New York Journal of Mathematics

New York J. Math. **4** (1998) 35–56.

Homology for Operator Algebras III: Partial Isometry Homotopy and Triangular Algebras

S. C. Power

Abstract. The partial isometry homology groups H_n defined in Power [17] and a related chain complex homology CH_* are calculated for various triangular operator algebras, including the disc algebra. These invariants are closely connected with K-theory. Simplicial homotopy reductions are used to identify both H_n and CH_n for the lexicographic products $A(G) \star A$ with $A(G)$ a digraph algebra and A a triangular subalgebra of the Cuntz algebra O_m. Specifically $H_n(A(G) \star A) = H_n(\Delta(G)) \otimes_{\mathbb{Z}} K_0(C^*(A))$ and $CH_n(A(G) \star A)$ is the simplicial homology group $H_n(\Delta(G); K_0(C^*(A)))$ with coefficients in $K_0(C^*(A))$.

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Taking the perspective that equivalence classes of projections in the stable algebra of a non-self-adjoint algebra A may be viewed as 0-simplexes one can often identify the resulting homology group $H_0(A)$ as $K_0(C^*(A))$. Analogously, viewing partial isometries in the stable algebra as 1-simplexes one can similarly formulate higher order homology group invariants for A. This was done recently in Power [17] with the intention of extending the limit homology groups introduced by Davidson and Power [3], for regular limit algebras, to subalgebras of general C*-algebras.

In the present paper we develop further these homology invariants together with a related chain complex homology CH_* also derived from partial isometries in the stable algebra. Our main purpose is to indicate methods for calculation mainly in

Received January 23, 1998.
Mathematics Subject Classification. 47D25, 46K50.
Key words and phrases. operator algebra, homology group, nonselfadjoint, Cuntz algebra.

ISSN 1076-9803/98

the setting of triangular algebras, for which $A \cap A^*$ is abelian. As we see there is a close connection with K-theory for operator algebras, both in terms of the formulation of the invariants and in their identification.

In appropriate contexts induced homology group homomorphisms together with symmetric homology scales seem to provide critical invariants for the position of subalgebras and for the classification of limit algebras. See for example Donsig and Power [5], [6] and Power [17], [18]. In [6] for example, we completely characterise the regular isomorphism classes of the so-called rigid systems of 4-cycle matrix algebras in terms of K_0, H_1 and scales in $K_0 \oplus H_1$. This indicates that partial isometry homology may be more accessible and appropriate than Hochschild cohomology at least in the common setting of algebras with a regular diagonal. At the same time it will be of interest to elucidate the connections between $H_*(A)$ and the Hochshild cohomology of operator algebras, as given in Gilfeather and Smith [7], [8], for example.

The partial isometry homology group $H_1(A)$ can be viewed as an obstruction to the cancellative triangulability of cycles of partial isometries. (In the sequel we shall restrict attention to homology groups arising from normalising partial isometries, this being appropriate for algebras with a regular maximal abelian self-adjoint subalgebra (masa).) To indicate this idea briefly, consider a partial isometry $2n$-cycle, by which we mean a $2n$-tuple $\{v_1, v_2, \ldots, v_{2n}\}$ with $v_{2n}^* v_{2n-1} v_{2n-2}^* \cdots v_2^* v_1 = v_1^* v_1$, and with appropriately matching initial and final projections, $v_1 v_1^* = v_2 v_2^*$ etc. Such a cycle is associated with a $2n$-sided polygonal directed graph (digraph) with alternating edge directions. It may be that for a particular such cycle in the stable algebra of A that one can add additional partial isometries from the stable algebra so that the totality has a digraph (with compositions of edges respecting operator multiplication) which provides a triangulation of the original $2n$-cycle graph. In this case the $2n$-cycle gives no contribution to $H_1(A)$. Thus, if partial isometry cycles are always triangulable in this way then $H_1(A)$ vanishes. This is the case for the disc algebra for example. However there is no general converse assertion because $H_1(A)$ may also vanish for reasons of cancellation, as in the case of some of the algebras of Theorem 2.

Theorem 1. *Let $A(\mathbb{D})$ be the disc algebra. Then $H_0(A(\mathbb{D})) = \mathbb{Z}$ and $H_n(A(\mathbb{D})) = 0$ for $n \geq 1$.*

For Theorem 2 below $H_n(\Delta(G))$ denotes the integral simpicial homology of the simplicial complex $\Delta(G)$ of the digraph algebra $A(G)$. We write $A_1 \star A_2$ for the triangular lexicographic product (see [14], [15]) relative to the natural direct sum decomposition $A_1 = (A_1 \cap A_1^*) + A_1^0$. Thus $A(G) \star A$, with A triangular, is simply the algebra

$$(A(G) \cap A(G)^*) \otimes A + A(G)^0 \otimes C^*(A),$$

which is also triangular if $A(G)$ is triangular. The remaining terminology is explained later in the text.

Triangular algebras have a distinguished (maximal abelian self-adjoint) diagonal and an associated family of normalising partial isometries. Accordingly we can define partial isometry homology group invariants based on this family. In general the problem of uniqueness of diagonals must be addressed. On the other hand lexicographic products do give diverse triangular algebras with computable homology.

Theorem 2. *Let $A(G)$ be a digraph algebra. Let TO_m^0 and TO_m be the refinement nest subalgebras of the algebraic Cuntz algebra O_m^0 and its closure in the Cuntz algebra O_m, respectively. Then $H_n(A(G) \star TO_m^0) = H_n(A(G) \star TO_m) = H_n(\Delta(G)) \otimes \mathbb{Z}_{m-1}$ for all $m \geq 1$ and $n \geq 0$.*

In the important special case of the 4-cycle digraph algebra

$$A(G) = \begin{bmatrix} \mathbb{C} & 0 & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{C} & 0 \\ 0 & 0 & 0 & \mathbb{C} \end{bmatrix}$$

the lexicographic product $\mathcal{A} = A(G) \star TO_m$ may be viewed as

$$A(G) \star TO_m = \begin{bmatrix} TO_m & 0 & O_m & O_m \\ 0 & TO_m & O_m & O_m \\ 0 & 0 & TO_m & 0 \\ 0 & 0 & 0 & TO_m \end{bmatrix}.$$

Whilst the K_0 group here is simply $K_0(\mathcal{A} \cap \mathcal{A}^*)$, which is $C(X, \mathbb{Z})$ with X a Cantor space, the homology group $H_1(\mathcal{A})$ is $\mathbb{Z}_{m-1}$.

There are two stages in the proof of Theorem 2. First we require a key analytical result which is of independent interest, Lemma 5.4, on the structure of normalising partial isometries in O_m and its stable algebra. The second stage — Steps 1, 2 and 3 of Section 6 — can be viewed as the identification of $H_n(\mathcal{A})$ through simplicial homotopy reductions of general normalising partial isometry complexes to elementary partial isometry complexes. In fact this direct approach is applicable in other contexts for which the normalising partial isometries are well-understood. This is the case for example for crossed products and semicrossed products of $C(X)$ with X a Cantor space [19].

In contrast to direct identifications one can also exploit the established machinery of simplicial homology transferred to partial isometry homology. This theme is taken up in the final section. Here a different but closely related form of partial isometry homology, CH_*, is defined in terms of the homology of a chain complex. This homology is more sensitive to torsion as we see in Theorem 7.2, the analogue of Theorem 2. Algebraic topology techniques are easily imported for CH_* and we illustrate this briefly here with the Mayer Vietoris sequence for regular pairs.

1. The Partial Isometry Homology $H_n(\mathcal{A}; \mathcal{C})$

First we define the stable partial isometry homology groups given in Power [17]. It should be borne in mind that the definition given below provides a natural way of recovering the simplicial homology of the digraph of a digraph algebra in purely algebraic terms. Moreover, by doing so in terms of partial isometries (rather than partial orderings of minimal projections for example) we obtain homology groups which give the "correct" limit groups in the case of the fundamental algebras $A(G) \otimes B$, with $A(G)$ a digraph algebra and B an approximately finite C*-algebra. Recall that a digraph algebra $A(G)$ is a unital subalgebra of a complex matrix algebra which is spanned by some of the matrix units of a self-adjoint matrix unit system for the matrix algebra. The digraph G for such an algebra has edges (j, i) associated with the matrix units $e_{i,j}$ that belong to the algebra.

Let $\mathcal{B}$ be a unital involutive algebra and let $\mathcal{C} \subseteq \mathcal{A}$ be unital subalgebras, with $\mathcal{C}$ self-adjoint. Usually we take $\mathcal{C}$ to be an abelian subalgebra or a maximal abelian subalgebra of $\mathcal{B}$. The stable algebra of $\mathcal{A}$ is the algebra $M_\infty(\mathcal{A})$ of finitely nonzero infinite matrices over $\mathcal{A}$, that is, the natural union of matrix algebras over $\mathcal{A}$. It is immediately clear from the definition below that the homology groups are stable in the sense that

$$H_n(\mathcal{A};\mathcal{C}) = H_n(M_\infty(\mathcal{A}); M_\infty(\mathcal{C})).$$

It is appropriate to consider stable homology since this leads to the natural connections with K-theory. Moreover the stable formulation allows the freedom necessary for the algebraic homotopy aguments in the proof of Theorem 2.

