\psi \subset W$ for an open neighborhood $\tilde{W} \subset W$ of the zero section in TX. Next let us consider the unique torsionfree metric connection on E. It defines a parallel transport $\tau_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ for every smooth path $\gamma : [0,1] \to X$. For every $v \in TX$ denote by $\tau_{\exp v}$ or just $\tau_{y,x}$ with $y = \exp v$ and $x = \varrho(v)$ the parallel transport along $[0,1] \ni t \mapsto \exp tv \in X$. Now the following microlocal lift $\mathcal{M}_\psi$ is well-defined:

$$\mathcal{M}_\psi : \mathcal{C}^\infty(U,E) \to \mathrm{S}^{-\infty}(U,TX,E), \qquad f \mapsto f^\psi, = \begin{cases} \psi(v)\,\tau^{-1}_{\exp v}(f(\exp v)) & \text{for } v \in W \\ 0 & \text{else.} \end{cases} \tag{7}$$

$\mathcal{M}_\psi$ is a linear but not multiplicative map between function spaces.

Over the tangent bundle TX we can define the Fourier transform as the following sheaf morphism:

$$\mathcal{F} : \mathrm{S}^{-\infty}(U,TX,E) \to \mathrm{S}^{-\infty}(U,T^*X,E), \qquad a \mapsto \hat{a}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{T_{\pi(\xi)}X} e^{-i<\xi,v>}\, a(v)\, dv. \tag{8}$$

We also have a reverse Fourier transform:

$$\mathcal{F}^{-1} : \mathrm{S}^{-\infty}(U,T^*X,E) \to \mathrm{S}^{-\infty}(U,TX,E), \qquad b \mapsto \check{b}(v) = \frac{1}{(2\pi)^{n/2}} \int_{T^*_{\pi(v)}X} e^{i<\xi,v>}\, b(\xi)\, d\xi. \tag{9}$$

It is easy to check that $\mathcal{F}^{-1}$ and $\mathcal{F}$ are well-defined and inverse to each other.

Composing $\mathcal{M}_\psi$ and $\mathcal{F}$ gives rise to the Fourier transform $\mathcal{F}_\psi = \mathcal{F} \circ \mathcal{M}_\psi$ on the Riemannian manifold X. $\mathcal{F}_\psi$ has the left inverse

$$\mathrm{S}^{-\infty}(U,T^*X,E) \to \mathcal{C}^\infty(U,E), \quad a \mapsto \mathcal{F}^{-1}(a)|_{0_U}, \tag{10}$$

where $|_{0_U}$ means the restriction to the natural embedding of U into T^*X as zero section.

By definition $\mathcal{M}_\psi$ and $\mathcal{F}_\psi$ do depend on the smooth cut-off function ψ, but this arbitrariness will only have minor effects on the study of pseudodifferential operators defined through $\mathcal{F}_\psi$. We will get back to this point later on in this section.

The exponential function exp gives rise to normal coordinates $z_x : V_x \to \mathbb{R}^d$, where $x \in V_x \subset X$, V_x open and

$$\exp_x^{-1}(y) \;=\; \sum_{k=1}^{d} z_x^k(y) \cdot e_k(y) \tag{11}$$

for all $y \in V_x$ and an orthonormal frame $(e_1, ..., e_d)$ of TV_x. Furthermore we receive bundle coordinates $(z_x, \zeta_x) : T^*V_x \to \mathbb{R}^{2d}$ and $(z_x, v_x) : TV_x \to \mathbb{R}^{2d}$. Note that for fixed $x \in X$ the map

$$V_x \times T^*V_x \ni (y,\xi) \mapsto (z_y(\pi(\xi)), \zeta_y(\xi)) \in \mathbb{R}^{2d} \tag{12}$$

is smooth and that by the Gauß Lemma

$$z_x^k(y) = -z_y^k(x) \tag{13}$$

for every $y \in V_x$.

Now let a be a smooth function defined on T^*U and polynomial in the fibers. Then locally $a = \sum_\alpha (a_{x,\alpha} \circ \pi)\, \zeta_x$ with respect to a normal coordinate system at $x \in U$ and functions $a_{x,\alpha} \in \mathcal{C}^\infty(U)$. Define the operator $A : \mathcal{D}(U) \to \mathcal{C}^\infty(U)$ by

$$f \mapsto Af \;=\; \mathrm{Op}_\psi(a)\,(f) \;=\; \left(U \ni x \mapsto \frac{1}{(2\pi)^{n/2}} \int_{T_x^* X} a(\xi)\, \hat{f^\psi}(\xi)\, d\xi \in \mathbb{C} \right) \tag{14}$$

and check that

$$\begin{aligned} Af\,(x) &= \sum_\alpha a_{x,\alpha}(x)\,(-i)^{|\alpha|}\, \frac{\partial^{|\alpha|}}{\partial {v_x}^\alpha} \left[\psi\,(f \circ \exp\,) \right] (0_x) \\ &= \sum_\alpha a_{x,\alpha}(x)\,(-i)^{|\alpha|}\, \frac{\partial^{|\alpha|}}{\partial {z_x}^\alpha} f\,(x) \end{aligned} \tag{15}$$

for every $x \in U$. Therefore Af is a differential operator independent of the choice of ψ.

However, if a is an arbitrary element of the symbol space $\mathrm{S}^\infty_{\rho,\delta}(U, \mathrm{Hom}(E,F))$, Eq. (14) still defines a continuous operator $A_\psi = \mathrm{Op}_\psi(a) : \mathcal{D}(U,E) \to \mathcal{C}^\infty(U,F)$, which is a pseudodifferential operator but independent of the cut-off function ψ only up to smoothing operators. Let us show this in more detail for the scalar case, i.e., where $E = F = X \times \mathbb{C}$. The general case is proven analogously.

First choose $\mu \in \mathbb{R}$ such that $a \in \mathrm{S}^\mu_{\rho,\delta}(U)$, and consider the kernel distribution $\mathcal{D}(U \times U) \to \mathbb{C}$, $(f \otimes g) \mapsto \, < A_\psi f, g > \, = \int_X g(x)\, A_\psi f(x)\, dx$ of A_ψ. It can be written in the following way:

(16)

$$\begin{aligned} < g, A_\psi f > \; &= \frac{1}{(2\pi)^n} \int_X \int_{T_x^* X} \int_{T_x X} \psi(v)\, g(x)\, f(\exp v)\, e^{-i<\xi,v>}\, a(\xi)\, dv\, d\xi\, dx \\ &= \frac{1}{(2\pi)^n} \int_X \quad \int_{T_x X} \int_{T_x^* X} \psi(v)\, g(x)\, f(\exp v)\, e^{-i<\xi,v>}\, a(\xi)\, d\xi\, dv\, dx; \end{aligned}$$

where the first integral is an iterated one, the second one an oscillatory integral. To check that Eq. (16) is true use a density argument or in other words approximate the symbol a by elements $a_k \in \mathrm{S}^{-\infty}(U)$ (in the topology of $\mathrm{S}^{\tilde\mu}_{\rho,\delta}(U)$ with $\tilde\mu > \mu$) and prove the statement for the a_k. The claim then follows from continuity of both sides of Eq. (16) with respect to a. Let $\tilde\psi : TX \to [0,1]$ be another cut-off function such that $\tilde\psi|_{\tilde O} = 1$ on a neighborhood $W' \subset W$ of the zero section of T^*X. We can assume $W' = \tilde W$ and claim the operator $A_\psi - A_{\tilde\psi}$ to be smoothing. For the proof it suffices to show that the oscillatory integral

$$K_{A_\psi - A_{\tilde\psi}}(v) \;=\; \frac{1}{(2\pi)^n} \int_{T^*_{\pi(v)} X} (\psi(v) - \tilde\psi(v))\, e^{-i<\xi,v>}\, a(\xi)\, d\xi, \quad v \in TU, \tag{17}$$

which gives an integral representation for the kernel of $A_\psi - A_{\tilde\psi}$ defines a smooth function $K_{A_\psi - A_{\tilde\psi}} \in \mathcal{C}^\infty(TU)$. The phase function $T^*_{\pi(v)}X \ni \xi \mapsto -\xi(v) \in \mathbb{R}$ has only critical points for $v = 0$. Hence for $v \neq 0$ there exists a (-1)-homogeneous vertical vector field L on T^*X such that $L\, e^{-i<\cdot,v>} = e^{-i<\cdot,v>}$. The adjoint $L^\dagger$ of L satisfies $\left(L^\dagger\right)^k a \in \mathrm{S}^{\mu-\rho k}_{\rho,\delta}(U)$, where $k \in \mathbb{N}$. As the amplitude

$$(v,\xi) \mapsto (\psi(v) - \tilde\psi(v))\, a(\xi)$$

vanishes on $W' \times_U T^*U$, the equation

$$K_{A_\psi - A_{\tilde\psi}}(v) = \frac{1}{(2\pi)^n} \int_{T^*_{\pi(v)}X} e^{-i<\xi,v>} \left(L^\dagger\right)^k a(\xi)\, d\xi \tag{18}$$

holds for any integer k fulfilling $\mu + \rho k < -\dim X$. Note that the integral in Eq. (18) unlike the one in Eq. (17) is to be understood in the sense of Lebesgue. Hence $K_{A_\psi - A_{\tilde\psi}} \in \mathcal{C}^\infty(TU)$ and $A_\psi - A_{\tilde\psi}$ is smoothing.

Let us now prove that A_ψ lies in the space $\Psi^\mu_{\rho,\delta}(U)$ of pseudodifferential operators of order μ and type (ρ,δ) over U. We have to check that A_ψ is pseudolocal and with respect to some local coordinates looks like a pseudodifferential operator on an open set of $\mathbb{R}^d$.

(*i*) A_ψ is pseudolocal: Let $u, v \in \mathcal{D}(U)$ and $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$. Then the integral kernel K of $\mathcal{C}^\infty(U) \ni f \mapsto v\, A_\psi(uf) \in \mathcal{C}^\infty(U)$ has the form

$$K(v) \;=\; \frac{1}{(2\pi)^n} \int_{T^*_{\pi(v)}X} \psi(v)\, a(\xi)\, v(\pi(v))\, u(\exp v)\, e^{-i<\xi,v>}\, d\xi. \tag{19}$$

There exists an open neighborhood $W' \subset W$ of the zero section in TX such that the amplitude $\psi\,(v\circ\pi)\,(u \circ \exp)\, a$ vanishes on $W' \times_U T^*X$. As for all nonzero $v \in TX$ the phase function $\xi \mapsto -<\xi, v>$ is noncritical, an argument similar to the one above for $K_{A_\psi - A_{\tilde\psi}}$ shows that K is a smooth function on TU, so $\mathcal{C}^\infty(U) \ni f \mapsto v\, A_\psi(uf) \in \mathcal{C}^\infty(U)$ is smoothing.

(*ii*) A_ψ is a pseudodifferential operator in appropriate coordinates: Choose for $x \in U$ an open neighborhood $U_x \subset U$ so small that any two points in U_x can be connected by a unique geodesic. In particular we assume that $U_x \times U_x$ lies in the range of $(\pi, \exp) : W \to X \times X$. Furthermore let us suppose that $\psi \circ z_y(\tilde y) = 1$ for all $y, \tilde y \in U_x$. Then we have for $f \in \mathcal{D}(U_x)$:

(20)

$$\begin{aligned}
&A_\psi f\,(y) \;= \\
&= \frac{1}{(2\pi)^{n/2}} \int_{T^*_yX} a(\xi)\, \hat{f^\psi}(\xi)\, d\xi \\
&= \frac{1}{(2\pi)^{n/2}} \int_{T^*_xX} a\Big(\underbrace{T^*_{z_x(y)} \exp_x^{-1}}_{= d_x(y)}(\xi)\Big) \Big|\det d_x(y)\Big|\, \mathcal{F}(f\circ\exp)\big(d_x(y)\,(\xi)\big)\, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{T^*_xX} \int_{T_yX} a\Big(d_x(y)\,(\xi)\Big) \Big|\det d_x(y)\Big|\, (f \circ \exp_y)(v)\, e^{-i<d_x(y)\,(\xi),v>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{T^*_xX} \int_{T_xX} a\Big(d_x(y)\,(\xi)\Big) \Big|\det d_x(y)\Big| \Big|\underbrace{\det\left(T\,(z_y \circ z_x^{-1}\right)}_{= Z_{y,x}}(v)\Big| \\
&\qquad\qquad (f\circ z_x^{-1})(v)\, e^{-i<d_x(y)\,(\xi), z_y\circ z_x^{-1}(v)>}\, dv\, d\xi
\end{aligned}$$

Now the phase function $(v,\xi) \mapsto -< d_x(y)(\xi), z_y \circ z_x^{-1}(v) >$ vanishes for $\xi \neq 0$ if and only if $v = z_x(y)$. Hence after possibly shrinking U_x the Kuranishi trick gives a smooth function $G_x : U_x \times T_xX \to \operatorname{Iso}\left(T^*_xX, T^*_yX\right) \subset \operatorname{Lin}\left(T^*_xX, T^*_yX\right)$ such that $< d_x(y)(\xi), z_y \circ z_x^{-1}(v) > = < G_x(y,v)(\xi), v - z_x(y) >$ for all $(y,v) \in U_x \times T_xX$ and

$\xi \in T_x^* X$. A change of variables in Eq. (20) then implies:

$$(21)\quad \begin{aligned} A_\psi f(y) &= \frac{1}{(2\pi)^n} \int_{T_x^* X} \int_{T_x X} a\Big(\Big[d_x(y)\, G_x^{-1}(y,v)\Big]\,(\xi)\Big)\, \Big|\det d_x(y)\Big|\, \Big|Z_{y,x}(v)\Big| \\ &\qquad\qquad \Big|\det G_x^{-1}(y,v)\Big|\, (f \circ z_x^{-1})(v)\, e^{-i<\xi, v - z_x(y)>}\, dv\, d\xi \\ &= \frac{1}{(2\pi)^n} \int_{T_x^* X} \int_{T_x X} \tilde{a}_x(y,v,\xi)\, (f \circ z_x^{-1})(v)\, e^{-i<\xi, v - z_x(y)>}\, dv\, d\xi, \end{aligned}$$

where $\tilde{a}_x(y,v,\xi) = a\Big(\Big[d_x(y)\, G_x^{-1}(y,v)\Big]\,(\xi)\Big)\, \Big|\det d_x(y)\Big|\, \Big|Z_{y,x}(v)\Big|\, \Big|\det G_x^{-1}(y,v)\Big|$ is a symbol lying in $\mathrm{S}^\mu_{\rho,\delta}(U_x \times U_x \times T_x X)$, as $\rho + \delta \geq 1$ and $\rho > \delta$. Hence A is a pseudodifferential operator with respect to normal coordinates over U_x of order μ and type (ρ, δ).

