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Irrational Numbers of Constant Type — A New Characterization

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Abstract. Given an irrational number α and a positive integer m, the distinct fractional parts of $\alpha, 2\alpha, \cdots, m\alpha$ determine a partition of the interval $[0,1]$. Defining $d_\alpha(m)$ and $d'_\alpha(m)$ to be the maximum and minimum lengths, respectively, of the subintervals of the partition corresponding to the integer m, it is shown that the sequence $\left(\frac{d_\alpha(m)}{d'_\alpha(m)}\right)_{m=1}^{\infty}$ is bounded if and only if α is of constant type. (The proof of this assertion is based on the continued fraction expansion of irrational numbers.)

Contents

1. Introduction

Let α be a real irrational number, and $\alpha - [\alpha] = \{\alpha\}$ be the fractional part of α (where $[\cdot]$ is the greatest integer function). For $k = 1, 2, \cdots, m$, consider the sequence of distinct points $\{k\alpha\}$ in $[0,1]$, arranged in increasing order:

$$0 < \{k_1\alpha\} < \cdots < \{k_j\alpha\} < \{k_{j+1}\alpha\} < \cdots < \{k_m\alpha\} < 1$$

where $1 \leq k_j \leq m$ for $j = 1, 2, \cdots, m$.

Let $d_\alpha(m)$ and $d'_\alpha(m)$ denote, respectively, the maximum and minimum lengths of the subintervals determined by the above partition of $[0,1]$. Using the continued fraction expansion of α (see Section 2), and the Three Distance Theorem (Theorem 1, Section 3), we obtain a new characterization of irrational numbers of constant type (defined as irrationals with bounded partial quotients). We show in

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Theorem 2 (The Main Theorem, Section 3), that the sequence $\left(\dfrac{d_\alpha(m)}{d'_\alpha(m)}\right)_{m=1}^{\infty}$ is bounded if and only if α is an irrational number of constant type.

Other characterizations of irrational numbers of constant type can be found in the survey article by J. Shallit [3]. In the investigation of certain dynamical systems, Theorem 2 is essential for the formulation of stability criteria for orbits of so-called quantum twist maps [2].

2. Basic Properties of Continued Fractions

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ denote the natural numbers, integers, rational numbers, and real numbers, respectively, and α denotes an irrational number. Proofs of the facts 1 and 2 below can be found in [1, p. 30].

Fact 1. *$\alpha \in \mathbb{R}\backslash\mathbb{Q}$ if and only if α has infinite (simple) continued fraction expansion:*

$$\alpha = [a_0; a_1, a_2, \cdots, a_n, \cdots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for $n \geq 1$. □

Definition 1. An irrational number, α, is of constant type provided there exists a positive number, $B(\alpha)$, such that $B(\alpha) = \sup_{n\geq 1}(a_n) < \infty$. (See reference [3].)

Fact 2. *Define integers p_n and q_n by:*

$$\begin{array}{llll} p_{-1} = 1 & ; \quad p_0 = a_0 & ; \quad p_n = a_n p_{n-1} + p_{n-2} & , \ n \geq 1 \\ q_{-1} = 0 & ; \quad q_0 = 1 & ; \quad q_n = a_n q_{n-1} + q_{n-2} & , \ n \geq 1 \end{array}$$

Then, for $n \geq 0$, $\gcd(p_n, q_n) = 1$, and $0 < q_1 < q_2 < \cdots < q_n < q_{n+1} < \cdots$. Furthermore, $(q_n\alpha - p_n)$ and $(q_{n+1}\alpha - p_{n+1})$ are of opposite sign for all $n \geq 0$. □

Note: $\left(\dfrac{p_n}{q_n}\right)_{n\geq 0}$ are called the principal convergents to α.

Lemma 1. *Define $\eta_n = |q_n\alpha - p_n|$. For all $n \geq 0$, $\eta_{n-1} = a_{n+1}\eta_n + \eta_{n+1}$, and hence, $\eta_n < \eta_{n-1}$.*

Proof. From Fact 2, we have

$$|q_{n-1}\alpha - p_{n-1}| = |(q_{n+1}\alpha - p_{n+1}) - a_{n+1}(q_n\alpha - p_n)|$$

The lemma follows from the fact that $a_n > 0$ for $n \geq 1$, and that $(q_n\alpha - p_n)$ and $(q_{n+1}\alpha - p_{n+1})$ have opposite signs. □

3. The Main Theorem

For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $m \in \mathbb{N}$, the fractional parts, $\{\alpha\}, \{2\alpha\}, \ldots, \{m\alpha\}$, define a partition, $P_\alpha(m)$, of $[0, 1]$:

$$0 = d_0 < d_1 < \cdots < d_j < d_{j+1} < \cdots < d_m < d_{m+1} = 1$$

The maximum and minimum lengths of the subintervals of $P_\alpha(m)$ are denoted, respectively, by

$$\begin{aligned} d_\alpha(m) &:= \max_{0\le i\le m}(d_{i+1}-d_i) \\ d'_\alpha(m) &:= \min_{0\le i\le m}(d_{i+1}-d_i) \end{aligned}$$

For the partition $P_\alpha(m)$, the differences $(d_{i+1}-d_i)$ can be completely characterized [4] in terms of $\eta_n = |q_n\alpha - p_n|$. Collecting the relevant results in reference [4], we have

Theorem 1 (Three Distance Theorem)**.** *Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $m \in \mathbb{N}$.*