A partial isometry u in $M_\infty(\mathcal{A})$ is an element for which u^*u is a (self-adjoint) projection and is said to be $M_\infty(\mathcal{C})$-*normalising*, or simply *normalising* if the context is clear, if ucu^* and u^*cu belong to $M_\infty(\mathcal{C})$ whenever c does. In particular, if p is a projection in $M_\infty(\mathcal{C})$ and pu is a partial isometry, then pu is also normalising.

Let $D \subseteq M_\infty(\mathcal{B})$ be star isomorphic to the matrix algebra $M_r(\mathbb{C})$, with full matrix unit system $\{u_{ij} : 1 \leq i, j \leq r\}$ consisting of normalising partial isometries. In particular each projection u_{ii} belongs to $M_\infty(\mathcal{A})$ and it follows that the subalgebra

$$A_D = D \cap M_\infty(\mathcal{A})$$

is spanned by the matrix units u_{ij} in $M_\infty(\mathcal{A})$. The associated pairs (j, i) form the edges of a (transitive relation) digraph. In particular A_D is (completely) isomorphic to a digraph algebra, and we refer to A_D as a digraph algebra of $\mathcal{A}$. More generally define the digraph algebras A_D when D is star isomorphic to a direct sum of full matrix algebras. Also it is convenient to refer to unital subdigraph algebras of A_D (those unital subalgebras given by a subsemigroup of the matrix units) also as *digraph algebras of* $\mathcal{A}$. Note that the partial matrix unit systems of these subalgebras must not only satisfy the obvious multiplicative relations but must also generate a complete matrix unit system for a finite dimensional C*-algebra.

It is through such algebras, or partial isometry complexes, together with associated regular inclusions and direct sums, that we define the partial isometry homology $H_n(\mathcal{A};\mathcal{C})$. At least, this is appropriate in the case of unital and sigma unital algebras.

Two digraph subalgebras $A_1 = A_{D_1}$ and $A_2 = A_{D_2}$ are said to be *equivalent* if there is a unitary element v in $M_\infty(\mathcal{A} \cap \mathcal{A}^*)$ (more precisely, in some sufficiently large matrix algebra over $(\mathcal{A} \cap \mathcal{A}^*)$, which is normalising, such that $vA_1v^* = A_2$.

To the resulting equivalence class $[A_D]$ of A_D there is a well-defined digraph G and simplicial complex $\Delta(G)$. This complex is obtained from the undirected graph $\overline{G}$ of G by including 0-simplexes $\langle v_i \rangle$ for the vertices v_i of $\overline{G}$ and t-simplexes for each complete subgraph of $\overline{G}$ with $t+1$ vertices.

Define the simplicial homology group $H_n([A_D])$ to be the usual simplicial homology group of $\Delta(G)$ with coefficients in $\mathbb{Z}$. The group $H_n(\mathcal{A};\mathcal{C})$ is defined as the quotient

$$(\sum_{[A_D]} \oplus H_n(([A_D]))/J_n$$

where J_n is a subgroup of the (restricted) direct sum associated with *inclusions* and *splittings* of the subalgebras A_D. Explicitly, J_n is generated by elements

$$-g \oplus \theta(g)$$

and

$$-h \oplus \theta_1(h) \oplus \theta_2(h),$$

where $g \in H_n([A_{D_1}])$ and $\theta : H_n([A_{D_1}]) \to H_n([A_{D_2}])$ is induced by an *inclusion* of matrix unit systems, and where $h \in H_n([A_D])$ and

$$\theta_1 + \theta_2 : H_n([A_D]) \to H_n([A_{D_1}]) \oplus H_n([A_{D_2}])$$

is the mapping induced by a *splitting* $u_{ij} = u^1_{ij} + u^2_{ij}$. By a splitting, we mean that there is a projection p in $M_\infty(C)$, dominated by the initial projection of $u_{1,1}$ such that $u^1_{ij} = u^1_{i1}pu^1_{1j}$ for all appropriate i, j. In view of the assumed normalising property of the u_{ij} the new systems $\{u^1_{ij}\}$ and $\{u^2_{ij}\}$ obtained in this way are also normalising.

One can also express $H_n(\mathcal{A}; \mathcal{C})$ as a universal object amongst groups G with families of embeddings $H_n(\Delta[A_D]) \to G$ respecting the splitting and inclusion induced maps.

In the case when $\mathcal{A}$ is a *triangular algebra* in the sense that $\mathcal{A} \cap \mathcal{A}^*$ is abelian it is convenient to define $H_n(\mathcal{A})$ to be the group $H_n(\mathcal{A}; \mathcal{A} \cap \mathcal{A}^*)$. This definition is particularly appropriate for operator algebras $\mathcal{A}$ in which $\mathcal{C}$ is a regular subalgebra in the sense that the normalising partial isometries in $\mathcal{A}$ generate $\mathcal{A}$.

The following basic result is in Section 2 of [17]. See also the parallel Theorem 7.1 below. The partial isometry homology $H_n(A(G))$ is defined to be $H_n(A(G); C)$ where C is any maximal abelian self-adjoint subalgebra of $A(G)$. This is well-defined since all such diagonal algebras C are unitarily equivalent. A convenient feature of triangular algebras is that we can employ the definition of the last paragraph and avoid problems of uniqueness of diagonals. In this regard there are already complications in the case of diagonals of approximately finite algebras. (See [4].)

Theorem 3. *The partial isometry homology of a digraph algebra $A(G)$ is naturally isomorphic to the simplicial homology of the simplicial complex $\Delta(G)$ of the digraph G.*

The homology scale

Let $A(G) \subseteq M_r(\mathbb{C})$ be a digraph algebra with diagonal subalgebra D_r and with homology groups $H_n(A(G); D_r)$ where $r = |G|$. Identify these groups with the simplicial homology groups $H_n(\Delta(G))$. We may define the *scale* of $H_n(A(G))$ as a subset determined by n-cycles which, in the following sense, lie in the complex $\Delta(G)$ for $A(G)$. For simplicity we take $n = 1$.

A 1-cycle is said to belong to the scale $\Sigma_1(A(G))$ of $H_1(A(G))$ if it has the form

$$\sum_{i=1}^{m_1} \delta_{1,i} + \cdots + \sum_{i=1}^{m_k} \delta_{k,i}$$

where each pair $\delta_{k,i}, \delta_{j,l}$ is disjoint if $k \neq j$, and each partial sum

$$\sum_{i=1}^{m_1} \delta_{k,i}$$

is a 1-cycle for which the 1-simplexes $\delta_{k,i}$ are essentially disjoint.

For a general pair $(\mathcal{A},\mathcal{C})$ define the *scale* of $H_n(\mathcal{A};\mathcal{C})$ to be the images of the scales for all the inclusions

$$H_n(A) \to H_n(\mathcal{A};\mathcal{C})$$

arising from digraph subalgebras A contained in $\mathcal{A}$ (rather than the stable algebra of $\mathcal{A}$).

As an illustration consider the digraph algebra

$$\mathcal{A} = \begin{bmatrix} M_{2,2} & M_{2,3} & 0 & M_{2,2} \\ 0 & M_{3,3} & 0 & 0 \\ 0 & M_{3,2} & M_{2,2} & M_{2,2} \\ 0 & 0 & 0 & M_{2,2} \end{bmatrix}$$

where $M_{2,3}$ is the space of 2×3 complex matrices. In this case $H_1(\mathcal{A}) = \mathbb{Z}$ and the scale is the subset $\{-2,-1,0,1,2\}$. (The reduced digraph of $\mathcal{A}$ is a 4-cycle and so, in the terminology of [5], $\mathcal{A}$ is a 4-cycle algebra.)

For the algebras of Theorem 2 the scales of the higher homology groups are improper in that they coincide with the groups themselves.

For another elementary example consider the tensor product algebra $\mathcal{A} \otimes C(X)$ with $\mathcal{A}$ the digraph algebra above and X a Cantor space. Then

$$H_1(\mathcal{A} \otimes C(X)) = H_1(\mathcal{A}) \otimes K_0(C(X)) = \mathbb{Z} \otimes C(X,\mathbb{Z}) = C(X,\mathbb{Z})$$

(see [17]) and the scale can be identified with the subset $C(X,\{-2,-1,0,1,2\})$.

The scale is a symmetric subset of the abelian group $H_n(\mathcal{A};\mathcal{C})$. That is, if g is an element then so too is $-g$. In many approximately finite contexts it is a generating subset. Also, as with the K_0 scale, the homology scales provide isomorphism invariants. In particular we remark that the scaled first homology group plays a key role in the regular classification of limits of cycle algebras. (See [5], [6].)