Our considerations now lead us to

Theorem 3.1. *Let X be Riemannian, E, F Riemannian or Hermitian vector bundles over X and* exp *the exponential function corresponding to the Levi-Civita connection on X. Then any smooth cut-off function $\psi : TX \to [0,1]$ gives rise to a linear sheaf morphism $\mathrm{Op}_\psi : \mathrm{S}^\infty_{\rho,\delta}(\cdot, \mathrm{Hom}(E,F)) \to \Psi^\infty_{\rho,\delta}(\cdot, E, F)$, $a \mapsto A_\psi$ defined by*

$$(22)\qquad A_\psi f\,(x) \;=\; \frac{1}{(2\pi)^{n/2}} \int_{T_x^* X} a(\xi)\, \hat{f^\psi}(\xi)\, d\xi,$$

where $a \in \mathrm{S}^\infty_{\rho,\delta}(U, \mathrm{Hom}(E,F))$, $f \in \mathcal{D}(U,E)$, $x \in U$ and $U \subset X$ open. This morphism preserves the natural filtrations of $\mathrm{S}^\infty_{\rho,\delta}(\cdot, \mathrm{Hom}(E,F))$ and $\Psi^\infty_{\rho,\delta}(\cdot, E, F)$. In particular it maps the subsheaf $\mathrm{S}^{-\infty}(\cdot, \mathrm{Hom}(E,F))$ of smoothing symbols to the subsheaf $\Psi^{-\infty}(\cdot, \mathrm{Hom}(E,F)) \subset \Psi^\infty_{\rho,\delta}(\cdot, \mathrm{Hom}(E,F))$ of smoothing pseudodifferential operators.

The quotient morphism $\overline{\mathrm{Op}} : (\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty})(\cdot, \mathrm{Hom}(E,F)) \to (\Psi^\infty_{\rho,\delta}/\Psi^{-\infty})(\cdot, E, F)$ is an isomorphism and independent of the choice of the cut-off function ψ.

Proof. The first part of the theorem has been shown above. So it remains to prove that $\overline{\mathrm{Op}} : (\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty})(U, \mathrm{Hom}(E,F)) \to (\Psi^\infty_{\rho,\delta}/\Psi^{-\infty})(U, E, F)$ is bijective for all open $U \subset X$. Let us postpone this till Theorem 4.2, where we will show that an explicit inverse of $\overline{\mathrm{Op}}$ is given by the complete symbol map introduced in the following section. □

By the above Theorem the effect of ψ for the operator Op_ψ is only a minor one, so from now on we will write Op instead of Op_ψ.

4. The symbol map

In the sequel denote by $\varphi : X \times T^*X|_O \to \mathbb{C}$ the smooth phase function defined by

$$(23)\qquad X \times T^*X|_O \ni (x,\xi) \;\mapsto <\xi, \exp^{-1}_{\pi(\xi)}(x)> \,=\, <\xi, z_{\pi(\xi)}(x)> \,\in \mathbb{C},$$

where O is the range of $(\varrho, \exp) : W \to X \times X$.

Theorem and Definition 4.1. *Let $A \in \Psi^\mu_{\rho,\delta}(U, E, F)$ be a pseudodifferential operator on a Riemannian manifold X and $\psi : T^*X \to [0,1]$ a cut-off function. Then* *the ψ-cut symbol of A with respect to the Levi-Civita connection on X is the section*

$$\sigma_\psi(A) = \sigma_{\psi,A} : T^*U \to \pi^*(\mathrm{Hom}(E,F)), \qquad \xi \mapsto \left[\Xi \mapsto A\left(\psi_{\pi(\xi)}(\cdot)\, e^{i\varphi(\cdot,\xi)}\, \tau_{(\cdot),\pi(\xi)}\Xi\right)\right](\pi(\xi)), \tag{24}$$

where for every $x \in X$ $\psi_x = \psi \circ \exp_x^{-1} = \psi \circ z_x$. $\sigma_\psi(A)$ is an element of $\mathrm{S}^\mu_{\rho,\delta}(U,X)$. The corresponding element $\overline{\sigma}(A)$ in the quotient $(\mathrm{S}^\mu_{\rho,\delta}/\mathrm{S}^{-\infty})(U,\mathrm{Hom}(E,F))$ is called the normal symbol of A. It is independent of the choice of ψ.

Proof. Let us first check that $\sigma_{\psi,A}$ lies in $\mathrm{S}^\mu_{\rho,\delta}(U,\mathrm{Hom}(E,F))$. It suffices to assume that E and F are trivial bundles, hence that A is a scalar pseudodifferential operator. We can find a sequence $(x_\iota)_{\iota\in\mathbb{N}}$ of points in U and coordinate patches $U_\iota \subset U$ with $x_\iota \in U_\iota$ such that there exist normal coordinates $z_\iota = z_{x_\iota} : U_\iota \to T_{x_\iota}X$ on the U_ι. We can even assume that the operator $A_\iota : \mathcal{D}(U_\iota) \to \mathcal{C}^\infty(U_\iota)$, $u \mapsto Au|_{U_\iota}$ induces a pseudodifferential operator on $T_{x_\iota}X$. In other words there exist symbols $a_\iota \in \mathrm{S}^\mu_{\rho,\delta}(O_\iota \times O_\iota, T^*_{x_\iota}X)$ with $O_\iota = z_\iota(U_\iota)$ such that A_ι is given by the following oscillatory integral:

$$A_\iota u(y) = \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota}X} \int\limits_{T_{x_\iota}X} a_\iota(z_\iota(y), v, \xi)\,(u \circ z_\iota^{-1})(v)\, e^{-i<\xi, v - z_\iota(y)>}\, dv d\xi. \tag{25}$$

Now let (ϕ_ι) be a partition of unity subordinate to the covering (U_ι) of U and for every index ι let $\phi_{\iota,1}, \phi_{\iota,2} \in \mathcal{C}^\infty(U)$ such that $\operatorname{supp}\phi_{\iota,1} \subset U_\iota$, $\operatorname{supp}\phi_\iota \cap \operatorname{supp}\phi_{\iota,2} = \emptyset$ and $\phi_{\iota,1} + \phi_{\iota,2} = 1$. Then the symbol $\sigma_{\psi,A}$ can be written in the following form:

$$\begin{aligned} \sigma_{\psi,A}(\zeta) &= \left[A\Big(\psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)\right](\pi(\zeta)) \\ &= \frac{1}{(2\pi)^n} \sum_\iota \left[\phi_{\iota,1}\, A_\iota\Big(\phi_\iota(\cdot)\, \psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)\right](\pi(\zeta)) \\ &\quad + K\Big(\psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)(\pi(\zeta)), \end{aligned} \tag{26}$$

where

$$K : \mathcal{D}(U) \to \mathcal{C}^\infty(U) \quad f \mapsto \sum_\iota \phi_{\iota,2}\, A_\iota(\phi_\iota\, f), \quad \zeta \in T^*X_\iota \tag{27}$$

is a smoothing pseudodifferential operator. By the Kuranishi trick we can assume that there exists a smooth function $G : O_\iota \times O_\iota \to \mathrm{Lin}\left(T^*_{x_\iota}X, T^*_y X\right)$ such that $G(v,w)$ is invertible for every $v, w \in O_\iota$ and

$$< G\left(z_\iota(y), T(z_\iota \circ z_y^{-1})(v)\right)(\xi), v > = < \xi, z_\iota \circ z_y^{-1}(v) - z_\iota(y) > \tag{28}$$

for $y \in U_\iota$, $v \in T_y X$ appropriate and $\xi \in T^*_{x_\iota} X$. Several changes of variables then give the following chain of oscillatory integrals for $\zeta \in T^* U_\iota$:

(29)

$$\begin{aligned}
&\Big[A_\iota \Big(\phi_\iota(\cdot)\, \psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)} \Big) \Big] (y) \\
&= \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota} X} \int\limits_{T_{x_\iota} X} \phi_{\iota,1}(y)\, a_\iota\big(z_\iota^{-1}(y), v, \xi\big)\ \phi_\iota(z_\iota^{-1}(v))\, \psi(z_y \circ z_x^{-1}(v)) \\
&\quad e^{i<\zeta, z_y \circ z_\iota^{-1}(v)>}\, e^{-i<\xi, v - z_\iota(y)>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota} X} \int\limits_{T_y X} \phi_{\iota,1}(y)\, a_\iota\big(z_\iota^{-1}(y), z_\iota \circ z_y^{-1}(v), \xi\big)\ \phi_\iota(z_y^{-1}(v))\, \psi(v) \\
&\quad e^{i<\zeta, v>}\, e^{-i<\xi, z_\iota \circ z_y^{-1}(v) - z_\iota(y)>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota} X} \int\limits_{T_y X} \phi_{\iota,1}(y)\, a_\iota\big(z_\iota^{-1}(y), z_\iota \circ z_y^{-1}(v), \xi\big)\ \phi_\iota(z_y^{-1}(v))\, \psi(v) \\
&\quad e^{i<\zeta, v>}\, e^{-i<G(z_\iota(y), T(z_\iota \circ z_y^{-1})(v))(\xi), v>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int\limits_{T^*_y X} \int\limits_{T_y X} \phi_{\iota,1}(y)\, a_\iota\big(z_\iota^{-1}(y), z_\iota \circ z_y^{-1}(v), G^{-1}\left(z_\iota(y), T(z_\iota \circ z_y^{-1})(v)\right)(\xi)\big) \\
&\quad \phi_\iota(z_y^{-1}(v))\, \psi(v) \left| \det G^{-1}\left(z_\iota(y), T(z_\iota \circ z_y^{-1})(v)\right) \right| e^{i<\zeta, v>}\, e^{-i<\xi, v>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota} X} \int\limits_{T_{x_\iota} X} b_\iota(z_\iota(y), v, \xi)\, e^{-i<\xi - T^*(z_y \circ z_\iota^{-1})(\zeta), v - z_\iota(y)>}\, dv\, d\xi,
\end{aligned}$$

where the smooth function b_ι on $O_\iota \times O_\iota \times T^*_{x_\iota} X$ is defined by

(30)

$$\begin{aligned}
&b_\iota(z_\iota(y), v, \xi)\ = \\
&= \phi_{\iota,1}(y)\, a_\iota\left(z_\iota^{-1}(y), z_\iota \circ z_y^{-1} \circ T(z_y \circ z_\iota^{-1})(v), G^{-1}\Big(z_\iota(y), v\Big) \circ \left(T^*(z_\iota \circ z_y^{-1})(\xi)\right)\right) \\
&\quad \phi_\iota(z_y^{-1}(T(z_y \circ z_\iota^{-1})(v)))\, \psi(T(z_y \circ z_\iota^{-1})(v)) \left| \det G^{-1}(z_\iota(y), v) \right|
\end{aligned}$$

and lies in $\mathrm{S}^\mu_{\rho,\delta}(O_\iota \times O_\iota, T^*_{x_\iota} X)$. Let $B_\iota \in \Psi^\mu_{\rho,\delta}(O_\iota, T_{x_\iota} X)$ be the pseudodifferential operator on O_ι corresponding to the symbol b_ι. The symbol σ_ι defined by $\sigma_\iota(v,\xi) = e^{-i<\xi,v>}\, B_\iota\left(e^{i<\xi,\cdot>}\right)$, where $v \in O_\iota$ and $\xi \in T^*_{x_\iota} X$, now is an element of $\mathrm{S}^\mu_{\rho,\delta}(O_\iota, T^*_{x_\iota} X)$ as one knows by the general theory of pseudodifferential operators on real vector spaces (see [7, 12, 5]). But we have

$$\Big[\phi_{\iota,1}\, A_\iota\Big(\phi_\iota(\cdot)\, \psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)\Big](\pi(\zeta)) \ = \ \sigma_\iota\left(z_\iota(\pi(\zeta)), T^*\left(z_{\pi(\zeta)} \circ z_\iota^{-1}\right)(\zeta)\right), \tag{31}$$

hence Eq. (26) shows $\sigma_{\psi,A} \in \mathrm{S}^\mu_{\rho,\delta}(U,X)$ if we can prove that $\sigma_{\psi,K} \in \mathrm{S}^{-\infty}(U,X)$ for $\sigma_{\psi,K}(\zeta) = K\Big(\psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)(\pi(\zeta))$. So let $k : U \times U \to \mathbb{C}$ be the kernel function of K and write $\sigma_{\psi,K}$ as an integral:

(32) $$\sigma_{\psi,K}(\zeta) = \int_U k(\pi(\zeta), y)\, \psi_{\pi(\zeta)}(y)\, e^{-i\varphi(y,\zeta)}\, dy.$$

Let further $\Xi_1, ..., \Xi_k$ be differential forms over U and $\mathcal{V}_1, ..., \mathcal{V}_k$ the vertical vector fields on T^*U corresponding to $\Xi_1, ..., \Xi_k$. Differentiating Eq. (32) under the integral sign and using an adjointness relation then gives

(33)
$$\begin{aligned}
&|\zeta|^{2m}\, \mathcal{V}_1 \cdot ... \cdot \mathcal{V}_k\, \sigma_{\psi,K}\,(\zeta) \ = \\
&= \int_U k(\pi(\zeta), y)\, \psi_{\pi(\zeta)}(y)\ < \Xi_1, z_{\pi(\zeta)}(y) > \ ... \ < \Xi_k, z_{\pi(\zeta)}(y) > \\
&\qquad i^{2mn+k} \frac{\partial^{2mn}}{\partial z_{\pi(\zeta)}^{(2m,...,2m)}} e^{-i<\zeta, z_{\pi(\zeta)}(\cdot)>}(y)\, dy \\
&= \int_U \left(\frac{\partial^{2mn}}{\partial z_{\pi(\zeta)}^{(2m,...,2m)}} \right)^{\dagger} \big(k(\pi(\zeta), \cdot)\, \psi_{\pi(\zeta)}(\cdot)\ < \Xi_1, z_{\pi(\zeta)}(\cdot) > \ ... \ < \Xi_k, z_{\pi(\zeta)}(\cdot) > \big)\, (y) \\
&\qquad i^{2mn+k} e^{-i<\zeta, z_{\pi(\zeta)}(y)>}\, dy.
\end{aligned}$$

Therefore $|\zeta|^{2m}\, \mathcal{V}_1 \cdot ... \cdot \mathcal{V}_k\, \sigma_{\psi,K}\,(\zeta)$ is uniformly bounded as long as $\pi(\zeta)$ varies in a compact subset of U. A similar argument for horizontal derivatives of $\sigma_{\psi,K}$ finally proves $\sigma_{\psi,K} \in \mathrm{S}^{-\infty}(U, X)$.