- (a) *m can be uniquely represented as $m = rq_k + q_{k-1} + s$, for some $k \ge 0$, $1 \le r \le a_{k+1}$, and $0 \le s < q_k$ (where a_k's are the partial quotients of α and q_k's are given in Fact 2).*
- (b) *For the partition $P_\alpha(m)$, there are $(r-1)q_k + q_{k-1} + s + 1$ subintervals of length η_k, $s+1$ subintervals of length $\eta_{k-1} - r\eta_k$, and $q_k - (s+1)$ subintervals of length $\eta_{k-1} - (r-1)\eta_k$, where the unique integers k, r and s are as in part (a).*

Remark 1. From Theorem 1, we observe

- (a) $\eta_{k-1} - r\eta_k = \eta_{k+1} + (a_{k+1} - r)\eta_k$, by Lemma 1
- (b) $\eta_{k-1} - (r-1)\eta_k = \eta_k + \eta_{k-1} - r\eta_k$
- (c) When $q_k = s+1$, there are no subintervals of length $\eta_{k-1} - (r-1)\eta_k$.

Corollary 1. *For $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, the maximum length, $d_\alpha(m)$, and minimum length, $d'_\alpha(m)$, of the subintervals of partition $P_\alpha(m)$, are given by:*

- (a) *When $q_k > s+1$,*

$$d_\alpha(m) = \begin{cases} \eta_{k+1} + \eta_k & , \ r = a_{k+1} \\ \eta_{k+1} + (a_{k+1} - r + 1)\eta_k & , \ r < a_{k+1} \end{cases}$$

When $q_k = s+1$,

$$d_\alpha(m) = \begin{cases} \eta_k & , \ r = a_{k+1} \\ \eta_{k+1} + (a_{k+1} - r)\eta_k & , \ r < a_{k+1} \end{cases}$$

- (b) *For all $q_k \ge s+1$,*

$$d'_\alpha(m) = \begin{cases} \eta_{k+1} & , \ r = a_{k+1} \\ \eta_k & , \ r < a_{k+1} \end{cases}$$

where k, r, s, a_k, and η_k are as in Theorem 1.

Proof. From Remark 1(a) and Lemma 1 we have,

$$\eta_{k-1} - r\eta_k = \begin{cases} \eta_{k+1} & < \ \eta_k \ , \ r = a_{k+1} \\ \eta_{k+1} + (a_{k+1} - r)\eta_k & > \ \eta_k \ , \ r < a_{k+1} \end{cases}$$

Now, the corollary follows from Theorem 1, Remark 1(b) and Remark 1(c). □

Theorem 2 (Main Theorem)**.** *Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $m \in \mathbb{N}$, and let $d_\alpha(m)$, $d'_\alpha(m)$ be, respectively, the maximum and minimum lengths of the subintervals of the partition $P^\alpha(m)$. The sequence $\left(\dfrac{d_\alpha(m)}{d'_\alpha(m)}\right)_{m=1}^{\infty}$ is bounded if and only if α is an irrational number of constant type.*

Proof. Let $m = rq_k + q_{k-1} + s$, where k, r, and s are the unique integers given by Theorem 1. From Corollary 1 and Lemma 1, we have

$$\frac{d_\alpha(m)}{d'_\alpha(m)} = \begin{cases} \epsilon + \dfrac{\eta_{k+2}}{\eta_{k+1}} + a_{k+2} & , \; r = a_{k+1} \\ \\ \epsilon + \dfrac{\eta_{k+1}}{\eta_k} + (a_{k+1} - r) & , \; r < a_{k+1} \end{cases}$$

where $\epsilon = 1$ for $q_k > s + 1$ and $\epsilon = 0$ for $q_k = s + 1$.

(a) If α is of constant type (Definition 1), then the partial quotients, a_n, of α, satisfy $a_n \leq B(\alpha) < \infty$ for all $n \geq 1$. Since $\dfrac{\eta_{j+1}}{\eta_j} < 1$ for all $j \geq 0$ (by Lemma 1), $\dfrac{d_\alpha(m)}{d'_\alpha(m)} < B(\alpha) + 2$ for all $m \in \mathbb{N}$. Hence, $\left(\dfrac{d_\alpha(m)}{d'_\alpha(m)}\right)_{m=1}^{\infty}$ is bounded.

(b) Suppose $\dfrac{d_\alpha(m)}{d'_\alpha(m)} < B_0$ where $0 < B_0 < \infty$ for all $m \in \mathbb{N}$. In particular, for $m = q_{k+1}$ [corresponding to $r = a_{k+1}$, $s = 0$], we have $\dfrac{d_\alpha(q_{k+1})}{d'_\alpha(q_{k+1})} = \epsilon + \dfrac{\eta_{k+2}}{\eta_{k+1}} + a_{k+2} < B_0$ for all $k \geq 0$. Hence, $a_{k+2} < B_0$ for all $k \geq 0$. Setting $B = \max\{B_0, a_1\}$, we have $a_n \leq B$ for all $n \geq 1$, and hence α is of constant type. □

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References

[1] S. Drobot, *Real Numbers*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964

[2] G. Karner, *On quantum twist maps and spectral properties of Floquet operators*, Ann. Inst. H. Poincaré A, **68** (1998), to appear.

[3] J. Shallit, *Real numbers with bounded partial quotients: a survey*, Enseign. Math., **38** (1992), 151–187.

[4] N. B. Slater, *Gaps and steps for the sequence $n\theta$ mod 1*, Proc. Camb. Phil. Soc., **63** (1967), 1115–1123.

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