2. Vanishing Homology

Cancellation

Let $(\mathcal{A}_1,\mathcal{C}_1),(\mathcal{A}_2,\mathcal{C}_2)$ be pairs as in Section 1. A *regular homomorphism* between such pairs is an algebra homomorphism $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\varphi(\mathcal{C}_1) \subseteq \mathcal{C}_2$ and φ maps the normaliser of $\mathcal{C}_1$ in $\mathcal{A}_1$ into the normaliser of $\mathcal{C}_2$ in $\mathcal{A}_2$. A star-extendible homomorphism $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ is one which is a restriction of a star homomorphism between the generated star algebras. In particular, such a map maps a partial matrix unit system in $M_\infty(\mathcal{A}_1)$ to one in $M_\infty(\mathcal{A}_2)$. Accordingly it is the star-extendible regular maps that induce natural group homomorphisms

$$H_n\varphi : H_n(\mathcal{A}_1;\mathcal{C}_1) \to H_n(\mathcal{A}_2;\mathcal{C}_2).$$

Suppose that $\alpha : \mathcal{A} \to \mathcal{A}$ is a regular star-extendible automorphism with respect to a regular masa $\mathcal{C}$ of $\mathcal{A}$. Let us write, simply, $id \oplus \alpha$ for the maps $M_{2^k} \otimes \mathcal{A} \to M_{2^{k+1}} \otimes \mathcal{A}$ given by $a \to a \oplus (\mathrm{id}_{M_{2^k}} \otimes \alpha)(a)$ for $k = 0,1,2,\ldots$. These maps are regular homomorphisms, with respect to the diagonal masas $D_{2^k} \otimes \mathcal{C}$, and we may form the algebraic direct limit

$$(\tilde{\mathcal{A}};\tilde{\mathcal{C}}) = \lim_{\to}((M_{2^k} \otimes \mathcal{A}; D_{2^k} \otimes \mathcal{C}), id \oplus \alpha).$$

It can be verified that

$$\begin{aligned} H_n(\tilde{\mathcal{A}};\tilde{\mathcal{C}}) &= \lim_{\rightarrow}(H_n(M_{2^k}\otimes\mathcal{A};D_{2^k}\otimes\mathcal{C}),H_n(id\oplus\alpha)) \\ &= \lim_{\rightarrow}(H_n(\mathcal{A};\mathcal{C}),H_n(id\oplus\alpha)). \end{aligned}$$

In particular if $H_n\alpha : H_n(\mathcal{A};\mathcal{C}) \to H_n(\mathcal{A};\mathcal{C})$ is the map $g \to -g$ then $(id\oplus\alpha)_*$ is the zero map and $H_n(\tilde{\mathcal{A}};\tilde{\mathcal{C}}) = 0$. These limit algebras illustrate how homology groups may vanish through cancellation.

For a concrete example we may take $\mathcal{A} \subseteq M_4(\mathbb{C})$ to be the fundamental 4-cycle matrix algebra (spanned by the diagonal matrix units e_{ii} and $e_{13}, e_{14}, e_{24}, e_{23}$) and let α be a reflection automorphism. If one considers the 4-cycle $\{v_1,v_2,v_3,v_4\} = \{e_{13},e_{14},e_{24},e_{23}\}$ in the first algebra $\mathbb{C}\otimes\mathcal{A}$ then no image in a subsequent superalgebra $M_{2^k}\otimes\mathcal{A}$ is triangulable (in the sense mentioned in the introduction and below). Nevertheless the 4-cycle provides a generator for $H_1(\mathbb{C}\otimes\mathcal{A};\mathbb{C}\otimes\mathcal{A}\cap\mathcal{A}^*) = \mathbb{Z}$, and its image in $M_2\otimes\mathcal{A}$ can be split, in our sense, as a direct sum of two 4-cycles of opposite orientation.

We remark that one can consider, more generally, direct limits $\lim\limits_{\rightarrow}(M_{n_k}\otimes\mathcal{A};D_{n_k}\otimes\mathcal{C})$ where each embedding is a direct sum of automorphisms of $M_{n_k}\otimes\mathcal{A}$ coming from automorphisms of a fixed digraph algebra $\mathcal{A}$. In this way one obtains a very wide family of subalgebras of C*-algebras with computable normalising partial isometry homology.

In the examples above cancellation is built in at the outset in the presentation of the algebras. The following simple example illustrates how one might have to be more creative in seeking homology cancellation or reduction.

Let $\mathcal{A} \subseteq M_2\otimes\mathcal{L}(H)$ be spanned by $e_{1,1}\otimes(\mathbb{C}+K)$, $e_{2,2}\otimes(\mathbb{C}+K)$, $e_{1,2}\otimes\mathcal{L}(H)$, that is

$$\mathcal{A} = \begin{bmatrix} \mathbb{C}+K & \mathcal{L}(H) \\ 0 & \mathbb{C}+K \end{bmatrix}$$

where $K \subseteq \mathcal{L}(H)$ is the ideal of compact operators and $\mathcal{L}(H)$ is the algebra of all operators on H, a separable Hilbert space. Consider the 4-cycle $\{u_1,\ldots,u_4\}$ where $u_i = e_{1,2}\otimes w_i, 1\le i\le 4$, and where $w_1^*w_1 = w_4^*w_4 = I$, $\;w_1w_1^* = w_2w_2^*$ has defect 1, $\;w_2^*w_2 = w_3^*w_3$ has defect 2, and $w_3w_3^* = w_4w_4^*$ has defect 3. We can indicate this 4-cycle data with the array

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

and the diagram

$$\begin{tikzcd} 1 & 0 \arrow[l] \arrow[dl] \\ 3 & 2 \arrow[l] \arrow[ul] \end{tikzcd}$$

The 4-cycle admits no trivial triangulation, in the sense that none of the operators $u_3^*u_2, u_2^*u_3, u_3^*u_4, u_4^*u_3$ belong to $\mathcal{A}$, as partial isometries in $\mathbb{C}+K$ have zero Fredholm index. For similar reasons, if $\mathcal{A}_1$ is a digraph algebra associated with $\{u_1,\ldots,u_4\}$ there is no multiplicity one inclusion $i : A_1 \to A_D$ for which $i_* = 0$.

However if A_2 is associated with a 4-cycle $\{v_1,\dots,v_4\}$ with defect array

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

then the direct sum $\{u_1\oplus v_1,\dots,u_4\oplus v_4\}$ is associated with the defect array

$$\begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix}.$$

Both these arrays have a constant column and so A_2 and the algebra A_3, for the direct sum cycle, have trivially triangulable 4-cycle graphs. In particular the inclusions $H_n(A_2)\to H_n(\mathcal{A};\mathcal{C})$, and $H_n(A_3)\to H_n(\mathcal{A};\mathcal{C})$ are the zero maps for $n>1$. A simple elaboration of this argument leads to $H_n(\mathcal{A})=H_n(\mathcal{A};\mathcal{C})=0$.

Triangulation

We have observed how $H_n(\mathcal{A};\mathcal{C})$ may vanish by virtue of cancellation giving rise to induced zero maps $i_*: H_n(A_{D_1})\to H_n(A_{D_2})$. Here the inclusion i necessarily is of multiplicity greater than one. It can also happen that $H_n(\mathcal{A};\mathcal{C})$ vanishes for more geometric reasons in the following sense. Suppose that for every digraph algebra A_{D_1} for the pair $(\mathcal{A},\mathcal{C})$ there is a containing digraph algebra A_{D_2}, with multiplicity one star-extendible regular inclusion $i: A_{D_1}\to A_{D_2}$, such that $H_n i=0$. For the case $n=1$ we can view the digraph $G(A_{D_2})$ as providing triangulations of the cycles in $G(A_{D_1})$. In this case $H_n(\mathcal{A};\mathcal{C})=0$.

We give two illustrations.

Let $(\mathcal{A},\mathcal{C})=(T_m,D_m)$ where $T_m\subseteq M_m(\mathbb{C})$ is the upper triangular subalgebra. Let $A_D\subseteq M_N\otimes T_m$ be an $M_N\otimes D_m$ normalising digraph algebra for $\mathcal{A}$, with connected digraph G and partial matrix unit system $\{u_{ij}:(i,j)\in E(G)\}$. The partial matrix unit system can be decomposed as a direct sum $u_{ij}=u_{ij}^{(1)}+\dots+u_{ij}^{(r)}$ (in the appropriate splitting sense) where each $u_{ij}^{(k)}$ has rank one. We show that for each associated digraph algebra $A_{D^{(k)}}$ there is a regular multiplicity one inclusion inducing a zero map on H_n for each $n\geq 1$. We may as well assume already that rank $u_{ij}=1$ for all i,j. Note that since each u_{ij} is normalising it follows that

$$(I_N\otimes e_{kk})u_{ij}(I_N\otimes e_{ll})=u_{ij}$$

for precisely one pair k,l with $1\leq k,l\leq m$. It now follows that $\{u_{ij}:(i,j)\in E(G)\}$ is unitarily equivalent to a subset of the standard matrix unit system for $M_N\otimes T_m$. In particular the inclusion $i: A_D\to M_N\otimes T_m$ is of the admissible kind specified in the definition of $H_n(\mathcal{A};\mathcal{C})$. Plainly $H_n i=0$ for $n\geq 1$, since $H_n([M_n\otimes T_m])=0$, and so $H_n(T_m;D_m)=0$ for $n\geq 1$.

For the second illustration, consider the disc algebra $A(\mathbb{D})$, viewed in the usual way, as a function algebra on the unit circle S^1. Let $\{u_1,\dots,u_{2n}\}$ be a $2n$-cycle of partial isometries in $M_{2n}\otimes A(\mathbb{D})$ of the form

$$u_k=v_k\otimes w_k$$

where $\{v_1,\dots,v_{2n}\}$ is the standard $2n$-cycle in M_{2n} given by $v_1=e_{12}, v_2=e_{32}, v_3=e_{34},\dots,v_{2n-1}=e_{2n-1,2n}, v_{2n}=e_{1,2n}$, and where $w_1,\dots w_{2n}$ are inner functions in $A(\mathbb{D})$ with $w_1w_3\dots w_{2n-1}=w_2w_4\dots w_{2n}$. The $2n$-cycle $\{u_1,\dots,u_{2n}\}$ together with the diagonal projections $e_{ii}\otimes 1$ spans a subalgebra, $A(w_1,\dots,w_{2n})$ say, which is completely isometrically isomorphic to the digraph algebra in $M_{2n}(\mathbb{C})$ associated with the standard $2n$-cycle. Furthermore, $A(w_1, \ldots, w_{2n})$ is a digraph algebra of $A(\mathbb{D})$ according to our definition.