Next we have to show that $\sigma_{\psi,A} - \sigma_{\tilde{\psi},A}$ is a smoothing symbol for any second cut-off function $\tilde{\psi} : T^*X \to [0,1]$. It suffices to prove that $\tilde{\sigma}_\iota$ with

(34) $$\tilde{\sigma}_\iota(\zeta) = A_\iota \left(\phi_\iota\, (\psi - \tilde{\psi}) \circ z_{\pi(\zeta)}\, e^{-i\varphi(\cdot,\zeta)} \right)$$

is smoothing for every $\phi_\iota \in \mathcal{D}(U_\iota)$. We can find a (-1)-homogeneous first order differential operator L on the vector bundle $T^*_{x_\iota}X \times T_{x_\iota}X \to T_{x_\iota}X$ such that for all $v \neq z_\iota(\pi(\zeta))$ and $\xi \in T^*_{x_\iota}X$ the equation

(35) $$L e^{-i<\cdot, v - z_\iota(\pi(\zeta))>}(\xi) \ = \ e^{-i<\xi, v - z_\iota(\pi(\zeta))>}$$

holds (see for example Grigis, Sjöstrand [5] Lemma 1.12 for a proof). Because $(\psi - \tilde{\psi}) \circ z_{\pi(\zeta)} \circ z_\iota^{-1}$ vanishes on a neighborhood of $z_\iota(\pi(\zeta))$, the following equational chain of iterated integrals is also true:

(36)

$$\begin{aligned}
\tilde\sigma_\iota(\zeta) &= \\
&= \frac{1}{(2\pi)^n}\int\limits_{T^*_{x_\iota}X}\int\limits_{T_{x_\iota}X} a_\iota\big(z_\iota^{-1}(y),v,\xi\big)\;\phi_\iota(z_\iota^{-1}(v))\;\Big(\psi-\tilde\psi\Big)\,(z_{\pi(\zeta)}\circ z_x^{-1}(v)) \\
&\qquad e^{i<\zeta,z_{\pi(\zeta)}\circ z_\iota^{-1}(v)>}\;e^{-i<\xi,v-z_\iota(\pi(\zeta))>}\,dv\,d\xi \\
&= \frac{1}{(2\pi)^n}\int\limits_{T^*_{x_\iota}X}\int\limits_{T_{x_\iota}X} a_\iota\big(z_\iota^{-1}(y),v,\xi\big)\;\phi_\iota(z_\iota^{-1}(v))\;\Big(\psi-\tilde\psi\Big)\,(z_{\pi(\zeta)}\circ z_x^{-1}(v)) \\
&\qquad e^{i<\zeta,z_{\pi(\zeta)}\circ z_\iota^{-1}(v)>}\;L^k\,e^{-i<\xi,v-z_\iota(\pi(\zeta))>}\,dv\,d\xi \\
&= \frac{1}{(2\pi)^n}\int\limits_{T^*_{x_\iota}X}\int\limits_{T_{x_\iota}X} \Big(\big(L^\dagger\big)^k a_\iota\Big)\big(z_\iota^{-1}(y),v,\xi\big)\;\phi_\iota(z_\iota^{-1}(v))\,\Big(\psi-\tilde\psi\Big)\,(z_{\pi(\zeta)}\circ z_x^{-1}(v)) \\
&\qquad e^{i<\zeta,z_{\pi(\zeta)}\circ z_\iota^{-1}(v)>}\;e^{-i<\xi,v-z_\iota(\pi(\zeta))>}\,dv\,d\xi.
\end{aligned}$$

Note that the last integral of this chain is to be understood in the sense of Lebesgue if $k\in\mathbb{N}$ is large enough. The same argument which was used for proving that the symbol $\sigma_{\psi,K}$ from Eq. (32) is smoothing now shows $\sigma_\iota\in \mathrm{S}^{-\infty}(U_\iota,X)$. □

After having defined the notion of a symbol, we will now give its essential properties in the following theorem.

Theorem 4.2. *Let $A\in\Psi^\mu_{\rho,\delta}(U,E,F)$ be a pseudodifferential operator on a Riemannian manifold X and $a=\sigma_{\psi,A}\in \mathrm{S}^\mu_{\rho,\delta}(U,\mathrm{Hom}(E,F))$ be its ψ-cut symbol with respect to the Levi-Civita connection. Then A and the pseudodifferential operator $\mathrm{Op}_\psi(a)$ defined by Eq. (22) coincide modulo smoothing operators. Moreover the sheaf morphisms $\overline{\sigma}:(\Psi^\infty_{\rho,\delta}/\Psi^{-\infty})(\cdot,E,F)\to(\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty})(\cdot,\mathrm{Hom}(E,F))$ and $\overline{\mathrm{Op}}:(\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty})(\cdot,\mathrm{Hom}(E,F))\to(\Psi^\infty_{\rho,\delta}/\Psi^{-\infty})(\cdot,E,F)$ are inverse to each other.*

Before proving the theorem let us first state a lemma.

Lemma 4.3. *The operator*

$$K:\mathcal{D}(U,E)\to\mathcal{C}^\infty(U,F),\quad f\mapsto\Big(U\ni x\mapsto A[(1-\psi_x)f]\,(x)\in F\Big) \tag{37}$$

is smoothing for any cut-off function $\psi:TX\to\mathbb{C}$.

Proof. For simplicity we assume that $E=F=X\times\mathbb{C}$. The general case follows analogously. Consider the operators $K_{a,\iota}:\mathcal{D}(U)\to\mathcal{C}^\infty(U)$ where $K_{a,\iota}f\,(x)=\phi_\iota(x)A[(1-\psi_a)f](x)$ for $x\in U$, $a\in U_\iota$. The open covering (U_ι) of U and a subordinate partition of unity (ϕ_ι) are chosen such that for every point $a\in U_\iota$ there exist normal coordinates z_a on U_ι. We can even assume that $\psi(z_a(y))=1$ for all $a,y\in U_\iota$. Hence $\mathrm{supp}\,(1-\psi_a)\cap\mathrm{supp}\,\phi_\iota=\emptyset$, and $K_{a,\iota}$ is smoothing, because A is pseudolocal. This yields a family of smooth functions $k_\iota:U\times U\to\mathbb{C}$ such that

$$K_{a,\iota}f\,(x)\;=\int\limits_U k_\iota(x,y)\,(1-\psi_a(y))\,f(y)\,dy. \tag{38}$$

Now the smooth function $k\in\mathcal{C}^\infty(U\times U)$ with $k(x,y)\;=\;\sum_\iota k_\iota(x,y)\,(1-\psi_x(y))$ is the kernel function of K. □

Proof of Theorem 4.2. Check that for an arbitrary cut-off function $\tilde{\psi} : TX \to \mathbb{C}$ one can find a cut-off function $\psi : TX \to \mathbb{C}$ such that the equation

$$\psi_x(y)\, f(y) \;=\; \frac{1}{(2\pi)^{n/2}} \int\limits_{T_x^*X} \psi_x(y)\, e^{i<\xi, z_x(y)>}\, \hat{f^{\tilde{\psi}}}(\xi)\, d\xi \tag{39}$$

holds for $x, y \in U$. Approximating the integral in Eq. (39) by Riemann sums, we can find a sequence $(i_k)_{k\in\mathbb{N}}$ of natural numbers, a family $(\xi_{k,\iota})_{k\in\mathbb{N}, 1\leq\iota\leq\iota_k}$ of elements $\xi_{k,\iota} \in T_x^*X$ and a sequence $(\epsilon_k)_{k\in\mathbb{N}}$ of positive numbers such that

$$\begin{aligned} &\lim_{k\to\infty} \epsilon_k \;=\; 0, \\ &[(\psi_x) f]\,(y) \;=\; \lim_{k\to\infty} \frac{\epsilon_k^n}{(2\pi)^{n/2}} \sum_{1\leq i\leq i_k} \psi_x(y)\, e^{i<\xi_{k,\iota}\,,\, z_x(y)>}\, \hat{f^{\tilde{\psi}}}(\xi_{k,\iota}). \end{aligned} \tag{40}$$

By an appropriate choice of the $\xi_{k,\iota}$ one can even assume that the series on the right hand side of the last equation converges as a function of $y \in U$ in the topology of $\mathcal{D}(U)$. As A is continuous on $\mathcal{D}(U)$, we now have

$$\begin{aligned} A[(\psi_x) f](x) \;&=\lim_{k\to\infty} \frac{\epsilon_k^n}{(2\pi)^{n/2}} \sum_{1\leq i\leq i_k} A\left[\psi_x(\cdot)\, e^{i<\xi_{k,\iota}\,,\, z_x(\cdot)>}\right](x)\, \hat{f^{\tilde{\psi}}}(\xi_{k,\iota}) \\ &= \lim_{k\to\infty} \frac{\epsilon_k^n}{(2\pi)^{n/2}} \sum_{1\leq i\leq i_k} \sigma_A(\xi_{k,\iota})\, \hat{f^{\tilde{\psi}}}(\xi_{k,\iota}) \\ &= \frac{1}{(2\pi)^{n/2}} \int\limits_{T_x^*X} \sigma_A(\xi)\, \hat{f^{\tilde{\psi}}}(\xi)\, d\xi \\ &= A_{\tilde{\psi}} f. \end{aligned} \tag{41}$$

On the other hand Lemma 4.3 provides a smoothing operator K such that for $f \in \mathcal{D}(U)$ and $x \in U$:

$$Af\,(x) \;=\; A[(\psi_x) f]\,(x) + Kf\,(x) \;=\; A_{\tilde{\psi}} f\,(x) + Kf\,(x). \tag{42}$$

Thus the first part of the theorem follows.

Now let $a \in \mathrm{S}^\mu_{\rho,\delta}(U, \mathrm{Hom}(E,F))$ be a symbol, and consider the corresponding pseudodifferential operator $A = A_\psi \in \Psi^\mu_{\rho,\delta}(U, E, F)$. After possibly altering ψ we can assume that there exists a second cut-off function $\tilde{\psi} : T^*X \to \mathbb{C}$ such that $\tilde{\psi}|_{\operatorname{supp}\psi} = 1$. Then we have for $\zeta \in T_x^*X$ and $\Xi \in E_x$

$$\begin{aligned} \sigma_{A,\psi}(\zeta)\Xi \;&= \frac{1}{(2\pi)^{n/2}} \int\limits_{T_x^*X} a(\xi)\, \mathcal{F}\left(\left(\psi_x(\cdot) e^{i\varphi(\cdot,\zeta)} \tau_{x,(\cdot)}\Xi\right)^{\tilde{\psi}}\right)(\xi)\, d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int\limits_{T_x^*X} a(\xi)\, \mathcal{F}\left(\psi_x(\cdot) e^{i\varphi(\cdot,\zeta)}\Xi\right)(\xi)\, d\xi \\ &= \quad \frac{1}{(2\pi)^n} \int\limits_{T_x^*X} \int\limits_{T_xX} a(\xi)\Xi\, \psi(v)\, e^{-i<\xi,v>}\, e^{i<\zeta,v>}\, dv\, d\xi, \end{aligned} \tag{43}$$

where the last integral is an iterated one. Next choose a smooth function $\phi : \mathbb{R} \to [0,1]$ such that $\operatorname{supp}\phi \subset [-2,2]$ and $\phi|_{[-1,1]} = 1$ and define $a_k \in \mathrm{S}^{-\infty}(U, \mathrm{Hom}(E,F))$

by $a_k(\xi) = \phi\left(\frac{|\xi|}{k}\right)$. Then obviously $a_k \to a$ in $\mathrm{S}^{\tilde{\mu}}_{\rho,\delta}(U\mathrm{Hom}(E,F),)$ for $\tilde{\mu} > \mu$. We can now write:

(44)

$$\begin{aligned}
\sigma_{A,\psi}(\zeta)\Xi &= \lim_{k\to\infty} \frac{1}{(2\pi)^n} \int\limits_{T_x^*X}\int\limits_{T_xX} a_k(\xi)\Xi\,\psi(v)\,e^{-i<\xi-\zeta,v>}\,dv\,d\xi \\
&= \lim_{k\to\infty} a_k(\zeta)\Xi - \frac{1}{(2\pi)^{n/2}} \int\limits_{T_xX} (1-\psi(v))\,\left[\mathcal{F}^{-1}a_k\right](v)\,e^{i<\zeta,v>}\,dv \\
&= a(\zeta) - \lim_{k\to\infty} \frac{1}{(2\pi)^{n/2}} \int\limits_{T_xX} (1-\psi(v))\,\left[\mathcal{F}^{-1}\left(\left(L^\dagger\right)^m a_k\right)\right](v)\,e^{i<\zeta,v>}\,dv \\
&= a(\zeta) + \lim_{k\to\infty} K_m(\zeta,a_k),
\end{aligned}$$

where $m\in\mathbb{N}$, L is a smooth (-1)-homogeneous first order differential operator on the vector bundle $T^*X\times_U \dot{T}U \to \dot{T}U$ such that $Le^{i<\cdot,v>}(\xi) = e^{i<\xi,v>}$ and

$$\text{(45)}\quad K_m(\zeta,a_k) \;=\; -\frac{1}{(2\pi)^{n/2}} \int\limits_{T_xX} (1-\psi(v))\,\left[\mathcal{F}^{-1}\left(\left(L^\dagger\right)^m a_k\right)\right](-v)\,e^{i<\zeta,v>}\,dv.$$

Note that $\dot{T}U$ consists of all nonzero vectors $v\in TU$ and that $\left(L^\dagger\right)^m a \in \mathrm{S}^{\tilde{\mu}-\rho m}_{\rho,\delta}$ for every $\tilde{\mu}\geq\mu$. Now fixing $\tilde{\mu}\geq\mu$ and choosing m large enough that $\rho m > \tilde{\mu} + 2(\mathrm{dim}X - 1)$ yields

(46)

$$\begin{aligned}
K_m(\zeta,a) &=: \lim_{k\to\infty} K_m(\zeta,a_k) \\
&= \lim_{k\to\infty} -\frac{1}{(2\pi)^n} \int\limits_{T_xX}\int\limits_{T_x^*X} \left(\left(L^\dagger\right)^m a_k\right)(\xi)\,(1-\psi(v))\,e^{-i<\xi-\zeta,v>}\,d\xi\,dv \\
&= -\frac{1}{(2\pi)^n} \int\limits_{T_xX}\int\limits_{T_x^*X} \left(\left(L^\dagger\right)^m a\right)(\xi)\,(1-\psi(v))\,e^{i<\zeta,v>}\,e^{-i<\xi,v>}\,d\xi\,dv
\end{aligned}$$

in the sense of Lebesgue integrals. As the phase function $\xi \mapsto -<\xi,v>$ is noncritical for every nonzero v and $1-\psi(v)$ vanishes in an open neighborhood of the zero section of TU, a standard consideration already carried out in preceeding proofs entails that $K_m(\cdot,a)$ is a smoothing symbol. Henceforth $\sigma_{A,\tilde{\psi}}$ and a differ by the smoothing symbol $K_m(\cdot,a)$. This gives the second part of the theorem. $\square$

In the following let us show that the normal symbol calculus gives rise to a natural transformation between the functor of pseudodifferential operators and the functor of symbols on the category of Riemannian manifolds and isometric embeddings.

First we have to define these two functors. Associate to any Riemannian manifold X the algebra $\mathcal{A}(X) = \Psi^\infty_{\rho,\delta}/\Psi^{-\infty}(X)$ of pseudodifferential operators on X modulo smoothing ones, and the space $\mathcal{S}(X) = \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(X)$ of symbols on X modulo smoothing ones. As Y is Riemannian, we have for any isometric embedding $f: X\to Y$ a natural embedding $f_*: T^*X \to T^*Y$. So we can define $\mathcal{S}(f): \mathcal{S}(Y)\to\mathcal{S}(X)$ to be the pull-back f^* of smooth functions on T^*Y via f_*. The morphism $\mathcal{A}(f):\mathcal{A}(Y)\to\mathcal{A}(X)$ is constructed by the following procedure. If f is submersive, i.e., a diffeomorphism onto its image, $\mathcal{A}(f)$ is just the pull-back of a pseudodifferential operator on Y to X via f. So we now assume that f is not submersive. Choose an open tubular neighborhood U of $f(X)$ in Y and a smooth cut-off function $\phi : U \to [0,1]$ with compact support and $\phi|_V = 1$ on a neighborhood $V \subset U$ of $f(X)$. As U is tubular there is a canonical projection $\Pi : U \to X$ such that $\Pi \circ f = id_X$. Now let the pseudodifferential operator P represent an element of $\mathcal{A}(Y)$. We then define $\mathcal{A}(f)(P)$ to be the equivalence class of the pseudodifferential operator

$$\mathcal{E}' \ni u \mapsto f^* \left[P\left(\phi(\Pi^* u)\right)\right] \in \mathcal{E}', \tag{47}$$

where $\mathcal{E}'$ denotes distributions with compact support. Note that modulo smoothing operators $f^* \left[P\left(\phi(\Pi^* u)\right)\right]$ does not depend on the choice of ϕ. With these definitions $\mathcal{A}$ and $\mathcal{S}$ obviously form contravariant functors from the category of Riemannian manifolds with isometric embeddings as morphisms to the category of complex vector spaces and linear mappings.

The following proposition now holds.

Proposition 4.4. *The symbol map σ forms a natural transformation from the functor $\mathcal{A}$ to the functor $\mathcal{S}$.*

Proof. As f is isometric, the relation

$$f \circ \exp \;=\; \exp \circ Tf \tag{48}$$

is true. But then the phasefunctions on X and Y are related by

$$\varphi_Y(y, f_*(\xi)) \;=\; \varphi_X(\Pi(y), \xi), \tag{49}$$

where $y \in Y$ and $\xi \in T^*X$ with y and $f(\pi(\xi))$ sufficiently close. Hence for a pseudodifferential operator P on Y

$$\left(f^* \sigma_Y(P)\right)(\xi) = \left[P\left(\psi_{f(\pi(\xi))}(\cdot)\, \mathrm{e}^{i\varphi_Y(\cdot, f_*\xi)}\right)\right]\left(f(\pi(\xi))\right) \tag{50}$$

and

$$\left(\sigma_X f^*(P)\right)(\xi) = \left[P\left(\psi_{f(\pi(\xi))}(\cdot)\, \phi(\cdot)\, \mathrm{e}^{i\varphi_X(\Pi(\cdot),\xi)}\right)\right](\pi(\xi)). \tag{51}$$

Eq. (49) now implies

$$f^* \sigma_Y(P) \;=\; \sigma_X f^*(P) \tag{52}$$

which gives the claim. □

5. The symbol of the adjoint

In the sequel we want to derive an expression for the normal symbol σ_{A^*} of the adjoint of a pseudodifferential operator A in terms of the symbol σ_A.

Let E, F be Hermitian (Riemannian) vector bundles over the Riemannian manifold X, and let $<\,,\,>_E$, resp. $<\,,\,>$, be the metric on E, resp. F. Denote by D^E, resp. D^F the unique torsionfree metric connection on E, resp. F, and by D the induced metric connection on $\mathrm{Hom}(E,F)$, resp. $\pi^*(\mathrm{Hom}(E,F))$. Next define the function $\rho : O \to \mathbb{R}^+$ by

$$\rho(x,y)\cdot \nu_x = \nu(y) \circ (T_v \exp_x \times \cdots \times T_v \exp_x), \quad x, y \in O, \quad v = \exp_x^{-1}(y), \tag{53}$$

where ν is the canonical volume density on T_xX and ν_x its restriction to a volume density on T_xX. Now we claim that the operator $\mathrm{Op}^*(a) : \mathcal{D}(U,F) \to \mathcal{C}^\infty(U,E)$ defined by the iterated integral

(54)
$$\begin{aligned}\mathrm{Op}^*(a)g(x) =&\frac{1}{(2\pi)^n}\int_{T_x^*X}\int_{T_xX} e^{-i<\xi,v>}\,\tau_{\exp v}^{-1}\left[a^*\left(T_v^*\exp_x^{-1}(\xi)\right)\,g(\exp v)\right]\tilde{\psi}(v)\\ &\rho^{-1}(\exp_x(v),x)\;dv\,d\xi\end{aligned}$$

is the (formal) adjoint of $\mathrm{Op}\,(a)$, where $a \in \mathrm{S}^\mu_{\rho,\delta}(U,\mathrm{Hom}(E,F))$, $U \subset X$ open and $\tilde{\psi}$ is the cut-off function $TX \ni v \mapsto \psi\left(\exp^{-1}_{\exp_x v}(x)\right) \in [0,1]$. Assuming $f \in \mathcal{D}(V,E)$, $g \in \mathcal{D}(U,F)$ with $V \subset U$ open, $V \times V \subset O$ and μ to be sufficiently small the following chain of (proper) integrals holds:

(55)
$$\begin{aligned}
&\int\limits_X < f(x), \mathrm{Op}^*(a)g\,(x) >_E \;d\nu(x) \;=\\
&=^1 \frac{1}{(2\pi)^n}\int\limits_X\int\limits_{T_xX\times T_x^*X} e^{i<\xi,v>}\; < f(x), \tau_{\exp v}\left[a^*\left(T_v^*\exp_x^{-1}(\xi)\right)\,g(\exp v)\right] >\\
&\qquad\qquad \tilde{\psi}(v)\,\rho^{-1}(\exp_x(v),x)\;d\omega_x^n(v,\xi)d\nu(x)\\
&=^2 \frac{1}{(2\pi)^n}\int\limits_X\int\limits_{T^*V} e^{i<T^*\exp_x(\xi),\exp_x^{-1}(\pi(\xi))>}\; < \tau_{\pi(\xi),x}f(x), a^*(\xi)\,g(\pi(\xi)) >\\
&\qquad\qquad \psi(\exp^{-1}_{\pi(\xi)}(x))\,\rho^{-1}(\pi(\xi),x)\;d\omega^n(\xi)\,d\nu(x)\\
&=^3 \frac{1}{(2\pi)^n}\int\limits_{T^*V}\int\limits_X e^{-i<\xi,\exp^{-1}_{\pi(\xi)}(x)>}\; < \tau_{\pi(\xi),x}f(x), a^*(\xi)\,g(\pi(\xi)) >\\
&\qquad\qquad \psi(\exp^{-1}_{\pi(\xi)}(x))\,\rho^{-1}(\pi(\xi),x)\;d\nu(x)\,d\omega^n(\xi)\\
&=^4 \frac{1}{(2\pi)^n}\int\limits_{T^*V}\int\limits_{T_{\pi(\xi)}X} e^{-i<\xi,w>}\; < a(\xi)\,\tau^{-1}_{\exp w}f(\exp w), g(x) > \;\psi(w)\,dw\,d\omega^n(\xi)\\
&= \frac{1}{(2\pi)^n}\int\limits_X\int\limits_{T_xX\times T_x^*X} e^{-i<\xi,w>}\; < a(\xi)\,\tau^{-1}_{\exp w}f(\exp w), g(x) > \;\psi(w)\,dw\,d\xi\,d\nu(x)\\
&= \int\limits_X < \mathrm{Op}\,(a)f(x), g(x) >_F \;d\nu(x)
\end{aligned}$$

In explanation of the above equalities, we have

1. ω_x, ω canonical symplectic forms on $T_xX \times T_x^*X$, resp. T^*X
2. by $T^*\exp_x : T^*V \to T_xX \times T_x^*X$ symplectic, $x \in V$
3. Gauß Lemma and Fubini's Theorem
4. coordinate transformation $V \ni x \mapsto w = \exp^{-1}_{\pi(\xi)}(x) \in T_{\pi(\xi)}X$.

Recall that $\mathrm{S}^{-\infty}(U,\mathrm{Hom}(E,F))$ is dense in any $\mathrm{S}^\mu_{\rho,\delta}(U,\mathrm{Hom}(E,F))$, $\mu \in \mathbb{R}$ with respect to the Fréchet topology of $\mathrm{S}^{\tilde{\mu}}_{\rho,\delta}(U,\mathrm{Hom}(E,F))$ if $\mu < \tilde{\mu}$. By using a partition of unity and the continuity of $\mathrm{S}^{\tilde{\mu}}_{\rho,\delta}(U, \mathrm{Hom}(E,F)) \ni a \mapsto \mathrm{Op}\,(a)f \in \mathcal{C}^\infty(U,F)$, resp. $\mathrm{S}^{\tilde{\mu}}_{\rho,\delta}(U, \mathrm{Hom}(E,F)) \ni a \mapsto \mathrm{Op}^*(a)g \in \mathcal{C}^\infty(U,E)$, for any $f \in \mathcal{D}(U,E)$ and $g \in \mathcal{D}(U,F)$, Eq. (55) now implies that

$$\int_X < f(x), \mathrm{Op}^*(a)g(x) >_E \; d\nu(x) = \int_X < \mathrm{Op}\,(a)f(x), g(x) >_F \; d\nu(x) \tag{56}$$

holds for every $f \in \mathcal{D}(U,E)$, $g \in \mathcal{D}(U,F)$ and $a \in \mathrm{S}^{\mu}_{\rho,\delta}(U)$ with $\mu \in \mathbb{R}$. Thus $\mathrm{Op}^*(a)$ is the formal adjoint of $\mathrm{Op}\,(a)$ indeed.

The normal symbol σ_{A^*} of $A^* = \mathrm{Op}^*(a)$ now is given by

(57)

$$\sigma_{A^*}(\xi)\,\Xi \;\sim\; [A^* g(\cdot,\xi,\Xi)]\,(\pi(\xi)) = \frac{1}{(2\pi)^n} \int_{T_xX} \int_{T^*_xX} e^{-i<\zeta-\xi,v>}$$
$$\tau^{-1}_{\exp v}\left[a^*\left(T^*_v \exp_x^{-1}(\zeta)\right)\,\tau_{\exp v}\Xi\right]\tilde{\psi}(v)\,\psi(v)\,\rho^{-1}(\exp_x(v),x)\;dv\,d\zeta,$$

where $\xi \in T^*X$, $x = \pi(\xi)$, $\Xi \in F_x$ and

$$g(y,\xi,\Xi) = \psi\left(\exp^{-1}_{\psi(\xi)}(y)\right)\, e^{i<\xi,\exp^{-1}_{\pi(\xi)}(y)>}\,\tau_{y,\pi(\xi)}\Xi \tag{58}$$

with $y \in X$. Thus by Theorem 3.4 of Grigis, Sjöstrand [5] the following asymptotic expansion holds:

$$\begin{aligned} \sigma_{A^*}(\xi) &\sim \sum_{\alpha\in\mathbb{N}^d} \frac{(-i)^{|\alpha|}}{\alpha!}\; \frac{\partial^{|\alpha|}}{\partial v^\alpha}\bigg|_{v=0} \frac{\partial^{|\alpha|}}{\partial \zeta^\alpha}\bigg|_{\zeta=\xi} \\ &\qquad \Big(\left[\tau^{-1}_{\exp v}\circ a^*\left(T^*_v \exp_x^{-1}(\zeta)\right)\circ\tau_{\exp v}\right]\rho^{-1}(\exp_x(v),x)\Big) \\ &= \sum_{\alpha,\beta\in\mathbb{N}^d} \frac{(-i)^{|\alpha+\beta|}}{\alpha!\,\beta!} \left[\frac{D^{|\alpha|+|\beta|}}{\partial z_x^\beta\,\partial\zeta_x^\alpha}\bigg|_\xi a^*\right]\left[\frac{\partial^{|\alpha|}}{\partial {z_x}^\alpha}\bigg|_x \rho^{-1}(\cdot,x)\right], \end{aligned} \tag{59}$$

where

$$\frac{D^{|\alpha|+|\beta|}}{\partial z_x^\beta\,\partial\zeta_x^\alpha}\bigg|_\xi a^* = \frac{\partial^{|\beta|}}{\partial v^\beta} a\bigg|_{v=0} \frac{\partial^{|\alpha|}}{\partial\zeta^\alpha} a\bigg|_\xi \left[\tau^{-1}_{\exp v}\circ a\left(T^*_v\exp_x^{-1}(\zeta)\right)\circ\tau_{\exp v}\right], \tag{60}$$

are symmetrized covariant derivatives of $a \in \mathrm{S}^\infty(X, \mathrm{Hom}(E,F))$ with respect to the apriorily chosen normal coordinates (z_x, ζ_x).

Next we will search for an expansion of ρ and its derivatives. For that first write

$$\frac{\partial}{\partial z_x^k} = \sum_{k,l} \theta^l_k(x,\cdot)\, e_l, \tag{61}$$

where $(e_1, ..., e_d)$ is the orthonormal frame of Eq. (11), and the θ^l_k are smooth functions on an open neighborhood of the diagonal of $X \times X$. Then we have the following relation (see for example Berline, Getzler, Vergne [1], p. 36):

$$\theta^k_l(x,y) = \delta^k_l - \frac{1}{6}\sum_{m,n} R^k{}_{mln}(y)\, z^m_x(y)\, z^n_x(y) + O(|z_x(y)|^3), \tag{62}$$

where ${R^k}_{mln}(y)$, resp. their "lowered" versions $R_{kmln}(y)$, are given by the coefficients of the curvature tensor with respect to normal coordinates at $y \in X$, i.e., $R\left(\frac{\partial}{\partial z_y^m}\right) = \sum_{k,l,n} {R^k}_{mln}(y) \frac{\partial}{\partial z_y^k} \otimes dz_y^l \otimes dz_y^n$. But this implies

$$\begin{aligned} \rho(x,y) &= \left|\det\left(\theta_k^l(x,y)\right)\right| \\ &= 1 - \frac{1}{6}\sum_k \sum_{m,n} {R^k}_{mkn}(x)\, z_x^m(y)\, z_x^n(y) + O(|z_x(y)|^3 \\ &= 1 - \frac{1}{6}\sum_{m,n} \mathrm{Ric}_{mn}(x)\, z_x^m(y)\, z_x^n(y) + O\left(|z_x(y)|^3\right). \end{aligned} \tag{63}$$

Using Eq. (13) $z_x^k(y) = -z_y^k(x)$ for x and y close enough we now get

$$\left.\frac{\partial}{\partial z_x^k}\right|_y \rho^{-1}(\cdot, x) = \frac{1}{3}\sum_l \mathrm{Ric}_{kl}(x)\, z_x^l(y) + O\left(|z_x(y)|^2\right), \tag{64}$$

$$\left.\frac{\partial^2}{\partial z_x^k \partial z_x^l}\right|_y \rho^{-1}(\cdot, x) = \frac{1}{3}\mathrm{Ric}_{kl}(x) + O\left(|z_x(y)|\right). \tag{65}$$

The above results are summarized by the following theorem.

Theorem 5.1. *The formal adjoint $A^* \in \Psi^\mu_{\rho,\delta}(U, F, E)$ of a pseudodifferential operator $A \in \Psi^\mu_{\rho,\delta}(U, E, F)$ between Hermitian (Riemannian) vector bundles E, F over an open subset U of a Riemannian manifold X is given by*

$$\begin{aligned} A^* g\,(x) \; = \mathrm{Op}^*(a) g\,(x) = \frac{1}{(2\pi)^n} \int_{T_x^* X} \int_{T_x X} e^{-i<\xi, v>} \\ \tau^{-1}_{\exp v}\left[a^*\left(T_v^* \exp_x^{-1}(\xi)\right) g(\exp v)\right] \tilde{\psi}(v)\, \rho^{-1}(\exp_x(v), x)\; dv\, d\xi \end{aligned}$$

where $x \in X$, $g \in \mathcal{D}(U, F)$ and $a = \sigma_A$ is the symbol of A. The symbol σ_{A^*} of A^* has asymptotic expansion

(66)

$$\begin{aligned} \sigma_{A^*}(\xi) \;\sim\; & a^*(\xi) \;+\; \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \geq 1}} \left.\frac{D^{2|\alpha|}}{\partial z^\alpha_{\pi(\xi)}\, \partial \zeta^\alpha_{\pi(\xi)}}\right|_\xi a^* \;- \\ & -\; \frac{1}{6} \sum_{\alpha \in \mathbb{N}^d} \sum_{k,l} (-i)^{|\alpha|} \left[\left.\frac{D^{2|\alpha|+2}}{\partial z^\alpha_{\pi(\xi)}\, \partial \zeta^\alpha_{\pi(\xi)}\, \partial \zeta^k_{\pi(\xi)}\, \partial \zeta^l_{\pi(\xi)}}\right|_\xi a^* \right] \mathrm{Ric}_{kl}(\pi(\xi)) \;+ \\ & +\; \sum_{\substack{\alpha, \beta \in \mathbb{N}^d \\ |\beta| \geq 3}} \frac{(-i)^{|\alpha\,+\,\beta|}}{\alpha!\, \beta!} \left[\left.\frac{D^{2|\alpha|+|\beta|}}{\partial z^\alpha_{\pi(\xi)}\, \partial \zeta^{\alpha+\beta}_{\pi(\xi)}}\right|_\xi a^* \right] \left[\left.\frac{\partial^{|\alpha|}}{\partial {z_{\pi(\xi)}}^\alpha}\right|_{\pi(\xi)} \rho^{-1}(\cdot, x) \right] \\ =\; & a^*(\xi) \;-\; i \sum_k \left.\frac{D^2}{\partial z^k_{\pi(\xi)}\, \partial \zeta^k_{\pi(\xi)}}\right|_\xi a^* \;-\; \frac{1}{2} \sum_{k,l} \left.\frac{D^4}{\partial z^k_{\pi(\xi)}\, \partial z^l_{\pi(\xi)}\, \partial \zeta^k_{\pi(\xi)}\, \partial \zeta^l_{\pi(\xi)}}\right|_\xi a^* \;- \\ & -\; \frac{1}{6} \sum_{k,l} \left[\left.\frac{D^2}{\partial \zeta^k_{\pi(\xi)} \partial \zeta^l_{\pi(\xi)}}\right|_\xi a^* \right] \mathrm{Ric}_{kl}(\pi(\xi)) \;+\; r(\xi) \end{aligned}$$

with $\xi \in T^*X$ *and* $r \in \mathrm{S}^{\mu-2(\rho-\delta)}_{\rho,\delta}(U, \mathrm{Hom}(E, F))$.

6. Product expansions

Most considerations of elliptic partial differential operators as well as applications of our normal symbol to quantization theory (see Pflaum [8]) require an expression for the product of two pseudodifferential operators A, B in terms of the symbols of its components. In the flat case on $\mathbb{R}^d$ it is a well known fact that σ_{AB} has an asymptotic expansion of the form

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \frac{\partial^{|\alpha|}}{\partial \zeta^\alpha} \sigma_A(x,\xi) \, \frac{\partial^{|\alpha|}}{\partial x^\alpha} \sigma_B(x,\xi). \tag{67}$$

Note that strictly speaking, the product AB of pseudodifferential operators is only well-defined, if at least one of them is properly supported. But this is only a minor set-back, as any pseudodifferential operator is properly supported modulo a smoothing operator and the above asymptotic expansion also describes a symbol only up to smoothing symbols.

In the following we want to derive a similar but more complicated formula for the case of pseudodifferential operators on manifolds.

Proposition 6.1. *Let* $A \in \Psi^\mu_{\rho,\delta}(U, E, F)$ *be a pseudodifferential operator between Hermitian* (*Riemannian*) *vector bundles* E, F *over a Riemannian manifold* X *and* $a = \sigma(A)$ *its symbol. Then for any* $f \in \mathcal{C}^\infty(U, E)$ *the function*

$$\sigma_{A,f} : T^*U \to F, \quad \xi \mapsto \left[A\left(\psi_{\pi(\xi)} f \, e^{i\varphi(\cdot,\xi)} \right) \right] (\pi(\xi)) \tag{68}$$

is a symbol of order μ *and type* (ρ, δ) *and has an asymptotic expansion of the form*

$$\sigma_{A,f}(\xi) \sim \sum_{\alpha} \frac{1}{i^{|\alpha|}\alpha!} \left[\left. \frac{D^{|\alpha|}}{\partial \zeta_{\pi(\xi)}{}^\alpha} \right|_\xi a \right] \left[\left. \frac{D^{|\alpha|}}{\partial z_{\pi(\xi)}{}^\alpha} \right|_{\pi(\xi)} f \right]. \tag{69}$$

Proof. Let $x = \pi(\xi)$, $N \in \mathbb{N}$ and $\psi : T^*X \to \mathbb{C}$ a cut-off function. Then the following chain of equations holds:

(70)

$$\begin{aligned}
\sigma_{A,f}(\xi) &= \left[A\left(\psi_{\pi(\xi)} f \, e^{i\varphi(\cdot,\xi)} \right) \right] (x) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{T^*_x X} a(\zeta) \, \left[\mathcal{F}\left(f^\psi \, e^{i<\xi,\cdot>} \right) \right] (\zeta) \, d\zeta \\
&= \frac{1}{(2\pi)^n} \int_{T^*_x X} \int_{T_x X} e^{i<\xi-\zeta,v>} \, a(\zeta) \, \tau^{-1}_{\exp v} f(\exp v) \, \psi(v) \, dv \, d\zeta \\
&= \frac{1}{(2\pi)^n} \int_{T^*_x X} \int_{T_x X} e^{-i<\zeta,v>} \, a(\xi+\zeta) \, \tau^{-1}_{\exp v} f(\exp v) \, \psi(v) \, dv \, d\zeta \\
&= \frac{1}{(2\pi)^n} \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \int_{T^*_x X} \int_{T_x X} e^{-i<\zeta,v>} \frac{D^{|\alpha|}}{\partial \zeta_x{}^\alpha} a(\xi) \, \zeta^\alpha \, \tau^{-1}_{\exp v} f(\exp v) \, \psi(v) \, dv \, d\zeta \\
&\quad + r_N(\xi) \\
&= \sum_{|\alpha| \leq N} \frac{1}{i^{|\alpha|} \, \alpha!} \left[\left. \frac{D^{|\alpha|}}{\partial \zeta_x{}^\alpha} \right|_\xi a \right] \cdot \left[\left. \frac{D^{|\alpha|}}{\partial z_x{}^\alpha} \right|_x f \right] + r_N(\xi).
\end{aligned}$$

Note that in this equation all integrals are iterated ones and check that with Taylor's formula r_N is given by

(71)
$$r_N(\xi) = \frac{1}{(2\pi)^n} \sum_{|\beta|=N+1} \frac{N+1}{\beta!} \int_{T_x^*X} \int_0^1 (1-t)^N \frac{D^{|\beta|}}{\partial {\zeta_x}^\beta} a(\xi + t\zeta)\, dt\, \zeta^\beta \left[\mathcal{F}\left(f^\psi\right)\right](\zeta)\, d\zeta.$$

Choosing N large enough, the Schwarz inequality and Plancherel's formula now entail

(72)
$$\begin{aligned} ||r_N(\xi)|| \leq \frac{1}{(2\pi)^{n/2}} \sum_{|\beta|=N+1} \frac{N+1}{\beta!} & \left|\left|\int_0^1 (1-t)^N \frac{D^{|\beta|}}{\partial {\zeta_x}^\beta} a(\xi + t(\cdot)) dt\right|\right|_{L^2(T_x^*X)} \\ & \left|\left| \frac{D^{|\beta|}}{\partial {v_x}^\beta}\ f^\psi|_{T_xX} \right|\right|_{L^2(T_xX)}. \end{aligned}$$