The algebra $A(w_1, \ldots, w_{2n})$, can be triangulated as follows.

Consider the natural inclusion

$$A(w_1, \ldots w_{2n}) \to A(w_1, \ldots, w_{2n}) \ \oplus \ \mathbb{C}e_{2n+1,2n+1}.$$

Let $w = w_1 w_3 \ldots w_{2n-1}$ and define

$$\begin{aligned} x_1 &= e_{1,2n+1} \otimes w, \\ x_2 &= e_{2,2n+1} \otimes \frac{w}{w_1}, \\ x_3 &= e_{3,2n+1} \otimes \frac{ww_2}{w_1}, \\ &\vdots \\ x_{2n} &= e_{2n,2n+1} \otimes \frac{ww_2}{w_1}.\frac{w_4}{w_3} \ldots \frac{w_{2n-2}}{w_{2n-1}} \end{aligned}$$

Note that in view of the equality $w_1 w_3 \ldots w_{2n-1} = w_2 w_4 \ldots w_{2n}$ it follows that $A(w_1, \ldots w_{2n})$ together with $x_1, \ldots, x_{2n}$ and $e_{2n+1,2n+1}$ span a digraph algebra, A^+ say. The digraph of A^+ is the cone over the digraph of A and so the inclusion $A(w_1, \ldots, w_{2n}) \to A^+$ induces the zero map on H_1.

3. The Proof of Theorem 1

We now turn to the proof that $H_n(A(\mathbb{D})) = 0$ for $n \geq 1$. The essential idea is the triangulation argument above, although this time we must consider matrix functions.

Let $D \subseteq M_N \otimes C(S^1)$ be a finite-dimensional C*-algebra with matrix unit system $\{u_{ij}\}$ normalising $M_N \otimes \mathbb{C}$. Without loss of generality assume that D is isomorphic to M_r and that A_D is spanned by $\{u_{ij} : (i,j) \in E(G)\}$ where G is a connected digraph with r vertices. We claim that there is a multiplicity one inclusion $D \to D^+$ where D^+ has matrix unit system $\{u_{ij} : 1 \leq i,j \leq r+1\}$, where $D^+ \subseteq M_{2N} \otimes C(S^1)$ and where $A_{D^+} = \{u_{ij} : (i,j) \in E(H)\}$ where H is a digraph containing G and all edges from the new vertex $r+1$ to vertices of G (labelled $1, \ldots, r$). That is H contains the cone over G.

This step will complete the proof since the simplicial complex of the cone has trivial higher order homology.

Define $v_{ij} \in M_2 \otimes M_N \otimes C(S^1)$ by $v_{ij} = e_{11} \otimes u_{ij}$, for $1 \leq i,j \leq r$. Let $v_{r+1,r+1} = e_{22} \otimes u_{r,r}$, $v_{r,r+1} = e_{12} \otimes u_{r,r}$, and consider the full matrix unit system for M_{r+1} which is generated by these matrix units and denoted $\{v_{ij} : 1 \leq i,j \leq r+1\}$. Consider the partial isometries $v_{i,r+1}$, for $1 \leq i \leq r$, which correspond to edges from vertex $r+1$ to the vertices of G. Each such partial isometry is a word in the set

$$\{e_{11} \otimes u_{ij}, e_{11} \otimes u_{ij}^* : 1 \leq i,j \leq r\} \cup \{v_{r,r+1}\}.$$

Moreover we can choose words of length at most r, in which no partial isometries are repeated. Thus, each $v_{i,r+1}$ has the form

$$e_{12} \otimes u_1 \ldots u_k$$

where $k \leq r$ and for each i with $1 \leq i \leq r$ either u_i or u_i^* belongs to $M_N \otimes A(\mathbb{D})$.

A partial isometry w in $M_N \otimes A(\mathbb{D})$ has the form

$$w = U_1 D_1 U_2 D_2 \dots U_t D_t U_{t+1}$$

where each U_k is a scalar partial isometry and where each D_k is a diagonal matrix of functions whose entries consist of a single Blaschke factor ϕ_k and constant functions. (This may be deduced from the well-known corresponding assertion for inner functions in $M_N \otimes A(\mathbb{D})$.) It follows that if $\Phi(z)$ is the inner function $\Phi(z) = I \otimes (\phi_1 \dots \phi_t)$ then

$$Aw^* B \Phi(z)$$

is a partial isometry in $M_N \otimes A(\mathbb{D})$ for all partial isometries A, B in $M_N \otimes A(D)$ for which Aw^*B is a partial isometry. Let $\Psi(z)$ be the product of the Blaschke factors associated with all the partial isometries u_{ij} in A_D. Define

$$v'_{ij} = \begin{bmatrix} I & 0 \\ 0 & \Psi(z) \end{bmatrix}^* v_{ij} \begin{bmatrix} I & 0 \\ 0 & \Psi(z) \end{bmatrix}$$

for $1 \leq i, j \leq r+1$. This is a matrix unit system for $D^+ = M_{r+1}$ with the desired properties.

4. K_0-regular Inclusion and Homotopy

We begin by considering $H_0(\mathcal{A}; \mathcal{C})$ and the following idea (see also [17]) will be useful.

The inclusion $\mathcal{C} \to \mathcal{A}$ is said to be *K_0-regular* if

(i) the induced map $K_0\mathcal{C} \to K_0(C^*\mathcal{A})$ is a surjection, and
(ii) whenever p, q are projections in $M_N \otimes \mathcal{C}$ which are unitarily equivalent in $M_N \otimes C^*\mathcal{A}$, then there is a digraph subalgebra for $\mathcal{A}$ with connected graph which contains projections p' and q' as minimal projections, where the $K_0(C^*(\mathcal{A}))$ classes $[p], [p'], [q], [q']$ all agree.

Note, for example, that $D_m \to T_m$, the diagonal algebra inclusion, is K_0-regular. Also, $\mathbb{C} \to A(\mathbb{D})$ is K_0-regular. We shall see that the masas for the triangular algebras of Theorem 2 have K_0-regular inclusions.

Proposition 4.1. *If $\mathcal{C} \to \mathcal{A}$ is a K_0-regular inclusion then $H_0(\mathcal{A}; \mathcal{C}) = K_0(C^*(\mathcal{A}))$.*

Proof. By the hypotheses

$$\sum_{[A_D]} \oplus \, H_0([A_D])$$

contains the subgroup

$$\sum_{[p] \in (K_0 C^*(A))_+} \oplus \, \mathbb{Z}$$

arising from the degenerate digraph algebras $\mathbb{C}p$ associated with projections p in $M_N \otimes \mathcal{C}$ for some N. Moreover, in view of the inclusion and splitting relations used in the definition of J_0 we see that

$$\begin{aligned} H_0(\mathcal{A};\mathcal{C}) &= (\sum_{[A_D]} \oplus H_0([A_D]))/J_0 \\ &= (\sum_{[p]\in(K_0C^*(A))_+} \oplus\mathbb{Z})/J_0 \\ &= K_0C^*(A). \end{aligned}$$

The second equality here is a consequence of the inclusion relations in J_0. The last equality holds since the splitting relations correspond to the semigroup relations for $K_0(C^*(A))_+$, and for any semigroup S the quotient $(\sum_{s\in S}\oplus\mathbb{Z})/R$, arising from the semigroup relations R, is the Grothendieck group of S. □

The assertions concerning H_0A for the triangular algebras of Theorem 2 follow from this proposition and the K_0-regularity of the diagonal inclusions discussed below.

We now consider a simple retraction procedure which will be useful for identifying the partial isometry homology of lexicographic products.

Let A_1, A_2 be digraph algebras for the pair $(\mathcal{A},\mathcal{C})$ for which there is a containing digraph algebra A with the following properties:

(i) The digraph $G(A)$ has vertices $\{v,w\}\cup\{v_1,\ldots,v_r\}$ and A_1 (respectively A_2) is the subalgebra of A determined by the full subgraph of $G(A)$ for the vertices $\{v\}\cup\{v_1,\ldots,v_r\}$ (respectively $\{w\}\cup\{v_1,\ldots,v_r\}$).

(ii) $(v,v_i)\in E(G(A))$ if and only if $(w,v_i)\in E(G(A))$ and $(v_i,v)\in E(G(A))$ if and only if $(v_i,w)\in E(G(A))$, and at least one of the edges (v,w) or (w,v) belongs to $E(G(A))$.

In this case we say that there is an *elementary homotopy* between A_1 and A_2 (and between A_2 and A_1). Since $\Delta(G(A_1))$ and $\Delta(G(A_2))$ are simplicial retractions of $\Delta(G(A))$ it follows that $H_n(\Delta(G(A_1))) = H_n(\Delta(G(A_2))) = H_n(\Delta(G(A)))$ for all $n\geq 0$. Moreover, since the inclusions $A_1\to A$, $A_2\to A$ induce simplicial homology isomorphisms it follows, in the notation of Section 1, that

$$(H_n([A_1])\oplus H_n([A_2])\oplus H_n([A]))/J_n = H_n([A_i])/J_n$$

for $i=1$ or 2. In this way we will be able to obtain reductions through inclusion relations corresponding to homotopy. More generally this reduction will also hold if A_1 and A_2 are *homotopic*, by which we mean that there is a finite chain of elementary homotopies connecting A_1 to A_2. Plainly there is a more general notion of homotopy, allowing for retractions, but the present usage suffices for the proof below.

5. The Cuntz Algebras and TO_m

The Cuntz algebra O_m is the universal C*-algebra generated by m isometries $S_1,\ldots,S_m$ with $S_1S_1^*+\cdots+S_mS_m^*=1$. In fact, any C*-algebra generated by a set of isometries satisfying this relation is isomorphic to O_m. It will be convenient to consider the specific representation on $L^2[0,1]$ generated by the natural isometries $S_1,\ldots,S_m$ where $S_iS_i^*$ is the orthogonal projection onto $L^2[(i-1)/m, i/m]$, for $1\leq i\leq m$. Specifically, for $f\in L^2[0,1]$,

$$(S_i f)(x) = \begin{cases} \sqrt{m} f(mx - (i-1)) & , \quad \text{for } i - 1 \le mx \le i \\ 0 & , \quad \text{otherwise.} \end{cases}$$

The (unclosed) star algebra generated by $S_1, \dots, S_m$ is also uniquely determined by any realisation, and we denote this algebra by O_m^0.

Recall the following basic facts from Cuntz [1]. Let W_k denote the words of length k in the letters $1, 2, \dots, n$. If $\mu = \mu_1 \dots \mu_k \in W_k$ then write $S_\mu = S_{\mu_1} \dots S_{\mu_k}$, $l(\mu) = k$ and let

$$d(\mu) = \frac{\mu_1 - 1}{m} + \cdots + \frac{\mu_k - 1}{m^k}.$$

Every word in the operators $S_1, \dots, S_m$ and their adjoints can be reduced to the form $S_\mu S_\omega^*$ for uniquely determined words μ, ω. For words of the same length, we have

Lemma 5.1. *Let $\mu, \omega \in W_k$. Then $S_\mu S_\omega^*$ is the natural partial isometry with initial space $L^2[d(\omega), d(\omega) + m^{-k}]$ and final space $L^2[d(\mu), d(\mu) + m^{-k}]$.*

Thus $\{S_\mu S_\omega^* : \mu, \omega \in W_k\}$ is a set of matrix units for a copy of the matrix algebra M_{m^k}. The union of these algebras will be denoted F^0, and the closed union, which is a UHF C*-algebra of type m^∞, will be denoted F. Also write C for the masa in F generated by $\{S_\mu S_\mu^* : \mu \in W_k\}$.

We now define the triangular algebra TO_m. Let $\mathcal{N}$ be the nest of projections in F^0 corresponding to the subspaces $L^2[0, i/m^k]$, for $1 \le i \le m^k, k = 1, 2, \dots$. Define

$$\begin{aligned} TF &= \{a \in F : (1-p)ap = 0, \ \forall\, p \in \mathcal{N}\}, \\ TO_m &= \{a \in O_m : (1-p)ap = 0, \ \forall\, p \in \mathcal{N}\}, \end{aligned}$$

and define TF^0 and TO_m^0 similarly. Then TF is a copy of the refinement algebra $\varinjlim(T_{m^k}, \rho)$ determined by the so-called refinement embeddings. (See [13].) For this reason we refer to TO_m as the refinement subalgebra of O_m. We can also think of the algebras TF and TO_m as the Volterra nest subalgebras of the realisations of F and O_m. Alternatively, the algebras TO_m and TO_m^0 can be described in purely intrinsic terms, as follows. This description will not be needed below, and we refer the reader to [11] for a proof.

Proposition 5.2. *If $l(\mu) \le l(\omega)$ then $S_\mu S_\omega^* \in TO_m$ if and only if $d(\mu) \le d(\omega)$. If $l(\mu) < l(\omega)$ then $S_\mu S_\omega^* \in TO_m$ if and only if*

$$d(\mu) + m^{-l(\mu)} \le d(\omega) + m^{-l(\omega)}$$

Furthermore, TO_m is generated as an operator algebra by the operators $S_\mu S_\omega^*$ in TO_m.

In particular, it follows that TO_m is generated by a subsemigroup of an inverse semigroup of normalising partial isometries. The same is true for the algebras $A(G) \star TO_m$.

The next two lemmas provide the purely C*-algebraic technical results that we need to understand normalising partial isometries and the normalising finite-dimensional C*-algebras associated with O_m. In brief they allow for a reduction to the case of standard matrix unit systems with matrix units that are orthogonal sums of the standard partial isometries $S_\mu S_\omega^*$.

Lemma 5.3 (Cuntz [1].). *Each operator a in the star algebra O_m^0 generated by $S_1, \ldots, S_m$ has a unique representation*

$$a = \sum_{i=1}^{N} (S_1^*)^i a_{-i} + a_0 + \sum_{i=1}^{N} a_i S_1^i$$

where $a_i \in F$ for each i. Moreover the linear maps E_i given by $E_i(a) = a_i$ extend to continuous contractive linear maps from O_m to F.

Lemma 5.4. (i) *If v is a C-normalising partial isometry in O_m then there is a partial isometry w in O_m^0 such that $v = cw$ where c is a partial isometry in C and w has the form*

$$w = \sum_{i=1}^{N} (S_1^*)^i v_{-i} + v_0 + \sum_{i=1}^{N} v_i S_1^i$$

where the sum is an orthogonal sum of partial isometries with each v_j equal to an orthogonal finite sum of partial isometries in $\{S_\mu S_\omega^* : \mu, \omega \in W_j\}$.

(ii) *If v is a $M_n \otimes C$-normalising partial isometry in $M_n \otimes O_m$ then there is a partial isometry w' such that $v = cw'$ where c is a partial isometry in $M_n \otimes C$ and where $w' = (w_{ij})_{i,j=1}^n$ with each w_{ij} a partial isometry as in* (i).

Let us say that two *standard* partial isometries v, w in O_m, by which we mean those of the form $S_\mu S_\omega^*$, are *disjoint* if for all projections p, q in C the equality $pvq = pwq$ implies $pvq = pwq = 0$. This is equivalent to the graphs of the partial homeomorphisms inducing v and w (as composition operators) having at most one point in common.

Proof of Lemma 5.4. Note that if the index $l(\mu) - l(\omega)$ for $S_\mu S_\omega^*$ differs from the index for $S_\rho S_\delta^*$ then these partial isometries are disjoint. Let $v \in N_C(O_m)$. It follows from Lemma 5.3 that there is a finite complex combination $w = \alpha_1 u_1 + \cdots + \alpha_n u_n$ of disjoint standard partial isometries such that $\|v - w\| < \frac{1}{4}$. Here the coefficients α_i are nonzero complex numbers.

We claim that there is a subset of $\{u_i\}$, which we may relabel as $u_1, \ldots, u_l$, consisting of partial isometries with orthogonal initial projections and orthogonal final projections such that $v = c(u_1 + \cdots + u_l)$ for some partial isometry c in C.

By disjointness there are projections p, q in C such that $pvq \neq 0$ and $pwq = \alpha_i p u_i q$ for some i. Relabel to arrange $i = 1$. Thus $\|pvq - \alpha_1 p u_1 q\| < \frac{1}{4}$. Also $|1 - |\alpha_1|| < \frac{1}{4}$ and so $\|v_1 - t\| < \frac{1}{2}$ where $v_1 = pvq$ and $t = pu_1 q$. In particular tt^* is a projection in C and $\|tt^* - v_1 t^*\| < \frac{1}{2}$. Thus $\|P v_1 t^* P^\perp\| < \frac{1}{2}$ for all projections P in C. Since $v_1 t^*$ is normalising it follows that $P v_1 t^* P^\perp = 0$ for all such P. Since C is a masa in O_m it follows that $v_1 t^*$ is a partial isometry, d_1 say, in C. Thus $v_1 = d_1 t$ with t a standard partial isometry and d_1 a partial isometry in C.

The partial isometry u_1 can be expressed as a strong operator topology sum $\sum_{i=1}^\infty p_i u_1 q_i$ of orthogonal partial isometries of the form above. In this way we deduce that $v(u_1 u_1^*) = cu_1$ where $c = \sum_{i=1}^\infty d_i$. Furthermore, since $v = cu_1 + (v - cu_1)$ is necessarily an orthogonal sum of two partial isometries it follows that $c = E_0(vu_1^*)$ and hence that c belongs to C.

Repeating the argument above obtain an orthogonal decomposition

$$v = v' + v''$$

where $v' = c_1u_1 + \cdots + c_lu_l$ with $c_i \in C$ for each l, where, after relabelling, $|1 - |\alpha_i\| < \frac{1}{4}$ precisely for $1 \le i \le l$. But now consider $v'' = v - v'$. This is a normalising partial isometry. Suppose that $v'' \neq 0$. Then (by disjointness again) there are projections p, q in C such that $pvq = pv''q \neq 0$ and $pwq = \alpha_j pu_j q$. Of necessity $j > l$ and $|1 - |\alpha_j|| \ge \frac{1}{4}$. Since $\|w - v\| < \frac{1}{4}$ we obtain $\|\alpha_j pu_j q - pv''q\| < \frac{1}{4}$ which, given the inequality for α_j, leads to the contradiction $\|pv''q\| \neq 1$

The proof of (ii) is similar. □

Although in this paper we focus on TO_m, we note that O_m has many other natural maximal triangular subalgebras.