But for $K \subset U$ compact and $t \in [0,1]$ there exist constants $D_{N,K}, \tilde{D}_{N,K} > 0$ such that

$$\left|\left| \frac{D^{|\beta|}}{\partial {\zeta_x}^\beta} a(\xi + t\zeta) \right|\right| \leq D_{N,K} (1 + ||\xi|| + ||\zeta||)^{\mu - \rho(N+1)} \tag{73}$$

and

(74)
$$\begin{aligned} \left|\left|\int_0^1 (1-t)^k \frac{D^{|\beta|}}{\partial {\zeta_x}^\beta} a(\xi + t(\cdot)) dt\right|\right|_{L^2(T_x^*X)} & \leq D_{N,K} \left|\left| (1 + ||\xi|| + ||\cdot||)^{\mu-\rho(N+1)} \right|\right|_{L^2(T_x^*X)} \\ & = \tilde{D}_{N,K}\, (1 + ||\xi||)^{n+\mu-\rho(N+1)} \end{aligned}$$

for all $x \in K$, $\xi, \zeta \in T_x^*X$ and all indices $\beta \in \mathbb{N}^d$ with $|\beta| = N+1$. Inserting this in (72) we can find a $C_{N,K} > 0$ such that for all $x \in K$ and $\xi \in T_x^*X$

(75)
$$\begin{aligned} & ||r_N(\xi)|| \leq \\ & \quad C_{N,K} \Big(1 + ||\xi||\Big)^{n+\mu-\rho(N+1)} \sum_{|\gamma| \leq N+1} \sup \left\{ \left|\left| \frac{D^{|\gamma|}}{\partial {z_x}^\gamma} f(y) \right|\right| \;:\; y \in \operatorname{supp} \psi_{\pi(\xi)} \right\}. \end{aligned}$$

Because of $\lim_{N\to\infty} \dim X + \mu + \rho(N+1) = -\infty$ and a similar consideration for the derivatives of $\sigma_{A,f}$ the claim now follows. □

Next let us consider two pseudodifferential operators $A \in \Psi^\infty_{\rho,\delta}(U, F, G)$ and $B \in \Psi^\infty_{\rho,\delta}(U, E, F)$ between vector bundles E, F, G over X. Modulo smoothing operators we can assume one of them to be properly supported, so that $AB \in \Psi^\infty_{\rho,\delta}(U, E, G)$ exists. Defining the extended symbol $b^{ext} = \sigma^{ext}_B : U \times T^*U \to \operatorname{Hom}(E, F)$ of B by

(76)
$$\begin{aligned} & b^{ext}(x,\xi)\Xi = \\ & \psi_{\pi(\xi)}(x)\, e^{-i\varphi(x,\xi)} \left[B\left(\psi_{\pi(\xi)}(\cdot)\, e^{i\varphi(\cdot,\xi)} \tau_{(\cdot),\pi(\xi)}\Xi\right)\right](x), \quad x \in U,\ (\xi,\Xi) \in \pi^*(E|_U) \end{aligned}$$

the symbol σ_{AB} of the composition AB is now given modulo $\mathrm{S}^{-\infty}(U, \mathrm{Hom}(E,G))$ by

$$\begin{aligned}\sigma_{AB}(\xi)\Xi &= \left[AB\left(\psi_{\pi(\xi)} e^{i\varphi(\cdot,\xi)}\tau_{(\cdot),\pi(\xi)}\Xi\right)\right](\pi(\xi)) \\ &= \left[A\left(\psi_{\pi(\xi)}\, e^{i\varphi(\cdot,\xi)}\, \sigma_B^{ext}(\cdot,\xi)\Xi\right)\right](\pi(\xi)).\end{aligned} \tag{77}$$

By Proposition 6.1 this entails that σ_{AB} has an asymptotic expansion of the form

$$\sigma_{AB}(\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{i^{|\alpha|}\alpha!} \left[\left.\frac{D^{|\alpha|}}{\partial\zeta_{\pi(\xi)}{}^{\alpha}}\right|_{\xi} \sigma_A\right] \left[\left.\frac{D^{|\alpha|}}{\partial z_{\pi(\xi)}{}^{\alpha}}\right|_{\pi(\xi)} \sigma_B^{ext}(\cdot,\xi)\right]. \tag{78}$$

As we want to find an even more detailed expansion for σ_{AB}, let us calculate the derivatives $\left.\frac{D^{|\alpha|}}{\partial z_{\pi(\xi)}{}^{\alpha}}\right|_{\pi(\xi)} \sigma_B^{ext}(\cdot,\xi)$ in the following proposition.

Proposition 6.2. *Let $B \in \Psi^{\mu}_{\rho,\delta}(U,E,F)$ be a pseudodifferential operator between Hermitian (Riemannian) vector bundles over the Riemannian manifold X and $b = \sigma(B)$ its symbol. Then for any $f \in \mathcal{C}^\infty(U,E)$ the smooth function*

$$\sigma_{B,f}^{ext} : U\times T^*U \to F, \quad (x,\xi)\mapsto \psi_{\pi(\xi)}(x)\, e^{-i\varphi(x,\xi)} \left[B\left(\psi_{\pi(\xi)} f\, e^{i\varphi(\cdot,\xi)}\right)\right](x) \tag{79}$$

is a symbol of order μ and type (ρ,δ) on $U\times T^*U$ and has an asymptotic expansion of the form

$$\begin{aligned}\sigma_{B,f}^{ext}(x,\xi) \sim & \sum_{k\in\mathbb{N}} \sum_{\substack{\beta,\beta_0,\ldots,\beta_k\in\mathbb{N}^n\\ \beta_0+\ldots+\beta_k=\beta\\ |\beta_1|,\ldots,|\beta_k|\geq 2}} \frac{i^{k-|\beta|}}{k!\beta_0!\cdot\ldots\cdot\beta_k!}\, \psi_{\pi(\xi)}(x) \left[\left.\frac{D^{|\beta|}}{\partial\zeta_x{}^{\beta}}\right|_{d_x\varphi(\ ,\xi)} b\right] \\ & \left[\left.\frac{D^{|\beta_0|}}{\partial z_x{}^{\beta_0}}\right|_x \psi_{\pi(\xi)} f\right] \left[\left.\frac{\partial^{|\beta_1|}}{\partial z_x{}^{\beta_1}}\right|_x \varphi(\cdot,\xi)\right]\cdot\ldots\cdot \left[\left.\frac{\partial^{|\beta_k|}}{\partial z_x{}^{\beta_k}}\right|_x \varphi(\cdot,\xi)\right].\end{aligned} \tag{80}$$

Furthermore the derivative $T^*U \ni \xi \mapsto \left[\frac{\partial^{|\alpha|}}{\partial z_{\pi(\xi)}{}^{\alpha}} \sigma_{B,f}^{ext}(\cdot,\xi)\right](\pi(\xi)) \in F$ with $\alpha\in\mathbb{N}^n$ is a symbol with values in F of order $\mu+\delta|\alpha|$ and type (ρ,δ) on T^*U and has an asymptotic expansion of the form

$$\begin{aligned}&\left[\frac{D^{|\alpha|}}{\partial z_{\pi(\xi)}{}^{\alpha}} \sigma_{B,f}^{ext}(\cdot,\xi)\right](\pi(\xi)) \sim \\ &\sim \sum_{k\in\mathbb{N}} \sum_{\substack{\tilde\alpha,\alpha_0,\ldots,\alpha_k\in\mathbb{N}^n\\ \tilde\alpha+\alpha_0+\ldots+\alpha_k=\alpha}} \sum_{\substack{\beta,\beta_0,\ldots,\beta_k\in\mathbb{N}^n\\ \beta_0+\ldots+\beta_k=\beta\\ |\beta_1|,\ldots,|\beta_k|\geq 2}} \frac{i^{k-|\beta|}\alpha!}{k!\tilde\alpha!\alpha_0!\cdot\ldots\cdot\alpha_k!\beta_0!\cdot\ldots\cdot\beta_k!} \\ &\left\{\left.\frac{D^{|\tilde\alpha|}}{\partial z_{\pi(\xi)}{}^{\tilde\alpha}}\right|_{\pi(\xi)} \left[\left.\frac{D^{|\beta|}}{\partial\zeta_{(-)}{}^{\beta}}\right|_{d_{(-)}\varphi(\ ,\xi)} b\right]\right\} \left\{\left.\frac{D^{|\alpha_0|}}{\partial z_{\pi(\xi)}{}^{\alpha_0}}\right|_{\pi(\xi)} \left[\left.\frac{D^{|\beta_0|}}{\partial z_{(-)}{}^{\beta_0}}\right|_{(-)} f\right]\right\}\cdot \\ &\left\{\left.\frac{\partial^{|\alpha_1|}}{\partial z_{\pi(\xi)}{}^{\alpha_1}}\right|_{\pi(\xi)} \left[\left.\frac{\partial^{|\beta_1|}}{\partial z_{(-)}{}^{\beta_1}}\right|_{(-)} \varphi(\cdot,\xi)\right]\right\}\cdot\ldots\cdot \left\{\left.\frac{\partial^{|\alpha_k|}}{\partial z_{\pi(\xi)}{}^{\alpha_k}}\right|_{\pi(\xi)} \left[\left.\frac{\partial^{|\beta_k|}}{\partial z_{(-)}{}^{\beta_k}}\right|_{(-)} \varphi(\cdot,\xi)\right]\right\}.\end{aligned}$$

Note 6.3. The differential operators of the form $\frac{D^{|\alpha|}}{\partial z_{\pi(\xi)}{}^{\alpha}}$ act on the variables denoted by $(-)$, the differential operators of the form $\frac{D^{|\beta|}}{\partial z_{(-)}{}^{\beta}}$ on the variables $(\cdot)$.

Proof of Proposition 6.2. Let $\eta \in \mathcal{C}^\infty(U \times U \times T^*U)$ be a smooth function such that

$$\eta(x,y,\xi) \; = \; \varphi(y,\xi) - \varphi(x,\xi) - < d_x\varphi(\ ,\xi), z_x(y) > \tag{81}$$

for $x, y \in \operatorname{supp} \psi_{\pi(\xi)}$ and such that $T^*U \ni \xi \mapsto \eta(x,y,\xi) \in \mathbb{C}$ is linear for all $x, y \in U$. Furthermore denote by $F \in \mathcal{C}^\infty(U \times U \times T^*U, E)$ the function $(x,y,\xi) \mapsto \psi_{\pi(\xi)}(y) f(y) e^{i\eta(x,y,\xi)}$. By the Leibniz rule and the definition of η we then have

$$\begin{aligned}\left[\frac{D^{|\beta|}}{\partial {z_x}^\beta}\bigg|_y F(x,\cdot,\xi)\right] = \sum_{0 \le k \le \frac{|\beta|}{2}} \; \sum_{\substack{\beta_0,\ldots,\beta_k \in \mathbb{N}^n \\ \beta_0+\ldots+\beta_k=\beta \\ |\beta_1|,\ldots,|\beta_k| \ge 2}} \frac{i^k}{k!\beta_0! \cdot \ldots \cdot \beta_k!} \left[\frac{D^{|\beta_0|}}{\partial {z_x}^{\beta_0}}\bigg|_y \psi_{\pi(\xi)} f\right] \cdot \\ \left[\frac{\partial^{|\beta_1|}}{\partial {z_x}^{\beta_1}}\bigg|_y \eta(x,\cdot,\xi)\right] \cdot \ldots \cdot \left[\frac{\partial^{|\beta_k|}}{\partial {z_x}^{\beta_k}}\bigg|_y \eta(x,\cdot,\xi)\right] \, e^{i\eta(x,y,\xi)}.\end{aligned} \tag{82}$$