Let $A \subseteq F$ be a maximal triangular subalgebra which contains the masa C. Let $\mathcal{A} \subseteq O_m$ be the set of operators a for which $E_0(a) \in A$ and $E_i(a) = 0$ for $i < 0$. Both $\mathcal{A}$ and its superalgebra $\mathcal{A} + F$ have, roughly speaking, the nature of an analytic subalgebra or semicrossed product in the sense of Muhly and Solel [9] and Peters [10], for example.

In general one expects a maximal triangular subalgebra to have trivial partial isometry homology groups for $n > 1$ and this is so for TF and TO_m by simple triangulation argument in the spirit of the next section. The analytic algebras $\mathcal{A}$ above ("bianalytic" is a more accurate designation) also have trivial higher homology. Nevertheless we now indicate a maximal triangular subalgebra A of $M_4 \otimes C(X)$ for which $H_1(A) = \mathbb{Z}$, and this can be used in the construction of more elaborate examples, with trivial centre for example, also with nonzero H_1.

Let X be a Cantor space and let U, V be open subsets with dense union and with $X/(U \cup V) = \{x\}$ where x is a point of closure of U and of V. Write $C(U)$ and $C(V)$ for the subalgebras of $C(X)$ supported on U and V and define

$$\mathcal{A} = \begin{bmatrix} C(X) & C(V) & C(X) & C(X) \\ C(U) & C(X) & C(X) & C(X) \\ 0 & 0 & C(X) & C(V) \\ 0 & 0 & C(U) & C(X) \end{bmatrix}.$$

This is a maximal triangular algebra. Over the point x in the maximal ideal space of the centre of $\mathcal{A}$, the local algebra for x is isomorphic to the 4-cycle algebra $A(D_4)$. This in turn leads to the fact that $H_1(\mathcal{A}) = \mathbb{Z}$.

6. The Proof of Theorem 2

Let $(\mathcal{A}, \mathcal{C}) = (A(G) \star TO_m, \mathbb{C}^{|G|} \otimes C)$. Clearly the $K_0(O_m)$ classes [1] and $[S_iS_i^*]$ coincide for $i = 1, \ldots, m$ and so $m[1] = [S_1S_1^*] + \cdots + [S_mS_m^*] = [1]$. Thus $(m-1)[1] = 0$. Moreover Cuntz [2] has shown that for $m > 1$ $K_0(O_m) = \mathbb{Z}/(m-1)$, with [1] as generator. In particular the inclusion $C \to O_m$ induces a K_0 group surjection. In fact the inclusion $\mathcal{C} \to \mathcal{A}$ is K_0-regular as we now show.

We may assume that G is connected. Let us say that a projection p in $M_N \otimes \mathbb{C}^{|G|} \otimes C$ is $\mathcal{A}$-connected to a projection p' if there is a digraph subalgebra for $\mathcal{A}$, with connected digraph, which contains p, p' as minimal projections. In particular, if $p \neq p'$ then these projections are orthogonal. Note first that each projection p in $M_N \otimes \mathbb{C}^{|G|} \otimes C$ is $\mathcal{A}$-connected to a projection p' in $M_N \otimes e_{1,1} \otimes C$. Accordingly, it will be enough to show that two orthogonal projections p', p'' in $M_N \otimes e_{1,1} \otimes C$, if unitarily equivalent in $M_N \otimes e_{1,1} \otimes O_m$, are $\mathcal{A}$-connnected. But $M_n \otimes \mathcal{A}$ contains

$M_N \otimes e_{1,1} \otimes TO_m$, and there exist partial isometries u, v in $M_N \otimes e_{1,1} \otimes TO_m$ with $uu^* = vv^*, u^*u = p'$ and $v^*v = p''$, and so p' and p'' are $\mathcal{A}$-connnected.

Since the standard partial isometries $S_\mu S_\omega^*$ are normalising the K_0-regularity of the inclusion $C \to \mathcal{A}$ follows.

Step 1. Reduction to standard digraph subalgebras.

Let D_1 be a finite-dimensional C*-algebra in the stable algebra $M_\infty(C^*(\mathcal{A}))$ with a normalising matrix unit system for $M_\infty(\mathcal{C})$. Then, for some integer N,

$$D_1 \subseteq M_N \otimes C^*(A(G)) \otimes O_m = M_N \otimes C^*(\mathcal{A}),$$

with matrix unit system normalising $M_N \otimes \mathbb{C}^{|G|} \otimes C$. Using Lemma 5.4 (ii) it follows from routine C*-algebra theory that D_1 is unitarily equivalent, by a unitary operator in $M_N \otimes\ \mathbb{C}^{|G|} \otimes C$ to a finite-dimensional C*-algebra D_2 with a matrix unit system which is decomposable, by splitting, as a *direct sum* of matrix unit systems each with matrix units v of the standard form $f_{kl} \otimes e_{ij} \otimes S_\mu S_\omega^*$, where (f_{kl}) is a standard system for $M_N, \{e_{ij}\}$ the standard system for $C^*(A(G))$ and with k, l, i, j, μ, ω depending on v. We say A_{D_2} is a *standard* digraph algebra for $\mathcal{A}$. Thus, if $\mathcal{D}$ is the sub-collection of digraph algebra classes $[A_D]$ associated with the elementary tensor systems then we have the first reduction

$$H_n(\mathcal{A}; \mathcal{C}) = (\sum_{\mathcal{D}} \oplus H_n([A_D]))/I_n$$

where I_n is the ideal in the direct sum (of the right hand side above) generated by the splitting and inclusion relations for the standard algebras A_D.

In fact this reduction follows in two stages. First note, as we did above, that A_{D_1} is equivalent to a subalgebra A_{D_2} of a standardised algebra A_D where the subalgebra inclusion map $A_{D_2} \to A_D$ is regular. Secondly, note that, for similar reasons,

$$J_n \cap (\sum_{\mathcal{D}} \oplus H_n([A_D]) = I_n$$

since a generator for the ideal J_n, arising from a splitting, or an inclusion, can be written as a sum of elements in I_n.

Step 2. Homotopies to equalise traces of diagonal matrix units.

Assume now that D is a standardised finite-dimensional C*-algebra matrix summand, with associated digraph subalgebra $A_D = D \cap (M_N \otimes \mathcal{A})$ with a matrix unit system of elementary tensors. Further reduction will be obtained by non-self-adjoint homotopy in the sense described earlier.

Claim: Suppose that the digraph of A_D is connected so that D is a full matrix algebra. Then A_D is homotopic to $A_{D'}$ where D' is also a full matrix subalgebra but with matrix units v of the form $f_{kl} \otimes e_{ij} \otimes S_\mu S_\omega^*$ where $l(\mu)$ and $l(\omega)$ are equal to a fixed integer (which is independent of v).

Consider first the subalgebras of A_D of the form

$$A_1 = (f_{kk} \otimes e_{ll} \otimes 1) A_D (f_{kk} \otimes e_{ll} \otimes 1)$$

for some self-adjoint matrix unit $v = f_{kk} \otimes e_{ll} \otimes S_\lambda S_\lambda^*$ for D. Then A_1 is a digraph subalgebra of $f_{kk} \otimes e_{ll} \otimes TO_m$ with matrix units of the form $f_{kk} \otimes e_{ll} \otimes w_{ij}$ where w_{ij} has the form $S_\mu S_\omega^*$ for various μ, ω.

In particular each w_{ii} is an interval projection for the projection nest in TO_m. First we consider homotopies to equalise the traces of these matrix units.

Since D is a full matrix algebra we know that these diagonal projections are equivalent in O_m. Order the projections w_{ii} as a family $\{w_1, w_2, \dots, w_r\}$ with the usual ordering of intervals of a projection nest. Also, relabel the w_{ij} for consistency so that $w_{ij} = w_i w_{ij} w_j$. Let u_1 be an interval projection of small trace which is the final projection of a partial isometry $z = S_\mu S_\omega^*$ in TO_m with initial projections w_1. We then see that A_1 is homotopic (and isomorphic) to the digraph algebra $\hat{A}_1$ in which w_1 is replaced by u_1 and each w_{1j} is replaced by zw_{1j}. Strictly speaking, according to our definition, this is not quite a homotopy since u_1 may not be orthogonal to w_1 and in this case we cannot consider the containing digraph algebra for the pair A_1, $\hat{A}_1$. However we are only interested in equivalence classes and it is a simple matter to replace $\hat{A}_1$ by an equivalent algebra in $M_{2N} \otimes A(G) \otimes O_m$ for which there does exist the necessary containing algebra.

Similarly we can, through an elementary homotopy, replace w_2 by an interval projection u_2 of the same trace as u_1, as long as the trace of u_1 is small enough. Repeating, obtain a homotopy between A_1 and (an isomorphic) digraph algebra A_1' with diagonal matrix units of the form $f_{kk} \otimes e_{ll} \otimes u_i$ with the projections u_i of the same trace.

Since $\mathcal{A}$ is the lexicographic product $A(G) \star TO_m$ there is in fact no obstruction to extending these homotopies to homotopies between A_D and $A_{D'}$ for some other standard matrix algebra D'. Indeed at the first stage if w is any matrix unit of the form

$$f_{k',k} \otimes e_{l',l} \otimes u$$

where $l' \neq l$, $e_{l,l'} \in A(G)$ and where $u = S_\mu S_\nu^*$ in O_m (but perhaps not in TO_m) and has final projection w_1, then simply replace u by zu. Similarly, if

$$f_{k',k} \otimes e_{l',l} \otimes v$$

is a matrix unit of A_D, where $l' \neq l$ and v has initial projection w_1, then we may replace v by vz^*. In this way extend the first elementary homotopy and it is clear that the subsequent homotopies can be extended in the same way.

Similarly we can continue with homotopies to equalise the traces of all the diagonal matrix units of A_D and this establishes the claim, and Step 2.

Note that the algebra $A_{D'}$ is necessarily a (multiplicity one) subalgebra of $M_N \otimes A(G) \star T_{m^r}$ for some r (equal to the common values of $l(\mu)$) and some matrix algebra M_N. Furthermore note that the algebras $M_N \otimes A(G) \star T_{m^r}$, $r = 0, 1, 2, \dots$, are all homotopic. In particular it follows that $H_n(\mathcal{A}; \mathcal{C})$ is finitely generated since it coincides with the subgroup $H_n([A(G) \star T_m])/I_n$.

In fact, in view of these homotopies, this subgroup is necessarily a quotient of $H_n(\Delta(G)) \otimes \mathbb{Z}_{m-1}$.

Step 3. The Isomorphism.

We have shown that given the special digraph subalgebra A_D with matrix unit system consisting of elementary tensors, and with connected digraph, there is a similar digraph subalgebra for $\mathcal{A}$ of the form

$$A_{D^+} = \begin{bmatrix} A_{D'} & ? \\ 0 & A_D \end{bmatrix}$$

where $A_{D'}$ is a subalgebra of $M_N \otimes A(G) \star T_{m^r}$ spanned by elementary tensors. We refer to this as a homotopy construction for A_D. Although the class $[A_{D'}]$ is not determined explicitly in the construction — there is freedom in the choice of r and in the choice of small interval projections — there is however a well-defined surjective map

$$\nu_{A_D} : H_n([A_D]) \to H_n(\Delta(G)) \otimes \mathbb{Z}_{m-1}$$

determined from the composition

$$\begin{aligned} H_n([A_D]) &\to H_n([A_{D'}]) \to H_n([M_N \otimes A(G) \star T_{m^r}]) \\ &\to H_n(\Delta(G)) \to H_n(\Delta(G)) \otimes \mathbb{Z}_{m-1} \end{aligned}$$

where the first map is the isomorphism induced by the inclusion in A_{D^+}, the second is induced by inclusion and the third is the natural indentification isomorphism.

A splitting $A_D \to A_{D_1} \oplus A_{D_2}$ of A_D need not yield digraph algebras A_{D_i} of the same type, with elementary tensor matrix unit systems (since, for example, $I - S_1S_1^*$ is not of the form $S_\mu S_\mu^*$ if $m \geq 3$). Nevertheless it is clear that one also has a homotopy construction for digraph algebras A_E of this type (with connected digraph) and well-defined maps ν_{A_E} as before.

We now note that the family $\{\nu_{A_E}\}$ (we can restrict to the case of connected digraphs for simplicity, although this is not necessary) respects splittings and inclusions in the following sense. If $A_{E_1} \to A_{E_2}$ is an inclusion of standard digraph algebras for $\mathcal{A}$ then the following diagram commutes

$$\begin{array}{ccc} H_n([A_{E_1}]) & \xrightarrow{i_*} & H_n([A_{E_2}]) \\ \nu_{A_{E_1}} \downarrow & & \nu_{A_{E_2}} \downarrow \\ H_n(\Delta(G)) \otimes \mathbb{Z}_{m-1} & \xrightarrow{id} & H_n(\Delta(G)) \otimes \mathbb{Z}_{m-1} \end{array}$$

and if $\theta : A_E \to A_{E_1} \oplus A_{E_2}$ is a splitting, with induced map

$$\theta_* : H_n([A_E]) \to H_n([A_{E_1}]) \oplus H_n([A_{E_2}])$$

then the following diagram commutes

$$\begin{array}{ccc} H_n([A_E]) & \xrightarrow{\theta_*} & H_n([A_{E_1}]) \oplus H_n([A_{E_2}]) \\ \nu_{A_E} \downarrow & & \nu_{A_{E_1}} + \nu_{A_{E_2}} \downarrow \\ H_n(\Delta(G)) \otimes \mathbb{Z}_{m-1} & \xrightarrow{id} & H_n(\Delta(G)) \otimes \mathbb{Z}_{m-1} \end{array}$$

By Step 1, $H_n(\mathcal{A};\mathcal{C})$ is the quotient group associated with the groups $H_n([A_E])$ and the family of splitting maps θ_* and inclusion maps i_*. Thus, in view of the above there is a surjection

$$H_n(\mathcal{A};\mathcal{C}) \to H_n(\Delta(G)) \otimes \mathbb{Z}_{m-1}$$

In view of the remarks at the end of Step 2, for example, this surjection must be an isomorphism. □

7. The Partial Isometry Chain Complex Homology

Whilst the definition of the groups $H_n(\mathcal{A};\mathcal{C})$ is suitable for the purpose of direct identifications for particular algebras it has the drawback that it does not present normalising partial isometry homology as the homology of a chain complex. With such a presentation one can more fully exploit the standard techniques of algebraic topology and we illustrate this below.

For a pair $\mathcal{A}, \mathcal{C}$, as before, we form the chain complex $(C_n(\mathcal{A};\mathcal{C}), \partial)$ obtained from the quotients $C_n(\mathcal{A};\mathcal{C})$ in the short exact sequences

$$0 \to Q_n \to \sum_{[A_D]} \oplus C_n([A_D]) \to C_n(\mathcal{A};\mathcal{C}) \to 0$$

where Q_n is the subgroup of the full n-chain group $\sum_{[A_D]} \oplus C_n([A_D])$ (restricted direct sum) associated with *inclusions* and *splittings* of the digraph spaces D. Explicitly, Q_n is generated by elements

$$-g \oplus \theta(g)$$

and

$$-h \oplus \theta_1(h) \oplus \theta_2(h),$$

where $g \in C_n([A_{D_1}])$ and $\theta : C_n([A_{D_1}]) \to C_n([A_{D_2}])$ is induced by an *inclusion* of matrix unit systems, and where $h \in C_n([A_D])$ and

$$\theta_1 + \theta_2 : C_n([A_D]) \to C_n([A_{D_1}]) \oplus C_n([A_{D_2}])$$

is the mapping induced by a *splitting* $u_{ij} = u_{ij}^1 + u_{ij}^2$.

The boundary operators $\partial : C_n([A_D]) \to C_{n-1}([A_D])$ respect inclusions and splittings and so induce group homomorphisms $Q_n \to Q_{n-1}$ and boundary operator $\partial_n : C_n(\mathcal{A};\mathcal{C}) \to C_{n-1}(\mathcal{A};\mathcal{C})$. We define the *($\mathcal{C}-$normalising)-partial isometry chain complex homology* of the pair $(\mathcal{A},\mathcal{C})$ to be the homology $CH_n(\mathcal{A};\mathcal{C})$ of the chain complex $(C_n(\mathcal{A};\mathcal{C}), \partial_n)$, where ∂_0 is the zero map.

Put another way, (Q_n, ∂) is a subcomplex of the direct sum of the chain complexes $(C_n([A_D]), \partial)$, the complex $(C_n(\mathcal{A};\mathcal{C}), \partial)$ is defined to be the associated quotient complex, and the homology CH_* is defined to be the homology of this quotient.

The following theorem, and the analogous Theorem 1, show that H_* and CH_* coincide for digraph algebras.

For a directed graph G of a digraph algebra define the *reduced graph G_r* to be the undirected graph obtained from G through an identification of the vertices v, w of G for which both of the edges (v,w) and (w,v) belong to G.

Theorem 7.1. *If $\mathcal{C}$ is a masa in a digraph algebra $A(G)$ then $CH_n(A(G);\mathcal{C})$ is naturally isomorphic to $H_n(\Delta(G))$ for all n.*

Proof. Let $A \subseteq M_k \otimes A(G)$ be a normalising digraph algebra for $A(G)$. The unitary equivalence class $[A]$ has a representative, which we may take to be A, with partial matrix unit system $\{u_{i,j}\}$ such that each $u_{i,j}$ is an orthogonal sum of standard matrix units $f_{l,m} \otimes e_{p,q}$ in $M_k \otimes A(G)$. Such choices of A give inclusion induced maps

$$i_A : C_n([A]) \to C_n([M_{k(A)} \otimes A(G)]).$$

Let

$$q_k : C_n([M_k \otimes A(G)]) \to C_n(\Delta(G_r))$$

be the natural surjections determined by the identification of equivalent 0-simplexes of $\Delta([M_k \otimes A(G)])$. Then the homomorphism

$$i = \sum_{[A]} \oplus q_{k(A)} \circ i_A : \sum_{[A]} \oplus C_n([A]) \to C_n(\Delta(G_r))$$

maps Q_n to zero and so induces a surjective group homomorphism

$$\theta : C_n(A(G);\mathcal{C}) \to C_n(\Delta(G_r)).$$

On the other hand any natural inclusion $\Delta(G_r) \to \Delta(\overline{G}) = \Delta([A(G)]$, with $\overline{G}$ the undirected graph, which gives a cross-section for q_1, induces a cross-section for θ. By the first paragraph above this cross-section is surjective and so θ is an isomorphism. Moreover θ determines a chain complex isomorphism and so the theorem follows. □

The chain complex homology need not coincide with $H_n(\mathcal{A};\mathcal{C})$. Perhaps the simplest way to see this is to consider a digraph algebra for which $\mathrm{Tor}(H_n(A(G)),\mathbb{Z}_2)$ is nonzero. Then $H_n(A(G)\otimes O_3; D_{|G|}\otimes C) = H_n(\Delta(G))\otimes\mathbb{Z}_2$, by Theorem 2.1 of [17]. On the other hand $CH_n(A(G)\otimes O_3; D_{|G|}\otimes C)$ is computable, by direct methods similar to the proof below, (or by a Kunneth formula), as the homology of the chain complex $(C_n(A(G)\otimes K_0(O_3)), \partial\otimes id)$. Accordingly, by the universal coefficient theorem, the chain complex homology has the extra torsion term. Such a difference also appears in the next theorem.