As $(x,y,\xi) \mapsto \left[\frac{\partial^{|\beta_j|}}{\partial {z_x}^{\beta_j}}\Big|_y \eta(x,\cdot,\xi)\right]$, $0 \le j \le k$ is smooth and linear with respect to ξ, the preceeding equation entails that $(x,y,\xi) \mapsto \left[\frac{D^{|\beta|}}{\partial {z_x}^{\beta}}\Big|_y F(x,\cdot,\xi)\right]$ is absolutely bounded by a function $(x,y,\xi) \mapsto C_\beta(x,y)\,|\xi|^{\frac{|\beta|}{2}}$, where $C_\beta \in \mathcal{C}^\infty(U \times U)$ and $C_\beta \ge 0$. On the other hand we have

$$\sigma^{ext}_{B,f}(x,\xi) \; = \; \psi_{\pi(\xi)}(x) \left[B\left(\psi_{\pi(\xi)}(\cdot) f(\cdot) e^{i\eta(\cdot,x,\xi)} \, e^{i<d_x\varphi(\ ,\xi),z_x(\cdot)>}\right)\right](x) \tag{83}$$

and by the proof of Proposition 6.1

$$\begin{aligned}\sigma^{ext}_{B,f}(x,\xi) &= \psi_{\pi(\xi)}(x) \sum_{|\beta| \le N} \frac{i^{|\beta|}}{\beta!} \left[\frac{D^{|\beta|}}{\partial {\xi_x}^\beta}\bigg|_{d_x\varphi(\ ,\zeta)} b\right] \left[\frac{D^{|\beta|}}{\partial {z_x}^\beta}\bigg|_x F(x,\cdot,\xi)\right] + r_N(x,\xi) \\ &= \psi_{\pi(\xi)}(x) \sum_{|\beta| \le N} \sum_{0 \le k \le \frac{|\beta|}{2}} \; \sum_{\substack{\beta_0,\ldots,\beta_k \in \mathbb{N}^n \\ \beta_0+\ldots+\beta_k=\beta \\ |\beta_1|,\ldots,|\beta_k| \ge 2}} \frac{i^{k-|\beta|}}{k!\beta_0! \cdot \ldots \cdot \beta_k!} \left[\frac{D^{|\beta|}}{\partial {\xi_x}^\beta}\bigg|_{d_x\varphi(\ ,\zeta)} b\right] \cdot \\ &\quad \left[\frac{D^{|\beta_0|}}{\partial {z_x}^{\beta_0}}\bigg|_x \psi_{\pi(\xi)} f\right] \cdot \left[\frac{\partial^{|\beta_1|}}{\partial {z_x}^{\beta_1}}\bigg|_x \varphi(\cdot,\xi)\right] \cdot \ldots \cdot \left[\frac{\partial^{|\beta_k|}}{\partial {z_x}^{\beta_k}}\bigg|_x \varphi(\cdot,\xi)\right] + r_N(x,\xi),\end{aligned} \tag{84}$$

where

$$\begin{aligned}||r_n(x,\xi)|| &\le C_{N,K}(1 + ||d_x\varphi(\ ,\xi)||)^{n+\mu-\rho(N+1)} \\ &\quad \sum_{|\gamma| \le N+1} \sup \left\{ \left|\left| \left[\frac{\partial^{|\gamma|}}{\partial {z_x}^\gamma}\bigg|_y F(x,\cdot,\xi)\right]\right|\right| \, : \, y \in \operatorname{supp} \psi_x \right\} \\ &\le D_{N,K}(1 + ||\xi||)^{n+\mu-(\rho-1/2)(N+1)}\end{aligned} \tag{85}$$

holds for compact $K \subset U$, $x \in K$, $\xi \in T^*X|_K$ and constants $C_{N,K}, D_{N,K} > 0$. Because $\rho > \frac{1}{2}$, we have $\lim_{N\to\infty} \dim X + \mu - (\rho - \frac{1}{2})(N+1) = -\infty$. By Eqs. (84) and (85) and similar relations for the derivatives of $\sigma^{ext}_{A,f}$ we therefore conclude that Eq. (80) is true. The rest of the claim easily follows from Leibniz rule. $\square$

The last proposition and Eq. (78) imply the next theorem.

Theorem 6.4. *Let $A, B \in \Psi^{\infty}_{\rho,\delta}(U,E,F)$ be two pseudodifferential operators between Hermitian (Riemannian) vector bundles E, F over a Riemannian manifold X. Assume that one of the operators A and B is properly supported. Then the symbol σ_{AB} of the product AB has the following asymptotic expansion.*

(86)

$$\sigma_{AB}(\xi) \sim \sum_{\alpha\in\mathbb{N}^d} \frac{i^{-|\alpha|}}{\alpha!} \left[\left.\frac{D^{|\alpha|}}{\partial {\zeta_{\pi(\xi)}}^{\alpha}}\right|_{\xi} \sigma_A \right] \left[\left.\frac{D^{|\alpha|}}{\partial {z_{\pi(\xi)}}^{\alpha}}\right|_{\pi(\xi)} \sigma_B \right] +$$

$$+ \sum_{k\geq 1} \sum_{\substack{\alpha,\tilde\alpha,\alpha_1,\ldots,\alpha_k\in\mathbb{N}^n \\ \tilde\alpha+\alpha_1+\ldots+\alpha_k=\alpha}} \sum_{\substack{\beta,\beta_1,\ldots,\beta_k\in\mathbb{N}^n \\ \beta_1+\ldots+\beta_k=\beta \\ |\beta_1|,\ldots,|\beta_k|\geq 2}} \frac{i^{k-|\alpha|-|\beta|}}{k!\cdot\tilde\alpha!\cdot\alpha_1!\cdot\ldots\cdot\alpha_k!\beta_1!\cdot\ldots\cdot\beta_k!}$$

$$\left[\left.\frac{D^{|\alpha|}}{\partial {\zeta_{\pi(\xi)}}^{\alpha}}\right|_{\xi} \sigma_A \right] \left\{ \left.\frac{D^{|\tilde\alpha|}}{\partial {z_{\pi(\xi)}}^{\tilde\alpha}}\right|_{\pi(\xi)} \left[\left.\frac{D^{|\beta|}}{\partial {\zeta_{(-)}}^{\beta}}\right|_{d_{(-)}\varphi(\ ,\xi)} \sigma_B \right] \right\} \cdot$$

$$\left\{ \left.\frac{\partial^{|\alpha_1|}}{\partial {z_{\pi(\xi)}}^{\alpha_1}}\right|_{\pi(\xi)} \left[\left.\frac{\partial^{|\beta_1|}}{\partial {z_{(-)}}^{\beta_1}}\right|_{(-)} \varphi(\cdot,\xi) \right] \right\} \cdot\ldots\cdot \left\{ \left.\frac{\partial^{|\alpha_k|}}{\partial {z_{\pi(\xi)}}^{\alpha_k}}\right|_{\pi(\xi)} \left[\left.\frac{\partial^{|\beta_k|}}{\partial {z_{(-)}}^{\beta_k}}\right|_{(-)} \varphi(\cdot,\xi) \right] \right\}.$$

A somewhat more explicit expansion up to second order is given by

Corollary 6.5. *If σ_A is a symbol of order μ and σ_B a symbol of order $\tilde\mu$ the coefficients in the asymptotic expansion of σ_{AB} are given up to second order by the following formula:*

$$\begin{aligned} \sigma_{AB}(\xi) = & \sigma_A(\xi)\cdot\sigma_B(\xi) - \\ & - i\sum_l \frac{D\sigma_A}{\partial\zeta_{\pi(\xi),l}}(\xi)\, \frac{D\sigma_B}{\partial z^l_{\pi(\xi)}}(\xi) - \frac{1}{2}\sum_{k,l} \frac{D^2\sigma_A}{\partial\zeta_{\pi(\xi),k}\partial\zeta_{\pi(\xi),l}}(\xi)\, \frac{D^2\sigma_B}{\partial z^k_{\pi(\xi)}\partial z^l_{\pi(\xi)}}(\xi) \\ & - \frac{1}{12}\sum_{k,l,m,n} \frac{D\sigma_A}{\partial\zeta_{\pi(\xi),n}}(\xi)\, \frac{D^2\sigma_B}{\partial\zeta_{\pi(\xi),l}\partial\zeta_{\pi(\xi),m}}(\xi)\, {R^k}_{mln}(\pi(\xi))\, \zeta_{\pi(\xi),k}(\xi) + r(\xi) \end{aligned} \tag{87}$$

where $\xi\in T^*U$, $x=\pi(\xi)$, $r\in \mathrm{S}^{\mu+\tilde\mu-3}_{\rho,\delta}(U)$ and the ${R^m}_{nkl}(y)$ are the coefficients of the curvature tensor with respect to normal coordinates at $y\in X$, i.e., $R\left(\frac{\partial}{\partial z^m_y}\right) = \sum_{k,l,n} {R^k}_{mln}(y)\frac{\partial}{\partial z^k_y}\otimes dz^l_y\otimes dz^n_y$.

Proof. First define the order of a summand $s = s^{\beta,\beta_1,\ldots,\beta_k}_{\alpha,\tilde\alpha,\alpha_1,\ldots,\alpha_k}$ in Eq. (86) by $\mathrm{ord}(s) = |\alpha-\tilde\alpha+\beta-k|$. Then it is obvious that $s\in \mathrm{S}^{\mu+\tilde\mu+\mathrm{ord}(s)\,(\rho-\delta)}_{\rho,\delta}(U)$. Therefore we have to calculate only the summands s with $\mathrm{ord}(s)\leq 2$. To achieve this let us calculate some coordinate changes and their derivatives. Recall the matrix-valued function $\left(\theta^k_l\right)$ of Eq. (61) and construct its inverse $\left(\bar\theta^k_l\right)$. In other words

$$\sum_k \theta^l_k(x,y)\,\bar\theta^k_m(x,y) = \delta^l_m \tag{88}$$

holds for every $l,m=1,\ldots,d$. We can now write

$$\left.\frac{\partial}{\partial z^k_y}\right|_z = \sum_{l,m} \theta^l_k(y,z)\,\bar\theta^m_l(x,z)\ \left.\frac{\partial}{\partial z^m_x}\right|_z \tag{89}$$

and receive the following formulas:

$$\begin{aligned}\frac{\partial z_x^k}{\partial z_y^l}(z) =&\delta_l^k \;-\; \frac{1}{6}\sum_{m,n} R^k{}_{mln}(y)\, z_x^m(z)\, z_x^n(z)\\ &+\;\frac{1}{6}\sum_{m,n} R^k{}_{mln}(x) z_x^m(z)\, z_x^n(z) \;+\; O\left(|z_x(z)+z_y(z)|^3\right),\end{aligned}\tag{90}$$

$$\begin{aligned}\frac{\partial^2 z_x^k}{\partial z_y^l \partial z_y^m}(z) =&-\;\frac{1}{6}\sum_{n} \left(R^k{}_{mln}(y)+R^k{}_{nlm}(y)\right)\; z_x^n(z)\\ &+\;\frac{1}{6}\sum_{n} \left(R^k{}_{mln}(x)+R^k{}_{nlm}(x)\right) z_x^n(z)\\ &+\; O\left(|z_x(z)+z_y(z)|^2\right),\end{aligned}\tag{91}$$

$$\frac{\partial}{\partial z_x^n}\Big|_{y=z}\frac{\partial^2 z_x^k}{\partial z_y^l \partial z_y^m}(y) = \frac{1}{6}\;\left(R^k{}_{mln}(x)+R^k{}_{nlm}(x)\right)\;+\;O\left(|z_x(z)|\right).\tag{92}$$

Next consider the functions

$$\varphi_{\alpha\beta}:\; T^*X\to\mathbb{R},\qquad \xi\mapsto \frac{\partial^{|\alpha|}}{\partial {z_{\pi(\xi)}}^{\alpha}}\Big|_{y=\pi(\xi)}\frac{\partial^{|\beta|}\varphi(\cdot,\xi)}{\partial^\beta {z_y}^\beta}(y)$$

which appear in the summands s of Eq. (86). One can write the $\varphi_{\alpha\beta}$ in the form

$$\varphi_{\alpha\beta}(\xi) = \sum_k \zeta_{\pi(\xi),k}(\xi)\;\frac{\partial^{|\alpha|}}{\partial {z_{\pi(\xi)}}^{\alpha}}\Big|_{y=\pi(\xi)}\frac{\partial^{|\beta|} z^k_{\pi(\xi)}}{\partial {z_y}^\beta}(y)\tag{93}$$

Thus by Eq. (91)

$$\varphi_{0\beta}(\xi) = 0\tag{94}$$

holds for every $\beta\in\mathbb{N}^d$ with $|\beta|\geq 2$. Furthermore check that

$$d_y\varphi(\cdot,\xi) = \sum_k \zeta_{\pi(\xi),k}(\xi)\; dz^k_{\pi(\xi)}\Big|_y\tag{95}$$

is true.

Now we are ready to calculate the summands s of order $\mathrm{ord}(s)\leq 2$ by considering the following cases.

(i) $\mathrm{ord}(s)=0$:

$$s(\xi)\;=\;\sigma_A(\xi)\,\sigma_B(\xi).\tag{96}$$

(ii) $\mathrm{ord}(s)=1$, $|\alpha|=1$, $k=0$, $\tilde\alpha=\alpha$:

$$s(\xi)\;=\;-\,i\sum_l \frac{D\sigma_A}{\partial\zeta_{\pi(\xi),l}}(\xi)\;\frac{D\sigma_B}{\partial z^l_{\pi(\xi)}}(\xi).\tag{97}$$

(iii) $\mathrm{ord}(s)=1$, $|\alpha|=0$, $k=1$, $|\beta|=2$: In this case $s=0$ because of Eq. (94).

(iv) $\mathrm{ord}(s)=2$, $|\alpha|=2$, $k=0$, $\tilde\alpha=\alpha$:

$$s(\xi)\;=\;-\,\frac{1}{2}\sum_{k,l}\frac{D^2\sigma_A}{\partial\zeta_{\pi(\xi),k}\partial\zeta_{\pi(\xi),l}}(\xi)\;\frac{D^2\sigma_B}{\partial z^k_{\pi(\xi)}\partial z^l_{\pi(\xi)}}(\xi).\tag{98}$$

(*v*) $\mathrm{ord}(s) = 2$, $|\alpha| = 1$, $k = 1$, $|\beta| = 2$: In this case $\tilde{\alpha} = 0$ by Eq. (94). Hence by Eq. (92)

$$(99) \quad s(\xi) = -\frac{1}{12} \sum_{k,l,m,n} \frac{D\sigma_A}{\partial \zeta_{\pi(\xi),n}}(\xi) \, \frac{D^2 \sigma_B}{\partial \zeta_{\pi(\xi),l} \partial \zeta_{\pi(\xi),m}}(\xi) \, R^k{}_{mln}(\pi(\xi)) \, \zeta_{\pi(\xi),k}(\xi).$$

Summing up these summands we receive the expansion of σ_{AB} up to second order. This proves the claim. □

The product expansion of Theorem 6.4 gives rise to a a bilinear map # on the sheaf $\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(\cdot, \mathrm{Hom}(F,G)) \times \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(\cdot, \mathrm{Hom}(E,F))$ by

$$(100) \quad \begin{aligned} \mathrm{S}^\infty_{\rho,\delta}(U, \mathrm{Hom}(F,G)) \times \mathrm{S}^\infty_{\rho,\delta}(U, \mathrm{Hom}(E,F)) &\to \mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty}(U, \mathrm{Hom}(E,G)), \\ (a,b) &\mapsto a\#b = \sigma_{\mathrm{Op}(a)\,\mathrm{Op}(b)} \end{aligned}$$

for all $U \subset X$ open and vector bundles E, F, G over X. The #-product is an important tool in a deformation theoretical approach to quantization (cf. Pflaum [8]). In the sequel it will be used to study ellipticity of pseudodifferential operators on manifolds.