Theorem 7.2. *Let $A(G)$ be a digraph algebra. Let TO_m^0 and TO_m be the refinement nest subalgebras of the algebraic Cuntz algebra O_m^0 and its closure in the Cuntz algebra O_m, respectively. Then for all $m \geq 1$ and $n \geq 0$ the groups $CH_n(A(G)\star TO_m^0)$ and $CH_n(A(G)\star TO_m)$ coincide with the simplicial homology group $H_n(\Delta(G);\mathbb{Z}_{m-1})$ with coefficients in $\mathbb{Z}_{m-1}$. In particular*

$$CH_n(A(G)\star TO_m) = (H_n(\Delta(G))\otimes\mathbb{Z}_{m-1}))\oplus(Tor(H_{n-1}(\Delta(G)),\mathbb{Z}_{m-1})).$$

Sketch of proof. Let $\mathcal{A} = A(G)*TO_m$. The case for TO_m^o is essentially the same. As in step one of the proof of Theorem 2 obtain the reduction

$$C_n(\mathcal{A};\mathcal{C}) = (\sum_{\mathcal{D}}\oplus C_n([A_D]))/IQ_n$$

where the sum is taken over classes $[A_D]$ associated with elementary tensor systems and where IQ_n is the ideal in this direct sum generated by splitting and inclusions.

For each A_D (with connected digraph) the existence of a homotopy with a digraph subalgebra of $M_N\otimes A(G)\star T_{m^r}$, for some r, leads to a group homomorphism

$$i_D : C_n([A_D]) \to C_n([M_N\otimes A(G)\star T_{m^r}]).$$

This homomorphism depends on the particular homotopy and on N and r. But as before the natural induced maps

$$\tilde{i}_D : C_n([A_D]) \to C_n(\Delta(G_r))\otimes\mathbb{Z}_{m-1}$$

are well defined and are chain maps. The sum

$$i = \sum_{\mathcal{D}}\oplus\,\tilde{i}_D : \sum_{\mathcal{D}}\oplus C_n([A_D]) \to C_n(\Delta(G_r))\otimes\mathbb{Z}_{m-1}$$

respects splittings and inclusions and so gives a chain complex surjection

$$i : (C_n(\mathcal{A};\mathcal{C}),\partial) \to (C_n(\Delta(G_r))\otimes\mathbb{Z}_{m-1},\partial\otimes\mathrm{id}).$$

On the other hand the homotopy reductions show that the cross section of i induced by a natural injection $\Delta(G_r) \to \Delta(G)$ is surjective. Thus i is a chain isomorphism. □

Mayer Vietoris Sequence

Let $\mathcal{C} \subseteq \mathcal{A}_1 \subseteq \mathcal{B}$, $\mathcal{C} \subseteq \mathcal{A}_2 \subseteq \mathcal{B}$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ are subalgebras of the star algebra $\mathcal{B}$. We say that the pair $\mathcal{A}_1, \mathcal{A}_2$ is a *regular pair* if every $M_n \otimes \mathcal{C}$-normalising partial isometry in $M_N \otimes (\mathcal{A}_1 + \mathcal{A}_2)$ can be decomposed as an orthogonal sum of normalising partial isometries in $M_N \otimes \mathcal{A}_1$ and $M_N \otimes \mathcal{A}_2$. There are many contexts providing regular pairs. We shall not go into the details of this but nevertheless we remark that Lemma 5.4 is useful for identifying regular pairs. Furthermore all pairs of subalgebras of an AF C*-algebra B which contain a fixed regular canonical masa are regular pairs. (See Chapter 4 of [13].)

Theorem 7.3. *If $\mathcal{C} \subseteq \mathcal{A}_i \subseteq \mathcal{B}$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ is a regular pair with sum $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ then we have the Mayer Vietoris sequence*

$$.. \to CH_n(\mathcal{A}_1 \cap \mathcal{A}_2; \mathcal{C}) \to CH_n(\mathcal{A}_1; \mathcal{C}) \oplus CH_n(\mathcal{A}_2; \mathcal{C}) \to$$

$$CH_n(\mathcal{A}; \mathcal{C}) \to CH_{n-1}(\mathcal{A}_1 \cap \mathcal{A}_2; \mathcal{C}) \to ..$$

The theorem follows immediately from the excision lemma below and standard algebraic topology. (See Chapter 6 of [20] for example.)

If $\mathcal{C} \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2$, with $(\mathcal{C}, \mathcal{A}_i)$ as before, then we define the *relative homology* $CH_n(\mathcal{A}_2, \mathcal{A}_1; \mathcal{C})$ as the homology of the quotient chain complex

$$(C_n(\mathcal{A}_2; \mathcal{C})/C_n(\mathcal{A}_1; \mathcal{C}), \partial).$$

In particular one has the long exact sequence for the connecting homomorphisms

$$d_n : CH_n(\mathcal{A}_2, \mathcal{A}_1; \mathcal{C}) \to CH_{n-1}(\mathcal{A}_1; \mathcal{C}).$$

Lemma 7.4 (Excision). *For a regular pair $\mathcal{A}_1, \mathcal{A}_2$ as above,*

$$CH_n(\mathcal{A}_1, \mathcal{A}_1 \cap \mathcal{A}_2; \mathcal{C}) = CH_n(\mathcal{A}_1 + \mathcal{A}_2, \mathcal{A}_2; \mathcal{C}).$$

Proof. It is sufficient to show that the inclusion induced chain complex map

$$(C_n(\mathcal{A}_1; \mathcal{C}) + C_n(\mathcal{A}_2; \mathcal{C}), \partial) \to (C_n(\mathcal{A}_1 + \mathcal{A}_2, ; \mathcal{C}), \partial)$$

induces an isomorphism of homology. However if A is a normalising digraph subalgebra of $\mathcal{A}_1 + \mathcal{A}_2$ then, since $\mathcal{A}_1, \mathcal{A}_2$ is a regular pair it follows, from a simple finiteness argument, that A has a splitting $A \to A_1 \oplus \cdots \oplus A_r$ where each A_i is a digraph subalgebra of $\mathcal{A}_1$ or $\mathcal{A}_2$. Thus the inclusion induced map above is already an isomorphism. □

By way of a simple application of the Mayer-Vietoris sequence in our context consider the nontriangular operator algebra

$$\mathcal{A} = \begin{bmatrix} F & 0 & O_m & O_m \\ 0 & F & O_m & O_m \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix}$$

contained in $M_4 \otimes O_m$, which can be viewed as the intersection $\mathcal{A}_1 \cap \mathcal{A}_2$ where $\mathcal{A}_1 = A(D_4) \otimes O_m$ and

$$\mathcal{A}_2 = \begin{bmatrix} F & O_m & O_m & O_m \\ 0 & F & O_m & O_m \\ 0 & 0 & F & O_m \\ 0 & 0 & 0 & F \end{bmatrix},$$

and let $\mathcal{C} \subseteq \mathcal{A}$ be the diagonal $\mathbb{C}^4 \otimes C$. It can be shown that, for reasons of cancellation, $CH_1(\mathcal{A}_2; \mathcal{C}) = 0$. The argument for this, which we leave to the reader, uses defect symmetrisation as in Section 2.

The sum $\mathcal{A}_1 + \mathcal{A}_2$ is $T_4 \otimes O_m$ and so in particular $H_n(\mathcal{A}_1 + \mathcal{A}_2; \mathcal{C}) = 0$, for $n > 0$, either by direct calculation or by a natural Kunneth formula. The pair $\mathcal{A}_1, \mathcal{A}_2$ is a regular pair and, with $\mathcal{C}$ suppressed, the Mayer-Vietoris sequence gives

$$CH_2(\mathcal{A}_1 + \mathcal{A}_2) \to CH_1(\mathcal{A}_1 \cap \mathcal{A}_2) \to CH_1(\mathcal{A}_1) \oplus CH_1(\mathcal{A}_2) \to H_1(\mathcal{A}_1 + \mathcal{A}_2)$$

which is

$$0 \to CH_1(\mathcal{A}_1 \cap \mathcal{A}_2) \to \mathbb{Z}_{m-1} \oplus 0 \to 0$$

Thus, in view of exactness we get, as expected, $CH_1(\mathcal{A}; \mathcal{C}) = \mathbb{Z}_{m-1}$.

Modifying $\mathcal{A}_2$, with TF replacing F, similar arguments show that the triangular algebra

$$\mathcal{E} = \begin{bmatrix} TF & 0 & O_m & O_m \\ 0 & TF & O_m & O_m \\ 0 & 0 & TF & 0 \\ 0 & 0 & 0 & TF \end{bmatrix}$$

also has $CH_1(\mathcal{E}; \mathcal{C}) = \mathbb{Z}_{m-1}$.

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Department of Mathematics and Statistics, Lancaster University, England
s.power@lancaster.ac.uk http://www.maths.lancs.ac.uk/~power/

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