7. Ellipticity and normal symbol calculus

The global symbol calculus introduced in the preceding paragraphs enables us to investigate the invertibility of pseudodifferential operators on Riemannian manifolds. In particular we are now able define a notion of elliptic pseudodifferential operators which is more general than the usual notion by Hörmander [7] or Douglis, Nirenberg [2].

Definition 7.1. A symbol $a \in \mathrm{S}^\infty_{\rho,\delta}(X, \mathrm{Hom}(E,F))$ on a Riemannian manifold X is called elliptic if there exists $b_0 \in \mathrm{S}^\infty_{\rho,\delta}(X, \mathrm{Hom}(F,E)))$ such that

$$(101) \qquad a\#b_0 - 1 \in \mathrm{S}^{-\varepsilon}_{\rho,\delta}(X) \quad \text{and} \quad b_0\#a - 1 \in \mathrm{S}^{-\varepsilon}_{\rho,\delta}(X)$$

for an $\varepsilon > 0$. A symbol $a \in \mathrm{S}^m_{\rho,\delta}(X, \mathrm{Hom}(E,F))$ is called elliptic of order m if it is elliptic and one can find a symbol $b_0 \in \mathrm{S}^m_{\rho,\delta}(X, \mathrm{Hom}(F,E))$ fulfilling Eq.(101).

An operator $A \in \Psi^\infty_{\rho,\delta}(X, E, F)$ is called elliptic resp. elliptic of order m, if its symbol σ_A is elliptic resp. elliptic of order m.

Let us give some examples of elliptic symbols resp. operators.

Example 7.2. (*i*) Let $a \in \mathrm{S}^m_{1,0}(X, \mathrm{Hom}(E,F))$ be a classical symbol, i.e., assume that a has an expansion of the form $a \sim \sum_{j\in\mathbb{N}} a_{m-j}$ with $a_{m-j} \in \mathrm{S}^{m-j}_{1,0}(X)$ homogeneous of order $m-j$. Further assume that a is elliptic in the classical sense, i.e., that the principal symbol a_m is invertible outside the zero section. By the homogeneity of a_m this just means that a_m is elliptic of order m. But then a and $\mathrm{Op}(a)$ must be elliptic of order m as well.

(*ii*) Consider the symbols $l : T^*X \to \mathbb{C}$, $\xi \mapsto ||\xi||^2$ and $a : T^*X \to \mathbb{C}$, $\xi \mapsto \frac{1}{1+||\xi||^2}$ of Example 2.3 (*ii*). Then the pseudodifferential operator corresponding to l is minus the Laplacian: $\mathrm{Op}(l) = -\Delta$. Furthermore the symbols l and $1+l$ are elliptic of order 2, and the relation $a\#(1+l) - 1$, $(1+l)\#a - 1 \in \mathrm{S}^{-1}_{1,0}(X)$ is satisfied. In case X is a flat Riemannian manifold, one can calculate directly that modulo smoothing symbols $a\#l = l\#a = 1$, hence $\mathrm{Op}(a)$ is a parametrix for the differential operator $\mathrm{Op}(1+l) = 1 - \Delta$.

(*iii*) Let $a_\varphi(\xi) = (1 + ||\xi||^2)^{\varphi(\pi(\xi))}$ be the symbol defined in Example 2.3 (*iii*), where $\varphi : X \to \mathbb{R}$ is supposed to be smooth and bounded. Then a is elliptic and $b(\xi) = (1 + ||\xi||^2)^{-\varphi(\pi(\xi))}$ fulfills the relations (101). In case φ is not locally constant, we thus receive an example of a symbol which is not elliptic in the sense of Hörmander [7] but elliptic in the sense of the above definition.

Let us now show that for any elliptic symbol $a \in \mathrm{S}^\infty_{\rho,\delta}(X, \mathrm{Hom}(E, F))$ one can find a symbol $b \in \mathrm{S}^\infty_{\rho,\delta}(X, \mathrm{Hom}(F, E))$ such that even

$$a\#b - 1 \in \mathrm{S}^{-\infty}(X, \mathrm{Hom}(F, F)) \quad \text{and} \quad b\#a - 1 \in \mathrm{S}^{-\infty}(X, \mathrm{Hom}(E, E)). \tag{102}$$

Choose b_0 according to Definition 7.1 and let $r = 1-a\#b_0$, $l = 1-b_0\#a \in \mathrm{S}^{-\infty}(X)$. Then the symbols

$$b_r = b_0\,(1 + r + r\#r + r\#r\#r + \dots) \quad \text{and} \quad b_l = (1 + l + l\#l + l\#l\#l + \dots)\,b_0$$

are well-defined and fulfill $a\#b_r = 1$ and $b_l\#a = 1$ modulo smoothing symbols. Hence $b_r - b_l \in \mathrm{S}^{-\infty}(X, \mathrm{Hom}(F, E))$, and $b = b_r$ is the symbol we were looking for. Thus we have shown the essential part for the proof of the following theorem.

Theorem 7.3. *Let $A \in \Psi^\infty_{\rho,\delta}(X, E, F)$ be an elliptic pseudodifferential operator. Then there exists a parametrix $B \in \Psi^\infty_{\rho,\delta}(X, F, E)$ for A, i.e., the relations*

$$A\,B - 1 \in \Psi^{-\infty}(X, F, F) \quad \textit{and} \quad B\,A - 1 \in \Psi^{-\infty}(X, E, E) \tag{103}$$

hold. If A is elliptic of order m, B can be chosen of order $-m$. On the other hand, if A is an operator invertible in $\Psi^\infty_{\rho,\delta}(X)$ modulo smoothing operators, then A is elliptic. In case X is compact an elliptic operator $A \in \Psi^\infty_{\rho,\delta}(X)$ is Fredholm.

Proof. Let $a = \sigma_A$. As a is elliptic one can choose b such that (102) holds. If a is elliptic of order m, the above consideration shows that b is of order $-m$. The operator $B = \mathrm{Op}(b)$ then is the parametrix for A. If on the other hand $A = \mathrm{Op}(a)$ has a parametrix $B = \mathrm{Op}(b)$, then $a\#b - 1$ and $b\#a - 1$ are smoothing, hence A is elliptic. Now recall that a pseudodifferential operator induces continuous mappings between appropriate Sobolev-completions of $\mathcal{C}^\infty(X, E)$, resp. $\mathcal{C}^\infty(X, F)$, and that with respect to these Sobolev-completions any smoothing pseudodifferential operator is compact. Hence the claim that for compact X an elliptic A is Fredholm follows from (103). □

Let us give in the following propositions a rather simple criterion for ellipticity of order m in the case of scalar symbols.

Proposition 7.4. *A scalar symbol $a \in \mathrm{S}^m_{\rho,\delta}(X)$ is elliptic of order m, if and only if for every compact set $K \subset X$ there exists $C_K > 0$ such that*

$$||a(\xi)|| \geq \frac{1}{C_K}||\xi||^m \tag{104}$$

for all $\xi \in T^*X$ with $\pi(\xi) \in K$ and $||\xi|| \geq C_K$.

Proof. Let us first show that the condition is sufficient. By assumption there exists a function $b \in \mathcal{C}^\infty(T^*X)$ such that for every compact $K \subset X$ there is $C_K > 0$ with

$$a(\xi) \cdot b\,(\xi) = 1 \quad \text{and} \quad ||b(\xi)|| \leq C_K||\xi||^{-m} \tag{105}$$

for $\xi \in T^*X$ with $\pi(\xi) \in K$ and $||\xi|| \geq C_K$. Hence $a \cdot b - 1 \in \mathrm{S}^{-\infty}(X)$. After differentiating the relation $r = a \cdot b - 1$ in local coordinates (x, ξ) of T^*X we receive

(106)
$$\sup_{\xi \in T^*X|_K} \left| \left(1 + ||\xi||^2\right)^{m/2} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^{|\beta|}}{\partial \xi^\beta} b(\xi) \right| \leq C \sup_{\xi \in T^*X|_K} \left| a(\xi) \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^{|\beta|}}{\partial \xi^\beta} b(\xi) \right| \leq$$
$$\leq C \sup_{\xi \in T^*X|_K} \left\{ |r(\xi)| + \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \beta_1 + \beta_2 = \beta \\ |\alpha_1 + \beta_1| > 0}} \left| \frac{\partial^{|\alpha_1|}}{\partial x^{\alpha_1}} \frac{\partial^{|\beta_1|}}{\partial \xi^{\beta_1}} a(\xi) \frac{\partial^{|\alpha_2|}}{\partial x^{\alpha_2}} \frac{\partial^{|\beta_2|}}{\partial \xi^{\beta_2}} b(\xi) \right| \right\},$$

hence by induction $b \in \mathrm{S}^{-m}_{\rho,\delta}(X)$ follows. By the product expansion Eq. (87) this implies

$$a\#b - 1 = r + t_1 \quad \text{and} \quad b\#a - 1 = r + t_2, \tag{107}$$

where $t_{1/2} \in \mathrm{S}^{-\varepsilon}_{\rho,\delta}(X)$ with $\varepsilon = \min\{(\rho - \delta), 2\rho - 1\} > 0$. Hence a is an elliptic symbol.

Now we will show the converse and assume the symbol a to be elliptic of order m. According to Theorem 7.3 there exists $b \in \mathrm{S}^{-m}_{\rho,\delta}(X)$ such that $\mathrm{Op}(b)$ is a parametrix for $\mathrm{Op}(a)$. But this implies by Eq. (87) that $ab - 1 \in \mathrm{S}^{-\varepsilon}_{\rho,\delta}$ for some $\varepsilon > 0$, hence $|a(\xi)b(\xi) - 1| < \frac{1}{2}$ for all $\xi \in T^*X|_K$ with $||\xi|| > C_K$, where $K \subset X$ compact and $C_K > 0$. But then

$$\frac{1}{2} < |a(\xi)b(\xi)| < C'_K |a(\xi)| ||\xi||^{-m}, \quad \xi \in T^*X|_K,\ ||\xi|| > C_K, \tag{108}$$

which gives the claim. □

In the work of Douglis, Nirenberg [2] a concept of elliptic systems of pseudodifferential operators has been introduced. We want to show in the following that this concept fits well into our framework of ellipticity. Let us first recall the definition by Douglis and Nirenberg. Let $E = E_1 \oplus ... \oplus E_K$ and $F = F_1 \oplus ... \oplus F_L$ be direct sums of Riemannian (Hermitian) vector bundles over X, and $A_{kl} \in \Psi^{m_{kl}}_{1,0}(X, E_k, F_l)$ with $m_{kl} \in \mathbb{R}$ be classical pseudodifferential operators. Denote the principal symbol of A_{kl} by $p_{kl} \in \mathrm{S}^{m_{kl}}_{1,0}(X, E_k, F_l)$. Douglis, Nirenberg [2] now call the system (A_{kl}) elliptic, if there exist homogeneous symbols $b_{kl} \in \mathrm{S}^{-m_{kl}}_{1,0}(X, E_k, F_l)$ such that

(109)
$$\sum_l p_{kl} \cdot b_{lk'} - \delta_{kk'} \in \mathrm{S}^{-1}_{1,0}(X, E_k, F_l) \quad \text{and} \quad \sum_k b_{lk} \cdot p_{kl'} - \delta_{ll'} \in \mathrm{S}^{-1}_{1,0}(X, E_k, F_l)$$

for all k, k', l, l'. The following proposition shows that $A = (A_{kl})$ is an elliptic pseudodifferential operator, hence according to Theorem 7.3 possesses a parametrix.

Proposition 7.5. *Let $E = E_1 \oplus ... \oplus E_K$ and $F = F_1 \oplus ... \oplus F_L$ be direct sums of Riemannian (Hermitian) vector bundles over X. Assume that $A = (A_{kl})$ with $A_{kl} \in \mathrm{S}^{m_{kl}}_{1,0}(X, E, F)$ comprises an elliptic system of pseudodifferential operators in the sense of Douglis, Nirenberg [2]. Then the pseudodifferential operator A is elliptic.*

Proof. Write $a_{kl} = \sigma_{A_{kl}} = p_{kl} + r_{kl}$ with $a_{kl} \in \mathrm{S}^{m_{kl}}_{1,0}(X, \mathrm{Hom}(E_k, F_l)$ and $r_{kl} \in \mathrm{S}^{m_{kl}-1}_{1,0}(X, \mathrm{Hom}(E_k, F_l)$. Furthermore let $a = (a_{kl}) \in \mathrm{S}^{\infty}_{1,0}(X, \mathrm{Hom}(E, F))$, $b = (b_{kl}) \in \mathrm{S}^{\infty}_{1,0}(X, \mathrm{Hom}(E, F))$ and $r = (r_{kl}) \in \mathrm{S}^{\infty}_{1,0}(X, \mathrm{Hom}(E, F))$. By Eq. (109) and the expansion Eq. (87) it now follows

$$\begin{aligned} a\#b - 1 &= (p\#b - 1) + r\#b \in \mathrm{S}^{-1}_{1,0}(X, \mathrm{Hom}(E, F)), \\ b\#a - 1 &= (b\#p - 1) + b\#r \in \mathrm{S}^{-1}_{1,0}(X, \mathrm{Hom}(E, F)). \end{aligned} \tag{110}$$

But this proves the ellipticity of A. □

So with the help of the calculus of normal symbols one can directly construct inverses to elliptic operators in the algebra of pseudodifferential operators on a Riemannian manifold. In particular after having once established a global symbol calculus it is possible to avoid lengthy considerations in local coordinates. Thus one receives a practical and natural geometric tool for handling pseudodifferential operators on manifolds.

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